Angular momentum and mass formulas for rotating stationary quasiblack holes

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We consider the quasiblack hole limit of a stationary body when its boundary approaches its own gravitational radius, i.e., its quasihorizon. It is shown that there exists a perfect correspondence between the different mass contributions and the mass formula for quasiblack and black holes in spite of the difference in derivation and meaning of the formulas in both cases. For extremal quasiblack holes the finite surface stresses give zero contribution to the total mass. Analogous properties are derived for the angular momentum.

I. INTRODUCTION

In [1] we found the mass formula for static quasiblack holes. There, we have defined a quasiblack hole as the limiting configuration of a body, either nonextremal or extremal, when its boundary approaches the body's own gravitational radius. This definition is an enlargement from a previous definition [2] (see also [3–7]), which applied to configurations with extremal matter (i.e., with mass density equal to charge density in appropriate units) and spherically symmetric, to a generic static case with no particular symmetry neither some specific matter. In particular, the conclusion made in [2], was that static quasiblack holes, where matter has finite surface stresses, should be extremal, where the condition of finiteness of stresses is important for the conclusion. Then, in [1] the appearance of infinite stresses was allowed, and the condition of extremal matter could be dropped, i.e., nonextremal matter was also admitted. Notably, the mass formula found in [1], also implies, when the derivation of the formula is taken into account, that for finite stresses, quasiblack holes must be extremal. For the static extremal case, the existence of a quasiblack hole requires electric charge, or some other form of repulsive matter, such as in gravitational monopoles [8,9]. Although, the extremal condition in the electrical case may be achieved if a tiny fraction, 10^{-18}, of neutral hydrogen loses its electron, the requirement of charge somewhat bounds the astrophysical significance of such quasiblack holes. Dropping the extremal condition for the matter, infinite stresses appear at the quasiblack hole threshold, which makes these nonextremal objects quite unphysical, although as argued in [1], consideration of such systems has at least a systematic interest since it helps to understand better the distinction between nonex-tremal and extremal limits, and the relationship between quasiblack holes and black holes. The related issue of gravitational collapse to a quasiblack hole state, always an important problem, was treated preliminary in [10].

Now, the rotational counterpart of quasiblack holes were found first by Bardeen and Wagoner back in 1971 [11]. They discussed rotating thin disks and found that for rotation less than extremal, the exterior metric does not yield a Kerr vacuum spacetime, but for extremal rotation (i.e., mass equal to angular momentum per unit mass) of the disk, and in this case only, the exterior metric is the extremal Kerr metric. Thus, they were the first to find a situation in which matter in equilibrium can approach its own horizon, now called a quasihorizon. Such systems are precisely quasiblack holes. Recently, these rotational counterparts of quasiblack holes were further considered by Meinel [12], although the term quasiblack hole was still not coined there (see also [13,14]). Rotating objects have astrophysical relevance, so it is certainly of interest to consider the rotating versions of quasiblack holes. For a distant observer, such rapidly rotating bodies would look almost indistinguishable from black holes.

The paper of Meinel [12] is an important development of the subject, and it contains a very strong claim that should be further explored. On the basis of an analysis of the mass formulas alone, Meinel [12] argued that the only suitable candidate to the role of a limiting configuration (i.e., a quasiblack hole, or a body that approaches its own gravitational radius) corresponds to the extremal case. So the conclusion made in our work [2], that static quasiblack holes should be extremal, relying heavily on the properties of the finiteness of the surface stresses that arise in the quasiblack hole limit, and also through the mass formula afterward [1], have a seemingly analogous statement in the rotating stationary extremal case. The conclusion drawn in [12] was inferred only from the formula for the mass, and moreover, surface stresses were not taken into account at

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all. However, with the know-how one can take from the static case [1,2], one is led to consider these stresses in order to be able to make more general statements. Moreover, we will see that without an appropriate account for the stresses the analysis would remain essentially incomplete. Thus, bearing in mind both the theoretical interest of stationary configurations on the threshold of the formation of a horizon and their potential astrophysical significance, one should also allow for surface stresses, either finite or infinite, in the stationary case.

