PHYSICAL REVIEW D **79,** 044008 (2009)

Topological interpretation of Barbero-Immirzi parameter

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(Received 6 December 2008; published 6 February 2009)

We set up a canonical Hamiltonian formulation for a theory of gravity based on a Lagrangian density made up of the Hilbert-Palatini term and, instead of the Holst term, the Nieh-Yan topological density. The resulting set of constraints in the time gauge are shown to lead to a theory in terms of a real $SU(2)$ connection which is exactly the same as that of Barbero and Immirzi with the coefficient of the Nieh-Yan term identified as the inverse of the Barbero-Immirzi parameter. This provides a topological interpretation for this parameter. Matter coupling can then be introduced in the usual manner, *without* changing the universal topological Nieh-Yan term.

DOI: 10.1103/PhysRevD.79.044008 PACS numbers: 04.20.Fy, 04.60.−m, 04.60.Ds, 04.60.Pp

I. INTRODUCTION

The Hilbert-Palatini Lagrangian for pure gravity is written in terms of the connection fields ω_μ^{IJ} and tetrad e_μ^I as independent field variables. Its Holst generalization is given in terms of the Lagrangian density [1]:

$$\mathcal{L} = \frac{1}{2} e \Sigma_{IJ}^{\mu\nu} R_{\mu\nu}{}^{IJ}(\omega) + \frac{\eta}{2} e \Sigma_{IJ}^{\mu\nu} \tilde{R}_{\mu\nu}{}^{IJ}(\omega), \qquad (1)$$

where

$$\Sigma_{IJ}^{\mu\nu} := \tfrac{1}{2}(e_I^\mu e_J^\nu - e_J^\mu e_I^\nu),$$
$$R_{\mu\nu}{}^{IJ}(\omega) := \partial_{[\mu}\omega_{\nu]}{}^{IJ} + \omega_{[\mu}{}^{IK}\omega_{\nu]K}{}^{J},$$
$$\tilde{R}_{\mu\nu}{}^{IJ}(\omega) := \tfrac{1}{2}\epsilon^{IJKL} R_{\mu\nu KL}(\omega).$$

The second term is the Holst term with η^{-1} as the Barbero-Immirzi parameter [2,3]. For $\eta = -i$, this Lagrangian density leads to the canonical formulation in terms of the self-dual Ashtekar connection, which is a *complex* $SU(2)$ connection [4]. For real η, we have a Hamiltonian formulation in terms of a *real* $SU(2)$ connection, which coincides with the Barbero formulation for $\eta = 1$ [2,5].

Inclusion of the Holst term does not change the classical equation of motion of the Hilbert-Palatini action; there is no dependence on η in the equations of motion. In fact, when the connection equation $\omega_\mu{}^{IJ} = \omega_\mu{}^{IJ}(e)$ is used, the Holst term is identically zero.

Adding matter in the generalized Lagrangian density (1) needs special care. In particular, when spin-$\frac{1}{2}$ fermions are included through minimal coupling, the classical equations of motion acquire a dependence on η [6]. However, it is possible to modify the Holst term in such a way that the equations of motion remain unchanged. Such modification for spin-$\frac{1}{2}$ fermionic matter and also those in the $N = 1, 2,$ and 4 supergravities have been obtained [7,8]. *When the connection equation of motion is used,* the modified Holst terms in each of these cases become total divergences involving Nieh-Yan invariant density and divergence of axial current densities involving the fermion fields. The modified Holst term used in these formulations *changes* with the matter content of the theory.

It has been suggested that the Barbero-Immirzi parameter should have a topological interpretation in the same manner as the θ parameter of QCD [9]. For this to be the case, η should be the coefficient of a term in the Lagrangian density which is a *topological density.* Since such a term would be a total derivative for *all* field configurations, the classical equations of motion would remain unaltered. Such a term would be universal in the sense that it would not change when any matter coupling to gravity is introduced. The Holst term in (1) or any of its modifications mentioned above do not have such a property.

In the four-dimensional gravity, there are three possible topological densities, namely, Pontryagin, Euler, and Nieh-Yan. The first two are quadratic in the curvature tensor. The Nieh-Yan density contains a term linear in $R_{\mu\nu}{}^{IJ}(\omega)$ and an R-independent term. This is shown below to be associated with the Barbero-Immirzi parameter.

The Nieh-Yan density is given by [10]

$$I_{\rm NY} = \epsilon^{\mu\nu\alpha\beta}[D_\mu(\omega)e_\nu^I D_\alpha(\omega)e_{I\beta} - \tfrac{1}{2}\Sigma_{\mu\nu}^{IJ} R_{\alpha\beta IJ}(\omega)],$$
$$D_\mu(\omega)e_\nu^I := \partial_\mu e_\nu^I + \omega_{\mu J}^I e_\nu^J. \qquad (2)$$

This is a topological density; that is, it is a total divergence:

$$I_{\rm NY} = \partial_\mu J^\mu_{\rm NY}(e,\omega),$$
$$J^\mu_{\rm NY}(e,\omega) := \epsilon^{\mu\nu\alpha\beta} e_\nu^I D_\alpha(\omega) e_{I\beta}. \qquad (3)$$

Note that, unlike the Pontryagin and Euler densities, the Nieh-Yan density *vanishes identically for a torsion-free connection.*

The classical equations of motion from the Lagrangian density containing the Hilbert-Palatini term as well as the Nieh-Yan density

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$$\mathcal{L} = \frac{1}{2} e \Sigma^{\mu\nu}_{IJ} R_{\mu\nu}{}^{IJ}(\omega) + \frac{\eta}{2} I_{\rm NY} \tag{4}$$

are the same as those from the Hilbert-Palatini Lagrangian alone. We shall demonstrate that the canonical Hamiltonian formulation based on this new Lagrangian density also leads to a theory of real $SU(2)$ connections, exactly the same as that emerging from the theory with the original Holst term. This in turn, for $\eta = 1$, is the Barbero formulation. Inclusion of matter now does not need any further modification, and equations of motion continue to be independent of η for all couplings. This also allows a direct interpretation of the η parameter as a topological parameter in a manner analogous to the θ parameter in QCD.

