Generation of fluctuations during inflation: Comparison of stochastic and field-theoretic approaches

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We prove that the stochastic and standard field-theoretical approaches produce exactly the same results for the amount of light massive scalar field fluctuations generated during inflation in the leading order of the slow-roll approximation. This is true both in the case for which this field is a test one and inflation is driven by another field, and the case for which the field plays the role of inflaton itself. In the latter case, in order to calculate the mean square of the gauge-invariant inflaton fluctuations, the logarithm of the scale factor a has to be used as the time variable in the Fokker-Planck equation in the stochastic approach. The implications of particle production during inflation for the second stage of inflation and for the moduli problem are also discussed. The case of a massless self-interacting test scalar field in de Sitter background with a zero initial renormalized mean square is also considered in order to show how the stochastic approach can easily produce results corresponding to diagrams with an arbitrary number of scalar field loops in the field-theoretical approach (explicit results up to four loops included are presented).

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I. INTRODUCTION

It has been known for a long time that light minimally coupled scalar fields typically have anomalously high vacuum expectation values for even powers of fields in the de Sitter background. In particular, the mean square of a free, massive and minimally coupled to gravity scalar field in the equilibrium, de Sitter invariant quantum state (the Bunch-Davies vacuum) is [1]

$$\langle \phi^2 \rangle \simeq \frac{3H^4}{8\pi^2 m^2} \gg H^2 \tag{1}$$

if $m^2 \ll H^2$. Here $H \equiv \dot{a}/a$ is the Hubble parameter and $a(t)$ is the scale factor of a Friedmann-Robertson-Walker (FRW) cosmological model. For the particular case $m = 0$ and $\langle \phi^2 \rangle = 0$ (or small) at the beginning of the de Sitter stage (set at $t = 0$ here), we have

$$\langle \phi^2 \rangle \simeq \frac{H^3 t}{4\pi^2} \tag{2}$$

which grows without bound [2–4]. This anomalous growth is an essentially infrared effect, it occurs due to field Fourier modes with wavelengths far exceeding the de Sitter event horizon. Thus, it is not a consequence of the Gibbons-Hawking effective temperature $T = H/2\pi$ [5] experienced by a point observer inside her/his de Sitter horizon. In particular, in contrast with temperaturelike effects which should be universal for fields of all spins, this effect occurs for minimally coupled light scalar fields and gravitons only (but not for, e.g., photons or light fermions).

Because of continuous creation of infrared modes and growth of their occupation number, the quantum scalar field can be split into a long wave (coarse-grained) component and a short-wave (perturbative) one. Then it can be proved that the former component effectively becomes quasiclassical, though random (i.e. all noncommutative parts of it may be neglected), and it experiences a random walk described by the stochastic inflation approach. In some specific case but beyond one-loop approximation, this approach was used already in [3]; see [6] for the rigorous derivation in the generic case when this scalar field is a slow-rolling inflaton itself and for a number of analytic nonperturbative results.[1] The case of a scalar field with the quartic self-interaction in the exact de Sitter space-time was first studied using this approach in [9], and a number of nonperturbative results beyond any finite number of loops in quantum field theory (QFT) perturbative expansion were obtained there.

However, it should be emphasized that, just because of the nonperturbative nature of the stochastic approach to inflation, it is based on a number of heuristic approximations. Therefore, it is very important to check, whenever possible, results obtained by its application using the standard perturbative QFT in curved space-time. Also, an inflationary space-time is not the exactly de Sitter one, $\dot{H} \neq 0$, that can often lead to a drastic change in conclu-

[1]A description of the growth (2) in the free massless case in terms of the Fokker-Planck equation was first considered in [7]. The stochastic approach was applied to the description of eternal inflation in [8].

sions. That is why our purpose here is to consider new applications of this approach to inflationary space-times with $\dot{H} \neq 0$, though with $|\dot{H}| \ll H^2$ (as required by observational data on a slope of the primordial power spectrum of scalar perturbations in the Universe), and to compare results obtained in this way with those directly following from perturbative QFT in curved space-time. Our paper contains several novel results referring both to inflationary space-times and to the exact de Sitter space-time beyond one-loop approximation. Also, our results shed new light on the much discussed problem of the choice of an independent time variable in the Langevin and Fokker-Planck equations in the stochastic approach. Finally, we show that in most cases the mean square of a scalar field at the last stages of inflation is very different from the instantaneous Bunch-Davies value (1)—even "eternal" inflation (which does occur in the inflationary models we consider for a sufficiently large initial value of an inflaton field) is not eternal enough for a light scalar field to reach equilibrium.

The paper is organized as follows. In Sec. II the growth of fluctuations of a light test field during inflation driven by a massive inflaton is investigated. We prove that the results obtained in [10,11] using QFT with adiabatic regularization in curved space-time can be obtained from a general diffusion equation with a noise term which has the same form as that in the Langevin equation for an inflaton field in the stochastic inflation approach [6]. In Sec. III more details are presented on how the mean square of a test scalar field with an arbitrary (though light, $m \ll H$) mass is obtained using adiabatic regularization in curved space-time with a slowly changing curvature. Generation of light scalar field fluctuations in inflationary models can be so strong as to dominate finally the classical energy density of an inflaton and to drive a second stage of inflation. As an example, in Sec. IV we derive the conditions under which such a second stage can occur for the $m^2\phi^2$ inflationary model, with or without a break between the two stages of inflation. In Sec. V we discuss the impact of this inflationary particle production on the moduli problem. We show that quantum moduli problems are worse than the classical one and we improve on previous investigations [12,13]. In Sec. VI we discuss the diffusion (Langevin) equation for inflaton fluctuations in the case of a generic chaotic power-law potential $V(\phi) \propto \phi^n$. Using the results obtained by QFT methods, we show that, if one is interested in metric fluctuations, this diffusion equation should be formulated in terms of the independent time variable $\ln a$ which is directly related to the number of e-folds during inflation N (this was among the possibilities envisioned in [6]). In Sec. VII it is shown that the stochastic method can also be used beyond the one-loop approximations for a self-interacting scalar field in the exact de Sitter space-time, again leading to the same results as obtained by QFT methods. We conclude in Sec. VIII and discuss the perturbative expansion of the diffusion equation in the Appendix.

