Quadrupole moment of slowly rotating fluid balls

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In this paper we use the second order formalism of Hartle to study slowly and rigidly rotating stars with focus on the quadrupole moment of the object. The second order field equations for the interior fluid are solved numerically for different classes of possible equations of state and these solutions are then matched to a vacuum solution that includes the general asymptotically flat axisymmetric metric to second order, using the Darmois-Israel procedure. For these solutions we find that the quadrupole moment differs from that of the Kerr metric, as has also been found for some equations of state in other studies. Further we consider the post-Minkowskian limit analytically. In the paper we also illustrate how the relativistic multipole moments can be calculated from a complex gravitational potential.

I. INTRODUCTION

Sources for the Kerr metric have been sought for long, but with little success. The only known exact solution that can be matched to the Kerr metric seems to be the rotating dust disk, with maximal allowed angular momentum, found by Neugebauer and Meinel [1]. Some numerical studies with different equations of state for the source find quadrupole moments differing from that of the Kerr metric; see, e.g., [2–5]. In some recent papers [6,7] it is shown that in the rigidly rotating case incompressible fluids and fluids with a polytropic equation of state cannot be sources of the Kerr metric in the post-Minkowskian limit. These results are in accordance with the general expectation that the ellipsoidal shape of the rotating fluid ball produces an extra contribution to the quadrupole moment which should also be present in the corresponding quadrupole moment of the external field [4,8]. For a general review of relativistic rotating stars see [9].

In the present work we use the second order formalism for slowly and rigidly rotating stars, developed by Hartle [10], to study the quadrupole moment and its deviation from that of the Kerr metric for different classes of possible equations of state. For some earlier applications of the formalism see, e.g., [2,3,11] and for a comparison with numerical solutions of the full Einstein equations see [4]. The relativistic multipole moments of the vacuum exterior metric up to order two are calculated using an algorithm developed in [12].

The field equations for the fluid region to second order in the small rotational parameter Ω will be solved numerically using fourth order Runge-Kutta. When imposing an equation of state this system can be rewritten as a first order system of ordinary differential equations for nine functions, but a subsystem for six of these functions will be sufficient to study for our purposes. Assuming regularity at the center the solutions will depend on three constants of integration, corresponding to zeroth order central density or pressure, the magnitude of the angular velocity and one more second order small constant. If the solutions are required to be asymptotically flat this second order constant will be determined in terms of the other constants. Because of scaling invariance in the angular velocity it may be given a fixed value in the numerical runs. Hence, given an equation of state, we need only vary the central pressure or density when scanning the solution space. The solutions are then matched to a second order axisymmetric vacuum solution using the Darmois-Israel procedure [13–15]. This metric includes the general second order asymptotically flat stationary axisymmetric vacuum solution as a special case.

We also consider the post-Minkowskian limit analytically by expanding the field equations in the small parameter $\lambda \equiv GM/r_1c^2$ and make a comparison with the results of [6,7].

The paper is organized as follows: In Sec. II the method is briefly described and the field equations are presented. Also the second order vacuum metric is given and its relativistic multipole moments up to order two are calculated. The matching procedure is described in Sec. III and the integration constants for the vacuum solution are solved for in terms of the values of the interior solution on the matching surface. In Sec. IV the equations are rewritten in a form suitable for numerical integration. The results of the numerical runs are given in Sec. V and finally a post-Minkowskian analysis is made in Sec. VI.

II. PRELIMINARIES

To second order the metric of a slowly rotating axisymmetric object, both in the interior fluid region and the outside vacuum region, can be written as

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$$ds^2 = (1 + 2h)A^2 dt^2 - (1 + 2m)\frac{1}{B^2}dr^2$$
$$- (1 + 2k)r^2[d\theta^2 + \sin^2\theta(d\varphi - \omega dt)^2], \quad (1)$$

where ω is first order and h, m and k are second order in the rotational parameter [10]. The requirements of regularity at the center and asymptotic flatness imply that the first order function ω depends on r only. The second order functions h, m, k can be given as

$$h = h_0 + h_2 P_2(\cos\theta), \qquad m = m_0 + m_2 P_2(\cos\theta),$$
$$k = k_2 P_2(\cos\theta), \quad (2)$$

where h_0, m_0 and h_2, m_2, k_2 are functions of r only and $P_2(x) = \frac{1}{2}(3x^2 - 1)$ is the second order Legendre polynomial. This result follows from reflection symmetry in the equatorial plane, from that the equations for h, m and k separate with the ansätze $h = \sum_{i=0}^{\infty} h_i(r)P_i(\cos\theta)$ etc., and from the fact that there are no inhomogeneous terms containing ω in the equations for h_i, k_i and m_i for $i > 2$. For more details see [10].

The matching of the two spacetime regions happens via the application of a coordinate transformation $\varphi \rightarrow \varphi + \Omega t$ in the fluid region. It means that the inner fluid region rotates with respect to the distant stationary observers with angular velocity Ω. This parameter Ω is considered to be the small expansion parameter with respect to which ω is first order and the other corrections h, m, k are second order. In addition to this, we can also rescale the interior time coordinate first by a zeroth order constant c_4, and then later by a second order small constant c_3, i.e. $t \rightarrow c_4(1 + c_3)t$.

The matter content of the interior is modeled by a perfect fluid

$$T_{ab} = (\rho + p)u_a u_b - pg_{ab}. \quad (3)$$

The coordinate system used in (1) is assumed to be comoving with the fluid, i.e. the 4-velocity is assumed to possess the form

$$u^a = (1/\sqrt{g_{00}}, 0, 0, 0) = ((1 - h)/A, 0, 0, 0) \quad (4)$$

which also implies that the shear of the fluid is zero so it rotates rigidly.

A. The field equations

In this subsection we list the field equations relevant to various orders. A similar system of equations with a slightly different choice of variables was given in [16]. If no equation of state is specified then the only equation one gets to zeroth order of the rotational parameter is the pressure isotropy condition $G^1{}_1 = G^2{}_2$ which reads as

$$B\frac{d^2A}{dr^2} + \frac{d(rA)}{dr}\frac{d(B/r)}{dr} + \frac{A}{r^2 B} = 0. \quad (5)$$

Making use of $G^0{}_0 = T^0{}_0$ and $G^1{}_1 = T^1{}_1$ the energy den-

sity and pressure of the nonrotating configuration reads as

$$\rho_0 = \frac{1}{r^2}\left[1 - \frac{d(rB^2)}{dr}\right], \qquad p_0 = \frac{1}{r^2}\left[\frac{B^2}{A^2}\frac{d(rA^2)}{dr} - 1\right]. \quad (6)$$

To first order in the rotation parameter the only relation follows from $G^3{}_0 = 0$

$$\frac{d}{dr}\left(r^4 \frac{B}{A}\frac{d\omega}{dr}\right) + 4r^3 \omega \frac{d}{dr}\left(\frac{B}{A}\right) = 0. \quad (7)$$

The second order Einstein equations yield the following conditions. From $G^1{}_2 = 0$ one gets

$$r\frac{d}{dr}(h_2 + k_2) + r(h_2 - m_2)\frac{1}{A}\frac{dA}{dr} - h_2 - m_2 = 0. \quad (8)$$

The pressure isotropy condition in the angular directions, $G^2{}_2 = G^3{}_3$, gives

$$6(h_2 + m_2) - r^4 \frac{B^2}{A^2}\left(\frac{d\omega}{dr}\right)^2 + 4r^3 \omega^2 \frac{B}{A}\frac{d}{dr}\left(\frac{B}{A}\right) = 0. \quad (9)$$

The equality of the pressure in the angular and radial directions, i.e. $G^1{}_1 = G^2{}_2$, gives two equations. After eliminating the derivative of h_2 using (8) one obtains from the $P_2(\cos\theta)$ part

$$2r\frac{B^2}{A}\frac{dA}{dr}\left(r\frac{dk_2}{dr} - m_2\right) - 2r^2 B h_2 \frac{d}{dr}\left(\frac{B}{r}\right) + m_2 - 4k_2$$
$$- 5h_2 - \frac{1}{3}r^4 \frac{B^2}{A^2}\left(\frac{d\omega}{dr}\right)^2 = 0, \quad (10)$$

while the θ-independent part gives an equation for m_0 and h_0.

The energy density function can be decomposed as $\rho = \rho_0 + \rho_2$, where $\rho_2 = \rho_{20} + \rho_{22}P_2(\cos\theta)$ and ρ_{20} and ρ_{22} are second order small functions of the coordinate r. The analogous decomposition of the pressure is similarly given by $p = p_0 + p_2 = p_0 + p_{20} + p_{22}P_2(\cos\theta)$. We get one more equation for m_0 and h_0 if we assume that the equation of state $\rho = \rho(p)$ is unchanged to second order. For details see [11].