In this paper, we thus consider stationary configurations with surface stresses which can either be finite or grow without bound when the rotating quasiblack hole is being formed. In this sense, the statements in [12] are generalized. We extend further the analysis and consider configurations not only with mass and angular momentum [12], as well as surface stresses, but also with electrical charge. We find the angular momentum and mass formulas for this general charged stationary configuration. We also elucidate what makes the extremal state in the stationary case a distinguished configuration.

As in the static case [1], the two issues, namely, the relation between surface stresses and the mass formula for quasiblack holes, on the one hand, and the mass formulas for quasiblack holes and pure black holes, on the other hand, are interconnected. We argue there is a close correspondence between the mass formulas for quasiblack holes and black holes in all cases, nonextremal and extremal, despite the fact that the physical nature of these objects (see [2]) and the derivation of the mass formula itself are quite different. This, of course, makes the encountered close relationship between the formulas nontrivial. Our analysis has also rather unexpected consequences for the general relativistic counterpart of the classical Abraham-Lorentz model for the electron, connected with the distinguished role played by the stationary quasihorizon in the extremal case.

II. METRIC FORM FOR ROTATING STATIONARY CONFIGURATIONS

A. Metric form and the definition of a stationary quasiblack hole

Let us have a distribution of matter in a gravitational field which does not depend on time. Put the four-dimensional spacetime metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, with μ and ν being spacetime indices, in the form

$$ds^2 = -N^2dt^2 + g_{ik}(dx^i + N^i dt)(dx^k + N^k dt), \quad (1)$$

where, we use 0 as a time index, and $i, k = 1, 2, 3$ as spatial indices. In addition, N and N^i are the lapse function and shift vector which depend in general on the spatial coordinates x^i.

From (1), the metric of a stationary axially symmetric system can be written in a useful form by putting $N^3 =$

$N^\phi = -\omega$, where ϕ is the azimuthal coordinate and ω an angular velocity, and the other N^i obey $N^i = 0$. We denote the radial coordinate by l and put the radial potential $g_{ll} = 1$. If we further define a cylindrical coordinate z, the metric can be written in the form

$$ds^2 = -N^2dt^2 + dl^2 + g_{zz}dz^2 + g_{\phi\phi}(d\phi - \omega dt)^2, \quad (2)$$

an axially symmetric form, where the metric coefficients depend on l and z [15].

In [1] we extended the definition of a quasiblack hole from the spherically symmetric and extremal case [2] to a generic static case. Now, we extend it further to a stationary spacetime. Several points of [1] are repeated with the reservation that now $g_{00} \neq -N^2$ due to the terms responsible for rotation. Namely, consider a configuration depending on a parameter ε such that (a) for small but nonzero values of ε the metric is regular everywhere with a nonvanishing lapse function N, and at most the metric contains only deltalike shells, (b) taking as ε the maximum value of the lapse function on the boundary N_B, then in the limit $\varepsilon \to 0$ one has that the lapse function $N \leq N_B \to 0$ everywhere in the inner region, (c) the Kretschmann scalar Kr remains finite in the quasihorizon limit. This latter property implies another important property which can be stated specifically, namely, (d) the area A of the two-dimensional boundary $l = $ const attains a minimum in the limit under consideration, i.e., $\lim_{\varepsilon \to 0} \frac{\partial A}{\partial l}\big|_{l^*} = 0$, where l^* is the value of l at the quasihorizon. In addition, now we also require that in the limit under discussion $\omega \to \omega_h = $ const everywhere in the inner region. Here ω_h corresponds to the angular velocity of a black hole to which the quasiblack hole metric tends outside. Without this property, the differential rotation inside would serve to distinguish a black hole and quasiblack hole metrics and, thus, the definition of a quasiblack hole would not have physical meaning. The constancy of ω_h is a known property of black holes and can be substantiated by the regularity of the curvature invariants [15]. It is worth also mentioning that the system under consideration can in general represent either a compact body with a well-defined junction to an electrovacuum solution, or a dispersed distribution of matter.