II. GROWTH OF LIGHT TEST FIELD FLUCTUATIONS DURING MASSIVE INFLATION IN STOCHASTIC APPROACH

Let us first consider the production of particles and fluctuations for a test quantum field χ with a small mass $m \ll H$. Thus, we neglect the χ energy density and pressure in the background FRW equations. Also, it is assumed that there is no Bose condensate of χ, $\langle\chi\rangle = 0$.

As an example, in this section we limit ourselves to the simplest case of inflaton with the quadratic potential. Some of the QFT results for this model have already been obtained in [10,11]. The Friedmann equation for the massive inflaton is

$$H^2 = \frac{\rho_\phi}{3M_{\rm pl}^2} = \frac{1}{3M_{\rm pl}^2}\left[\frac{\dot{\phi}^2}{2} + m^2\frac{\phi^2}{2}\right] \simeq \frac{m^2}{M_{\rm pl}^2}\frac{\phi^2}{6}, \quad (3)$$

where $8\pi G = M_{\rm pl}^{-2}$. In the slow-roll approximation, the inflationary trajectory is well approximated by

$$H = H(t) \simeq H_0 - \frac{m^2}{3}(t - t_0), \quad (4)$$

$$\phi(t) \simeq \phi_0 - \sqrt{\tfrac{2}{3}} m M_{\rm pl}(t - t_0), \quad (5)$$

$$a(t) \simeq a_0 \exp\left[\frac{3}{2m^2}(H_0^2 - H^2)\right], \quad (6)$$

where the subscript 0 denotes the beginning of inflation (these expressions were first presented in [14] in the context of a closed bouncing FRW universe). In the rest of the paper, we set $a_0 = 1$ for simplicity. The equation for inhomogeneous Fourier modes of χ is

$$\ddot{\chi}_k + 3H\dot{\chi}_k + \left[\frac{k^2}{a^2} + m_\chi^2\right]\chi_k = 0, \quad (7)$$

and we denote $m_\chi^2 = \alpha m^2$. Let us first discuss the case $m_\chi = m$, or $\alpha = 1$, in order to see the relation between renormalized QFT results and the stochastic approach. This case was studied in [10] where inflaton fluctuations in a rigid space-time were considered. For the value of $\langle\chi^2\rangle$ renormalized by adiabatic subtraction, one obtains in the leading order:

$$\langle\chi^2\rangle_{\rm REN} \simeq \frac{H^2}{4\pi^2}\log a, \quad (8)$$

This result agrees with the solution of the differential equation for $\langle\chi^2\rangle_{\rm REN}$:

$$\frac{d\langle\chi^2\rangle_{\rm REN}}{dt} + \frac{2m^2}{3H(t)}\langle\chi^2\rangle_{\rm REN} = \frac{H^3(t)}{4\pi^2} \quad (9)$$

[see e.g. Eq. (10) of paper [3]] with H not constant in time, but having the time evolution given by Eq. (4). In the scope of the stochastic approach, the right-hand side of Eq. (4) arises from a noise term in the Langevin equation for the large-scale part of χ, see Eq. (36) below, coarse-grained over the physical 3D volume $\sim(\epsilon H)^{-3}$ which slowly expands during inflation. Here $\epsilon \ll 1$ is an auxiliary parameter which enters into the definition of the coarse-graining scale $k_{cg} = \epsilon a H$ in momentum space. Since we are interested in the leading order of the slow-roll approximation, terms of order $\dot{H}/H^2$ coming either from taking an explicit time derivative of H or from solutions of the wave equation (7) at the coarse-graining scale may be neglected. With the same accuracy, one may alternatively take the physical coarse-graining scale k_{cg}/a to be exactly constant during inflation and equal to $\epsilon H(t_f)$, where t_f denotes the end of inflation.

The general solution to Eq. (9) in the background (4) is indeed

$$\langle\chi^2\rangle_{\rm REN} = CH^2 + \frac{H^2}{4\pi^2}\log a = CH^2 + \frac{3H^2}{8m^2\pi^2}(H_0^2 - H^2), \tag{10}$$

where C is an integration constant and we have used Eq. (6) in the second expression. For a long stage of inflation, when $H(t) \ll H_0$, we have [10]

$$\langle\chi^2\rangle_{\rm REN} \simeq CH^2 + \frac{3H^2}{8m^2\pi^2}H_0^2 = \langle\chi^2(t_0)\rangle_{\rm REN}\frac{H^2}{H_0^2} + \frac{3H^2}{8m^2\pi^2}H_0^2. \tag{11}$$

The regime $H(t) \ll H_0$ occurs only when the number of e-folds is $N \approx N_0 = 3H_0^2/(2m^2)$.

The stochastic equation (9) can be easily extended to a generic m_χ not coinciding with the inflaton mass m:

$$\frac{d\langle\chi^2\rangle_{\rm REN}}{dt} + \frac{2m_\chi^2}{3H(t)}\langle\chi^2\rangle_{\rm REN} = \frac{H^3(t)}{4\pi^2}. \tag{12}$$

Its solution is

$$\langle\chi^2\rangle_{\rm REN} = C_1H^{2\alpha} + \frac{3H^4}{8\pi^2m^2(\alpha-2)} = C_2H^{2\alpha} + \frac{3H^{2\alpha}}{8\pi^2m^2(2-\alpha)}(H_0^{4-2\alpha} - H^{4-2\alpha}), \tag{13}$$

where C_1, C_2 are integration constants. For the particular marginal value $\alpha = 2$, Eq. (13) should be replaced by

$$\langle\chi^2\rangle_{\rm REN} = C_2H^4 - \frac{3H^4}{4\pi^2m^2}\log\left(\frac{H}{H_0}\right). \tag{14}$$

It is important to note that $\langle\chi^2\rangle_{\rm REN}$ is different from the instantaneous Bunch-Davies equilibrium value $3H^4/(8\pi^2m_\chi^2)$ in Eq. (1) for any α. Since $H(t)$ decreases with time, it is only for $\alpha \gg 2$ that $\langle\chi^2\rangle_{\rm REN}$ may approach (but not become exactly equal to) the instantaneous Bunch-Davies vacuum value at the last stage of inflation.

