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Scalar resonances in a unitary $\pi\pi$ S-wave model for $D^+ \to \pi^+\pi^-\pi^+$

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We propose a model for $D^+ \to \pi^+\pi^-\pi^+$ decays following experimental results which indicate that the two-pion interaction in the S wave is dominated by the scalar resonances $f_0(600)/\sigma$ and $f_0(980)$. The weak decay amplitude for $D^+ \to R\pi^+$, where R is a resonance that subsequently decays into $\pi^+\pi^-$, is constructed in a factorization approach. In the S wave, we implement the strong decay $R \to \pi^+\pi^-$ by means of a scalar form factor. This provides a unitary description of the pion-pion interaction in the entire kinematically allowed mass range $m^2_{\pi\pi}$ from threshold to about 3 GeV^2. In order to reproduce the experimental Dalitz plot for $D^+ \to \pi^+\pi^-\pi^+$, we include contributions beyond the S wave. For the P wave, dominated by the $\rho(770)^0$, we use a Breit-Wigner description. Higher waves are accounted for by using the usual isobar prescription for the $f_2(1270)$ and $\rho(1450)^0$. The major achievement is a good reproduction of the experimental $m^2_{\pi\pi}$ distribution, and of the partial as well as the total $D^+ \to \pi^+\pi^-\pi^+$ branching ratios. Our values are generally smaller than the experimental ones. We discuss this shortcoming and, as a by-product, we predict a value for the poorly known $D \to \sigma$ transition form factor at $q^2 = m^2_\pi$.

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I. INTRODUCTION

In 2001, the E791 Collaboration found very strong evidence for a light and broad scalar-isoscalar resonance in $D^+ \to \pi^+\pi^-\pi^+$ decays [1], confirming the existence of the $f_0(600)$ (also referred to as the σ). This pioneering work was soon followed by the authentication of another elusive scalar, the $K_0^*(800)$ (or κ), in $D^+ \to K^-\pi^+\pi^+$ [2]. In the past, several analyses of $\pi\pi$ scattering data already claimed the presence of the σ in the form of a pole close to threshold and with a large imaginary part [3–5]. However, its manifestation in scattering is subtle and one can consider the E791 experiment as the first solid empirical evidence for this resonance. The understanding of this pole in $\pi\pi$ scattering has improved considerably thanks to chiral perturbation theory (ChPT) [6] and to dispersion relations, namely, Roy's equations [7] (for recent works see, for instance, [8–10]). In 2006, the σ pole was obtained from a theoretical analysis combining these two ingredients and yielding the accurate result $\sqrt{s_\sigma} = (441^{+18}_{-16} - i272^{+9}_{-12.5})$ MeV [11]. Furthermore, in the last few years, experimental evidences for the σ in other processes such as $J/\psi \to \omega\pi^+\pi^-$ [12] and, notably, from new analyses of $D^+ \to \pi^+\pi^-\pi^+$ decays [13,14] have been published. Thus, at present, one can safely state that the σ is the lightest resonance in the hadronic spectrum.

However, in spite of all theoretical progress, a comprehensive description of the reaction $D^+ \to \pi^+\pi^-\pi^+$ is still to be accomplished. The reasons are twofold. First, the c-quark mass lies in an intermediate range of energy, between the realm of light quarks (u, d and s) and that of heavy quarks (b and t). Although, on the one hand, decays of light mesons such as the kaon can be treated within the framework of ChPT and, on the other, decays of the B can be calculated within QCD factorization [15], heavy quark effective theory [16] and soft collinear effective theory [17], no such rigorous framework exists for the D. Second, one deals with a three-body final state which renders a full treatment of mesonic final state interactions (FSIs) most involved.

In the search of a more sound theoretical framework to treat this type of reaction, experimentalists usually fit the Dalitz plot for the three-body decay with the isobar model. Schematically, the model consists of a trial amplitude of the form

$$\mathcal{M} = \alpha_{\rm NR} e^{i\phi_{\rm NR}} + \sum_{i=1}^{n} \alpha_i e^{i\phi_i} \mathcal{A}_i, \qquad (1)$$

where the first term on the right-hand side corresponds to a nonresonant background and the sum runs over all the n resonances that contribute to the decay. In Eq. (1), the parameters α_i and ϕ_i are real constants and the subamplitudes $\mathcal{A}_i$ (depending on invariant masses), whose main

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ingredients are relativistic Breit-Wigner functions (BWs), represent the propagation and decay of each resonance. In the course of the analysis, E791 discovered that the usual set of $\pi\pi$ resonances was not sufficient to yield a good fit and, therefore, the addition of a new BW was required. Leaving its mass and width free to float, an improvement of the least square function χ^2 was obtained with the parameter values $m_\sigma = (478^{+24}_{-23} \pm 17)$ MeV and $\Gamma_\sigma = (324^{+42}_{-41} \pm 21)$ MeV. These values, however, depend strongly on the specific BW chosen to fit the Dalitz plot and, in the literature, one finds various choices of BW propagators. Moreover, this simple description for a very broad resonance close to a threshold, like the σ, has many deficiencies that are discussed, for instance, in Ref. [18].

On the theoretical side, a possible way of improving Eq. (1) consists in replacing the BW functions with expressions based on the knowledge of scattering amplitudes. This procedure respects unitarity and reveals the relation between scattering and production experiments. Using a description of the S-wave two-body FSIs within a unitarized ChPT framework [19], Oller has proposed, in Ref. [18], a modified version of Eq. (1). Albeit successful, this model does not tackle the weak vertex of the reaction and, consequently, the constants α_i and ϕ_i in Eq. (1) remain fit parameters.