B. Other discussions

For a situation in which the body's surface approaches the would-be horizon (quasihorizon), we take advantage of the asymptotics of the lapse function N and the function ω near the horizon [15]. Then, for the nonextremal case, approximating the metric in the outer region by that of a black hole, we have the following relations:

$$N = \kappa l + O(l^3), \quad (3)$$

and

$$\omega = \omega_h + O(l^2), \quad (4)$$

where κ is the surface gravity at the horizon obeying $\kappa = \text{constant} \neq 0$, and ω_h is the horizon value of ω obeying $\omega_h = \text{constant}$. For the extremal case $\kappa = 0$ similarly to the static case and $N \sim \exp(-Bl)$, $B = \text{constant}$ [1]. For the ultraextremal case [1], one has $N \sim l^{-n}$ and $\kappa = 0$. In both these two cases we assume that near the horizon

$$\omega = \omega_h + a_1 N + a_2 N^2 + \cdots, \qquad (5)$$

where ω_h and $a_1, a_2, \ldots$ are constants.

Two reservations are in order. First, the relevance of the asymptotics in Eq. (5) in our context should follow from the analysis of the near-horizon behavior of the scalars (such as Ricci, Kretschmann, and other scalars), composed out of the curvature components. Such an analysis was performed in [15] for the nonextremal case, only partially for the ultraextremal one, and not at all for the extremal case. Strictly speaking, the necessity of the asymptotics (5) was not proved formally in [15] for extremal and ultra-extremal horizons and remains a gap to be filled. However, its derivation represents a formal problem on its own that would take us far afield. Therefore, we simply assume the validity of the Taylor expansion given in Eq. (5). Second, Eq. (5) is assumed to be an expansion with respect to the quasihorizon for the outer region. On the other hand, we assume (as explained at the end of Sec. II A) that $\omega \to \omega_h = \text{const}$ everywhere in the inner region. As a result, there is a jump of the normal derivative $\frac{\partial \omega}{\partial l}$ in the quasihorizon limit for the nonextremal case. This is similar to what happens to the lapse function [1].

III. THE ANGULAR MOMENTUM AND MASS FORMULAS FOR THE STATIONARY CASE: A STATIONARY AXIALLY SYMMETRIC ROTATING CONFIGURATION SPACETIME

If the matter is joined onto a vacuum spacetime, then one has to be careful and use the junction condition formalism [16,17]. The angular momentum and mass of the matter distribution can be written as integrals over the region occupied by matter and fields. Defining T^ν_μ as the stress-energy tensor, the momentum J_i relative to a coordinate x^i is given by

$$J_i = - \int T^0_i \sqrt{-g} d^3 x, \qquad (6)$$

where g is the determinant of the metric $g_{\mu\nu}$. When the coordinate x^i is angular and cyclic ϕ, say, then J_ϕ is an angular momentum and one puts $J_\phi \equiv J$ (see, e.g., [18]). The mass of the matter distribution can be written as an integral over the region occupied by matter and fields. It is given by the Tolman formula [19] (see also [18,20])

$$M = \int (-T^0_0 + T^k_k) \sqrt{-g} d^3 x. \qquad (7)$$

This is the starting point of our analysis. We discuss these integrals for an axially symmetric rotating matter distribu-

tion in an axially symmetric rotating spacetime. In summary, we consider the stationary case, generalizing the static case discussed in a previous paper [1]. For the angular momentum and mass formulas for black holes, rather than quasiblack holes, see [21–24] and, particularly, [25] for the angular momentum formula.

A. The various angular momenta and masses

We consider the stationary case, with axial symmetry. We assume that the body has a well-defined quasiblack hole limit.

1. Total angular momentum, and total mass

Let us have a distribution of matter and a gravitational field which do not depend on time. Note also from Eq. (2) that $\sqrt{-g} = N\sqrt{g_3}$, where g_3 is the determinant of the spatial part of the metric (2), i.e., is the determinant of the metric on the hypersurface $t = \text{constant}$. We consider first the angular momentum. Then from Eq. (6), the total angular momentum J is given by

$$J = - \int T^0_\phi N \sqrt{g_3} d^3 x. \qquad (8)$$

Then, the total value of the angular momentum (8) can be split into three contributions, the inner, the surface, and the outer, such that

$$J_{\text{tot}} = J_{\text{in}} + J_{\text{surf}} + J_{\text{out}}. \qquad (9)$$

Next, we consider the mass, which can be written as an integral over the region occupied by matter and fields,

$$M = \int (-T^0_0 + T^k_k) N \sqrt{g_3} d^3 x. \qquad (10)$$

From Eq. (10) it is again convenient here to compose the linear split of the total mass into three different contributions, the inner, the surface mass, and the outer masses, such that

$$M_{\text{tot}} = M_{\text{in}} + M_{\text{surf}} + M_{\text{out}}. \qquad (11)$$

Note that for the outer mass a long-range electromagnetic field may be present.