When $H(t) \ll H_0$ and $\alpha < 2$,

$$\langle\chi^2\rangle_{\rm REN} \simeq \langle\chi^2(t_0)\rangle_{\rm REN}\frac{H^{2\alpha}}{H_0^{2\alpha}} + \frac{3H^{2\alpha}H_0^{4-2\alpha}}{8m^2\pi^2(2-\alpha)}. \tag{15}$$

In the limit $\alpha \ll 2$ (but $\alpha \neq 0$), χ renormalized fluctuations—and therefore the χ energy density—depend nontrivially on the duration of inflation [10,11]. Note that only for $\alpha = 1$ (i.e. $m_\chi = m$) and at the beginning of inflation, $\langle\chi^2\rangle \sim H_0^3t$ as occurs in the exact de Sitter space-time.

This shows the occurrence of a new characteristic scale in the problem, namely $\sqrt{-\dot{H}}$. In contrast to the de Sitter space-time, where H is the only scale present, in inflation we have also $\sqrt{-\dot{H}}$, i.e. $\frac{m}{\sqrt{3}}$ in the model under consideration. Fields which are light compared to the Hubble parameter are subsequently divided in two classes: the ones with $m_\chi > \sqrt{-3\dot{H}}$ and the ones with $m_\chi < \sqrt{-3\dot{H}}$. Their corresponding particle production rates look different.

A homogeneous solution of Eq. (12) given by the term containing C_1 in Eq. (13) is also the slow-roll solution for $\chi_{\rm cl}^2$, with $\chi_{\rm cl}$ being the classical homogeneous field $\chi(t)$. In the case $\alpha < 2$, this part of the general solution becomes negligible with respect to generated quantum fluctuations as inflation develops if $\langle\chi^2(t_0)\rangle_{\rm REN} \ll 3H_0^4/(8\pi^2m^2(2-\alpha))$ [see Eq. (15)]. In particular, if zero initially, $\langle\chi^2\rangle_{\rm REN}$ becomes larger than the instantaneous Bunch-Davies equilibrium value when

$$\frac{H}{H_0} < \left(\frac{\alpha}{2-\alpha}\right)^{1/[2(2-\alpha)]}. \tag{16}$$

The energy density of χ becomes comparable to that of the inflaton and cannot be neglected further on when

$$\frac{\rho_\chi}{\rho_\phi} \geq \frac{m_\chi^2\langle\chi^2\rangle_{\rm REN}}{6H^2M_{\rm pl}^2} \sim 1. \tag{17}$$

This never happens for $\alpha \geq 1$, but for $\alpha < 1$ backreaction effects become important when

$$H(t) \leq H(t_\star) = H_0\left(\frac{H_0\sqrt{\alpha}}{4\pi\sqrt{2-\alpha}M_{\rm pl}}\right)^{1/(1-\alpha)}. \tag{18}$$

Note that the dominant term in ρ_χ is $m^2\langle\chi^2\rangle_{\rm REN}$, the χ kinetic and gradient energy are subleading. Also, we always consider $H_0 < \gamma M_{\rm pl}$ with $\gamma = \mathcal{O}(1)$.

III. RENORMALIZATION APPROACH

We now wish to present the renormalized $\langle\chi^2\rangle_{\rm REN}$ for a generic mass m_χ obtained using dimensional regularization and adiabatic subtraction. Such a calculation goes beyond [10], which addressed the first case in the previous section ($\alpha = 1$) only.

Let us consider solutions of the wave equation (7) which give time-dependent coefficients in the expansion of the quantum field χ in terms of the Fock annihilation and creation operators for each Fourier mode $\mathbf{k}$, $k = |\mathbf{k}|$. We use a set of approximate solutions of Eq. (7), chosen so as to separate the ultraviolet (UV henceforth) and infrared (IR henceforth) domains for $k > \bar{\epsilon} aH$ and $k < \bar{\epsilon} aH$, respectively (here $\bar{\epsilon} < 1$ need not coincide with the parameter ϵ in the definition of the coarse-graining scale introduced above).

The UV solution whose contribution will mostly be removed by the adiabatic regularization can be written in the generic form,

$$\chi_k^{\rm UV} = \frac{1}{a^{3/2}}\left(\frac{\pi\lambda}{4H}\right)^{1/2} H_\nu^{(1)}\left(\frac{\lambda k}{aH}\right), \tag{19}$$

for suitable functions ν and λ (whose values are normally close to $3/2$ and 1, respectively, which are the values in the lowest order of the slow-roll approximation).

The most important contribution is the far IR one. Such a solution can be computed in the same approximation as used in our previous work [11], where a match at the moment t_k when $k = \bar{\epsilon} a(t_k) H(t_k)$ with the UV solution is imposed, together with the requirement of being a solution of (7) in the deep IR limit $k \to 0$. In the leading order, we choose the form

$$\chi_k^{\rm IR} = \frac{1}{a^{3/2}(t)}\left(\frac{\pi\lambda}{4H(t)}\right)^{1/2}\left(\frac{H(t_k)}{H(t)}\right)^x H_{3/2}^{(1)}\left(\frac{\lambda k}{a(t)H(t)}\right), \tag{20}$$

where the parameter x can be fixed using the second constraint. Note for the later use that at the moment t_k one has

$$H(t_k) = H_0\sqrt{1 + 2\frac{\dot{H}}{H_0^2}\log\frac{k}{\bar{\epsilon} H(t_k)}} \simeq H_0\sqrt{1 + 2\frac{\dot{H}}{H_0^2}\log\frac{k}{\bar{\epsilon} H_0}}. \tag{21}$$

Substituting Eq. (20) into Eq. (7) and considering the deep IR region, one finds that $x = 1 - \alpha$.