In a quantum framework, it is also possible to arrive at the canonical formulation based on the Lagrangian density (4) starting from the Hilbert-Palatini canonical formulation by rescaling the wave functional by $\exp\{\frac{i\eta}{2} \times \int d^3x J^t_{\rm NY}(e,\omega)\}$. Mercuri has used this approach to derive the canonical formulation containing the Barbero-Immirzi parameter for a theory with spin-$\frac{1}{2}$ fermions [11]. This demonstrated, for the first time, the role of Nieh-Yan density as the source of the quantization ambiguity reflected by the Barbero-Immirzi parameter. However, in this analysis the connection equation of motion has been used to express the $J^t_{\rm NY}(e,\omega)$ in terms of the fermions. It is desirable to carry out this procedure, retaining the $J^t_{\rm NY}$ as in Eq. (3) in terms of the original geometric variables. Such a method then can be applied directly to a theory of gravity with or without matter.

In this paper, we work within a classical framework. In Sec. II, we describe the Hamiltonian formulation based on the Lagrangian density (4) closely following the analysis carried out by Sa [5] for the Hilbert-Palatini gravity with the Holst term. In Sec. III, we discuss the matter couplings, in particular, the case of Dirac fermions. Coupling of any other matter can be done in an analogous and straightforward manner. Section IV contains a few concluding remarks.

II. HAMILTONIAN ANALYSIS

We propose the Lagrangian density for pure gravity to be that given in Eq. (4), rewritten as

$$\begin{aligned}\mathcal{L} &= \frac{1}{2} e \Sigma^{\mu\nu}_{IJ} R_{\mu\nu}{}^{IJ}(\omega) + \frac{\eta}{2}[e \Sigma^{\mu\nu}_{IJ} \tilde{R}_{\mu\nu}{}^{IJ}(\omega) \\ &\quad + \epsilon^{\mu\nu\alpha\beta} D_\mu(\omega) e^I_\nu D_\alpha(\omega) e_{I\beta}] \\ &= \frac{1}{2} e \Sigma^{\mu\nu}_{IJ} R^{(\eta)IJ}_{\mu\nu}(\omega) + \frac{\eta}{2} \epsilon^{\mu\nu\alpha\beta} D_\mu(\omega) e^I_\nu D_\alpha(\omega) e_{I\beta},\end{aligned} \tag{5}$$

where $R^{(\eta)IJ}_{\mu\nu}(\omega) := R_{\mu\nu}{}^{IJ}(\omega) + \eta \tilde{R}_{\mu\nu}{}^{IJ}(\omega)$ and we have used the identities

$$\begin{aligned}\Sigma^{\mu\nu}_{IJ} \tilde{R}_{\mu\nu}{}^{IJ}(\omega) &= \tilde{\Sigma}^{\mu\nu}_{IJ} R_{\mu\nu}{}^{IJ}(\omega), \\ e\tilde{\Sigma}^{\mu\nu}_{IJ} &:= \frac{e}{2} \epsilon_{IJKL} \Sigma^{\mu\nu KL} = -\frac{1}{2} \epsilon^{\mu\nu\alpha\beta} \Sigma_{\alpha\beta IJ}.\end{aligned} \tag{6}$$

Introducing the notation $t^a_I := \eta \epsilon^{abc} D_b(\omega) e_{Ic}$ and $\epsilon^{abc} := \epsilon^{tabc}$, the 3 + 1 decomposition is expressed as

$$\begin{aligned}\mathcal{L} &= e \Sigma^{ta}_{IJ} R^{(\eta)IJ}_{ta}(\omega) + \frac{e}{2} \Sigma^{ab}_{IJ} R^{(\eta)IJ}_{ab}(\omega) \\ &\quad + t^a_I (D_t(\omega) e^I_a - D_a(\omega) e^I_t).\end{aligned} \tag{7}$$

Defining $\omega^{(\eta)IJ}_a := \omega^{IJ}_a + \eta \tilde{\omega}^{IJ}_a$ and $\Sigma^{(\eta)ta}_{IJ} := \Sigma^{ta}_{IJ} + \eta \tilde{\Sigma}^{ta}_{IJ}$, we get

$$\begin{aligned}\mathcal{L} &= e \Sigma^{ta}_{IJ} \partial_t \omega^{(\eta)IJ}_a + \omega^{IJ}_t D_a(\omega)(e \Sigma^{(\eta)ta}_{IJ}) \\ &\quad + \frac{e}{2} \Sigma^{ab}_{IJ} R^{(\eta)IJ}_{ab}(\omega) + t^a_I \partial_t e^I_a + \omega_t{}^{IJ} t^a_I e_{aJ} \\ &\quad + e^I_t D_a(\omega) t^a_I - \partial_a (t^a_I e^I_t + e \Sigma^{(\eta)ta}_{IJ} \omega^{IJ}_t).\end{aligned} \tag{8}$$

We parametrize the tetrad fields as

$$\begin{aligned} e^I_t &= \sqrt{eN} M^I + N^a V^I_a, \qquad e^I_a = V^I_a; \\ M_I V^I_a &= 0, \qquad M_I M^I = -1,\end{aligned} \tag{9}$$

and then the inverse tetrad fields are

$$\begin{aligned} e^t_I &= -\frac{M_I}{\sqrt{eN}}, \qquad e^a_I = V^a_I + \frac{N^a M_I}{\sqrt{eN}}; \\ M^I V^a_I &:= 0, \qquad V^I_a V^b_I := \delta^b_a, \\ V^I_a V^a_J &:= \delta^I_J + M^I M_J.\end{aligned} \tag{10}$$

Defining $q_{ab} := V^I_a V_{bI}$ and $q := \det q_{ab}$ leads to $e := \det(e^I_\mu) = Nq$. We may thus trade the 16 tetrad fields with the 9 fields $V^a_I (M^I V^a_I = 0)$, the 3 fields $M^I (M_I M^I = -1)$, and the 4 fields N and N^a.