Motivated by the study of CP violation, more detailed versions of Eq. (1) were produced in the context of three-body hadronic B decays, in which both the weak and strong interactions are treated: $B \to \pi\pi\pi$ was considered in Refs. [20,21] whereas $B \to K\pi\pi$ and $B \to K\bar{K}K$ were treated in Refs. [22–24]. In these works, the weak decay is evaluated with the help of the effective weak Hamiltonian within QCD factorization whereas the hadronic FSIs are taken into account by means of unitary $K\pi$ and $\pi\pi$ form factors constrained by scattering data and ChPT. These models confront data very well and are precisely the basis of the approach we follow in the present work. Our main purpose is to test the description of $\pi^+\pi^-$ pairs in a relative S-wave state in $D^+ \to \pi^+\pi^-\pi^+$.

The decay amplitude for $D^+ \to \pi^+\pi^-\pi^+$ is constructed as follows. We assume that the three-body decay is always mediated by a resonance R as suggested by experimental analyses [1,13,14]. Then, for the $\pi^+\pi^-$ pairs in an S-wave state, denoted hereafter $(\pi^+\pi^-)_S$, we factorize the decay amplitude $D^+ \to R\pi^+$ using the effective weak Hamiltonian within naïve factorization. Afterwards, the three-body $(\pi^+\pi^-)_S\pi^+$ final state is constructed from the intermediate $R\pi^+$ state employing the $\pi\pi$ scalar form factors introduced in Ref. [22]. This ensures that our description of FSIs is unitary and includes the coupling to the $K\bar{K}$ state. The form factor is based on the experimental scattering phase shifts $\delta_{\pi\pi}$ and $\delta_{K\bar{K}}$ previously studied in Ref. [5]. As usual, we work within the quasi-two-body approximation in which interactions of the remaining π^+ with the $(\pi^+\pi^-)_S$ pair are neglected. This procedure is repeated for $\pi^+\pi^-$ in a P wave, which is well approximated by the $\rho(770)^0$ resonance. Finally, two higher mass resonances, the $f_2(1270)$ and the $\rho(1450)^0$, are included phenomenologically using the isobar model. Our final amplitude is then fitted to the signal function employed by the E791 Collaboration [1]; this comparison is carried out following a scheme presented in Ref. [18].

The present paper completes the description briefly reported in Ref. [25] and is organized as follows. In Sec. II, we discuss the effective weak Hamiltonian and the weak amplitudes employed in our description for the S and P waves. The construction of the three-body final state is presented in Sec. III, results for our fits are displayed in Sec. IV and a summary and discussions are given in Sec. V.

II. WEAK AMPLITUDES

Our phenomenological description of weak decays involving S- or P-wave resonances is based on the effective weak Hamiltonian $\mathcal{H}_{\rm eff}$, which is obtained by integrating out the heavy degrees of freedom of the standard model (SM) Lagrangian. The Hamiltonian is written as an operator product expansion (OPE) and reads

$$\mathcal{H}^{\Delta C=1}_{\rm eff} = \frac{G_F}{\sqrt{2}} \sum_i V_{\rm CKM} C_i(\mu)\hat{O}_i(\mu) + {\rm H.c.}, \qquad (2)$$

where $G_F = 1.16637(1) \times 10^{-5}\ {\rm GeV}^{-2}$ [26] is the Fermi decay constant, $V_{\rm CKM}$ are products of Cabibbo-Kobayashi-Maskawa (CKM) matrix elements, $C_i(\mu)$ are Wilson coefficients and $\hat{O}_i(\mu)$ local operators entering the OPE. Furthermore, μ is the renormalization scale which, in our case, is taken to be of order m_c. The operators $\hat{O}_i(\mu)$ represent the local four-quark weak interaction in the effective theory while $C_i(\mu)$ describe the hard- or short-distance physics and are calculated perturbatively in the full theory, in this case the SM. The contributions of all particles with mass $m > \mu = m_c$, such as the heavier b and t quarks and the W bosons, are included in $C_i(\mu)$.

In the present case, we need to consider the Cabibbo suppressed transition $c \to du\bar{d}$. Consequently, the tree level matrix elements are governed by the coefficient $V^{\rm tree}_{\rm CKM} = V_{cd}V^*_{ud} \equiv \lambda_d$ which, in Wolfenstein's parametrization [27], is of $\mathcal{O}(\lambda)$, where $\lambda = 0.2257$ [26]. In principle, our amplitudes receive contributions from strong penguins as well but, using the unitarity of the CKM matrix, one sees that these are governed by $V^P_{\rm CKM} = V_{cb}V^*_{ub}$, which is of $\mathcal{O}(\lambda^5)$. Thus, penguin amplitudes are strongly CKM suppressed and can safely be neglected. This suppression contrasts with the situation found in analogous B decays where, since one explores a different sector of the CKM matrix, penguin operators give rise to sizable contributions [20].

Taking into account only tree operators, the matrix element of $\mathcal{H}_{\rm eff}$ for the decay $D^+ \to R\pi^+$ can be written as

$$\langle R\pi^+|\mathcal{H}_{\rm eff}|D^+\rangle = \frac{G_F}{\sqrt{2}}\lambda_d \sum_{i=1}^{2} C_i(\mu)\langle R\pi^+|\hat{O}_i|D^+\rangle(\mu), \tag{3}$$

where the operators $\hat{O}_1$ and $\hat{O}_2$ are given by

$$\begin{aligned}\hat{O}_1 &= \bar{d}\gamma_\nu(1-\gamma_5)c\bar{u}\gamma^\nu(1-\gamma_5)d,\\ \hat{O}_2 &= \bar{u}\gamma_\nu(1-\gamma_5)c\bar{d}\gamma^\nu(1-\gamma_5)d.\end{aligned} \tag{4}$$

To deal with the matrix element $\langle R\pi^+|\hat{O}_i|D^+\rangle(\mu)$ of Eq. (3), we assume that factorization at leading order (in $\Lambda_{\rm QCD}/m_c$ and α_s) holds:

$$\begin{aligned}\langle R\pi^+|\mathcal{H}_{\rm eff}|D^+\rangle = \frac{G_F}{\sqrt{2}}\lambda_d[&a_1(\mu)\langle\pi^+|\bar{u}\gamma_\nu(1-\gamma_5)d|0\rangle\\ &\times\langle R|\bar{d}\gamma^\nu(1-\gamma_5)c|D^+\rangle\\ &+a_2(\mu)\langle R|\bar{d}\gamma_\nu(1-\gamma_5)d|0\rangle\\ &\times\langle\pi^+|\bar{u}\gamma^\nu(1-\gamma_5)c|D^+\rangle].\end{aligned} \tag{5}$$