B. Equations of state

To complete the system one more equation, e.g., an equation of state

$$\rho = \rho(p), \quad (11)$$

has to be specified. In this paper we will consider equations of state in the form

$$\rho = d_1 p + d_2\left(\frac{p}{p_c}\right)^{1/\gamma} + d_3. \quad (12)$$

Here p_c is the central pressure. Equation (12) includes some of the more used approximations for the equation of state of dense stars, like Newtonian polytropes, relativ-

istic polytropes with $d_1 = 1/(\gamma - 1)$, linear ones as well as the incompressible case.

We use units such that the speed of light $c = 1$ and $8\pi G = 1$. If then the unit of length is taken as α meter, the relation between our units and SI units are given by

$$\rho_{SI} = 5.358 \times 10^{25} \frac{\rho}{\alpha^2} \text{ kg/m}^3,$$

$$x_{SI} = 4.813 \times 10^{42} \frac{x}{\alpha^2} \text{ kgm}^{-1}\text{s}^{-2},$$

$$d_{1SI} = d_1, \tag{13}$$

where $x = \rho c^2$, p, d_2 or d_3.

Note that m_0 and h_0 do not appear in Eqs. (5) and (7)–(11) and hence this subsystem for A, B, ω, m_2, k_2 and h_2 decouples. Notice also that the equations can be solved

order by order and that the equations contain m_2 only algebraically. In Sec. V the system will be reformulated as a coupled system of six first order ordinary differential equations. Because of the requirement of a regular center the solutions to this subsystem will only depend on three constants of integration.

C. Vacuum metric

In the exterior vacuum region we will use a frame adapted to the asymptotically nonrotating observer. Solving the field equations detailed in Sec. II A by imposing $p = \rho = 0$ the metric functions for the vacuum region are given as follows [10,17][1]

$$A^2 = B^2 = 1 - 2M/r, \qquad \omega = \frac{2aM}{r^3}, \qquad h_0 = -m_0 = \frac{1}{r - 2M}\left(\frac{a^2 M^2}{r^3} + c_2\right),$$

$$h_2 = 3c_1 r(2M - r)\log\left(1 - \frac{2M}{r}\right) + a^2\frac{M}{r^4}(M + r) + 2c_1\frac{M}{r}(3r^2 - 6Mr - 2M^2)\frac{r - M}{2M - r} + \left(1 - \frac{2M}{r}\right)r^2 q_1, \tag{14}$$

$$k_2 = 3c_1(r^2 - 2M^2)\log\left(1 - \frac{2M}{r}\right) - a^2\frac{M}{r^4}(2M + r) - 2c_1\frac{M}{r}(2M^2 - 3Mr - 3r^2) + (2M^2 - r^2)q_1, \qquad m_2 = 6a^2\frac{M^2}{r^4} - h_2.$$

In this approximation, the slowly rotating solution is characterized by the mass M, the first order small rotation parameter a, and the second order small constants c_1, c_2 and q_1. When q_1 takes the value zero the metric is known to be the general asymptotically flat stationary and axisymmetric vacuum metric to second order (see, e.g., [18]). It can be easily checked that the solution is of Petrov-type D only if both c_1 and q_1 are zero. The metric is then equivalent to the Kerr metric to second order with mass $M \rightarrow M - c_2$.

When $q_1 \neq 0$ the metric cannot be asymptotically flat. It is important to keep in mind, however, that without the inclusion of this constant the matching conditions on the zero-pressure surface are overdetermined in general [17,19].

D. Multipole moments of the vacuum metric

The notion of relativistic gravitational multipole moments for static asymptotically flat vacuum spacetimes was developed in [20] by Geroch and later extended to the stationary case in [21,22] by Hansen and Thorne. These moments are defined on the 3-space of the timelike Killing trajectories.

A 3-space $(\mathcal{M}, h)$ with positive definite metric h is said to be asymptotically flat if it can be conformally mapped to a manifold $(\tilde{\mathcal{M}}, \tilde{h})$ with the following properties

(i) $\tilde{\mathcal{M}} = \mathcal{M} \cup \Lambda$, where Λ is a single point,

(ii) $\tilde{\Omega}|_\Lambda = \tilde{\Omega}_{,i}|_\Lambda = 0$, $\tilde{D}_i\tilde{D}_j\tilde{\Omega}|_\Lambda = \tilde{h}_{ij}|_\Lambda$

where $\tilde{h}_{ij} = \tilde{\Omega}^2 h_{ij}$.

From the timelike Killing vector field K^a one constructs the two scalar functions $f = K^a K_a$ and ψ, where the later

is obtained from the curl of K^a, $\psi_{,a} = \epsilon_{abcd}K^b K^{c;d}$, that is, a gradient due to the vacuum equations. The complex gravitational potential then reads

$$\xi = \frac{1 - \mathcal{E}}{1 + \mathcal{E}} \tag{15}$$

in terms of the Ernst potential $\mathcal{E} = f + i\psi$. It is given the conformal weight $-1/2$, so that $\tilde{\xi} = \tilde{\Omega}^{-1/2}\xi$. For axisymmetric spacetimes the metric is completely determined by the value of the potential on the axis of symmetry [23].

The multipole tensors on the 3-space of timelike Killing trajectories, with coordinates x^i, $i = 1, 2, 3$ and metric given by the projection operator $h_{ab} = -f g_{ab} + K_a K_b$, are then defined recursively as

$$P^{(0)}(x^i) = \xi, \qquad P_j^{(1)}(x^i) = \xi_{,j},$$

$$P_{k_1 k_2 \ldots k_{n+1}}^{(n+1)}(x^i) = D_{\langle k_{n+1}} P_{k_1 \ldots k_n\rangle}^{(n)} - \frac{1}{2}n(2n - 1)$$
$$\times R_{\langle k_1 k_2} P_{k_3 \ldots k_{n+1}\rangle}^{(n-1)}, \tag{16}$$

where $\langle k_1 \ldots k_{n+1}\rangle$ denotes the symmetric and trace-free part. D_i and R_{ij} are the covariant derivative and Ricci tensor, respectively, with respect to the 3-metric h_{ij} [21]. The $\tilde{P}_{k_1 \ldots k_n}^{(n)}$ are defined correspondingly in terms of the tilded quantities.

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[1]To make the norm of the timelike Killing field equal to unity at spatial infinity, the expression of h_0 in [17] has been slightly modified through a second order coordinate transformation of the time coordinate.

For axisymmetric spacetimes the multipole moments are given entirely in terms of the scalar moments defined as

$$P_n = \frac{1}{n!} \tilde{P}^{(n)}_{k_1 \dots k_n} n^{k_1} \dots n^{k_n}|_\Lambda \qquad (17)$$

in terms of the axis vector n^i [21].

In, e.g., [12,24], algorithms for calculating the multipole moments of a stationary axisymmetric spacetime are developed. We here use the method of [12] to calculate the moments up to second order. First, the metric is transformed to the canonical form

$$ds^2 = f(dt - \tilde{\omega} d\varphi)^2 - f^{-1}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2], \qquad (18)$$

where f, γ and $\tilde{\omega}$ are functions of ρ and z, through the coordinate transformation

$$\rho = Ar\sin\theta\left[1 + h_0 + h_2 + k_2 - \frac{3}{2}\sin^2\theta(h_2 + k_2)\right],$$

$$z = \left[(r - M)\left(1 + 2h_0 - \frac{1}{2}\sin^2\theta(h_2 + 2k_2 - m_2)\right) + r^2A^2\left(h_{0,r} - \frac{1}{2}\sin^2\theta(h_{2,r} + k_{2,r})\right)\right]\cos\theta, \qquad (19)$$

that holds to second order.

From the timelike Killing vector $K^a = \delta_0^a$ we find the two potentials

$$f = K^a K_a = g_{00} = (1 + 2h)A^2 - r^2\omega^2\sin^2\theta \qquad (20)$$

and

$$\psi = -\frac{2aM}{r^2}\cos\theta, \qquad (21)$$

obtained from the curl of K^a.