In order to compute the renormalized value of $\langle\chi^2\rangle$ at the same scale as is used in the stochastic approach, we note that the contribution of IR modes should be taken up to $k = \epsilon a(t) H(t)$ (the UV ones cancel due to adiabatic subtraction). Then it is natural to define t_k by using $\bar{\epsilon} = \epsilon$ in Eq. (21) in order to avoid terms like $\log\bar{\epsilon}/\epsilon$. We shall therefore make this choice.

As mentioned above, the leading contribution to the renormalized value of $\langle\chi^2\rangle$ is given by the integration over IR modes. In the leading order of the slow-roll approximation, one obtains the following result:

$$\langle\chi^2\rangle_{\rm REN} \simeq \langle\chi^2\rangle_{\rm IR} \simeq \frac{H^{2\alpha}(t)}{4\pi^2}\int_{\log l}^{\log \epsilon a(t) H(t)} H^{2-2\alpha}(t_k) d\log k, \tag{22}$$

with $l = \epsilon H_0$ and where the expansion of the Bessel function for a small argument is performed. Thus, adiabatic subtraction cancels the UV contribution more and more accurately as inflation develops. This defines a dynamical cutoff, and the IR part below this cutoff soon becomes dominant. Taking the time derivative of Eq. (22), one obtains

$$\frac{d}{dt}\langle 0|\chi^2|0\rangle_{\rm REN} = \frac{H^3(t)}{4\pi^2} + 2\alpha\frac{\dot{H}}{H}\langle 0|\chi^2|0\rangle_{\rm REN}. \tag{23}$$

This coincides with Eq. (12) for a generic mass $m_\chi^2 = \alpha m^2$ since the direct computation of the integral in Eq. (22) in the slow-roll approximation leads to the second expression in Eq. (13) with $C_2 = 0$. Therefore, the two methods of doing this calculation agree.

IV. CONSEQUENCES OF PARTICLE PRODUCTION: SECOND STAGE OF INFLATION

Let us discuss in more detail how the backreaction of χ may become important. If $\alpha \geq 1$, then the backreaction of χ will never be important during inflation [and afterwards, too, if the inflaton does not decay faster than χ during (p)reheating]. So, we consider $\alpha < 1$ in the following.

Let us introduce a new quantity $\beta = m^2 M_{\rm pl}^2/H_0^4$. Backreaction of χ becomes important before or after the end of inflation driven by the inflaton ϕ if $H(t_*)$ in Eq. (18) is greater or smaller than m (which is the scale of the end of inflation in the $m^2\phi^2$ inflationary model), respectively. Therefore, χ fluctuations become important before the end of inflation if

$$\left(\frac{H_0}{m}\right)^{2(1-\alpha)}\frac{H_0^2}{16\pi^2 M_{\rm pl}^2}\frac{\alpha}{2-\alpha} > 1. \tag{24}$$

Thus, for $\alpha \ll 1$ there is a second stage of inflation without a break if

$$\frac{1}{\beta} = \frac{H_0^4}{m^2 M_{\rm pl}^2} > \frac{32\pi^2}{\alpha}, \tag{25}$$

and with a break if the $<$ holds in the equation above. A second stage of inflation without a break starts in the presence of a light field with mass m_χ in the range

$$1 \gg \alpha = \frac{m_\chi^2}{m^2} > 72\pi^2\frac{M_{\rm pl}^2}{m^2 N_0^2} \to m_\chi^2 > 72\pi^2\frac{M_{\rm pl}^2}{N_0^2}, \tag{26}$$

which is a very broad range for large enough N_0. A second stage of inflation with a break would occur for smaller m_χ. In such a case the slow roll of the inflaton field practically ends, but inflation would be sustained by quantum fluctuations of χ corresponding to a much lower value for $|\dot{H}| \ll m^2/3$ (see also [15–17] for examples of inflation as a whole, or its second stage driven by quantum fluctuations of a scalar field). Note that a second stage of inflation can occur even if curvature during the first stage of inflation is low, $H_0 \ll M_{\rm pl}$, usually after a break in this case. Thus, a second stage of inflation seems to be very probable if the inflaton is not the lightest field around. Also, even without the second stage, χ may play a role of a curvaton [18–21].

V. CONSEQUENCES OF PARTICLE PRODUCTION: IMPACT ON THE MODULI PROBLEM

On considering χ as a modulus, the above calculations allow a better estimate of the moduli abundance produced by the expansion during inflation. The previous section shows that the moduli can even drive a second inflationary stage if $\alpha < 1$ and the energy stored will dominate before the end of inflation [see Eq. (18)]. We focus here on the case for which χ does not dominate during inflation.

In order to estimate the ratio n_χ/s (where $n_\chi = \rho_\chi/m_\chi$ and s are the number of χ particles and the entropy of produced particles, respectively), which is required to be less than 10^{-12}, we proceed as in [13]. We assume the immediate thermalization for the inflaton and χ after the accelerated expansion:

$$\rho_\phi = 3H^2 M_{\rm pl}^2 \simeq \frac{\pi^2 g T^4}{30}, \qquad s = \frac{2\pi^2 g T^3}{45}, \tag{27}$$

where T is the reheating temperature and g is the number of species (for instance $g = 106.75$ for the standard model [22]).