Next, using the identity

$$\Sigma^{ab}_{IJ} = 2Ne \Sigma^{t[a}_{IK} \Sigma^{b]t}_{JL} \eta^{KL} + N^{[a} \Sigma^{b]t}_{IJ} \tag{11}$$

and dropping the total space derivative terms

$$\mathcal{L} = e \Sigma^{ta}_{IJ} \partial_t \omega^{(\eta)IJ}_a + t^a_I \partial_t e^I_a - NH - N^a H_a - \tfrac{1}{2} \omega^{IJ}_t G_{IJ}, \tag{12}$$

where $2e\Sigma^{ta}_{IJ} = -\sqrt{q} M_{[I} V^a_{J]}$, $t^a_I := \eta \epsilon^{abc} D_b(\omega) V_{Ic}$, and

$$H = 2e^2 \Sigma^{ta}_{IK} \Sigma^{tb}_{JL} \eta^{KL} R^{(\eta)IJ}_{ab}(\omega) - \sqrt{q} M^I D_a(\omega) t^a_I, \tag{13}$$

$$H_a = e \Sigma^{tb}_{IJ} R^{(\eta)IJ}_{ab}(\omega) - V^I_a D_b(\omega) t^b_I, \tag{14}$$

$$G_{IJ} = -2D_a(\omega)(e \Sigma^{(\eta)ta}_{IJ}) - t^a_{[I} V_{J]} a. \tag{15}$$

Introduce the fields

$$\begin{aligned} E^a_i &:= 2e \Sigma^{ta}_{0i}, \qquad \chi_i := -M_i / M^0, \\ A^i_a &:= \omega^{(\eta)0i}_a - \chi_j \omega^{(\eta)ij}_a, \qquad \zeta^i := -E^a_j \omega^{(\eta)ij}_a.\end{aligned} \tag{16}$$

In terms of these, we have $2e\Sigma^{ta}_{ij} = -E^a_{[i}\chi_{j]}$ and $e\Sigma^{ta}_{IJ}\partial_t\omega^{(\eta)IJ}_a = E^a_i\partial_t A^i_a + \zeta^i\partial_t\chi^i$, and the Lagrangian density is

$$\mathcal{L} = E^a_i\partial_t A^i_a + \zeta^i\partial_t\chi^i + t^a_I\partial_t V^I_a - NH - N^a H_a - \tfrac{1}{2}\omega^{IJ}_t G_{IJ}, \tag{17}$$

where, now we need to reexpress H, H_a, and G_{IJ} ($G^i_{\rm boost} := G_{0i}$, $G^i_{\rm rot} := \frac{1}{2}\epsilon^{ijk}G_{jk}$) in terms of these new fields:

$$G^i_{\rm boost} = -\partial_a(E^a_i - \eta\epsilon^{ijk}E^a_j\chi_k) + E^a_{[i}\chi_{k]}A^k_a + (\zeta^i - \chi\cdot\zeta\chi^i) - t^a_{[0}V_{i]a}, \tag{18}$$

$$G^i_{\rm rot} = \partial_a(\epsilon^{ijk}E^a_j\chi_k + \eta E^a_i) + \epsilon^{ijk}(A^j_a E^a_k - \zeta_j\chi_k - t^a_j V^k_a), \tag{19}$$

$$H_a = E^b_i[R^{(\eta)0i}_{ab}(\omega) - \chi_j R^{(\eta)ij}_{ab}(\omega)] - V^I_a D_b(\omega)t^b_I \tag{20}$$

$$\begin{aligned} &= E^b_i\partial_{[a}A^i_{b]} + \zeta^i\partial_a\chi^i - V^I_a\partial_b t^b_I + t^b_I\partial_{[a}V^I_{b]} \\ &\quad - \frac{1}{1+\eta^2}[E^b_{[i}\chi_{l]}A^l_b + (\zeta_i - \chi\cdot\zeta\chi_i) - t^b_{[0}V_{i]b} - \eta\epsilon^{ijk}(A^j_b E^b_k - \zeta_j\chi_k - t^b_j V^k_b)]A^i_a \\ &\quad - \frac{1}{1+\eta^2}\left[\frac{1}{2}\epsilon^{ijk}(\eta G^k_{\rm boost} + G^k_{\rm rot}) - \chi^i(G^j_{\rm boost} - \eta G^j_{\rm rot})\right]\omega^{(\eta)ij}_a, \end{aligned} \tag{21}$$

$$\begin{aligned} H &= -E^a_k\chi_k H_a - \frac{1}{2}(1-\chi\cdot\chi)E^a_i E^b_j R^{(\eta)ij}_{ab}(\omega) - (E^a_k\chi_k V^I_a + \sqrt{q}M^I)D_b(\omega)t^b_I \\ &= -E^a_k\chi_k H_a + (1-\chi\cdot\chi)\left[E^a_i\partial_a\zeta_i + \frac{1}{2}\zeta_i E^a_i E^b_j\partial_a E^j_b\right] + \frac{1-\chi\cdot\chi}{2(1+\eta^2)}\zeta_i[-G^i_{\rm boost} + \eta G^i_{\rm rot}] - (E^a_k\chi_k V^I_a + \sqrt{q}M^I)\partial_b t^b_I \\ &\quad - \frac{1-\chi\cdot\chi}{1+\eta^2}\left[\frac{1}{2}E^a_{[i}E^b_{j]}A^i_a A^j_b + E^a_i A^i_a\chi\cdot\zeta + \eta\epsilon^{ijk}\zeta_i A^j_a E^a_k + \frac{3}{4}(\chi\cdot\zeta)^2 - \frac{3}{4}(\zeta\cdot\zeta) + \frac{1}{2}\zeta_i t^a_{[0}V_{i]a} - \frac{\eta}{2}\zeta_i\epsilon^{ijk}t^a_j V^k_a\right] \\ &\quad + \frac{1-\chi\cdot\chi}{1+\eta^2}\left[\frac{1}{\sqrt{E}}A^i_a t^a_i + \frac{1}{2}V^i_a(\zeta\cdot\chi t^a_i - \chi_i\zeta_j t^a_j + \eta\epsilon^{ijk}\zeta_j t^a_k)\right] \\ &\quad + \frac{1-\chi\cdot\chi}{1+\eta^2}\left[-\frac{1}{\sqrt{E}}\chi_i t^b_j + \frac{\eta}{2\sqrt{E}}\epsilon^{ijk}t^b_k + (1+\eta^2)E^a_i\partial_a E^b_j + E^a_i\chi_j E^b_m A^m_a - \eta\epsilon^{imn}E^a_m E^b_n A^j_a \right. \\ &\quad \left. - \frac{\eta}{4}(\epsilon^{ijm}E^b_n + \epsilon^{ijn}E^b_m)\chi_m\zeta_n\right]u^{ij}_b + \frac{1-\chi\cdot\chi}{2(1+\eta^2)}(\chi_m\chi_n - \delta_{mn})E^a_j E^b_i u^{im}_a u^{jn}_b. \end{aligned} \tag{22}$$