We introduce the coordinates $\bar{\rho} = \rho/(\rho^2 + z^2)$, $\bar{z} = z/(\rho^2 + z^2)$. A suitable conformal factor is then $\tilde{\Omega} = \bar{r}^2 \equiv \bar{\rho}^2 + \bar{z}^2$ and hence $\tilde{\xi} = (1/\bar{r})\xi$. As shown in [12], the first (up to $n = 3$) scalar moments P_n are given by the coefficients m_n in the expansion

$$\tilde{\xi}(\bar{\rho} = 0) = \Sigma_{n=0}^\infty m_n \bar{z}^n \qquad (22)$$

of $\tilde{\xi}$ on the axis, that in the original coordinates corresponds to $\theta = 0$ (or π). Expansion of ξ on the axis in the original coordinates gives to second order in the rotational parameter

$$\xi = \frac{M}{r - M} + i\frac{Ma}{(r - M)^2} - \frac{M^2 a^2}{r(r - M)^3} - (h_0 + h_2)A^2\left(\frac{r}{r - M}\right)^2. \qquad (23)$$

From (19) one finds that $r = z + M$ to zeroth order, which is sufficient for the three last terms in (23) since they are already first and second order. The first term needs to be expanded to second order in the rotational parameter, giv-

ing

$$\frac{M}{r - M} = \frac{M}{z - 2h_0(r - M) - r^2A^2h_{0,r}}$$

$$= \frac{M}{z} + \frac{M}{z^2}[2h_0(r - M) + r^2A^2h_{0,r}], \qquad (24)$$

where once again $r = z + M$ can be used in all second order functions. Finally, expressing $\tilde{\xi}$ in $\bar{z}$, one obtains to second order

$$\tilde{\xi} = M - c_2 + iMa\bar{z} - M\left(a^2 + \frac{16}{5}M^4c_1\right)\bar{z}^2. \qquad (25)$$

Hence the mass is given by $M - c_2$, the angular momentum by $J = Ma$ and the quadrupole moment by $Q = -M(a^2 + \frac{16}{5}M^4c_1)$. We will be interested in the relative deviation of the quadrupole moment from that of the Kerr metric

$$\frac{\Delta Q}{Q} \equiv \frac{Q - Q_{\text{Kerr}}}{Q_{\text{Kerr}}} = \frac{16M^4c_1}{5a^2}. \qquad (26)$$

An expansion of the exterior metric for large r gives the following leading terms of g_{00} (with $q_1 = 0$)

$$g_{00} = 1 - \frac{2M(1 - \frac{c_2}{M})}{r} + \frac{2MP_2(\cos\theta)(a^2 + \frac{16}{5}M^4c_1)}{r^3}, \qquad (27)$$

i.e., the associated Newtonian quadrupole moment reads as (cf., e.g., [25])

$$Q_{11} = Q_{22} = -Q_{33}/2 = 2M\left(a^2 + \frac{16}{5}M^4c_1\right) \qquad (28)$$

in an asymptotically Cartesian system with the 3-axis along the axis of rotation. Hence it is, up to a factor of 4, the same as the relativistic moment.

III. MATCHING

We here briefly describe the matching procedure, in which the Darmois-Israel junction conditions [13,14] are used. For more details the reader is referred to [11] and also to [15] for a general discussion on the matching of axisymmetric bodies in second order perturbation theory. In the fluid region, the matching surface $\mathcal{S}$ is defined by the condition of vanishing pressure, $p = 0$. In the limit of no rotation, the matching surface is the three-dimensional cylinder $r = r_1$. For slow rotation the equation of the matching surface $\mathcal{S}$ is

$$r = r_1 + \xi \qquad (29)$$

with

$$\xi = -\left[\frac{p_{20} + p_{22}P_2(\cos\vartheta)}{\frac{dp_0}{dr}}\right]_{|r=r_1} \equiv \xi_0 + \xi_2 P_2(\cos\vartheta), \qquad (30)$$

where the constants ξ_0 and ξ_2 are given by

$$\xi_0 = \frac{1}{12rB\frac{dA}{dr}\frac{d}{dr}\left(\frac{A}{B}\right)}\left[12rA^2\frac{dh_0}{dr} - 12m_0\frac{d}{dr}(rA^2)\right.$$
$$\left.\left. + r^4\left(\frac{d\omega}{dr}\right)^2\right]\right|_{r=r_1}$$

and

$$\xi_2 = -\frac{(3A^2h_2 + r^2\omega^2)}{3A\frac{dA}{dr}}\bigg|_{r=r_1}. \qquad (31)$$

In the vacuum exterior region suitable hypersurfaces for matching are determined by

$$r = r_1 + \chi \equiv \chi_0 + \chi_2 P_2(\cos\vartheta), \qquad (32)$$

where χ_0 and χ_2 are constants to be determined by the matching conditions [26].

In order to find isometric embeddings of the matching surface $\mathcal{S}$ in the vacuum and fluid domains, we equate with each other the respective induced metrics $ds_{(v)}^2$, ds^2 and induced extrinsic curvatures $K^{(v)}$, K

$$ds_{(v)}^2|_\mathcal{S} = ds^2|_\mathcal{S}, \qquad K^{(v)}|_\mathcal{S} = K|_\mathcal{S}, \qquad (33)$$

where K is defined by

$$K \equiv K_{ab}dx^a dx^b \equiv h_a{}^c h_b{}^d n_{(c;d)}dx^a dx^b \qquad (34)$$

in terms of the unit normal n_a of the matching surface $\mathcal{S}$ and the projection operator $h_a{}^b = n_a n^b + \delta_a^b$. To adjust the different coordinate systems with each other we apply a rigid rotation in the fluid region by setting $\varphi \to \varphi + \Omega t$ where Ω is a constant. Then we rescale the interior time coordinate t by $t \to c_4(1 + c_3)t$ with further zeroth and second order constants c_4 and c_3 to be determined from the matching conditions.

From the zeroth order matching conditions we get the following relations (all functions here and in the following first and second order relations are evaluated at the zeroth order matching surface $r = r_1$):

$$M = \frac{1}{2}r_1(1 - B^2), \qquad c_4 = \frac{B}{A}, \qquad (35)$$

$$r_1 = \frac{A}{2B^2\frac{dA}{dr}}(1 - B^2). \qquad (36)$$

To first order we solve for a and Ω

$$a = \frac{Br_1^3}{3A(B^2 - 1)}\frac{d\omega}{dr}, \qquad \Omega = \frac{r_1}{3}\frac{d\omega}{dr} + \omega. \qquad (37)$$

From the second order equations we can solve for c_1, c_2, c_3, q_1, χ_0 and χ_2 as

$$c_1 = \frac{B^2}{9r_1^2A^2(B^2 - 1)^6}\left[r_1^4B^2(B^4 - 3)\left(\frac{d\omega}{dr}\right)^2\right.$$
$$\left. + 36A^2h_2(1 - B^4) + 72A^2B^2(h_2 + k_2)\right], \qquad (38)$$

$$c_2 = \frac{\xi_0}{2}\left(B^2 - 1 + 2r_1B\frac{dB}{dr}\right) - r_1B^2m_0 - \frac{r_1^5}{36}\frac{B^2}{A^2}\left(\frac{d\omega}{dr}\right)^2, \qquad (39)$$

$$c_3 = \frac{r_1^4}{36A^2}\left(\frac{d\omega}{dr}\right)^2 + \frac{c_2}{r_1B^2} - h_0, \qquad (40)$$

$$q_1 = \frac{1}{9r_1^2A^2(B^2 - 1)^6}\left\{18k_2A^2(B^4 - 1)(B^4 - 8B^2 + 1)\right.$$
$$+ 216A^2B^2 \ln B[2B^2(h_2 + k_2) + (1 - B^4)h_2]$$
$$+ 36h_2A^2B^2(B^2 - 1)(B^4 + B^2 - 8)$$
$$+ r_1^4\left(\frac{d\omega}{dr}\right)^2 B^2[(B^2 - 1)(2 + 11B^2 - 7B^4)$$
$$\left. + 6B^2(B^4 - 3)\ln B]\right\}, \qquad (41)$$

$$\chi_0 = \xi_0 \quad \text{and} \quad \chi_2 = \xi_2, \qquad (42)$$

where ξ_0 and ξ_2 are obtained from (30) and (31). Note that we did *not* impose the equation of state (11) when calculating the matching conditions. Hence an appropriate matching can be done, i.e. the vacuum metric in Sec. II C is general enough for describing the exterior of any axisymmetric rigidly rotating perfect fluid ball up to second order.

IV. EQUATIONS AND BOUNDARY VALUES

In this section we provide a reformulation of the field equations to a form more suitable for numerical integration. By doing this we can get higher precision at the origin where apparent singularities arise, moreover the freely specifiable constants are identified more easily this way.