In this context, the new coefficients $a_1(\mu)$ and $a_2(\mu)$ which arise are expressed in terms of the Wilson coefficients $C_1(\mu)$ and $C_2(\mu)$ and of the number of colors $N_c = 3$ as

$$\begin{aligned}a_1(\mu) &= C_1(\mu) + \frac{1}{N_c}C_2(\mu),\\ a_2(\mu) &= C_2(\mu) + \frac{1}{N_c}C_1(\mu).\end{aligned} \tag{6}$$

We then have products of nonperturbative hadronic matrix elements that assume a decomposition in terms of Lorentz invariant form factors. At this point, a remark is in order. Since m_c is smaller than m_b by roughly a factor of 3, nonfactorizable contributions of order $\Lambda_{\rm QCD}/m_c$ are larger than in B decays. Hence, the factorization approximation may be less reliable for D physics. Factorization, however, was applied successfully to two-body D decays in the seminal papers by Bauer, Stech and Wirbel [28]. Later, it was used in the 1980s [29] and 1990s [30] to describe three-body hadronic D decays and, more recently, in 2001, Dib and Rosenfeld [31] have used the factorization approximation to treat the $D^+\to\sigma\pi^+$ decay in order to obtain, from the then novel E791 data, the $\sigma\pi\pi$ coupling and the form factor for the transition $D\to\sigma$. Finally, the transition $D\to f_0(980)$ was recently studied employing the same factorization scheme [32]. Thus, in spite of the complications introduced by the c mass scale, this approach is a reasonable starting point for a phenomenological analysis. Moreover, it enables the description to benefit from the treatment of mesonic FSIs developed in the context of B decays.[1]

[1]This is the reason why we do not adopt the effective mesonic Lagrangian of Refs. [33,34].

A. $D^+\to\sigma\pi^+$ and $D^+\to f_0(980)\pi^+$ decays

From the available experimental analyses it has become clear that the $\pi\pi$ S wave gives the most important contribution to the decay $D^+\to\pi^+\pi^-\pi^+$. The σ was found to account for almost 50% of the decays whereas the $f_0(980)$ accounts only for about 6% [1]. This picture remains unchanged with the more recent analyses of FOCUS [13] and CLEO [14] (see Table III for the values extracted from these papers). In addition, there are indications for a sizable component arising from the higher mass scalar resonances $f_0(1370)$ and $f_0(1500)$, although the data analyses are not conclusive. The E791 analysis includes only one state, the $f_0(1370)$, whereas the CLEO Collaboration claims that both are necessary to yield a good fit within the isobar model. In our model, the σ and the $f_0(980)$ enter explicitly in the amplitude while one higher mass state close to 1500 MeV is accounted for as a pole in the $(\pi^+\pi^-)_S$ form factor, discussed in Sec. III.

The exact nature of scalar mesons, i.e. whether their wave function is dominated by $\bar{q}q$ or $\bar{q}^2q^2$ states, or even glueballs, is not yet elucidated. In the case of the σ, the situation is even more obscure, since the issue of whether it is a preexisting quark state or dynamically generated by the strong S-wave $\pi\pi$ interactions is still under debate. Therefore, the inclusion of the σ in our amplitude is far from obvious. On the other hand, we know that the σ is an isoscalar state and no indications of a strange quark component in its wave function exists. Thus, we assign to the σ the minimal quark content compatible with these two facts, namely, $(u\bar{u}+d\bar{d})/\sqrt{2}$. Concerning the quark content of the $f_0(980)$, it is well established that it has a non-negligible strange component. Higher Fock states, e.g. $\bar{q}^2q^2$ and $\bar{q}^2q^2g$, may also be important to achieve a comprehensive description of its wave function.[2] In the context of heavy meson decays, however, there are indications that the $\bar{q}q$ component should be dominant [35]. Here, we consider the $f_0(980)$ as pure $\bar{q}q$ while allowing for an admixture of $s\bar{s}$:

$$f_0 = \frac{1}{\sqrt{2}}(u\bar{u}+d\bar{d})\sin\theta_{\rm mix} + s\bar{s}\cos\theta_{\rm mix}, \tag{7}$$

where $\theta_{\rm mix}$ is the mixing angle. With these definitions, we can compute the weak amplitude for $D^+\to\sigma\pi^+$ and $D^+\to f_0(980)\pi^+$ decays.

The matrix elements $\langle R|\bar{q}\gamma_\mu(1-\gamma_5)q|0\rangle$, where $R=\sigma$ or $f_0(980)$ and $q=u$, d or s, vanish by C invariance. Hence, the decay amplitude has no a_2 contribution [see Eq. (5)]. The annihilation topology is likely to be neglected since it contains form factors of light mesons evaluated at high momentum ($q^2=m_D^2$) [31]. The weak amplitude is then purely proportional to a_1. Finally, we note that, in the next section, we construct the three-body final state from the intermediate $\sigma\pi^+$ and $f_0\pi^+$ states with the help of the

[2]For a more complete discussion see Ref. [32].

$\pi\pi$ scalar form factor, thereby properly taking into account the strong FSIs in this channel.