2. Inner angular momentum and mass

As in the static case [1], one has for a quasiblack hole that $N_B \to 0$, where N_B is the value of N at the boundary as well as $N \to 0$ for the whole inner region. So, the inner contribution to the angular momentum vanishes in the quasiblack hole limit due to the factor N, i.e.,

$$J_{\text{in}} = 0. \qquad (12)$$

For the same reasons, and analogously to the static case, the inner contribution to the mass vanishes:

$$M_{\text{in}} = 0. \qquad (13)$$

3. Surface angular momentum and mass

Now consider the contribution of the surface to the angular momentum and mass. First, the angular momentum. One has

$$J_{\text{surf}} = - \int_{\text{surface}} T^0_\phi N \sqrt{g_3} d^3 x. \tag{14}$$

Defining γ as

$$\gamma = -\frac{1}{2N^2} g_{\phi\phi} \frac{\partial \omega}{\partial l}, \tag{15}$$

we can put Eq. (14) in the form

$$J_{\text{surf}} = \frac{1}{8\pi} \int \gamma N d\sigma, \tag{16}$$

where $d\sigma$ is the two-dimensional surface spanned by $t =$ constant, $l =$ constant. Now, for the pure black hole case, the angular momentum of the horizon is equal to [25]

$$J_{\text{h}} = -\frac{1}{8\pi} \int_{\text{horizon}} \xi^{\mu;\nu} d\sigma_{\mu\nu}, \tag{17}$$

where the integration is taken over the horizon surface with element $d\sigma_{\mu\nu}$, and ξ^μ are the components of the rotational Killing vector ξ, which is given by $\xi = \frac{\partial}{\partial \phi}$, and a semicolon denotes covariant derivative (see, e.g., [25]). One can now show that in the quasiblack hole limit, Eq. (16) reduces to Eq. (17). Indeed, taking a cross section of the metric (2) such that $t =$ constant and $l =$ constant, and developing expression (17) explicitly, one finds that in the quasiblack hole limit (16) coincides exactly with (17), so that

$$J_{\text{surf}} = J_{\text{h}}, \tag{18}$$

where J_{h} should now be interpreted as the angular momentum of the quasiblack hole. For the nonextremal case it is finite and, in general, nonzero. For the extremal case it is also finite and in general nonzero, as it follows from (5) and from $N \sim \exp(-Bl)$ as $l \rightarrow \infty$. Only in some special extremal configurations the surface stresses vanish (see, e.g., the example of the spherically symmetric static system composed of extremal charged dust in [6] and references therein). For the ultraextremal case, defined above, assuming the validity of the asymptotic expansion (5) one finds that the surface contribution to the angular momentum vanishes.

Now consider the contribution of the surface to the mass

$$M_{\text{surf}} = \int_{\text{surface}} (-T^0_0 + T^k_k) N \sqrt{g_3} d^3 x. \tag{19}$$

As in the static case there are deltalike contributions, given by

$$S^\nu_\mu = \int T^\nu_\mu dl, \tag{20}$$

where the integral is taken across the shell. Define α as

$$\alpha = 8\pi (S^a_a - S^0_0). \tag{21}$$

Then, from a combination of the equations above, we get

$$M_{\text{surf}} = \frac{1}{8\pi} \int \alpha N d\sigma, \tag{22}$$

where $d\sigma$ is the surface element. Now, one also has the relationship $8\pi S^\nu_\mu = [[K^\nu_\mu]] - \delta^\nu_\mu [[K]]$, where K^ν_μ is the extrinsic curvature tensor, $[[\ldots]] = [(\ldots)_+ - (\ldots)_-]$, subscripts "+" and "−" refer to the outer and inner sides, respectively (see, e.g., [16,17]). Also, $K_{\mu\nu} = -n_{\mu;\nu}$, where at the boundary surface $N =$ constant, and the normal unit vector is $n_\mu \sim N_{;\mu}$. Thus, $\alpha = -[[2K^0_0]]$, and further calculations give