A. Integrating the zeroth order field equation

In order to simplify (5) it is convenient to redefine the functions A, h, m and k in terms of the function ν as

$$A = e^\nu, \qquad h = \tilde{h}e^{-2\nu}, \qquad m = \tilde{m}e^{-2\nu}, \qquad k = \tilde{k}e^{-2\nu}. \qquad (43)$$

The equations simplify considerably due to the fact that only the derivative of ν will appear. Hence we introduce the function z by

$$\frac{z}{B} = r\frac{d\nu}{dr} + 1. \qquad (44)$$

Then the zeroth order equation (5) becomes first order in z and algebraic in B [27]

$$Br\frac{dz}{dr} + 2B^2 + z^2 - 4Bz + 1 = 0, \tag{45}$$

furthermore, the pressure of the nonrotating configuration (6) takes the form

$$p_0 = \frac{1}{r^2}(2Bz - B^2 - 1). \tag{46}$$

B. Series expansion around a regular center

For sufficiently regular configurations close to the center the metric coefficients can be given as power series in r. Assuming that the central pressure and density are finite it follows that $B(0) = z(0) = 1$. The assumption of smoothness of the configurations at the symmetry center, in the spacetime sense, implies that the odd coefficients in the expansions of the basic variables are zero.

Hence, assuming a smooth center in the spacetime sense, the odd powers will be omitted hereafter. Plugging the expressions

$$
\begin{aligned}
B &= 1 + b_1 r^2 + b_2 r^4 + \ldots, \\
z &= 1 + z_1 r^2 + z_2 r^4 + \ldots, \\
\omega &= \omega_0 + \omega_1 r^2 + \omega_2 r^4 + \ldots, \\
\tilde{h}_2 &= h_2^{(0)} + h_2^{(1)} r^2 + h_2^{(2)} r^4 \ldots, \\
\tilde{m}_2 &= m_2^{(0)} + m_2^{(1)} r^2 + m_2^{(2)} r^4 \ldots, \\
\tilde{k}_2 &= k_2^{(0)} + k_2^{(1)} r^2 + k_2^{(2)} r^4 \ldots
\end{aligned}
\tag{47}
$$

into the field equations together with the equation of state (11) justifies then that all coefficients can be given in terms of z_1, ω_0 and $h_1 \equiv h_2^{(1)}$.

To zeroth order one obtains

$$h_2^{(0)} = m_2^{(0)} = k_2^{(0)} = 0, \tag{48}$$

then to second order

$$
\begin{aligned}
m_2^{(1)} &= k_2^{(1)} = -h_2^{(1)} \equiv -h_1, \\
\omega_1 &= \frac{2}{5}\omega_0(z_1 - 3b_1), \qquad b_1 = -\frac{1}{6}\rho_{|2z_1}
\end{aligned}
\tag{49}
$$

and finally to fourth order

$$b_2 = -\frac{b_1^2}{2} + \frac{1}{10}(z_1^2 + 3b_1^2 - 4b_1z_1)\frac{d\rho}{dp}\bigg|_{2z_1},$$

$$z_2 = b_1 z_1 - \frac{z_1^2}{2} - b_1^2,$$

$$h_2^{(2)} = \frac{(3z_1^2 - 16z_1b_1 + 8b_1^2 - 10b_2)(\omega_0^2 + 3h_1)}{42(z_1 - b_1)}$$

$$\qquad + \frac{\omega_0^2}{21}(17b_1 - 5z_1), \tag{50}$$

$$k_2^{(2)} = \frac{\omega_0^2}{6}(z_1 - 3b_1) + \frac{h_1}{2}(b_1 - z_1) - h_2^{(2)},$$

$$m_2^{(2)} = \frac{2\omega_0^2}{3}(z_1 - 3b_1) - h_2^{(2)},$$

$$\omega_2 = \frac{\omega_0}{70}(z_1^2 - 36z_1b_1 + 74b_1^2 - 50b_2).$$

The central density and pressure are given by

$$\rho_{0c} = -6b_1, \qquad p_{0c} = 2z_1. \tag{51}$$

This shows that for realistic configurations $b_1 < 0$ and $z_1 > 0$, consequently the $z_1 - b_1$ term in the denominator of $h_2^{(2)}$ is nonvanishing.

C. System of differential equations

Motivated by the results of the previous section it is advantageous to define the new dependent variables β, ζ, $\tilde{\omega}$, $\hat{\omega}$, $\hat{h}$, $\hat{k}$ and $\hat{m}$ through

$$B = 1 + r\beta, \qquad z = 1 + r\zeta, \qquad \omega = \omega_0 + r\tilde{\omega},$$

$$\omega_{,r} = \tilde{\omega} + \hat{\omega}, \qquad \tilde{h}_2 = r\hat{h}, \qquad \tilde{k}_2 = r(r^2\hat{k} - \hat{h}), \tag{52}$$

$$\tilde{m}_2 = r(r^2\hat{m} - \hat{h}).$$

The closed subsystem of Eqs. (5) and (7)–(11) then gives six first order differential equations for the quantities β, ζ, $\tilde{\omega}$, $\hat{\omega}$, $\hat{h}$ and $\hat{k}$, while $\hat{m}$ can be solved for algebraically. The equations for the zeroth order quantities are given by

$$\frac{d\zeta}{dr} = -\frac{1}{rB}(-\zeta - 3r\zeta\beta + r\zeta^2 + 2r\beta^2), \tag{53}$$

$$\frac{d\beta}{dr} = -\frac{1}{2rB}(r\rho_0 + 4\beta + 3r\beta^2), \tag{54}$$

where the density ρ_0 is given by equation of state $\rho_0 = \rho(p_0)$ with the pressure given by

$$p_0 = \frac{1}{r}(2\zeta + 2r\zeta\beta - r\beta^2). \tag{55}$$

The equations for the first order quantities $\tilde{\omega}$ and $\hat{\omega}$ are

$$\frac{d\tilde{\omega}}{dr} = \frac{\hat{\omega}}{r}, \tag{56}$$

$$\frac{d\hat{\omega}}{dr} = -\frac{1}{rB}\left[r(4\omega_0 + 5r\tilde{\omega} + r\hat{\omega})\frac{d\beta}{dr} + 4(2\beta - \zeta)\omega_0 \right.$$
$$\left. + (14r\beta + 4 - 5r\zeta)\tilde{\omega} + (7r\beta + 5 - r\zeta)\hat{\omega} \right],$$

$$(57)$$

whereas the ones for the second order quantities $\hat{h}$ and $\hat{k}$ are long and hence are not given here. The corresponding homogeneous equations, given by putting $\omega = 0$ in (8)–(10), are

$$\frac{d\hat{h}_h}{dr} = \frac{2r\hat{k}_h - \beta\hat{h}_h}{r(\beta - \zeta)B} + \frac{\hat{h}_h}{\beta - \zeta}\frac{d\beta}{dr}, \quad (58)$$

$$\frac{d\hat{k}_h}{dr} = \frac{r\hat{k}_h(2r\zeta - 5r\beta - 3) + 2\hat{h}_h(\beta - \zeta)}{r^2 B}. \quad (59)$$

These two last equations will be used when adjusting the parameters so that the solutions become asymptotically flat.

Boundary conditions at $r = 0$ are given as

$$\frac{d\beta}{dr} = b_1, \qquad \frac{d\zeta}{dr} = z_1,$$

$$\frac{d\tilde{\omega}}{dr} = \frac{d\hat{\omega}}{dr} = \frac{2}{5}\omega_0(z_1 - 3b_1), \qquad \frac{d\hat{h}}{dr} = h_1, \quad (60)$$

$$\frac{d\hat{k}}{dr} = \frac{\omega_0^2}{6}(z_1 - 3b_1) + \frac{h_1}{2}(b_1 - z_1).$$

The constants $b_1 = -\rho_{0c}/6$ and $z_1 = p_{0c}/2$ are related through the equation of state (11), implying that we have three independent constants of integration (apart from possible constants in the equation of state). One of this (h_1) will be determined through the matching conditions on the zero-pressure surface in terms of the others when the requirement of asymptotic flatness is imposed.

There is a rescaling associated to the rescaling of the rotational parameter ω_0, following the rule $\omega_0 \rightarrow \gamma\omega_0$, which induces the transformation

$$\omega, \tilde{\omega}, \hat{\omega} \rightarrow \gamma\omega, \gamma\tilde{\omega}, \gamma\hat{\omega}, \qquad \hat{h}, \hat{m}, \hat{k} \rightarrow \gamma^2\hat{h}, \gamma^2\hat{m}, \gamma^2\hat{k},$$
$$\beta, \zeta, r \rightarrow \beta, \zeta, r. \quad (61)$$

Because of this scale invariance of the equations the constant ω_0 can be fixed. All other configurations with a given equation of state and specified central pressure can be obtained by rescaling.