The relevant form factor for the transition $D \to \sigma$, denoted by $F_0^{D\to\sigma}(q^2)$, is defined as[3]

$$q^\mu \langle \sigma(p)|\bar{d}\gamma_\mu(1-\gamma_5)c|D^+(P_D)\rangle = -i(m_D^2 - m_\sigma^2)\tilde{F}_0^{D\to\sigma}(q^2), \quad (8)$$

where $q^2 = (P_D - p)^2$ and $\tilde{F}_0^{D\to\sigma}(q^2) = F_0^{D\to\sigma}(q^2)/\sqrt{2}$. Furthermore, with the usual decomposition one has

$$\langle \pi^+(p)|\bar{u}\gamma^\mu(1-\gamma_5)d|0\rangle = if_\pi p^\mu. \quad (9)$$

It is then straightforward, once factorization has been applied, to write the amplitude for $D^+ \to \sigma\pi^+$ with the help of Eq. (5):

$$\mathcal{A}(D\to\sigma\pi^+) = \langle \sigma\pi^+|\mathcal{H}_{\rm eff}|D^+\rangle = \frac{G_F}{\sqrt{2}}\lambda_d a_1(m_c) f_\pi(m_D^2 - m_\sigma^2)\tilde{F}_0^{D\to\sigma}(m_\pi^2). \quad (10)$$

This expression coincides with the one found in Ref. [31] and can easily be compared with the amplitude for $B \to \sigma\pi$ from Ref. [20] as well. However, one should note that in Refs. [20,31], the normalization factor $1/\sqrt{2}$ is not included in the σ wave function. Finally, we have computed the strong penguin contributions to Eq. (10) and we have checked explicitly that they can be neglected due to CKM suppression.

For the decay $D^+ \to f_0(980)\pi^+$ we obtain the following result:

$$\mathcal{A}(D^+\to f_0\pi^+) = \frac{G_F}{\sqrt{2}}\lambda_d a_1(m_c) f_\pi(m_D^2 - m_{f_0}^2)\tilde{F}_0^{D\to f_0}(m_\pi^2), \quad (11)$$

where we have defined

$$\tilde{F}_0^{D\to f_0}(m_\pi^2) = \frac{\sin\theta_{\rm mix}}{\sqrt{2}} F_0^{D\to f_0}(m_\pi^2). \quad (12)$$

B. $D^+ \to \rho(770)^0\pi^+$ decay

The evaluation of the amplitude $\mathcal{A}(D^+ \to \rho(770)^0\pi^+)$ from Eq. (5) is quite analogous to the previous ones except that the contribution of the color suppressed tree diagram does not vanish. The quark content in this case is $\rho^0 = (u\bar{u} - d\bar{d})/\sqrt{2}$ [for simplicity we denote by ρ^0 the $\rho(770)^0$]. In the color allowed term we need to consider the transition $D \to \rho$ which can be parametrized in terms of form factors using the general $P \to V$ amplitude, where P and V represent a pseudoscalar and a vector meson, respectively:

$$q^\mu \langle V(p_V)|j_\mu^V - j_\mu^A|P(p_P)\rangle = -i2m_V(\epsilon^*\cdot q)A_0^{P\to V}(q^2). \quad (13)$$

Here $q = p_P - p_V$, ϵ^* is the vector meson polarization. Furthermore, j_μ^V and j_μ^A represent bilinear vector and axial-vector quark currents, respectively. In the color suppressed topology, we need the transition $D \to \pi$ obtained from the general $P_1 \to P_2$ transition

$$\langle P_2(p'_P)|j_\mu^V - j_\mu^A|P_1(p_P)\rangle = \left[(p_P + p'_P)_\mu - \frac{m_1^2 - m_2^2}{q^2}q_\mu\right] \times F_1(q^2) + \frac{m_1^2 - m_2^2}{q^2}q_\mu F_0(q^2). \quad (14)$$

We define the vector decay constant as

$$\langle V|j_\mu^V|0\rangle = m_V f_V \epsilon_\mu^*, \quad (15)$$

where we have included the factor m_V to have a decay constant f_V with the dimension of energy.[4] With these definitions the result reads

$$\mathcal{A}(D^+ \to \rho^0\pi^+) = -2(\epsilon^*\cdot q)\eta^0, \quad (16)$$

where

$$\eta^0 = \frac{G_F}{2}\lambda_d m_\rho[a_1(m_c)f_\pi A_0^{D\to\rho}(m_\pi^2) + a_2(m_c)f_\rho F_1^{D\to\pi}(m_\rho^2)], \quad (17)$$

and $q = (P_D - p_\rho) = p_\pi$. The minus sign in Eq. (16) arises from the fact that the $d\bar{d}$ component of the ρ^0 is the one that intervenes. This result has the same structure as the one found in Ref. [20] for the $B^- \to \rho^0\pi^-$ decay.

III. HADRONIC FINAL STATE INTERACTIONS

In this section we describe the construction of the three-pion final state from the weak amplitudes [Eqs. (10), (11), and (16)]. This is done in order to take into account the FSIs and the three-body phase space. By FSIs, we mean the mesonic interactions in the final state after hadronization.

It is important to define the kinematics we employ. We are considering the generic amplitude $\mathcal{M} \equiv \mathcal{M}(D^+ \to \pi^+\pi^-\pi^+)$ with four-momenta labeled as follows:

$$D^+(p_D) \to \pi^+(p_1)\pi^-(p_-)\pi^+(p_2). \quad (18)$$

[3] Throughout this paper, we employ the decompositions and definitions of Refs. [20,21] for the hadronic form factors, unless otherwise stated.

[4] This last definition is not the same as in Refs. [20,21].