In the above, $E^i_a := \sqrt{E}V^i_a$ is the inverse of E^a_i, i.e., $E^i_a E^b_i = \delta^b_a$, $E^i_a E^a_j = \delta^i_j$, and $E^{-1} = q(M^0)^2$ equals $\det E^a_i$. Furthermore, we have also set $u^{ij}_a := \omega^{(\eta)ij}_a - \frac{1}{2}E^{[i}_a\zeta^{j]}$. Notice that $E^b_i u^{ij}_b = 0$. The six independent fields in u^{ij}_a may be parametrized in terms of a symmetric matrix M^{ij} as $u^{ij}_a := \frac{1}{2}\epsilon^{ijk}E^l_a M^{kl}$ [5].

We have replaced the original 16 tetrad fields with 16 new fields: E^a_i, χ_i, N, and N^a. In place of the original 24 connection fields ω^{IJ}_μ, we use the new set of 24 fields A^i_a, ζ_i, M^{kl}, ω^{ij}_t, and ω^{0i}_t. The fields V^I_a and t^a_I are not independent; these are given in terms of the fundamental fields as $V^I_a = v^I_a$ and $t^a_I = \tau^a_I$, where

$$v^0_a := -\frac{1}{\sqrt{E}}E^i_a\chi_i, \qquad v^i_a := \frac{1}{\sqrt{E}}E^i_a, \tag{23}$$

$$\begin{aligned} \tau^a_0 &:= \eta\epsilon^{abc}D_b(\omega)V_{0c} \\ &= \eta\sqrt{E}E^a_m\left[G^m_{\rm rot} - \frac{\chi_l}{2}\left(\frac{2f_{ml} + N_{ml}}{1+\eta^2} + \epsilon_{mln}G^n_{\rm boost}\right)\right], \end{aligned} \tag{24}$$

$$\begin{aligned} \tau^a_k &:= \eta\epsilon^{abc}D_b(\omega)V_{ck} \\ &= -\frac{\eta}{2}\sqrt{E}E^a_m\left[\frac{2f_{mk} + N_{mk}}{1+\eta^2} + \epsilon_{kmn}G^n_{\rm boost}\right], \end{aligned} \tag{25}$$

where

$$\begin{aligned} 2f_{kl} &:= \epsilon_{ijk}E^a_i[(1+\eta^2)E^l_b\partial_a E^b_j + \chi_j A^l_a] \\ &\quad + \eta(E^a_l A^k_a - \delta^{kl}E^a_m A^m_a - \chi_l\zeta_k) + (l \leftrightarrow k), \end{aligned} \tag{26}$$

$$\begin{aligned} N_{kl} &:= \epsilon^{ijk}(\chi_m\chi_j - \delta_{mj})E^a_i u^{lm}_a + (l \leftrightarrow k) \\ &= (\chi\cdot\chi - 1)(M_{kl} - M_{mm}\delta_{kl}) + \chi_m\chi_n M_{mn}\delta_{kl} \\ &\quad + \chi_l\chi_k M_{mm} - \chi_m(\chi_k M_{ml} + \chi_l M_{mk}). \end{aligned} \tag{27}$$

We can upgrade V^I_a and t^a_I as independent fields through terms containing the Lagrange multiplier fields ξ^a_I and ϕ^I_a in the Lagrangian density:

$$\mathcal{L} = E^a_i \partial_t A^i_a + \zeta^i \partial_t \chi^i + t^a_I \partial_t V^I_a - \mathcal{H},$$
$$\mathcal{H} := NH + N^a H_a + \tfrac{1}{2}\omega^{IJ}_t G_{IJ} + \xi^a_I (V^I_a - v^I_a) + \phi^I_a (t^a_I - \tau^a_I), \tag{28}$$

where v^I_a and τ^a_I are defined in Eqs. (23)–(25). We have 24 pairs of canonically conjugate independent field variables (E^a_i, A^j_b), (ζ^i, χ^j), and (t^a_I, V^I_a). The remaining fields, namely, N, N^a, ω^{IJ}_t, ξ^a_I, ϕ^I_a, and M^{kl}, have no conjugate momenta since in the Lagrangian their velocities do not appear. Preservation of these constraints (vanishing of the variation of the Hamiltonian with respect to the fields) leads to the secondary constraints. From the variations with respect to fields ω^{0i}_t, ω^{ij}_t, N^a, N, ξ^a_I, and ϕ^I_a, we get the constraints

$$G^i_{\text{boost}} \approx 0, \qquad G^i_{\text{rot}} \approx 0; \qquad H_a \approx 0, \qquad H \approx 0; \tag{29}$$

$$V^I_a - v^I_a \approx 0, \qquad t^a_I - \tau^a_I \approx 0. \tag{30}$$

From the variation with respect to M^{kl} or equivalently u^{ij}_a, we get

$$\frac{\delta\mathcal{H}}{\delta M^{kl}}\delta M^{kl} \approx \frac{N\delta H}{\delta M^{kl}}\delta M^{kl} = \frac{N(1-\chi\cdot\chi)}{2(1+\eta^2)}\Big[(\eta t^a_k - \epsilon^{ijk}\chi_i t^a_j)V^l_a + f_{kl} + \frac{1}{2}N_{kl}\Big]\delta M^{kl} \approx 0.$$

This leads to

$$(\eta t^a_k - \epsilon^{ijk}\chi_i t^a_j)V^l_a + f_{kl} + \tfrac{1}{2}N_{kl} + (k \leftrightarrow l) \approx 0. \tag{31}$$