We first consider $\alpha \gg 2$ and obtain

$$\frac{n_\chi}{s} \simeq \frac{5g}{96 \times 10^2} \frac{T^5}{m_\chi M_{\rm pl}^4}. \tag{28}$$

Because of strong positive dependence on the reheating temperature, the requirement $n_\chi/s < 10^{-12}$ can easily be satisfied.

We now consider $\alpha < 2$, the case for which particle production is very different from the one extrapolated from the exact de Sitter space-time. The energy density stored in χ for $H \ll H_0$ is

$$\rho_\chi \simeq m_\chi^2 \frac{\langle \chi^2 \rangle_{\rm REN}}{2} \simeq \alpha \frac{3H^{2\alpha} H_0^{4-2\alpha}}{16\pi^2(2-\alpha)}, \tag{29}$$

and we obtain

$$\frac{n_\chi}{s} \simeq \frac{3}{64\pi^2} \frac{\alpha T}{m_\chi} \frac{H^{2\alpha-2}}{2-\alpha} \frac{H_0^{4-2\alpha}}{M_{\rm pl}^2} \propto T^{4\alpha-3}. \tag{30}$$

This equation replaces Eq. (4.16) of [13].[2] Note that the dependence on T is rather peculiar: for $\alpha > 3/4$, it grows with T, while for $\alpha < 3/4$, it decreases with T. For $\alpha \sim 0$, the ratio is

$$\frac{n_\chi}{s} \sim \alpha \frac{270}{128\pi^4} \frac{H_0^4}{m_\chi g T^3}. \tag{31}$$

Therefore the problem of light moduli cannot be solved by lowering the reheating temperature in the $m^2\phi^2$ model and is much worse than expected in [13].

[2] Equation (4.16) of [13] agrees with Eq. (30) for $\alpha = 1$ only; nevertheless, it is used for any α. Also, the growth law (2) is used there which is valid for $\alpha = 1$ only as pointed above.

VI. INFLATON FLUCTUATIONS AND STOCHASTIC APPROACH FOR GAUGE-INVARIANT FLUCTUATIONS

An inflaton has an effective mass which is much smaller than the Hubble parameter during inflation. Therefore, the above approach should hold for inflaton fluctuations, too. However, such a case differs from the previous one, since scalar metric fluctuations should be taken into account in addition to field (inflaton) ones. In other words, the quantity which is quantized is a linear combination of field and metric fluctuations. As in [11], we choose the gauge in which inflaton fluctuations coincide with the gauge-invariant Mukhanov variable which is canonically quantized [23]. Then our results will be valid in any gauge, if a gauge-invariant variable is considered.

The time evolution of the renormalized inflaton fluctuations for the $m^2\phi^2$ inflationary model has been already obtained in [11,24] using a perturbative QFT analysis of the Einstein equations. To the lowest order in the slow-roll and in long-wavelength approximations, equations for fluctuations in the first [11] and second [24] order are

$$3H\delta\dot\phi^{(1)} + (m^2 + 6\dot H)\delta\phi^{(1)} \approx 0, \tag{32}$$

$$3H\delta\dot\phi^{(2)} + (m^2 + 6\dot H)\delta\phi^{(2)} \approx \frac{\dot\phi}{2H}\frac{m^2}{M_{\rm pl}^2}(\delta\phi^{(1)})^2. \tag{33}$$

We now observe that these two equations can be obtained order by order from the expansion around a classical solution, $\phi(t, \mathbf{x}) = \phi_{\rm cl}(t) + \delta\phi^{(1)} + \delta\phi^{(2)}$, of the equation:

$$\frac{d\phi}{dN} = -\frac{V_\phi}{3H^2}, \tag{34}$$

where $N = \log(a/a_0)$ is the e-fold number from the beginning of inflation. This suggests that the equation (34) is valid to all orders in $\delta\phi \equiv \phi - \phi_{\rm cl}(t)$ in the context of the approximation used. Returning to the time evolution and using $dN = Hdt$, one has

$$\frac{1}{H}\frac{d}{dt}\delta\phi = -\left[\frac{d}{d\phi}\left(\frac{V_\phi}{3H^2}\right)\right]_{\phi=\phi_{\rm cl}}\delta\phi - \frac{1}{2}\left[\frac{d^2}{d\phi^2}\left(\frac{V_\phi}{3H^2}\right)\right]_{\phi=\phi_{\rm cl}}(\delta\phi)^2 + \mathcal{O}((\delta\phi)^3)$$
$$\simeq -\left[\frac{V_{\phi\phi}}{3H^2} - \left(\frac{V_\phi}{3H^2 M_{\rm pl}}\right)^2\right]_{\phi=\phi_{\rm cl}}\delta\phi - \frac{1}{2}\left[\frac{V_{\phi\phi\phi}}{3H^2} - \frac{V_{\phi\phi}V_\phi}{3H^4M_{\rm pl}^2} + \frac{2V_\phi^3}{27H^6M_{\rm pl}^4}\right]_{\phi=\phi_{\rm cl}}(\delta\phi)^2 + \mathcal{O}((\delta\phi)^3), \tag{35}$$

where $V_\phi = \frac{dV}{d\phi}$ and so on. The quadratic inflaton potential leads to the result in Eqs. (32) and (33) after expanding $\delta\phi = \delta\phi^{(1)} + \delta\phi^{(2)}$ and taking the coefficients in the brackets to the lowest order in slow roll. The same method may be used for an arbitrary inflaton potential $V(\phi)$.