Since in the final state we have two identical π^+, the amplitude has to be symmetric under the exchange $p_1 \leftrightarrow p_2$. What is more, in order to work with a Lorentz invariant Dalitz plot, it is convenient to define three invariant combinations of momenta:

$$s = (p_1 + p_2)^2, \qquad t = (p_- + p_1)^2, \qquad u = (p_- + p_2)^2. \tag{19}$$

They correspond to the usual Mandelstam variables with

$$s + t + u = m_D^2 + 3m_{\pi^\pm}^2, \tag{20}$$

so that only two of them are independent. Resonances occur only in the t and u channels since there are no isospin 2 resonances. We denote by $\mathcal{A}_R(u, t)$ the amplitude for a decay mediated by a resonance R in the u channel. The final symmetric result $\mathcal{M}_R(u, t)$ is hence obtained by summing

$$\mathcal{M}_R(u, t) = \mathcal{A}_R(u, t) + \mathcal{A}_R(t, u). \tag{21}$$

A. S wave

The FSIs are taken into account by means of the $\pi\pi$ scalar form factor. This method was introduced in Ref. [36] and was later applied to $B \to \pi\pi\pi$ decays in the vicinity of the σ pole in Ref. [20] using the form factors obtained in the context of unitarized ChPT [19]. In Refs. [22,23] a similar description was used to describe the $f_0(980)$ in $B \to \pi\pi K$ and $B \to K\bar{K}K$ employing a different set of form factors that rely on a previous analysis of $\pi\pi$ and $K\bar{K}$ scattering data [5]. These form factors are unitary and contain both the $\pi\pi$ and $K\bar{K}$ channels. Here, we briefly summarize the model since all the details can be found in Ref. [20] and in the appendix of Ref. [23].

The two-pion scalar form factor $\Gamma^n(x)$ that is relevant to our work is defined as [36]

$$\langle 0|\bar{n}n|\pi(p)\pi(p')\rangle = \sqrt{2}B_0\Gamma^n(x), \tag{22}$$

where $\bar{n}n = (\bar{u}u + \bar{d}d)/\sqrt{2}$, B_0 is proportional to quark condensate $B_0 = -\langle 0|\bar{q}q|0\rangle/f_\pi^2$ and $x = (p + p')^2$. Since we want to describe $D^+ \to \pi^+\pi^-\pi^+$ and we have defined in Eq. (10) the amplitude for the transition $D^+ \to \sigma\pi^+$, we need to introduce a function $\Pi_{\sigma\pi\pi}(x)$ that describes the σ propagation and decay,[5] i.e. the final state interactions. The full amplitude $D^+ \to \sigma\pi^+ \to (\pi^+\pi^-)_S\pi^+$ is given by

$$\mathcal{A}_\sigma(u, t) = \mathcal{A}(D^+ \to \sigma\pi^+)\Pi_{\sigma\pi\pi}(u).$$

The $\sigma \to (\pi^+\pi^-)_S$ decay is described without resorting to BW expressions. It can be obtained from the complex conjugate of Eq. (22) assuming that, close to the σ pole, this resonance gives the dominant contribution to $\Gamma^n(x)$. We have [20]

$$\Pi_{\sigma\pi\pi}(x) = \sqrt{\frac{2}{3}}\frac{B_0}{\langle\sigma|\bar{n}n|0\rangle}\Gamma^{n*}(x) = \chi_\sigma\Gamma^{n*}(x). \tag{23}$$

The normalization constant

$$\chi_\sigma = \sqrt{\frac{2}{3}}\frac{B_0}{\langle\sigma|\bar{n}n|0\rangle} \tag{24}$$

is, in principle, unknown as it depends on the matrix element $\langle\sigma|\bar{n}n|0\rangle$.

To further clarify the meaning of $\Pi_{\sigma\pi\pi}(x)$, it is convenient to consider its analogue in a BW framework

$$\Pi^{\rm BW}_{\sigma\pi\pi}(x) = \frac{g_{\sigma\pi\pi}}{m_\sigma^2 - x - im_\sigma\Gamma(x)}, \tag{25}$$

where $g_{\sigma\pi\pi}$ is the coupling between the σ and the pions. We can obtain an estimate for χ_σ comparing expressions (23) and (25) at $x = m_\sigma^2$. We obtain

$$\chi_\sigma|\Gamma^{n*}(m_\sigma^2)| = \frac{g_{\sigma\pi\pi}}{m_\sigma\Gamma(m_\sigma^2)}. \tag{26}$$

We take the central values $m_\sigma = 478$ MeV and $\Gamma_\sigma = 324$ MeV from the E791 fit and use $g_{\sigma\pi\pi} = 2.52$ GeV [37]. This yields, with the form factor of Ref. [22],

$$\chi_\sigma \approx 29\ \text{GeV}^{-1}. \tag{27}$$

From the weak amplitude given in Eq. (10) and from the expression of $\Pi_{\sigma\pi\pi}(x)$ the amplitude for $D^+ \to \sigma\pi^+ \to (\pi^+\pi^-)_S\pi^+$ reads

$$\mathcal{A}_\sigma(u, t) = \frac{G_F}{\sqrt{2}}\lambda_d a_1(m_c) f_\pi(m_{D^+}^2 - m_\sigma^2) \times \tilde{F}_0^{D\to\sigma}(m_\pi^2)\chi_\sigma\Gamma^{n*}(u), \tag{28}$$

where the t dependence is implicit. Similarly for the intermediate resonance $f_0(980)$, one has

$$\mathcal{A}_{f_0}(u, t) = \frac{G_F}{\sqrt{2}}\lambda_d a_1(m_c) f_\pi(m_{D^+}^2 - m_{f_0}^2) \times \tilde{F}_0^{D\to f_0}(m_\pi^2)\chi_{f_0}\Gamma^{n*}(u). \tag{29}$$

For the weak decay amplitude we make use of Eq. (11) and one should note that the normalization, denoted χ_{f_0}, which by virtue of Eq. (26) is proportional to the coupling $g_{f_0\pi\pi}$, differs from the one found in Eq. (28).