$$\alpha = \frac{2}{N}\left[\left(\frac{\partial N}{\partial l}\right)_+ - \left(\frac{\partial N}{\partial l}\right)_-\right] + \frac{1}{N^2} g_{\phi\phi}(\omega - \omega_{\text{h}}) \frac{\partial \omega}{\partial l}, \tag{23}$$

and so

$$M_{\text{surf}} = \frac{1}{4\pi} \int_{\text{surf}} \left[\left[\left(\frac{\partial N}{\partial l}\right)_+ - \left(\frac{\partial N}{\partial l}\right)_-\right] + \frac{2}{N} g_{\phi\phi} \omega \frac{\partial \omega}{\partial l}\right] d\sigma. \tag{24}$$

Now, as our surface approaches the would-be horizon, i.e., the quasihorizon, we take advantage of the asymptotics near the horizon of the lapse function N and of the function ω. Thus, taking into account expression (16) and the asymptotics (3) and (4) in the nonextremal case, or Eq. (5) in the extremal or ultraextremal cases, we obtain

$$M_{\text{surf}} = \frac{\kappa A_{\text{h}}}{4\pi} + 2\omega_{\text{h}} J_{\text{h}}, \tag{25}$$

where κ is the surface gravity of the quasiblack hole. So, in relation to the contribution of the surface stresses to the mass, what was said in the static case [1] applies here to the first term of Eq. (25). Namely, in the nonextremal case the stresses are infinite but their contribution is finite and nonzero, in the extremal case they are finite but their contribution vanishes, and in the ultraextremal case the stresses themselves are zero, so the contribution to the mass is zero as well. As far as the second, new, term in (25) is concerned, it follows that the surface contribution is nonzero for the nonextremal and extremal cases but vanishes in the ultraextremal one. Note also that, although for the nonextremal ($\kappa \neq 0$) case on one hand and for the extremal and ultraextremal ($\kappa = 0$) cases on the other, we have used different asymptotics of the metric coefficients near the quasihorizon, the smooth limiting transition $\kappa \rightarrow 0$ can be made in the formula (25) for the surface contribution as a whole, surely. Since Eq. (25) shows clearly that one cannot ignore surface stresses contribution in the nonextremal case, the analysis in [12] is incomplete. It omits from the very beginning just the most important feature of nonextremal configurations in their confronta-

tion with the extremal ones. This means the final conclusion of [12] hangs in midair. That is, one could naïvely think that one could simply restrict oneself to the case of vanishing stresses but in the problem under discussion this is impossible. Indeed, we have just seen that the stresses enter the mass formulas via the quantity α, so in the case of vanishing stresses M_{surf} would also vanish. But this does not happen.

4. Outer angular momentum and mass

The outer angular momentum is given generically by the expression

$$J_{\text{out}} = -\int_{\text{outer}} T_\phi^0 N \sqrt{g_3} d^3 x. \tag{26}$$

The outer mass is given generically by the expression

$$M_{\text{out}} = \int_{\text{outer}} (-T_0^0 + T_k^k) N \sqrt{g_3} d^3 x. \tag{27}$$

Further, we may split M_{out} into an electromagnetic part $M_{\text{out}}^{\text{em}}$, and a nonelectromagnetic part, $M_{\text{out}}^{\text{matter}}$, say, for the case of dirty black holes or dirty quasiblack holes, exactly in the manner as it was already done in [22], and obtain $M_{\text{out}} = M_{\text{out}}^{\text{em}} + M_{\text{out}}^{\text{matter}}$. Since $M_{\text{out}}^{\text{em}} = \varphi_h Q$ (see [1] for details), where φ_h is the electric potential on the horizon in the case of black holes, and the electric potential on the quasihorizon in the case of quasiblack holes, and Q is the corresponding electric charge, one finds

$$M_{\text{out}} = \varphi_h Q + M_{\text{out}}^{\text{matter}}. \tag{28}$$

B. The angular momentum and mass formulas

Putting all together, for the quasiblack hole case, and recalling that J_{in} goes to zero, the total angular momentum is equal to

$$J = J_h + J_{\text{out}}. \tag{29}$$

In vacuum, if matter is absent or negligible outside, we have only J_h; i.e., the total angular momentum is the quasiblack hole angular momentum.