Using constraints (30), and the expressions (23)–(25) for τ^a_I, v^I_a, Eq. (31) implies

$$(\eta\epsilon^{ijk}\chi_i + \delta_{kj})(2f_{jl} + N_{jl}) + \eta(1+\eta^2)(\delta^{kl}\chi_m G^m_{\text{boost}} - \chi_l G^k_{\text{boost}}) + (k \leftrightarrow l) \approx 0.$$

Using (29), this in turn implies the constraint

$$2f_{kl} + N_{kl} \approx 0, \tag{32}$$

where f_{kl} and N_{kl} are given in (26) and (27). This constraint can be solved for M_{kl}. Furthermore, it implies, from the definitions (24) and (25), that $\tau^a_I \approx 0$ and hence

$$t^a_I \approx 0. \tag{33}$$

Implementing this constraint then reduces the Hamiltonian density to

$$\mathcal{H} = NH + N^a H_a + \tfrac{1}{2}\omega^{IJ}_t G_{IJ}, \tag{34}$$

where now

$$G^i_{\text{boost}} = -\partial_a(E^a_i - \eta\epsilon^{ijk}E^a_j\chi_k) + E^a_{[i}\chi_{k]}A^k_a + (\zeta^i - \chi\cdot\zeta\zeta^i) \approx 0, \tag{35}$$

$$G^i_{\text{rot}} = \partial_a(\epsilon^{ijk}E^a_j\chi_k + \eta E^a_i) + \epsilon^{ijk}(A^j_a E^a_k - \zeta_j\chi_k) \approx 0, \tag{36}$$

$$H_a = E^b_i \partial_{[a}A^i_{b]} + \zeta_i\partial_a\chi_i - \frac{1}{1+\eta^2}[E^b_{[k}\chi_{l]}A^l_b + \zeta_i - \chi\cdot\zeta\chi^i - \eta\epsilon^{ijk}(A^j_a E^a_k - \zeta_j\chi_k)]A^i_a - \frac{1}{1+\eta^2}\Big[\frac{1}{2}\epsilon^{ijk}(\eta G^k_{\text{boost}} + G^k_{\text{rot}}) - \chi^i(G^j_{\text{boost}} - \eta G^j_{\text{rot}})\Big]\omega^{(\eta)ij}_a \approx 0, \tag{37}$$

$$H = -E^a_k\chi_k H_a + (1-\chi\cdot\chi)\Big[E^a_i\partial_a\zeta_i + \frac{1}{2}\zeta_i E^a_i E^b_j \partial_a E^j_b\Big] + \frac{(1-\chi\cdot\chi)}{2(1+\eta^2)}\zeta_i[-G^j_{\text{boost}} + \eta G^j_{\text{rot}}] - \frac{(1-\chi\cdot\chi)}{1+\eta^2}\Big[\frac{1}{2}E^a_{[i}E^b_{j]}A^i_a A^j_b + E^a_i A^i_a \chi\cdot\zeta + \eta\epsilon_{ijk}\zeta_i A^j_a E^a_k + \frac{3}{4}(\chi\cdot\zeta)^2 - \frac{3}{4}(\zeta\cdot\zeta)\Big] + \frac{(1-\chi\cdot\chi)}{2(1+\eta^2)}\Big[f_{kl}M^{kl} + \frac{1}{4}(\chi\cdot\chi - 1)(M^{kl}M^{kl} - M^{kk}M^{ll}) + \frac{1}{2}\chi_k\chi_l(M^{pp}M^{kl} - M^{kp}M^{lp})\Big] \approx 0. \tag{38}$$

In the last equation we have M^{kl} given by the constraint $2f_{kl} + N_{kl} = 0$, which can be solved as

$$(1-\chi\cdot\chi)M_{kl} = 2f_{kl} + (\chi_m\chi_n f_{mn} - f_{mm})\delta_{lk} + (\chi_m\chi_n f_{mn} + f_{mm})\chi_k\chi_l - 2\chi_m(\chi_l f_{mk} + \chi_k f_{ml}). \tag{39}$$

This is the same set of equations as those obtained by Sa [5] in his analysis of the action containing the Holst term.

We may fix the boost gauge transformations (time gauge) by imposing $\chi^i \approx 0$ which together with the $G^i_{\text{boost}} \approx 0$ forms a second class pair. Solving the boost constraint with $\chi^i = 0$ yields

$$\zeta_i = \partial_a E^a_i. \tag{40}$$

In this gauge we then recover a canonical Hamiltonian formulation in terms of real $SU(2)$ gauge fields A^i_a which reduces to the Barbero formulation for $\eta = 1$ [5].

To summarize, like the Holst term, the Nieh-Yan term leads to an $SU(2)$ gauge theoretic formulation. But it is only the coefficient of the Nieh-Yan term that has a topological character.

III. MATTER COUPLING

As stated earlier, the matter can now be coupled to gravity in a straightforward manner. As an example, we consider a spin-$\frac{1}{2}$ Dirac fermion with its usual *minimal* coupling to gravity. The Lagrangian density is[1]

$$\mathcal{L} = \frac{1}{2} e \Sigma^{\mu\nu}_{IJ} R_{\mu\nu}{}^{IJ}(\omega) + \frac{\eta}{2} I_{\rm NY} + \frac{ie}{2}[\bar{\lambda}\gamma^\mu D_\mu(\omega)\lambda - \overline{D_\mu(\omega)\lambda}\gamma^\mu\lambda], \tag{41}$$

where

$$D_\mu(\omega)\lambda := \partial_\mu \lambda + \tfrac{1}{2}\omega_{\mu IJ}\sigma^{IJ}\lambda,$$
$$\overline{D_\mu(\omega)\lambda} := \partial_\mu \bar{\lambda} - \tfrac{1}{2}\bar{\lambda}\omega_{\mu IJ}\sigma^{IJ}.$$

Notice that, unlike earlier attempts of setting up a theory of fermions and gravity with the Barbero-Immirzi parameter [7,8] where the Holst term was modified to include an additional nonminimal term for the fermions, the Lagrangian density here containing the Nieh-Yan density does not require any further modification; just the usual minimal fermion terms suffice. This is so because the Nieh-Yan term is topological.