How does one rederive these results in the stochastic approach? The consideration above suggests that one has to choose the time variable $N = \int H(t)dt$ in the Langevin stochastic equation for the large-scale part of ϕ for this aim. Then it acquires the form [see Eq. (55) of [6]]

$$\frac{d\phi}{dN} = -\frac{V_\phi}{3H^2} + \frac{f}{H}, \qquad \langle f(N_1)f(N_2)\rangle = \frac{H^4}{4\pi^2}\delta(N_1 - N_2), \tag{36}$$

where $H^2(t) = V(\phi(t))/3M_{\rm pl}^2$ in this (leading) order of the slow-roll approximation, thus, H may be considered as a function of ϕ. As proved in [6] (see also [9]), though formally the noise $f(N)$ is an operator quantity containing the Fock annihilation and creation operators a_k and $a_k^\dagger$ for modes with $k = \epsilon aH$, its values in different points of space and for different N are commutative. Thus, it is equivalent to some classical noise whose distribution function appears to be Gaussian since the quantum noise f is linear in a_k and $a_k^\dagger$. Correspondingly, the large-scale part of the quantum inflaton field $\hat\phi$ is equivalent to a classical stochastic field ϕ with some normalized probability distribution $\rho(\phi, N)$, $\int \rho(\phi, N)d\phi = 1$, in the sense of equality of all possible classical and quantum expectation values:

$$\langle F[\hat\phi]\rangle_{\rm REN} = \int \rho(\phi)F[\phi]d\phi, \tag{37}$$

where F means any functional which may include time and spatial derivatives of any order.[3]

Following [6], the number of e-folds N was considered as a time variable in the stochastic Langevin equation in a number of papers, e.g. in [28] and most recently in [29], while in many other ones the proper time t was used, e.g. in [25,30–32]. This usage of different time variables should not be mixed with the invariance of all physical results with respect to a (deterministic) time reparametrization $t \to f(t)$ which is trivially satisfied after taking the corresponding change in the metric lapse function into account. In contrast, the transformation from t to N made using the stochastic function $H(\phi(t))$ leads to a physically different stochastic process with another probability distribution. Our new statement in this paper following the arguments above is that one *should* use the N variable when calculating mean squares of any quantity containing metric fluctuations like the Mukhanov variable $\delta\phi$ or the gauge-invariant metric perturbation ζ. Otherwise, incorrect results would be obtained using the stochastic approach which would then not coincide with those obtained using perturbative QFT methods. This statement is further supported by exact nonperturbative results (valid to all orders of metric perturbations) from the general δN formalism which relates the value of ζ after inflation to the difference in the number of e-folds N in different points of space—it was first used in [3] for one inflaton field, see Eq. (17) of this paper, and then generalized to multicomponent inflation in [33,34], see also [35,36] for its recent developments.

On the other hand, the usage of the t variable is natural if one is interested e.g. in differences of the local duration of inflation in proper time in different points of space which, in principle, may be measured using a spatial distribution of a phase of the wave function of a heavy particle with a mass $m \gg H$. Thus, the choice of a proper time variable in the stochastic equation is not an absolute one but is dictated by the physical nature of "clocks" relevant to observable effects. In our case, N is the "clock." For example, if a rather small ($\sim 10\%$) contribution from the integrated Sachs-Wolfe effect due to the existence of dark energy (or a cosmological constant) in the present Universe is neglected, a large angle anisotropy of the cosmic microwave background temperature for multipoles $l < 50$ is just given by $\Delta N \equiv N - \langle N\rangle$ at the last scattering surface: $\Delta T/T = -\Delta N/5$, so that a larger local amount of inflation produces a negative spot in the cosmic microwave background temperature map (here $\langle\,\rangle$ means averaging over the 2-sphere—the last scattering surface).

Now, in order to compare with perturbative QFT results, let us assume that fluctuations in ϕ are still much less than the classical background value $\phi_{\rm cl}(N)$: $|\delta\phi| \equiv |\phi - \phi_{\rm cl}| \ll |\phi_{\rm cl}|$. In the rest of this section, we shall denote by H and ϕ their classical background values. Then the noise term in Eq. (36) may be considered as a perturbation. In the first order, after expanding the first term in the right-hand side of Eq. (36) in powers of $\delta\phi$, we get the following equation for $\delta \equiv \delta\phi^{(1)}$ (the super-

[3]Note that the so-called volume-weighted averaging [25], for which ρ is not normalizable and thus loses the sense of a probability distribution, is nowhere used in this paper, as well as in [6,9]. For this reason, criticism of this approach contained e.g. in the recent paper [26] has no application to our results. See [27] for the most elaborate proposal of how to cure problems of the volume-weighted approach.

script 1 denotes the order of expansion):

$$\frac{d\delta}{dN} + 2M_{\rm pl}^2\left(\frac{H'}{H}\right)'\delta = \frac{f}{H}, \tag{38}$$

where the prime denotes the derivative with respect to ϕ. Multiplying both sides of Eq. (38) by δ, averaging and using the relation $\langle f\delta\rangle = H^3/8\pi^2$, we obtain the equation for $u \equiv \langle(\delta\phi^{(1)})^2\rangle$ corresponding to the one-loop approximation of QFT in curved space-time:

$$\frac{du}{dN} + 4M_{\rm pl}^2\left(\frac{H'}{H}\right)'u = \frac{H^2}{4\pi^2}. \tag{39}$$

Equation (39) is valid for any potential $V(\phi)$ satisfying the slow-roll conditions. Its generic solution is

$$u = -\frac{H'^2}{8\pi^2 H^2 M_{\rm pl}^2}\int \frac{H^5}{H'^3}d\phi, \tag{40}$$

where the integration variable has been changed from N to ϕ using the slow-roll relation $dN = -H/(2H'M_{\rm pl}^2)d\phi$. Applying this solution to the case $V(\phi) = m^2\phi^2/2$, $H = m\phi/(\sqrt{6}M_{\rm pl})$ and assuming $u(0) = 0$, we obtain

$$u = \frac{H_0^6 - H^6}{8\pi^2 m^2 H^2}, \tag{41}$$

that just coincides with the result derived in [11] using QFT methods. Also, applying Eq. (40) to power-law inflation where $a(t) \sim t^p$ with $p \gg 1$ and H exponentially depends on ϕ, we easily recover the correct leading renormalized value of u obtained in [37].