Thus far, we have considered only a small energy range around the resonance poles. In other works, this prescription for the S wave was always used only within a limited region of the spectrum [20,22]. We want, however, to describe the whole Dalitz plot for $D^+ \to \pi^+\pi^-\pi^+$ where the $\pi\pi$ invariant mass ranges from $2m_{\pi^\pm} < \sqrt{s} < (m_{D^+} - m_{\pi^\pm})$, i.e. between 280–1700 MeV. To this aim, an ansatz is required to provide us with an expression for the entire S wave. Since the amplitudes (28) and (29) are both proportional to $\Gamma^{n*}(u)$, we propose the following amplitude for the S wave:

[5] $\Pi_{\sigma\pi\pi}(x)$ corresponds to $\Gamma_{\sigma\pi\pi}(x)$ of Ref. [20].

$$\mathcal{A}_S(u,t) = \frac{G_F}{\sqrt{2}}\lambda_d a_1(m_c) f_\pi (m_{D^+}^2 - u)\chi_{\text{eff}}\Gamma^{n*}(u). \quad (30)$$

In the last equation, χ_{eff} is a new normalization constant that encompasses all the form factors and normalizations for the scalar resonances. In addition, we have replaced the terms $(m_{D^+}^2 - m_R^2)$ by $(m_{D^+}^2 - u)$; this u dependence suppresses the contribution of higher mass resonances [22]. It is not easy to obtain a good estimate for χ_{eff} since it receives contributions from all the scalar-isoscalar states. The following lower bound of χ_{eff} results from Eqs. (28)–(30) from the $f_0(980)$ which is a well established resonance. Indeed, we have

$$\chi_{\text{eff}} > \frac{\sin\theta_{\text{mix}}}{\sqrt{2}} F_0^{D\to f_0}(m_\pi^2)\chi_{f_0}.$$

Using $\tilde{F}_0^{D\to f_0}(m_\pi^2) = 0.215$, the average value from the two models of Ref. [32], and $\chi_{f_0} = 28.9\ \text{GeV}^{-1}$ [23] one then gets the estimate

$$\chi_{\text{eff}} > 6.2\ \text{GeV}^{-1}. \quad (31)$$

From Eqs. (21) and (30) we can construct our final expression for the $D^+ \to (\pi^+\pi^-)_S\pi^+$ amplitude:

$$\mathcal{M}_S(u,t) = \frac{G_F}{\sqrt{2}}\lambda_d a_1(m_c) f_\pi \chi_{\text{eff}}[(m_{D^+}^2 - u)\Gamma^{n*}(u) + (m_{D^+}^2 - t)\Gamma^{n*}(t)]. \quad (32)$$

The last expression has only one parameter that can be considered as unknown: χ_{eff}. As far as $\Gamma^{n*}(x)$ is concerned, with $x = u, t$, we employ the form factor of Ref. [22]. It is unitary and takes into account the coupling to the $K\bar{K}$ channel.[6] This form factor is obtained within an on-shell approximation and can be written explicitly in terms of the S-wave scattering phase shifts $\delta_{\pi\pi}(x)$ and $\delta_{K\bar{K}}(x)$ as

$$\Gamma^{n*}(x) = \frac{1}{2}\Big[R_{\pi\pi}^n(x)(1 + \eta(x)e^{2i\delta_{\pi\pi}(x)}) - iR_{K\bar{K}}^n(x) \times \sqrt{\frac{k_2}{k_1}}\sqrt{1-\eta^2(x)}e^{i[\delta_{\pi\pi}(x)+\delta_{K\bar{K}}(x)]}\Big], \quad (33)$$

where $\eta(x)$ is the inelasticity for $\pi\pi$ scattering, $k_1 = \sqrt{x/4 - m_\pi^2}$ and $k_2 = \sqrt{x/4 - m_K^2}$. For $\delta_{\pi\pi}(x)$, $\delta_{K\bar{K}}(x)$ and $\eta(x)$, we employ the results of Ref. [5]. In addition, Eq. (33) depends on the production functions $R_{\pi\pi}^n(x)$ and $R_{K\bar{K}}^n(x)$ introduced in Ref. [36], which result from a matching to the ChPT expansion of $\Gamma^n(x)$. In practice, these functions can simply be written as $R_i^n(x) = c_i + d_i x$. The real coefficients c_i and d_i, which depend on the low energy constants of ChPT, were determined in Ref. [36] and updated in Ref. [38]. However, the validity of the production functions beyond $\sim$1.2 GeV is not guaranteed. To circumvent this problem, we introduce in our form factor a cutoff x_{cut} above which we saturate the $R_i^n(x)$, namely,

$$R_i^n(x) = \begin{cases} c_i + d_i x & \text{for } x < x_{\text{cut}}, \\ c_i + d_i x_{\text{cut}} & \text{for } x > x_{\text{cut}}. \end{cases} \quad (34)$$

This procedure does not affect the unitarity of the form factor but, of course, it introduces an additional parameter x_{cut} in the model. Our fits are done for different values of x_{cut} in order to carefully ascertain the dependence on this parameter.

B. *P* wave

The construction of the three-pion final state from the intermediate $\rho^0\pi^+$ state is similar to that done for the S wave. However, in the case of the ρ^0 it is not crucial to employ the vector form factor of the pion to describe the FSIs. Since the ρ^0 is a relatively narrow resonance, far from threshold, that strongly dominates the corresponding form factor, the BW description gives a good approximation. From the coupling of the ρ^0 to the pair $\pi^+\pi^-$ defined as $\langle\pi^+(q_+)\pi^-(q_-)|\rho^0\rangle = g_\rho(q_- - q_+)$ [20] and with the use of Eq. (16) we have

$$\mathcal{A}_{\rho^0}(u,t) = \eta_0(t-s)\Pi_{\rho\pi\pi}(u), \quad (35)$$

where η_0 is given in Eq. (17) and s, t and u are defined in Eq. (19). The factor $(t-s)$ comes from the sum over the polarizations of the ρ^0 and the function $\Pi_{\rho\pi\pi}(u)$ is defined by

$$\Pi_{\rho\pi\pi}(u) = \frac{g_\rho}{m_\rho^2 - u - im_\rho\Gamma_\rho(u)}. \quad (36)$$

For the running width $\Gamma_\rho(u)$ we take the usual relativistic prescription

$$\Gamma_\rho(u) = \frac{m_\rho}{\sqrt{u}}\Gamma_\rho^{\text{tot}}\left(\frac{p(u)}{p(m_\rho^2)}\right)^3, \quad (37)$$