We expand the fermion terms as

$$\begin{aligned}\mathcal{L}(F) &:= \frac{ie}{2}[\bar{\lambda}\gamma^\mu D_\mu(\omega)\lambda - \overline{D_\mu(\omega)\lambda}\gamma^\mu\lambda] \\ &= [\partial_t\bar{\lambda}\Pi - \bar{\Pi}\partial_t\lambda] - NH(F) - N^a H_a(F) \\ &\quad - \frac{1}{2}\omega_t^{IJ}G_{IJ}(F),\end{aligned} \tag{42}$$

where $\bar{\Pi}$ and Π are canonically conjugate momenta fields associated with λ and $\bar{\lambda}$, respectively. Explicitly,[2]

$$\begin{aligned}\bar{\Pi} &= -\frac{ie}{2}\bar{\lambda}\gamma^t = \frac{i\sqrt{q}}{2}M_I\bar{\lambda}\gamma^I, \\ \Pi &= -\frac{ie}{2}\gamma^t\lambda = \frac{i\sqrt{q}}{2}M_I\gamma^I\lambda,\end{aligned} \tag{43}$$

$$G^{IJ}(F) = \bar{\Pi}\sigma^{IJ}\lambda + \bar{\lambda}\sigma^{IJ}\Pi, \tag{44}$$

$$H_a(F) = \overline{D_a(\omega)\lambda}\Pi - \bar{\Pi}D_a(\omega)\lambda, \tag{45}$$

$$H(F) = (-2e\Sigma^{ta}_{IJ})[\overline{D_a(\omega)\lambda}\sigma^{IJ}\Pi + \bar{\Pi}\sigma^{IJ}D_a(\omega)\lambda]. \tag{46}$$

Incorporating these fermionic terms in the pure gravity Lagrangian density given in Eq. (28), we write the full Lagrangian density as

$$\begin{aligned}\mathcal{L} &= E^a_i\partial_t A^i_a + \zeta^i\partial_t\chi_i + t^a_I\partial_t V^I_a + \partial_t\bar{\lambda}\Pi - \bar{\Pi}\partial_t\lambda \\ &\quad - NH' - N^a H'_a - \tfrac{1}{2}\omega_t^{IJ}G'_{IJ} - \xi^a_I(V^I_a - v^I_a) \\ &\quad - \phi^I_a(t^a_I - \tau^a_I),\end{aligned} \tag{47}$$

where now

$$G'_{IJ} = G^{IJ} + G^{IJ}(F), \qquad H'_a = H_a + H_a(F), \qquad H' = H + H(F), \tag{48}$$

with G^{IJ}, H_a, and H as the contributions from the pure gravity sector as given by Eqs. (13)–(15) or equivalently by Eqs. (19)–(22).

The various quantities above can then be rewritten in terms of the basic fields as

$$\begin{aligned}G'^i_{\rm boost} &= -\partial_a(E^a_i - \eta\epsilon^{ijk}E^a_j\chi_k) + E^a_{[i}\chi_{k]}A^k_a \\ &\quad + (\zeta^i - \chi\cdot\zeta\chi^i) - t'^a_{[0}V_{i]a} + [\bar{\Pi}(1+i\eta\gamma_5)\sigma_{0i}\lambda \\ &\quad + \bar{\lambda}(1+i\eta\gamma_5)\sigma_{0i}\Pi];\end{aligned} \tag{49}$$

$$\begin{aligned}G'^i_{\rm rot} &= \partial_a(\epsilon^{ijk}E^a_j\chi_k + \eta E^a_i) + \epsilon^{ijk}(A^j_a E^a_k - \zeta_j\chi_k - t'^a_j V^k_a) \\ &\quad + [\bar{\Pi}(i\gamma_5 - \eta)\sigma_{0i}\lambda + \bar{\lambda}(i\gamma_5 - \eta)\sigma_{0i}\Pi];\end{aligned} \tag{50}$$

$$\begin{aligned}H'_a &= E^b_i\partial_{[a}A^i_{b]} + \zeta_i\partial_a\chi_i - \partial_b(t'^b_I V^I_a) + t'^b_I\partial_a V^I_b + [\partial_a\bar{\lambda}(1+i\eta\gamma_5)\Pi - \bar{\Pi}(1+i\eta\gamma_5)\partial_a\lambda] - [\bar{\lambda}\sigma_{0i}\Pi + \bar{\Pi}\sigma_{0i}\lambda]A^i_a \\ &\quad - \frac{1}{1+\eta^2}[E^b_{[i}\chi_{l]}A^l_b + \zeta_i - \chi\cdot\zeta\chi^i - t'^b_{[0}V_{i]b} - \eta\epsilon^{ijk}(A^j_a E^b_k - \chi_j\zeta_k - t'^b_j V^k_b)]A^i_a \\ &\quad - \frac{1}{1+\eta^2}\left[\frac{1}{2}\epsilon^{ijk}(\eta G'^k_{\rm boost} + G'^k_{\rm rot}) - \chi^i(G'^j_{\rm boost} - \eta G'^j_{\rm rot})\right]\omega^{(\eta)ij}_a;\end{aligned} \tag{51}$$

[1] Our Dirac matrices satisfy the Clifford algebra: $\gamma^I\gamma^J + \gamma^J\gamma^I = 2\eta^{IJ}$, $\eta^{IJ} := {\rm diag}(-1, 1, 1, 1)$. The chiral matrix $\gamma_5 := i\gamma^0\gamma^1\gamma^2\gamma^3$ and $\sigma^{IJ} := \frac{1}{4}[\gamma^I, \gamma^J]$.