The rescaling of the radial coordinate $r \rightarrow \alpha r$ induces a change of the parameters in the equation of state, (12), given by

$$d_1 \rightarrow d_1, \qquad d_2, d_3, p_{0c} \rightarrow \frac{d_2}{\alpha^2}, \frac{d_3}{\alpha^2}, \frac{p_{0c}}{\alpha^2}. \quad (62)$$

The dependent variables then scale as

$$\beta, \zeta, \tilde{\omega}, \hat{\omega}, \hat{m}, \hat{k} \rightarrow \frac{\beta}{\alpha}, \frac{\zeta}{\alpha}, \frac{\tilde{\omega}}{\alpha}, \frac{\hat{\omega}}{\alpha}, \frac{\hat{m}}{\alpha}, \frac{\hat{k}}{\alpha},$$
$$\omega, \omega_0, \hat{h} \rightarrow \omega, \omega_0, \alpha\hat{h}. \quad (63)$$

Using this scale invariance we can set one of the constants in the equation of state to a fixed value.

D. Asymptotically flat solutions

Asymptotically flat solutions can be obtained by considering suitable linear combinations of homogeneous and particular solutions for h_2 and k_2, or equivalently for $\hat{h}$ and $\hat{k}$. These are solutions of the linear equations (58) and (59) and their corresponding inhomogeneous equations, respectively. Since we solve the equations order by order, and hence $\hat{k}$ and $\hat{h}$ do not appear in the lower order equations, we can add any homogeneous solution to a given particular solution, i.e., $\hat{k} = \hat{k}_p + C\hat{k}_h$, $\hat{h} = \hat{h}_p + C\hat{h}_h$, where C is an arbitrary constant and the subscripts p and h refer to particular and homogeneous solutions, respectively. A solution to the field equations is asymptotically flat iff $q_1 = 0$. Now the expression (41) is also linear in k_2 and h_2, or equivalently in $\hat{k}$ and $\hat{h}$, it has a structure like

$$q_1 = \alpha_1 + \alpha_2\hat{k} + \alpha_3\hat{h}. \quad (64)$$

Hence

$$q_1 = \alpha_1 + \alpha_2(\hat{k}_p + C\hat{k}_h) + \alpha_3(\hat{h}_p + C\hat{h}_h)$$
$$= \alpha_1 + \alpha_2\hat{k}_p + \alpha_3\hat{h}_p + C(\alpha_2\hat{k}_h + \alpha_3\hat{h}_h)$$
$$\equiv q_{1p} + Cq_{1h} \quad (65)$$

and $q_1 = 0$ can be obtained by choosing $C = -q_{1p}/q_{1h}$. Also the constant c_1 (38), that is used to calculate the quadrupole moment, has a similar structure, and hence the resulting c_1 is given by $c_1 = c_{1p} + Cc_{1h}$.

V. NUMERICAL SOLUTIONS

In this section we consider solutions with equations of state (12), with special attention to the quadrupole moments and their deviation from that of the Kerr metric. In particular we determine the quantity $\Delta Q/Q$ defined in (26). The system was solved using fourth order Runge-Kutta. In scanning the parameter space, due to the scaling invariance, without loss of generality, we fixed $\omega_0 = 0.1$ and the method for obtaining asymptotically flat solutions, described in Sec. IV D below, indirectly fixes the value of h_1. Hence, for a given equation of state, we only need to vary the central pressure, $2z_1$. When considering different equations of state we use the scale invariance (62) and (63) to fix one of the constants d_2 or d_3. When performing the corresponding rescaling $r \rightarrow \alpha r$ the quantity $\Delta Q/Q$ is invariant.

The integrations were carried out until the zero-pressure surface was reached. The accuracy of the code was checked for the Wahlquist solution [28].

A. Incompressible fluids (interior Schwarzschild)

This case was already considered in [3]. The numerically calculated quantity $\Delta Q/Q$, defined in (26), as a function of the central pressure is depicted in Fig. 1. In Fig. 2 the same quantity is given as function of the radius of the corresponding nonrotating configuration. When the central pressure approaches infinity and the radius the Buchdahl limit $\frac{9M}{4}$, then $\Delta Q/Q$ approaches 0.0212. Because of the scaling invariance, (62) and (63), this limiting value is independent of the density.

B. Linear equations of state

Linear equations of state

$$\rho = d_1 p + d_3 \qquad (66)$$

are of interest not only because they could approximate some compact objects, but also since in this class one might have a chance of finding exact solutions, at least to zeroth order.

An exact spherically symmetric perfect fluid solution is given by the Whittaker metric [29], with equation of state $\rho = -3p + \mu_0$. Its metric is

$$ds^2 = f_0 dt^2 - \frac{dr^2}{f_0(1 - \kappa^2 \mu_0 r^2/2)} - r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

where

$$f_0 = 1 + \frac{1}{\kappa^2}\left[1 - \arcsin\left(\kappa\sqrt{\frac{\mu_0}{2}}r\right)\sqrt{\frac{2}{\kappa^2 \mu_0 r^2} - 1}\right], \quad (67)$$
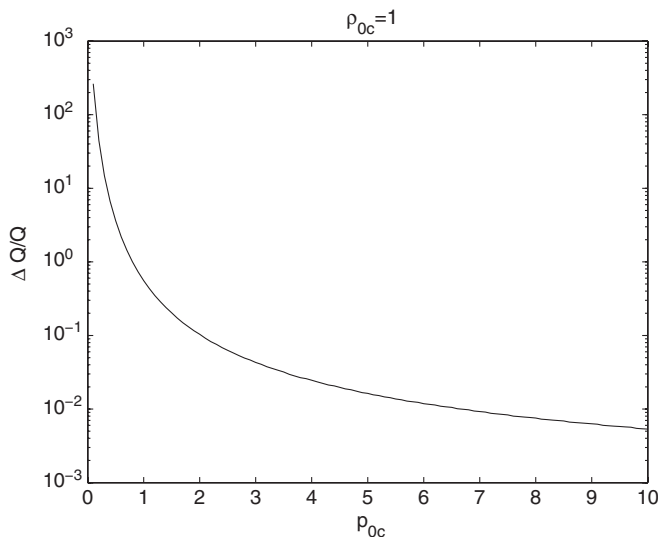

FIG. 1.   The quantity $\Delta Q/Q$ as function of central pressure for a sequence of fluid balls with constant density $\rho = 1$.
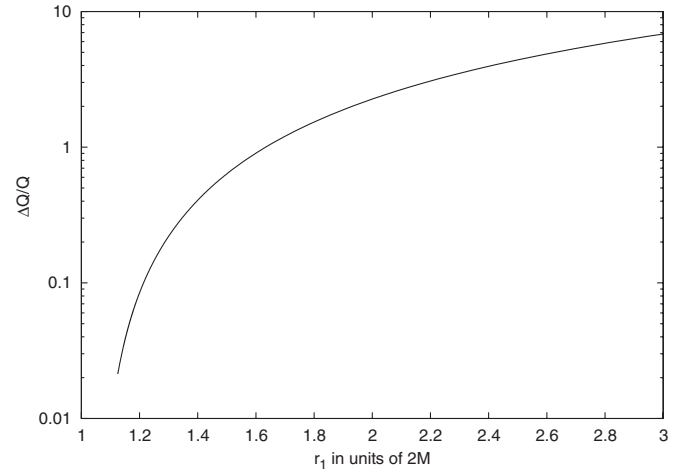

FIG. 2.   The quantity $\Delta Q/Q$ as function of radius $r_1/2M$ for a sequence of fluid balls with constant density $\rho = 1$.

and the pressure is

$$p = \frac{1}{2}\mu_0(1 - \kappa^2 f_0). \qquad (68)$$

It is the nonrotating limit of the Wahlquist metric [28], that cannot be matched to an asymptotically flat vacuum region [17,30]. However, there is another rotating generalization of Whittaker that can be matched, although its closed form is not known [31].

In the limit when the central pressure goes towards $\mu_0/2$, corresponding to $\kappa \to 0$, the Whittaker metric approaches the anti-de Sitter spacetime and the radius r_1 blows up to infinity as $\sqrt{2/\mu_0}/\kappa$, but $r_1/2M$ goes as $1 + 4\kappa^2/\pi^2$, i.e. the surface approaches the event horizon. This is not in conflict with the Buchdahl limit [32], since this spacetime does not meet the usual physical requirements. Even if the these are weaker than in Buchdahl's original work, see, e.g., [33], the central density becomes negative in the anti-de Sitter limit. For the corresponding second order rotating and asymptotically flat configuration we find that the quadrupole moment approaches that of the Kerr metric in this limit.