In a similar way, recalling that M_{in} goes to zero, we find the total mass is equal to

$$M = \frac{\kappa A_h}{4\pi} + 2\omega_h J_h + \varphi_h Q + M_{\text{out}}^{\text{matter}}. \tag{30}$$

Equation (30) is the mass formula for stationary quasiblack holes. But on closer inspection it is nothing else than the mass formulas for black holes [21–25]. Note that for the extremal case, the term $\frac{\kappa A_h}{4\pi}$ in Eq. (30) goes to zero, since κ is zero. In vacuum, if matter is absent or negligible outside, we return to

$$M_h = \frac{\kappa A_h}{4\pi} + 2\omega_h J_h + \varphi_h Q, \tag{31}$$

which is Smarr's formula, but now for quasiblack holes;

i.e., Eq. (31) is equal to a formula first found by Smarr for Kerr-Newman black holes [24]. Here we see it holds good for rotating quasiblack holes as well. It is also worth noting that in our approach we did not restrict ourselves to a compact body rotating with a constant angular velocity in vacuum as it was done in [12]. Instead, we have admitted all types of rotation, including differential and rigid rotations, as well as matter distribution outside the quasihorizon. Now, in the context of the uniqueness theorems, it is specially interesting to trace how the configuration of a self-gravitating rotating body approaches an outer vacuum Kerr-Newman metric. In this context, by allowing infinite surface stresses we can conjecture, resorting to the uniqueness theorems (see, e.g., [25]) and continuity arguments between a horizon and a quasihorizon with outer vacua, that the generic Kerr-Newman metric, and so the Kerr metric, is an outer metric for some type of matter that allows infinite stresses. In addition, Eqs. (30) and (31) reduce to the static case considered in [1] for $\omega_h = 0$.

Consider, as an example, the case where there is only rotation and no electrical field nor matter in the outer region. Thus, the exterior to the quasiblack hole is described by the extremal Kerr metric. Then $\kappa = 0$, and $M_h = 2\omega_h J_h$. Also see [1] for the case $\omega_h = 0$ and the example for the charged static case.

Thus, we have traced how the total mass of a quasiblack hole, which can be defined at asymptotical infinity as usual [26,27], is distributed among different terms including the contribution from the quasihorizon. We have found perfect correspondence with the black hole case.

IV. CONCLUSIONS

There are three main topics and conclusions that can be taken out of our results: (i) With rotation and charge the problem of a self-consistent analog of an elementary particle in general relativity is much more interesting than without rotation. If one wants a classical model for the electron, one certainly should look for including rotation; see [1,28,29] for the static case (see also [26,27]). As a byproduct, we have obtained that an extremal quasiblack hole can serve as a classical model of an Abraham-Lorentz electron in that both the inner and surface contribution of nonelectromagnetic forces vanish. In doing so, we showed that one may weaken the requirement of vanishing surface stresses since the finite stresses have zero contribution to the total mass. (ii) Here we have traced how the limiting transition from a stationary configuration to the quasiblack hole state reveals itself in the mass formula, going thus beyond the static case [1] and beyond what was found in [12] for a particular set of stationary configurations (see also [13,14]). It turns out that the perfect one-to-one correspondence between the different contributions for the total mass of a quasiblack hole and the mass formula for black holes persists in the generic stationary case. In particular, the inner contribution to the total mass vanishes

in the quasiblack hole limit (it is absent in the black hole case from the very beginning). The contribution of the surface stresses corresponds just to the contribution from the horizon surface of a black hole. This is not trivial, since the corresponding terms have quite different origins. In the quasiblack hole case they are due to the boundary between both sides of the surface. Meanwhile, in the black hole case only one (external) side is relevant and the integrand over this surface has nothing to do with the expression for surface stresses. Nonetheless, both terms coincide in the limit under discussion. Similar results were obtained for the angular momentum of the rotating configurations. As bodies with rotation occur widely in nature, the results obtained may have astrophysical implications. (iii) The difference between nonextremal and extremal quasiblack holes consists in that in the first case the surface stresses give finite contribution to the total mass, but become infinite, while in the second case they give zero contribution to the total mass, but are finite. As far as the mass is concerned, in the nonextremal case the surface of a quasiblack hole appears in a way similar to a membrane in the membrane paradigm setup [30], whereas in the extremal one we have in general a ''membrane without membrane'' [31]. The system with infinite stresses was rejected in [2], since it looks unphysical, and thus in [2] only extremal black holes were considered. However, consideration of such systems helps in understanding better the relationship between quasiblack holes and black holes and the distinction between nonextremal and extremal limits. With its astrophysical as well as theoretical importance, the rotating case, as was discussed here, acquires added relevance.