[2] The fermions are Grassmann-valued, and the functional differentiation is done on the left factor which accounts for the signs in the definitions of the conjugate momenta in (43).

$$
\begin{aligned}
H' = & -E^a_k\chi_k H'_a - (E^a_k\chi_k V^I_a + \sqrt{q}M^I)\partial_b t'^{b}_I + (1-\chi\cdot\chi)\Big[E^a_i\partial_a\zeta_i + \frac{1}{2}\zeta_i E^a_i E^b_j\partial_a E^j_b\Big] \\
& - \frac{(1-\chi\cdot\chi)}{1+\eta^2}\Big[\frac{1}{2}E^a_{[i}E^b_{j]}A^i_a A^j_b + E^a_i A^i_a\chi\cdot\zeta + \eta\epsilon_{ijk}\zeta_i A^j_a E^a_k + \frac{3}{4}(\chi\cdot\zeta)^2 - \frac{3}{4}(\zeta\cdot\zeta)\Big] \\
& + \frac{(1-\chi\cdot\chi)}{2(1+\eta^2)}[\zeta_i - 2V^i_b(t^b_0 - t'^b_0)][-G'^i_{\rm boost} + \eta G'^i_{\rm rot} - t'^a_{[0}V_{i]a} + \eta\epsilon_{ijk}t'^a_j V^k_a] \\
& + \frac{(1-\chi\cdot\chi)}{\sqrt{E}(1+\eta^2)}\Big[t'^b_m A^m_b + \frac{1}{2}E^i_b t'^b_{[i}\chi_{j]}\zeta_j + \frac{\eta}{2}\epsilon_{ijk}t'^b_i E^j_b\zeta_k\Big] - 2e\Sigma^{ta}_{IJ}[\partial_a\bar\lambda(1+i\eta\gamma_5)\sigma^{IJ}\Pi + \bar\Pi(1+i\eta\gamma_5)\sigma^{IJ}\partial_a\lambda] \\
& + E^a_k\chi_k[\partial_a\bar\lambda(1+i\eta\gamma_5)\Pi - \bar\Pi(1+i\eta\gamma_5)\partial_a\lambda] - 2e\Sigma^{ta}_{IJ}[-\bar\lambda\sigma_{0l}\sigma^{IJ}\Pi + \bar\Pi\sigma^{IJ}\sigma_{0l}\lambda]A^l_a \\
& - E^a_k\chi_k[\bar\Pi\sigma_{0l}\lambda + \bar\lambda\sigma_{0l}\Pi]A^l_a + \frac{(1-\chi\cdot\chi)}{2(1+\eta^2)}\Big[(\eta t'^a_k - \epsilon^{ijk}\chi_i t'^a_j)V^a_l + f_{kl} + (1+\eta^2)J_{kl} + \frac{1}{4}N_{kl}(M)\Big]M^{kl}, \qquad (52)
\end{aligned}
$$

where, as earlier, $2e\Sigma^{ta}_{0i} = E^a_i$, $2e\Sigma^{ta}_{ij} = -E^a_{[i}\chi_{j]}$, and f_{kl} and $N_{kl}(M)$ are given by Eqs. (26) and (27), respectively. Also,

$$
\begin{aligned}
t'^a_I &:= t^a_I - \eta e\Sigma^{ta}_{IJ}\bar\lambda\gamma_5\gamma^J\lambda \\
&= t^a_I + \frac{i\eta}{\sqrt{q}}e\Sigma^{ta}_{IJ}[M^J(\bar\Pi\gamma_5\lambda - \bar\lambda\gamma_5\Pi) \\
&\quad + 2M_L(\bar\Pi\gamma_5\sigma^{LJ}\lambda + \bar\lambda\gamma_5\sigma^{LJ}\Pi)], \qquad (53)
\end{aligned}
$$

$$
\begin{aligned}
2J_{kl} &:= \frac{1}{2\sqrt{E}}\bar\lambda\gamma_5\Big(\chi_k\gamma_l + \chi_l\gamma_k + 2\delta_{kl}\frac{M^I\gamma_I}{M^0}\Big)\lambda \\
&= \frac{i}{2}(\delta_{kl} + M_k M_l)(\bar\Pi\gamma_5\lambda - \bar\lambda\gamma_5\Pi) \\
&\quad + iM_l M^J(\bar\Pi\gamma_5\sigma_{Jk}\lambda + \bar\lambda\gamma_5\sigma_{Jk}\Pi) + (k\leftrightarrow l). \qquad (54)
\end{aligned}
$$

The Hamiltonian density now reads

$$
\begin{aligned}
\mathcal{H} &= NH' + N^a H'_a + \tfrac{1}{2}\omega^{IJ}_t G'_{IJ} + \xi^a_I(V^I_a - v^I_a) \\
&\quad + \phi^I_a(t^a_I - \tau^a_I). \qquad (55)
\end{aligned}
$$

The constraints associated with the fields N^a, N, ω^{0i}_t, ω^{ij}_t, ξ^a_I, and ϕ^I_a, respectively, are

$$
H'_a \approx 0, \qquad H' \approx 0, \qquad G'^i_{\rm boost} \approx 0, \qquad G'^i_{\rm rot} \approx 0, \qquad (56)
$$

$$
V^I_a - v^I_a \approx 0, \qquad t^a_I - \tau^a_I \approx 0. \qquad (57)
$$

The remaining fields M^{kl}, from $\frac{\delta H'}{\delta M^{kl}}\delta M^{kl} \approx 0$, lead to the constraint

$$
\begin{aligned}
(\eta t'^a_k - \epsilon^{ijk}\chi_i t'^a_j)V^l_a + f_{kl} + \tfrac{1}{2}N_{kl} + (1+\eta^2)J_{kl} \\
+ (k\leftrightarrow l) \approx 0. \qquad (58)
\end{aligned}
$$

Using $t^a_I \approx \tau^a_I$, we write

$$
\begin{aligned}
t'^a_k &\approx -\frac{\eta}{2}\sqrt{E}E^a_l\Big[\frac{2f_{kl} + N_{kl}}{1+\eta^2} + 2J_{kl} + \epsilon_{kln}G'^n_{\rm boost}\Big], \\
t'^a_0 &\approx \eta\sqrt{E}E^a_l\Big[G'^l_{\rm rot} - \frac{\chi_k}{2}\Big(\frac{2f_{kl} + N_{kl}}{1+\eta^2} + 2J_{kl} \qquad (59) \\
&\quad + \epsilon_{kln}G'^n_{\rm boost}\Big)\Big].
\end{aligned}
$$

Using (59) in (58) leads to

$$
2f_{kl} + N_{kl} + 2(1+\eta^2)J_{kl} \approx 0, \qquad (60)
$$

generalizing the constraint (32) of the pure gravity case. This in turn implies

$$
t'^a_I \approx 0, \qquad (61)
$$

corresponding to the constraint (33) for pure gravity. Implementing this constraint along with those in (57) reduces the Hamiltonian density to

$$
\mathcal{H} = NH' + N^a H'_a + \tfrac{1}{2}\omega^{IJ}_t G'_{IJ}, \qquad (62)
$$

where the final set of constraints is obtained from Eqs. (49)–(52) by substituting $t'^a_I = 0$ and dropping the terms containing $G'^i_{\rm boost}$ and $G'^i_{\rm rot}$ in H'_a and H'. The M_{kl} is given by the solution of the constraint (60).