Similar results hold for all configurations with $d_1 < -1$. For all these $\rho_{0c} = -p_{0c}$ when $p_{0c} = d_3/(-d_1 - 1)$, and since the anti-de Sitter spacetime is a solution to our zeroth order equations and the solution is uniquely given by central pressure and density this is the resulting spacetime. Since the zero-pressure surface is further and further pushed outward when approaching the limit $p_{0c} = d_3/(-d_1 - 1)$, all these configurations also become infinite in extension. The ratio $r_1/2M$ once again tends towards one, and we also reobtain that the quadrupole moment approaches that for Kerr. Even if these spacetimes are quite unphysical, a study of them still is helpful in understanding which conditions are needed for a successful matching to the Kerr metric.

When performing the numerical runs, due to the scaling invariance, (62) and (63), we put $d_3 = 1$. Negative values of d_3 are excluded if the surface density should be larger than or equal to zero, and configurations with $d_3 = 0$ are not finite in size [34]. The quantity $\Delta Q/Q$ for a sequence of values for d_1 are given in Fig. 3. As seen, for more realistic configurations with $d_1 \gtrsim 1$ it differs significantly from zero.

C. Polytropes

1. Newtonian polytropes

In a recent paper [7] it was shown that slowly and rigidly rotating polytropes cannot be sources of the Kerr metric in the post-Minkowskian limit. Here we numerically find that slowly and rigidly rotating polytropes with arbitrary strength of the gravitational field cannot be matched to Kerr either.

We use the equation of state $\rho = d_2(\frac{p}{p_c})^{1/\gamma}$ obtained by putting $d_1 = d_3 = 0$ in (12). Because of the scaling invariance, (62) and (63), we put $d_2 = 1$. In Fig. 4 $\Delta Q/Q$ is shown as a function of the central pressure for a sequence of values of γ, including the physically interesting cases $\gamma = 4/3$ and $\gamma = 5/3$. As shown in [34] the configurations are finite in size for $\gamma > 1.2295$ ($1/\gamma < 0.7695$) and infinite for $\gamma < 1.2$ ($1/\gamma > 0.833$). In the interval in between the central pressure determines whether the configuration is finite or not. As seen $\Delta Q/Q$ never seems to approach zero. Furthermore, for physically reasonable configurations with $p_c \leq \rho_c = 1$, the quantity differs significantly from zero.

2. Relativistic polytropes

For a fluid with one type of constituent particles, a relativistic polytrope is given by $p = Cn^\gamma$, where C is a constant, in terms of the particle density n and the poly-

FIG. 4. The quantity $\Delta Q/Q$ as a function of central pressure for a sequence of Newtonian polytopes with equation of state $\rho = (\frac{p}{p_c})^{1/\gamma}$.

tropic index γ. This equation is suitable, e.g., to describe an ideal degenerate neutron gas. Using the energy conservation equation for a perfect fluid it is then easy to show that the equation of state becomes

$$\rho = \frac{1}{\gamma - 1}p + d_2 p^{1/\gamma} \qquad (69)$$

in terms of the pressure p. As seen the equation of state approaches a linear equation of state for large pressures and a Newtonian polytrope for low pressures. For a discussion of relativistic polytropes see, e.g., [35]. The results of the numerical runs are given in Fig. 5. As expected the results are similar to those of the Newtonian polytropes.
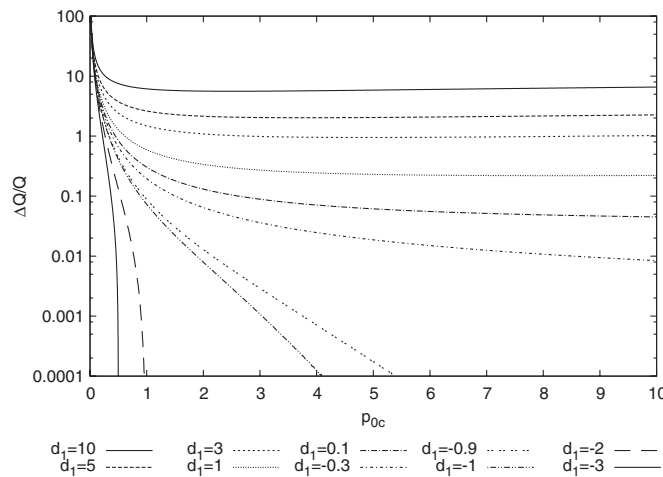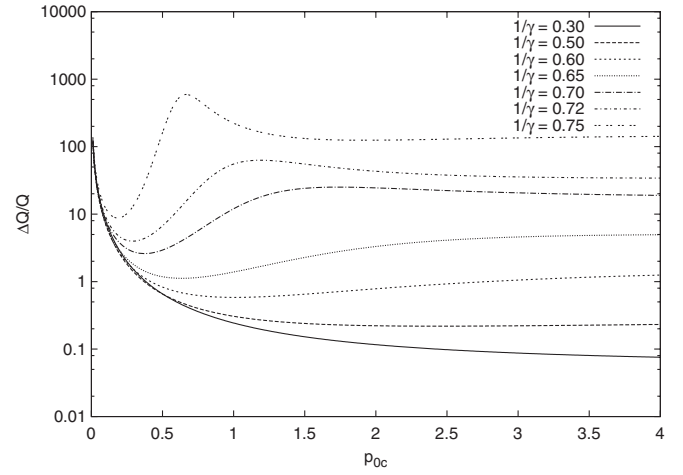

FIG. 3. The quantity $\Delta Q/Q$ as a function of central pressure for a sequence of fluid balls with linear equation of state $\rho = d_1 p + 1$.
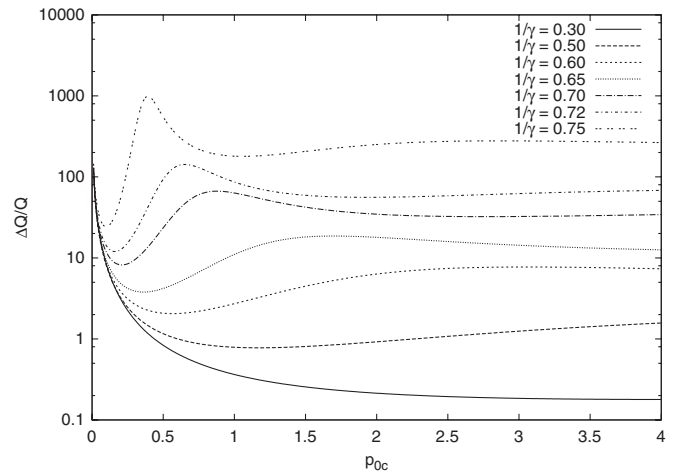

FIG. 5. The quantity $\Delta Q/Q$ as a function of central pressure for a sequence of relativistic polytropes with equation of state $\rho = \frac{1}{\gamma - 1}p + (\frac{p}{p_c})^{1/\gamma}$.

D. Combined linear and polytropic

When adding a constant to the Newtonian polytropes in Fig. 4 typically the local minima and maxima get less pronounced or disappear and for large central pressures the quadrupole moments get closer to that of the incompressible case, but never smaller. For combined linear and polytropic equations of state the curves become more similar to those of the relativistic polytropes in Fig. 5, and when there is additative constant the limiting value of the quadrupole moment for large pressures once again gets closer to that of the incompressible case.

VI. THE POST-MINKOWSKIAN LIMIT

Since we cannot solve the equations in the fluid region analytically, we look at the weak gravity limit by making an expansion in the small parameter $\lambda = M/r_1$, or in SI units GM/r_1c^2. Here M is the mass of the fluid ball and r_1 is its radius. A similar study, using global harmonic coordinates and the Lichnerowicz matching conditions [36], was performed in [6,7].

From an expansion of the equations in this parameter it turns out that the pressure is one order higher in λ than the density. This can also be understood in a Newtonian context where the virial theorem applied to a spherically symmetric configuration gives $\langle p \rangle = k\lambda\langle \rho \rangle$, where k is a constant of order unity and $\langle p \rangle$ and $\langle \rho \rangle$ are the average values of pressure and energy density. By considering the balance between gravitational and centrifugal forces on the equator of a rotating Newtonian fluid ball one obtains for the angular velocity that $\omega < \sqrt{\lambda}/r_1$, so it seems reasonable to let ω go as $\sqrt{\lambda}$. This is also consistent with that ω or its derivatives only appear quadratically in (9) and (10).