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[1] J. P. S. Lemos and O. B. Zaslavskii, Phys. Rev. D **78,** 124013 (2008).

[2] J. P. S. Lemos and O. B. Zaslavskii, Phys. Rev. D **76,** 084030 (2007).

[3] J. P. S. Lemos and O. B. Zaslavskii, Phys. Rev. D **78,** 024040 (2008).

[4] J. P. S. Lemos and E. J. Weinberg, Phys. Rev. D **69,** 104004 (2004).

[5] J. P. S. Lemos and V. T. Zanchin, J. Math. Phys. (N.Y.) **47,** 042504 (2006).

[6] J. P. S. Lemos and V. T. Zanchin, Phys. Rev. D **77,** 064003 (2008).

[7] O. B. Zaslavskii, Phys. Rev. D **70,** 104017 (2004).

[8] A. Lue and E. J. Weinberg, Phys. Rev. D **60,** 084025 (1999).

[9] A. Lue and E. J. Weinberg, Phys. Rev. D **61,** 124003 (2000).

[10] C. Barcelo, S. Liberati, S. Sonego, and M. Visser, Phys. Rev. D **77,** 044032 (2008).

[11] J. M. Bardeen and R. V. Wagoner, Astrophys. J. **167,** 359 (1971).

[12] R. Meinel, Classical Quantum Gravity **23,** 1359 (2006).

[13] R. Meinel, Ann. Phys. (Leipzig) **13,** 600 (2004).

[14] R. Meinel, in *Proceedings of the Eleventh Marcel Grossmann Meeting on General Relativity*, edited by H. Kleinert and R. T. Jantzen (World Scientific, Singapore, 2008), p. 2234; arXiv:gr-qc/0612011.

[15] A. J. M. Medved, D. Martin, and M. Visser, Phys. Rev. D **70,** 024009 (2004).

[16] W. Israel, Nuovo Cimento B **44,** 1 (1966).

[17] C. W. Misner, K. S. Thorne, and J. A. W. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Sec. 21.13.

[18] E. Poisson, *A Relativist's Toolkit* (Cambridge University Press, Cambridge, England, 2004), Sec. 4.3.

[19] R. Tolman, *Relativity, Thermodynamics and Cosmology* (Dover, New York, 1987), Sec. 92 (first published in 1934, by Clarendon Press).

[20] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Butterworth-Heinemann, Oxford, 1980), 4th ed., Sec. 105.

[21] J. M. Bardeen, B. Carter, and S. W. Hawking, Commun. Math. Phys. **31,** 161 (1973).

[22] B. Carter, in *Black Holes*, edited by C. DeWitt and B. S. DeWitt (Gordon and Breach, New York, 1973), p. 57.

[23] J. M. Bardeen, in Black Holes (Ref. [22]), p. 241.

[24] L. Smarr, Phys. Rev. Lett. **30,** 71 (1973).

[25] V. P. Frolov and I. D. Novikov, *Black Hole Physics: Basic Concepts and New Developments* (Kluwer Academic, Amsterdam, 1998), Sec. 12.2.

[26] R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **120,** 321 (1960).

[27] A. P. Lundgren, B. S. Schmekel, and J. W. York, Phys. Rev. D **75,** 084026 (2007).

[28] A. V. Vilenkin and P. I. Fomin, Nuovo Cimento Soc. Ital. Fis. **45A,** 59 (1978).

[29] O. B. Zaslavskii, Phys. Rev. D **70,** 104017 (2004).

[30] K. S. Thorne, R. H. Price, and D. A. Macdonald, *Black Holes: The Membrane Paradigm* (Yale University Press, Yale, 1986).

[31] J. A. Wheeler, *Geometrodynamics* (Academic Press, New York, 1962).