Time gauge

We may now make the gauge choice $\chi_i = 0$ and solve the boost constraint $G'^i_{\rm boost} = 0$ to obtain

$$
\zeta_i = \partial_a E^a_i - i\eta[\bar\Pi\gamma_5\sigma_{0i}\lambda + \bar\lambda\gamma_5\sigma_{0i}\Pi]. \qquad (63)
$$

Thus we have a canonical Hamiltonian formulation for a theory of gravity with fermions in terms of real $SU(2)$ gauge fields A^i_a with the following constraints:

$$
\begin{aligned}
G'^{i}_{\rm rot} &= \eta\partial_a E^a_i + \epsilon^{ijk}A^j_a E^a_k + i[\bar{\Pi}\gamma_5\sigma_{0i}\lambda + \bar{\lambda}\gamma_5\sigma_{0i}\Pi] \approx 0;\\
H'_a &= E^b_i\partial_{[a}A^i_{b]} + [\partial_a\bar{\lambda}(1+i\eta\gamma_5)\Pi - \bar{\Pi}(1+i\eta\gamma_5)\partial_a\lambda]\\
&\quad - \frac{1}{1+\eta^2}[\partial_a E^a_i - \eta\epsilon_{ijk}A^j_b E^b_k\\
&\quad - i\eta(\bar{\Pi}\gamma_5\sigma_{0i}\lambda + \bar{\lambda}\gamma_5\sigma_{0i}\Pi)]A^i_a \approx 0;\\
H' &= \left[E^a_i\partial_a\zeta_i + \frac{1}{2}\zeta_i E^a_i E^b_j\partial_a E^j_b\right]\\
&\quad - \frac{1}{1+\eta^2}\left[\frac{1}{2}E^a_{[i}E^b_{j]}A^i_a A^j_b + \eta\epsilon_{ijk}\zeta_i A^j_a E^a_k - \frac{3}{4}\zeta\cdot\zeta\right]\\
&\quad + 2E^a_i[\partial_a\bar{\lambda}(1+i\eta\gamma_5)\sigma_{0i}\Pi + \bar{\Pi}(1+i\eta\gamma_5)\sigma_{0i}\partial_a\lambda]\\
&\quad + E^a_i[\bar{\lambda}\sigma_{il}\Pi + \bar{\Pi}\sigma^{il}\lambda]A^l_a\\
&\quad + \frac{1}{2(1+\eta^2)}\Big[\{f_{kl} + (1+\eta^2)J_{kl}\}M^{kl}\\
&\quad - \frac{1}{4}(M^{kl}M^{kl} - M^{kk}M^{ll})\Big] \approx 0,
\end{aligned} \tag{64}
$$

where ζ^i are given by (63) and

$$M^{kl} = 2[f_{kl} + (1+\eta^2)J_{kl}] - \delta_{kl}[f_{mm} + (1+\eta^2)J_{mm}], \tag{65}$$

with

$$
\begin{aligned}
2f_{kl} &= (1+\eta^2)\epsilon^{ijk}E^a_i E^l_b\partial_a E^b_j + \eta(E^a_k A^l_a - \delta^{kl}E^a_m A^m_a)\\
&\quad + (k\leftrightarrow l),\\
2J_{kl} &= i\delta_{kl}[\bar{\Pi}\gamma_5\lambda - \bar{\lambda}\gamma_5\Pi].
\end{aligned} \tag{66}
$$

This completes our discussion of a fermion minimally coupled to gravity including the Nieh-Yan term. This analysis can now be extended in an analogous manner to a theory with any matter content with any couplings.

IV. CONCLUSIONS

We have demonstrated that inclusion of Nieh-Yan topological density in the Lagrangian density of a theory of gravity allows us, in the time gauge, to describe gravity in terms of a real $SU(2)$ connection. The set of constraints so obtained in the Hamiltonian formulation, for $\eta = 1$, is the same as that in the Barbero formulation. For other real values of this parameter, we have the Immirzi formulation with Barbero-Immirzi parameter $\gamma = \eta^{-1}$. Thus the parameter η has a similar interpretation as the θ parameter of QCD. Like the topologically nontrivial vacuum structure of QCD, which reflects itself in terms of presence of the θ parameter, the η parameter in the theory of gravity should indicate a rich vacuum structure of gravity which needs further and thorough investigation.

Like the θ term in QCD, the Nieh-Yan term in gravity is also universal; i.e., it does not need to be changed when various kinds of matter are coupled to the theory. We have discussed this in detail for spin-$\frac{1}{2}$ matter coupled to gravity. For other matter, for example, in the theories involving an antisymmetric tensor gauge field, and also theories of supergravity, the same Nieh-Yan topological term allows a description in terms of a theory of a real $SU(2)$ gauge connection in the time gauge. This is to be contrasted with the case of Holst modification of Hilbert-Palatini action, where for different matter couplings, the corresponding Holst term in the Lagrangian density needs to be changed on a case by case basis so as to keep the equations of motion unaltered [7,8]. It is worth emphasizing that the Nieh-Yan density is entirely made up of geometric quantities while the modified Holst terms contain matter fields as well. The two get related only after using the connection equation of motion.

In a complete theory of gravity, besides the Nieh-Yan topological term, we need to include two other topological terms: the Pontryagin density and the Euler density. This introduces two additional topological parameters associated with such topological terms, besides the parameter η we have discussed here. Any quantum theory of gravity should have all of these three CP-violating topological couplings.

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