We first consider the post-Minkowskian expansion of the spherically symmetric system. The functions A and B are expanded as

$$A = 1 + A_1 + A_2 + \ldots, \qquad B = 1 + B_1 + B_2 + \ldots, \tag{70}$$

where subscripts refer to the order in λ. The freedom of rescaling the time coordinate was used to put the zeroth order term in A to one. We do a similar expansion of the pressure $p = p_1 + p_2 + \ldots$ and density $\rho = \rho_1 + \rho_2 + \ldots$. Using the field equation (5) we first obtain that p_1 is a constant. We put this constant to zero so that a zero-pressure surface can be obtained. Then

$$B_1 = -r\frac{dA_1}{dr}. \tag{71}$$

The first nonvanishing terms of the density and pressure are

$$\rho_1 = \frac{2}{r^2}\frac{d}{dr}\left(r^2\frac{dA_1}{dr}\right), \tag{72}$$

$$p_2 = p_c - \left(\frac{dA_1}{dr}\right)^2 - 4\int\frac{1}{r}\left(\frac{dA_1}{dr}\right)^2 dr. \tag{73}$$

To complete the system we have to impose an equation of state $p = p(\rho)$, giving an equation for A_1. The differentiated version of this equation reads as

$$\frac{dp(\rho_1)}{d\rho_1}\frac{d\rho_1}{dr} = -\rho_1\frac{dA_1}{dr}, \tag{74}$$

where ρ_1 is given by (72). We note that the total mass M of the configuration will be of the order λ, while the radius r_1 will be of zeroth order.

The function ω, which is first order in the rotational parameter Ω, has the post-Minkowskian expansion

$$\omega = \omega_{1/2} + \omega_{3/2} + \ldots, \tag{75}$$

where the indices correspond to the appropriate fractional orders of the quantities in λ. From the field equation (7) we obtain that $\omega_{1/2}$ is a constant. This corresponds to a rigid rotation of the system. It is necessary to keep $\omega_{1/2}$ nonzero, since this will provide the nonlinear source term in the higher order equations.

The quantities second order in the rotational parameter, i.e. h_2, m_2 and k_2, are expanded as

$$h_2 = h_{21} + h_{22} + \ldots, \qquad m_2 = m_{21} + m_{22} + \ldots,$$
$$k_2 = k_{21} + k_{22} + \ldots, \tag{76}$$

where the second index at each new quantity indicates the order in the post-Minkowskian parameter λ. To first order in λ Eqs. (8)–(10) yield $m_{21} = k_{21} = -h_{21}$ and

$$\frac{d^2h_{21}}{dr^2} + \frac{2}{r}\frac{dh_{21}}{dr} - \frac{1}{2}\left(\frac{d\rho_1}{dA_1} + \frac{12}{r^2}\right)h_{21} = \frac{\omega_{1/2}^2 r^2}{6}\frac{d\rho_1}{dA_1}, \tag{77}$$

where

$$\frac{d\rho_1}{dA_1} = 2\frac{d^3A_1}{dr^3}\Big/\frac{dA_1}{dr} - \frac{4}{r^2} + \frac{4}{r}\frac{d^2A_1}{dr^2}\Big/\frac{dA_1}{dr}. \tag{78}$$

We note that a particular solution to Eq. (77) is given by $h_{21p} = -\omega_{1/2}^2 r^2/3$.

Using the condition that q_1, given by Eq. (41), should vanish (to first order in λ) for an asymptotically flat solution and that the corresponding c_1 is given by $c_1 = c_{1p} - c_{1h}q_{1p}/q_{1h}$ as described in Sec. IV D one obtains

$$c_1 = -\frac{5\omega_{1/2}^2}{48r_1^5(\frac{dA_1}{dr})|_{r=r_1}^5}\frac{\frac{d}{dr}\log(h_{21h}/(r^4\frac{dA_1}{dr}))|_{r=r_1}}{\frac{d}{dr}\log(rh_{21h}/\frac{dA_1}{dr})|_{r=r_1}}. \tag{79}$$

Here h_{21h} is a regular nonzero solution of the homogeneous version of (77). Note that in general c_1 goes to infinity as $1/\lambda^4$.

From the matching condition (35) it follows that

$$\frac{dA_1}{dr}\Big|_{r=r_1} = \frac{M}{r_1^2}, \tag{80}$$

where we assume that the mass M is positive. In [11] it is

shown that the rotating fluid ball is oblate in shape if

$$k_{2|r=r_1} + \frac{\xi_2}{r_1} < 0, \tag{81}$$

where ξ_2 is given by (31). Substitution of $k_{21} = -h_{21} = \omega_{1/2}^2 r^2/3 - h_{21h}$ in (81) together with (80) gives that the fluid ball is oblate iff $h_{21h|r=r_1} > 0$ to first order in λ.

For the incompressible case the equations can be integrated completely and give for the asymptotically flat case

$$A_1 = \frac{\rho_1 r^2}{12} + a_0, \qquad B_1 = -\frac{\rho_1 r^2}{6},$$

$$\omega_{3/2} = \frac{\rho_1 r^2 \omega_{1/2}}{5}, \qquad h_{21} = \frac{r^2 \omega_{1/2}^2}{2}, \tag{82}$$

$$p_2 = p_c - \frac{\rho_1^2 r^2}{12}.$$

The integration constant a_0 can be transformed away by a first order (in λ) rescaling of the time coordinate. The choice $a_0 = -\rho_1 r_1^2/4$ makes the constant c_4 in (35) equal to unity. Since $h_{21h|r=r_1} = 5\omega_{1/2}^2 r_1^2/6 > 0$ the ball is oblate. From this solution the radius r_1 of the fluid ball and the parameters M and a of the exterior metric are given by

$$r_1 = \frac{2\sqrt{3p_c}}{\rho_1}, \qquad M = \frac{4}{3} \frac{(3p_c)^{3/2}}{\rho_1^2},$$

$$a = -\frac{24}{5} \frac{p_c \omega_{1/2}}{\rho_1^2}, \tag{83}$$

respectively. The value of c_1 for asymptotically flat metrics is in this case

$$c_1 = \frac{5}{1024} \left(\frac{\rho_1}{p_c}\right)^5 \omega_{1/2}^2 \tag{84}$$

and the relative difference to the Kerr quadrupole moment is

$$\frac{\Delta Q}{Q} = \frac{25\rho_1}{16p_c} \tag{85}$$

which diverges as $1/\lambda$. By scaling the input parameters ρ_{0c}, p_{0c} and ω_0 in the numerical code used in the previous section as λ, λ^2 and $\lambda^{1/2}$, respectively, these values are also obtained numerically in the limit $\lambda \to 0$.

For Newtonian polytropes, $p = p_c(\frac{\rho}{\rho_c})^\gamma$, where ρ_c and p_c are the central density and pressure, respectively, the integration of (74) gives the following equation for A_1:

$$\rho = 2\frac{d^2 A_1}{dr^2} + \frac{4}{r}\frac{dA_1}{dr} = k(a_{1r_1} - A_1)^{1/(\gamma-1)} \tag{86}$$

with $a_{1r_1} = A_1(r_1)$ being the value of A_1 at the zero-pressure surface and

$$k = \left(\frac{\rho_c^\gamma}{p_c}\frac{(\gamma - 1)}{\gamma}\right)^{1/(\gamma-1)}. \tag{87}$$

The homogeneous version of (77) then becomes

$$\frac{d^2 h_{21h}}{dr^2} + \frac{2}{r}\frac{dh_{21h}}{dr}$$
$$- \frac{1}{2}\left(\frac{-k}{\gamma - 1}(a_{1r_1} - A_1)^{(2-\gamma)/(\gamma-1)} + \frac{12}{r^2}\right)h_{21h} = 0. \tag{88}$$

From (86) it follows that

$$\frac{d^2 A_1}{dr^2} \Big/ \frac{dA_1}{dr}\bigg|_{r=r_1} = -2/r_1 \tag{89}$$

on the zero-pressure surface $r = r_1$. Substitution of this and (80) into (79) now gives

$$c_1 = -\frac{5}{48}\frac{r^5\omega_{1/2}^2}{M^5}\frac{(\frac{dh_{21h}}{dr}/h_{21h} - 2/r)}{(\frac{dh_{21h}}{dr}/h_{21h} + 3/r)}\bigg|_{r=r_1}. \tag{90}$$

Let $h_{21h} \neq 0$ be the solution of (88) that makes the spacetime asymptotically flat. Substitution of $h_{21} = -\omega_{1/2}^2 r^2/3 + h_{21h}$ in (41) then gives

$$q_1 = \frac{1}{15r^2}\left(9h_{21h} + 3r\frac{dh_{21h}}{dr} - 5\omega_{1/2}^2 r^2\right)\bigg|_{r=r_1} = 0 \tag{91}$$

from which the denominator of (90) is

$$\left(\frac{dh_{21h}}{dr}/h_{21h} + \frac{3}{r}\right)\bigg|_{r=r_1} = \frac{5\omega_{1/2}^2}{3h_{21h}}. \tag{92}$$

Note that $h_{21h} = 0$ would give a nonzero q_1.