[6]In a two-coupled channel ($\pi\pi$, $K\bar{K}$) description of the final state interactions for D^+ decays, the FSIs are incorporated in the form factor via the following unitary equation system [Eq. (11) of Ref. [22]]:

$$\Gamma_i^{n*}(x) = R_i^n(x) + \sum_{j=1}^{2}\langle k_i|R_j^n(x)G_j(x)T_{ij}(x)|k_j\rangle,$$

where $|k_i\rangle$ and $|k_j\rangle$ represent the wave functions of two mesons in the momentum space and the indices $i, j = 1, 2$ refer to the $\pi\pi$ and $K\bar{K}$ channels, respectively. The matrix T is the two-body scattering matrix and the functions $G_j(x)$ are the free Green's functions. With this definition, the form factor of Eq. (33) corresponds to $\Gamma_1^{n*}(x)$. The driving terms entering in these equations are given by production functions $R_i^n(x)$ representing the meson-meson formation from $q\bar{q}$ pairs. Further details can be found in Ref. [22].

where $p(u) = \sqrt{u/4 - m_\pi^2}$ and $\Gamma_\rho^{\rm tot}$ is the total decay width. The final expression for the amplitude of the decay mediated by the ρ^0 is then

$$\mathcal{M}_{\rho^0}(u, t) = \eta_0[(t - s)\Pi_{\rho\pi\pi}(u) + (u - s)\Pi_{\rho\pi\pi}(t)]. \tag{38}$$

IV. RESULTS

A. Parameter values

Since the experimental situation of the P wave in $D^+ \to \rho^0\pi^+$ is less controversial than that for the S wave, this channel can be used to ascertain the quality of the model. With Eq. (38) we can calculate the branching ratio for the $D^+ \to \rho^0\pi^+$ decay and compare it to the experimental average [26]

$$\mathcal{B}^{\rm PDG}(D^+ \to \rho^0\pi^+, \rho^0 \to \pi^+\pi^-) = (8.2 \pm 1.5) \times 10^{-4}.$$

For the numerical input needed we take $a_1(m_c) = 1.15$ [39], $a_2(m_c) = -0.25$ [39], $f_\pi = 130.4$ MeV [26], $f_\rho = 0.209$ GeV [23], $F_1^{D\to\pi}(m_\rho^2) = 0.8$ [40], $A_0^{D\to\rho}(m_\pi^2) \approx A_0^{D\to\rho}(0) = 0.75$ [41], and $g_\rho = 5.8$ [20]. Using the standard formula for the three-body decay rate [26] and taking into account the symmetry factor 1/2 we have, using Eq. (38),

$$\mathcal{B}(D^+ \to \rho^0\pi^+, \rho^0 \to \pi^+\pi^-) = 8.63 \times 10^{-4}. \tag{39}$$

Our branching ratio is rather sensitive to the value of the form factor $A_0^{D\to\rho}(m_\pi^2)$ which has an uncertainty of about 20% at $q^2 = 0$ [41]. Therefore, the theoretical error associated with our result is large. However, since the central values for the parameters yield a result in agreement with the experimental average, we keep these values. Thus, the amplitude $\mathcal{M}_{\rho^0}(u, t)$ serves as a benchmark to the determination of the other parameters of our model.

B. Fits to E791 signal function

Since we do not have the real data at our disposal, the fit procedure consists in reproducing the E791 signal function and comparing our model to it. To this aim, we closely follow the method of Ref. [18]. Aitala *et al.* in Ref. [1] used in their best fit the trial amplitude Eq. (1) with six resonances, namely, the σ, the $f_0(980)$ and $f_0(1370)$ whose quantum numbers are $I^G(J^P) = 0^+(0^+)$, the $\rho(770)^0$ and $\rho(1450)^0$ with $1^+(1^-)$ and, finally the $f_2(1270)$ with $0^+(2^+)$. Let us recall that the amplitude has also a complex constant $\alpha_{\rm NR}e^{i\phi_{\rm NR}}$ identified with the nonresonant background. With these ingredients, the signal function can be written

$$\mathcal{M}^{\rm E791}(u, t) = \alpha_{\rm NR}e^{i\phi_{\rm NR}} + \sum_{i=1}^{6} \alpha_i e^{i\phi_i}[\mathcal{A}_i^{\rm E791}(u, t) + \mathcal{A}_i^{\rm E791}(t, u)], \tag{40}$$

where the individual amplitudes $\mathcal{A}_i^{\rm E791}(u, t)$ are modeled as

$$\mathcal{A}_i^{\rm E791}(u, t) = F_D^J(u) \times F_i^J(u) \times \Omega_i^J(u, t) \times {\rm BW}_i(u). \tag{41}$$

In the last expression, $F_D^J(u)$ and $F_i^J(u)$ are Blatt-Weisskopf damping factors that depend on the spin J of the resonance, $\Omega_i^J(u, t)$ are angular factors and ${\rm BW}_i(u)$ are the Breit-Wigner propagators. The full expressions for these functions are rather lengthy and can be found in the original paper [1] or, more detailed, in Ref. [18]. From Eq. (40), and employing the fit results for α_i and ϕ_i [1] as well as the values for masses and widths used by the E791 Collaboration we are able to reproduce the signal function. Then, we generate a Dalitz plot with points separated by 0.05 GeV and normalize this plot to the number of observed signal events, 1124. Since the Dalitz plot is symmetric under $u \leftrightarrow t$ it is sufficient to fit only half of the plot.