In the Appendix, the mean value of a second order field fluctuation $\langle\delta\phi^{(2)}\rangle$ is calculated and also conditions of the validity of the perturbation theory are reviewed.

To emphasize the difference, note that if we repeat the same procedure using the stochastic equation (36) written in terms of the independent variable t, $dt = dN/H$, we would obtain a different result [25,32]:

$$\tilde{u} = -\frac{H'^2}{8\pi^2 M_{\rm pl}^2}\int \frac{H^3}{H'^3}d\phi, \tag{42}$$

which, in particular, reduces to $3(H_0^4 - H^4)/(16\pi^2 m^2)$ for massive inflation. As already explained above, the discrepancy between (40) and (42) is not surprising and reflects the fact that different stochastic processes are considered, which is why we wrote $\tilde{u}$ instead of u in Eq. (42).

The general result presented in Eq. (40) for the renormalized value of the average squared gauge-invariant massive inflaton fluctuation has been explicitly verified by the authors in the context of QFT with renormalization obtained on employing the adiabatic subtraction prescription. In particular, we have considered a family of chaotic inflationary models with potentials $V(\phi) = \frac{\lambda}{n}\kappa^{n-4}\phi^n$ and obtained for such models results similar to those presented in Sec. II, but valid for gauge-invariant fluctuations. Equation (40) has been found to give renormalized values in complete agreement with our perturbative QFT calculations in the slow-roll approximation.

VII. FOUR-LOOP CALCULATION FOR A MASSLESS SELF-INTERACTING TEST SCALAR FIELD IN THE STOCHASTIC APPROACH

In the previous sections, only the mean value of $\langle(\delta\phi^{(1)})^2\rangle$ corresponding to one scalar loop in external background curved space-time was calculated (with small metric fluctuations taken into account also for the case of an inflaton scalar field). However, as was shown in [6,9], the stochastic approach can reproduce QFT results for any finite number of scalar loops and even beyond (e.g. results obtained using instantons). As an example, let us consider a massless self-interacting test scalar field χ with the potential $V(\chi) = \lambda\chi^4/4$ in the exact de Sitter space-time with the curvature H_0 following [9]. Then $N = H_0 t$, and it makes no difference which time variable is used in the stochastic Langevin equation (36). It is straightforward to construct the corresponding Fokker-Planck equation for a normalized probability distribution $\rho(\chi, N)$:[4]

$$\frac{\partial\rho}{\partial N} = \frac{\lambda}{3H_0^2}\frac{\partial(\rho\chi^3)}{\partial\chi} + \frac{H_0^2}{8\pi^2}\frac{\partial^2\rho}{\partial\chi^2}. \tag{43}$$

Multiplying Eq. (43) by χ^n where n is an even integer and integrating over χ from $-\infty$ to ∞, we obtain the following recurrence relation (cf. also [38]):

$$\frac{d}{dN}\langle\chi^n\rangle = -\frac{n\lambda}{3H_0^2}\langle\chi^{n+2}\rangle + \frac{n(n-1)H_0^2}{8\pi^2}\langle\chi^{n-2}\rangle. \tag{44}$$

Then solving Eq. (44) iteratively beginning from $n = 2$ with the initial conditions $\langle\chi^n(0)\rangle = 0$ for all n, we find

$$\begin{aligned}
\langle\chi^2\rangle_{\rm REN} &= \frac{H_0^2 N}{4\pi^2}(1 + \alpha_1\lambda X^2 + \alpha_2\lambda^2 X^4 + \alpha_3\lambda^3 X^6 + \cdots),\\
\langle\chi^4\rangle_{\rm REN} &= 3\left(\frac{H_0^2 N}{4\pi^2}\right)^2(1 + \beta_1\lambda X^2 + \beta_2\lambda^2 X^4 + \cdots),\\
\langle\chi^6\rangle_{\rm REN} &= 15\left(\frac{H_0^2 N}{4\pi^2}\right)^3(1 + \gamma_1\lambda X^2 + \cdots),
\end{aligned} \tag{45}$$

where $X = N/(2\pi)$ and

$$\begin{aligned}
\alpha_1 &= -\frac{2}{3} \qquad \alpha_2 = \frac{4}{5} \qquad \alpha_3 = -\frac{424}{315}\\
\beta_1 &= -2 \qquad \beta_2 = \frac{212}{45} \qquad \gamma_1 = -4.
\end{aligned} \tag{46}$$

The result for α_1 agrees with the perturbative QFT computations in [39,40] (see also [41]). Since the λ-independent term in the expression for $\langle\chi^2\rangle_{\rm REN}$ corre-

[4]This one point distribution function should not be confused with the energy density ρ_χ of a test χ field considered in Secs. II, III, IV, and V.

sponds to one scalar loop and each next power of λ requires one more scalar loop, the total result for $\langle\chi^2\rangle_{\rm REN}$ presented in (45) and (46) requires consideration of diagrams up to four loops included in the perturbative QFT treatment.

VIII. CONCLUSIONS

We have studied the application of the stochastic approach to inflationary space-times with $\dot{H} \neq 0$ and compared its results to those obtained using the standard QFT in curved space-time, extending the results of [10,11]. We can summarize our main results by the following points:

(1) On considering a test field χ with mass m_χ in $m^2\phi^2$ inflation, we have shown that the instantaneous Bunch-Davies expectation value $\langle\chi^2\rangle = 3H^4(t)/(8\pi^2 m_\chi^2)$ is never reached. It may be approached at the end of inflation for $m_\chi \gg m$ only. If $m_\chi \ll m$, the value of $\langle\chi^2\rangle$ at the end of inflation is quite different from that extrapolated from the exact de Sitter space-time.

(2) We have analyzed implications of the particle production peculiar to $m^2\phi^2$ inflation. The moduli problem is more serious than in the classical counterpart and also with respect to previous quantum investigations [13].