With the identifications $\gamma \equiv 1 + 1/n$,

$$A_1 \equiv -\lambda\Phi + a_{1r_1}, \qquad h_{21h} \equiv -\frac{\lambda}{2}\phi_2 \tag{93}$$

in terms of the functions Φ (that corresponds to the Newtonian potential for the nonrotating configuration) and ϕ_2 used in [7], and a rescaling of the r coordinate $r \equiv \sqrt{2/\bar{k}}s$, with $\bar{k} = k\lambda^{n-1}$, Eqs. (86) and (88) read as

$$\frac{d^2\Phi}{ds^2} + \frac{2}{s}\frac{d\Phi}{ds} + \Phi^n = 0 \tag{94}$$

and

$$\frac{d^2\phi_2}{ds^2} + \frac{2}{s}\frac{d\phi_2}{ds} + \left(n\Phi^{n-1} - \frac{6}{s^2}\right)\phi_2 = 0, \tag{95}$$

respectively. Hence these are exactly the same as Eqs. (23) and (41) in the paper [7] by Martín *et al.* By rescaling the time coordinate to first order in λ the central data for their function Φ can be met and hence we may rely on their theorem on page 10. From their proof the following inequality holds for regular solutions ϕ_2 of (95)

$$\frac{\phi_2'(s_1)}{\phi_2(s_1)} - \frac{2}{s_1} < 0, \tag{96}$$

with $\phi_2' \equiv d\phi_2/ds$ and $s_1 = \sqrt{\bar{k}/2}r_1$, or in our notation

$$\frac{dh_{21h}/dr}{h_{21h}}\bigg|_{r=r_1} - \frac{2}{r_1} < 0. \tag{97}$$

That $\phi_2(s_1) \neq 0$ and hence also $h_{21h}(r_1) \neq 0$ is guaranteed by their lemma. But (97) is actually exactly one of the factors in the numerator of (90) and hence we conclude that $c_1 \neq 0$.

Substitution of (92) in (90) now gives that $c_1 > 0$ iff $h_{21h|r=r_1} > 0$, or equivalently if the fluid ball is oblate in shape. Hence $\Delta Q/Q = 16M^4c_1/5a^2 > 0$ for oblate Newtonian polytropes.

VII. CONCLUSIONS

It seems that generically the modulus of the quadrupole moment for slowly and rigidly rotating perfect fluid balls is larger than that of the Kerr metric. Also, typically it becomes large for small central pressures. For the incompressible case the ratio $\Delta Q/Q$ is a monotonically decreasing function of central pressure. In the limit, when central pressure goes to infinity, it approaches a small, but positive value. Newtonian polytropes show a slightly more complicated behavior, and $\Delta Q/Q$ possesses a minimum for $\gamma < 2$. However, these minimal values are always larger than 0.2. The relativistic polytropes behave in a similar way, and the values of the minima are even higher. For linear equations of state something interesting happens. When the constant d_1 is less than -1, the configuration becomes infinite in extent for a finite central pressure. In this limit the zero-pressure surface approaches the event horizon and $\Delta Q/Q$ goes to zero. Also the central regions of these fluids approach the anti-de Sitter spacetime. All of these solution are quite unphysical since they have a central region with negative density. For the more realistic values of d_1, i.e. $d_1 > 1$, the curves for $\Delta Q/Q$ show minima, but with values larger than 0.2.

In a post-Minkowskian approximation the field equations simplify, but still the equation of state can make the problem analytically unsolvable. Generically $\Delta Q/Q$ seems to go to infinity as λ^{-1}. For the incompressible case the equations can be solved completely and the result agrees with [6]. We further verify the result of [7] that Newtonian polytropes cannot be matched to Kerr and also show that $\Delta Q/Q > 0$ if the rotating configuration is oblate. The differences in approach were essentially that they used global harmonic coordinates and the Lichnerowicz matching conditions, whereas we used the Darmois-Israel matching procedure. The equivalence of the Lichnerowicz and Darmois-Israel matching conditions has been demonstrated in the post-Minkowskian limit for the cases of Wahlquist and polytropes by Cuchí et al.; see [37].

[1] G. Neugebauer and R. Meinel, Astrophys. J. **414**, L97 (1993).
[2] J. B. Hartle and K. S. Thorne, Astrophys. J. **153**, 807 (1968).
[3] S. Chandrasekhar and J. C. Miller, Mon. Not. R. Astron. Soc. **167**, 63 (1974).
[4] E. Berti, F. White, A. Maniopoulou, and M. Bruni, Mon. Not. R. Astron. Soc. **358**, 923 (2005).
[5] W. G. Laarakkers and E. Poisson, Astrophys. J. **512**, 282 (1999).
[6] J. A. Cabezas, J. Martín, A. Molina, and E. Ruiz, Gen. Relativ. Gravit. **39**, 707 (2007).
[7] J. Martín, A. Molina, and E. Ruiz, Classical Quantum Gravity **25**, 105019 (2008).
[8] W. C. Hernandez, Jr., Phys. Rev. **159**, 1070 (1967).
[9] N. Stergioulas, Living Rev. Relativity **6**, 3 (2003), http://www.livingreviews.org/lrr-2003-3.
[10] J. B. Hartle, Astrophys. J. **150**, 1005 (1967).
[11] M. Bradley, D. Eriksson, G. Fodor, and I. Rácz, Phys. Rev. D **75**, 024013 (2007).
[12] G. Fodor, C. A. Hoenselaers, and Z. Perjés, J. Math. Phys. (N.Y.) **30**, 2252 (1989).
[13] G. Darmois, in *Mémorial de Sciences Mathématiques*

(Gauthier-Villars, Paris, 1927), Vol. 25, Chap. 5.
[14] W. Israel, Nuovo Cimento B **44**, 1 (1966).
[15] M. A. H. MacCallum, M. Mars, and R. Vera, Phys. Rev. D **75**, 024017 (2007).
[16] G. Fodor, in *Relativity Today. Proceedings of the Sixth Hungarian Relativity Workshop*, edited by A. Kiadó (Hoenselaers & Perjés, Budapest, 2002); arXiv:gr-qc/0612080.
[17] M. Bradley, G. Fodor, M. Marklund, and Z. Perjés, Classical Quantum Gravity **17**, 351 (2000).
[18] M. Bradley, G. Fodor, and Z. Perjés, Classical Quantum Gravity **17**, 2635 (2000).
[19] M. Mars and J. M. M. Senovilla, Mod. Phys. Lett. A **13**, 1509 (1998).
[20] R. Geroch, J. Math. Phys. (N.Y.) **11**, 2580 (1970).
[21] R. O. Hansen, J. Math. Phys. (N.Y.) **15**, 46 (1974).
[22] K. S. Thorne, Rev. Mod. Phys. **52**, 299 (1980).
[23] W. Simon and R. Beig, J. Math. Phys. (N.Y.) **24**, 1163 (1983); F. J. Ernst, in *Solutions of Einstein's Equations, Techniques and Results*, edited by C. Hoenselaers and W. Dietz (Springer, Heidelberg, 1984).
[24] T. Bäckdahl and M. Herberthson, Classical Quantum Gravity **22**, 3585 (2005).

[25] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon Press, Oxford, 1975) 4th ed..

[26] W. Roos, Gen. Relativ. Gravit. **7**, 431 (1976).

[27] G. Fodor, arXiv:gr-qc/0011040.

[28] H. D. Wahlquist, Phys. Rev. **172**, 1291 (1968).

[29] J. M. Whittaker, Proc. R. Soc. A **306**, 1 (1968).

[30] P. Sarnobat and C. A. Hoenselaers, Classical Quantum Gravity **23**, 5603 (2006).

[31] E. Mukumu, Master's thesis, Umeå University, 2008.

[32] H. A. Buchdahl, Phys. Rev. **116**, 1027 (1959).

[33] H. Andréasson, J. Differ. Eqs. **245**, 2243 (2008).

[34] U. S. Nilsson and C. Uggla Ann. Phys. (N.Y.) **286**, 292 (2000).

[35] J. M. Heinzle, N. Röhr, and C. Uggla, Classical Quantum Gravity **20**, 4567 (2003).

[36] A. Lichnerowicz, *Théories Relativistes de la Gravitation et de l'Electromagnetisme* (Masson, Paris, 1955).

[37] J. E. Cuchí, A. Molina, and E. Ruiz, in Proceedings of ERE, Salamanca, Spain, 2008, http://www.usal.es/~ere2008/website/modules/tinyd0/content/pdf/Cuchi.pdf.