(3) Concerning gauge-invariant inflaton fluctuations, we have clarified why the stochastic Langevin equation for the large-scale part of an inflaton field should be formulated using the number of e-folds $N = \ln(a/a_0)$ as an independent time variable, if one is interested in any result regarding gauge-invariant inflaton fluctuations and metric perturbations. The mean square of inflaton fluctuations calculated in this way has been shown to coincide with the earlier result of [11] obtained using standard perturbative methods.

(4) In the inflationary space-times studied here, neither test fields nor inflaton mean square expectation value admit a static equilibrium solution.

(5) The equivalence between the stochastic and the standard field-theoretic approaches works beyond the one-loop approximation to QFT in curved space-time.

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APPENDIX

In this Appendix, we shall denote by H and ϕ their classical background values; we also use $u \equiv \langle(\delta\phi^{(1)})^2\rangle$. Starting from Eq. (36), and performing a perturbative expansion, it is easy to derive for the average second order fluctuation $\langle\delta\phi^{(2)}\rangle$ a time evolution also governed by (see also [28])

$$\frac{d}{dt}\langle\delta\phi^{(2)}\rangle = \frac{H^3}{16\pi^2}\left(\frac{V_\phi}{V}\right) - \left(\frac{1}{3H}V_{\phi\phi} + 2\frac{\dot{H}}{H}\right)\langle\delta\phi^{(2)}\rangle + \frac{1}{2} \times\left[-\frac{1}{3H}V_{\phi\phi\phi} + \left(\frac{1}{H}V_{\phi\phi} + 4\frac{\dot{H}}{H}\right)\frac{V_\phi}{V}\right]u. \tag{A1}$$

The general solution of Eq. (A1) with the initial condition $\delta\phi(t = t_i) = 0$ is given by

$$\langle\delta\phi^{(2)}\rangle = \left(\frac{V_\phi}{V}\right)\int_{t_i}^{t} dt'\left(\frac{V}{V_\phi}\right)\left\{\frac{H^3}{16\pi^2}\left(\frac{V_\phi}{V}\right) + \frac{1}{2} \times\left[-\frac{1}{3H}V_{\phi\phi\phi} + \left(\frac{1}{H}V_{\phi\phi} + 4\frac{\dot{H}}{H}\right)\frac{V_\phi}{V}\right]u\right\}. \tag{A2}$$

In the particular case of a chaotic inflation with $V(\phi) = \frac{m^2}{2}\phi^2$ we obtain Eq. (41) for first order and

$$\langle\delta\phi^{(2)}\rangle = \frac{\dot{\phi}}{32\pi^2 m^2 H M_{\rm pl}^2}\left[\frac{H_0^6 - H^6}{H^2} - 3(H_0^4 - H^4)\right], \tag{A3}$$

for second order. Note that the solution (41) clearly agrees with the solution (13) for suitable initial conditions and $\alpha = -1$.

Now we wish to study the validity of the perturbative expansion by considering the ratio $\frac{\langle\delta\phi^{(2)}\rangle}{\sqrt{u}}$ and $\frac{\sqrt{u}}{\phi}$. Using the former results one obtains

$$\frac{\langle\delta\phi^{(2)}\rangle}{\sqrt{u}} = -\frac{1}{8\pi\sqrt{3}}\frac{1}{M_{\rm pl}}\frac{\frac{1}{H^2}(H_0^6 - H^6) - 3(H_0^4 - H^4)}{(H_0^6 - H^6)^{1/2}} \tag{A4}$$

$$\frac{\sqrt{u}}{\phi} = \frac{1}{4\pi\sqrt{3}}\frac{1}{M_{\rm pl}}\frac{1}{H^2}(H_0^6 - H^6)^{1/2}. \tag{A5}$$

Thus, as we can see, those two ratios differ only by a factor 2 for the leading term toward the end of inflation. To see when the perturbative expansion is no longer useful, we can use the variable $\tilde{N}$ defined as the number of e-folds away from the maximum value $N_{\rm max} = N_0 = \log\frac{a_{\rm max}}{a(t_i)} = \frac{3}{2}\frac{H_0^2}{m^2}$, namely,

$$N_{\rm max} - \tilde{N} = \log\frac{a(t)}{a(t_i)} \rightarrow \tilde{N} = \frac{3}{2}\frac{H^2}{m^2}. \tag{A6}$$

Using this variable one obtains, to leading order, for the ratio (A4) the following result:

$$\frac{\langle \delta\phi^{(2)}\rangle}{\sqrt{u}} = -\frac{\sqrt{2}}{24\pi}\frac{m}{M_{\rm pl}}\frac{N_{\rm max}}{\tilde{N}}(N_{\rm max}-\tilde{N})^{1/2}. \qquad (A7)$$

If we require that the absolute value of this ratio be less than one we obtain, under the condition $768\pi^2\frac{M_{\rm pl}^2}{H_0^2}\gg 1$, the following approximate constraint:

$$\tilde{N} \geq \frac{3\sqrt{3}}{48\pi}\frac{H_0^3}{M_{\rm pl}m^2}. \qquad (A8)$$

If we consider the end of inflation when $H = m$ this corresponds to $\tilde{N} = 3/2$, so in order to have a perturbative expansion with small terms for all of the duration of inflation we obtain, on considering the condition (A8) with $\tilde{N} = 3/2$, the following condition on H_0:

$$H_0 < \left(\frac{24\pi}{\sqrt{3}}M_{\rm pl}m^2\right)^{1/3}, \qquad (A9)$$

similarly on considering the ratio (A5) one obtains the condition

$$H_0 < \left(\frac{12\pi}{\sqrt{3}}M_{\rm pl}m^2\right)^{1/3}. \qquad (A10)$$

The above correspond, for the particular numerical value $M_{\rm pl} = 10^5$ m, respectively, to $H_0 < 163.28$ m and $H_0 < 129.596$ m. Such bounds are in agreement with previous investigations [24].

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