Our complete amplitude for the S wave is given by Eq. (32) and the decay mediated by the ρ^0 is described by Eq. (38). With only these two contributions, however, it is not possible to achieve a reasonable reproduction of the E791 signal function. We have to include the two other resonances that give sizable contributions to the fit: the $f_2(1270)$ and the $\rho(1450)$ (denoted for simplicity by f_2 and ρ', respectively). This is done with the help of the isobar model, using the same expressions as those of [1]. We leave free, in our fit, the corresponding magnitudes α_{f_2} and $\alpha_{\rho'}$ and phases ϕ_{f_2} and $\phi_{\rho'}$. The final amplitude for the decay $D^+ \to \pi^+\pi^-\pi^+$ reads then

$$\mathcal{M}(u, t) = \mathcal{M}_S(u, t) + \mathcal{M}_{\rho^0}(u, t) + \mathcal{M}_{\rho'}(u, t) + \mathcal{M}_{f_2}(u, t), \tag{42}$$

where $\mathcal{M}_S(u, t)$ is given by Eq. (32) and $\mathcal{M}_{\rho^0}(u, t)$ by Eq. (38). For $\mathcal{M}_{\rho'}(u, t)$ we use

$$\mathcal{M}_{\rho'}(u, t) = \alpha_{\rho'}e^{i\phi_{\rho'}}[\mathcal{A}_{\rho'}^{\rm E791}(u, t) + \mathcal{A}_{\rho'}^{\rm E791}(t, u)], \tag{43}$$

where the explicit form of $\mathcal{A}_{\rho'}$ is given in Eq. (41). Analogously, for $\mathcal{M}_{f_2}(u, t)$ we use

$$\mathcal{M}_{f_2}(u, t) = \alpha_{f_2}e^{i\phi_{f_2}}[\mathcal{A}_{f_2}^{\rm E791}(u, t) + \mathcal{A}_{f_2}^{\rm E791}(t, u)]. \tag{44}$$

We shall refer to this model as "model A."

The amplitude equation (42) receives contributions from five different resonances. These are the three scalars σ, $f_0(980)$ and $f_0(1500)$ that appear as poles in our S-wave

TABLE I. Fits of model A, Eq. (42), to E791 signal function Eq. (40). Uncertainties are solely statistical.

	$\sqrt{x_{\rm cut}} = 1.0$ GeV	$\sqrt{x_{\rm cut}} = 1.2$ GeV	$\sqrt{x_{\rm cut}} = 1.4$ GeV
$\chi_{\rm eff}$ [GeV^{-1}]	6.5 ± 0.3	6.1 ± 0.3	6.0 ± 0.3
$\alpha_{f_2} \times 10^5$	(4.7 ± 0.5)	(4.2 ± 0.4)	(4.2 ± 0.4)
ϕ_{f_2} (rd)	-6.03 ± 0.18	-6.11 ± 0.20	-6.18 ± 0.20
$\alpha'_\rho \times 10^6$	(2.2 ± 0.8)	(2.6 ± 0.7)	(3.0 ± 0.6)
$\phi_{\rho'}$ (rd)	-0.57 ± 0.27	-0.30 ± 0.20	-0.31 ± 0.17
χ^2/d.o.f.	0.20	0.22	0.22

form factor Eq. (33) [22], the P-wave resonances $\rho(770)$ and $\rho(1450)$ and the D wave represented by the $f_2(1270)$. Note that we do not include a nonresonant (NR) amplitude. The necessity for such an amplitude is controversial. The E791 Collaboration has shown that the inclusion of the σ reduces the contribution of the NR background to less than 10%. More recently, the CLEO Collaboration did not find any significant evidence for the NR amplitude and an upper limit of 3.5% was established [14]. In addition, the NR amplitude can be energy dependent and the simple complex constant evenly spread over the whole phase space may be an unreliable model. Most importantly, the $\pi\pi$ scalar form factor $\Gamma^{n*}(x)$ already includes the NR contributions to $\pi\pi$ scattering which could generate a double counting of the background.

Only five free parameters occur in Eq. (42). Four of them (α_{f_2}, ϕ_{f_2}, $\alpha_{\rho'}$ and $\phi_{\rho'}$) arise from the isobar model description of the $f_2(1270)$ and $\rho(1450)$. The fifth is the real constant $\chi_{\rm eff}$ introduced in Eq. (30). Therefore, the relative weak phase of the S wave with respect to the ρ^0 is fixed, as well as the strong phases. In Table I we display the results for fits to the E791 signal function. For the functions $R^n_{\pi\pi}(x)$ and $R^n_{K\bar{K}}(x)$ of Eq. (33) we use the updated values obtained in Ref. [38]. We show the results for three fits in which we vary the value of the cutoff $x_{\rm cut}$ introduced in Eq. (34). The fit parameters do not depend much on $x_{\rm cut}$. When quoting final values, we take the fit with $\sqrt{x_{\rm cut}} = 1.2$ GeV and include an uncertainty due to the dependence on this cutoff. In Fig. 1 we show both the E791 signal function and the result of the fit with $\sqrt{x_{\rm cut}} = 1.2$ GeV displayed as Dalitz plots. The projection of these two functions is compared in Fig. 2.

Concerning the parameters of the fit, from Table I we see that the normalization constant for the S wave, $\chi_{\rm eff}$, is well determined and confirms our expectations derived in Eq. (31). The magnitudes of the $f_2(1270)$ and of the $\rho(1450)$ are well constrained by the fit as well. On the other hand, the phases for these higher mass resonances are not well determined. Finally, since $\mathcal{M}_{f_2}(u,t)$ and $\mathcal{M}_{\rho'}(u,t)$ in our amplitude equation (42) are exactly the same as in the function we are fitting to, namely, Eq. (40), the interpretation of the χ^2/d.o.f. as a measurement of the quality of the fit is not reliable.

From Figs. 1 and 2, one sees that the main discrepancy in the fit comes from the σ region. This is most probably due to the off-shell effects that are not included in the form factor given by Eq. (33). The omission of these effects in $\Pi_{\sigma\pi\pi}(x)$ may lead to an underestimation of the σ peak [42]. The model gives a much better description of the ρ^0 and the $f_0(980)$ peaks, both very prominent in the projection. In the higher energy region, where the $f_2(1270)$ the $\rho(1450)$ and higher mass scalar states are present, our

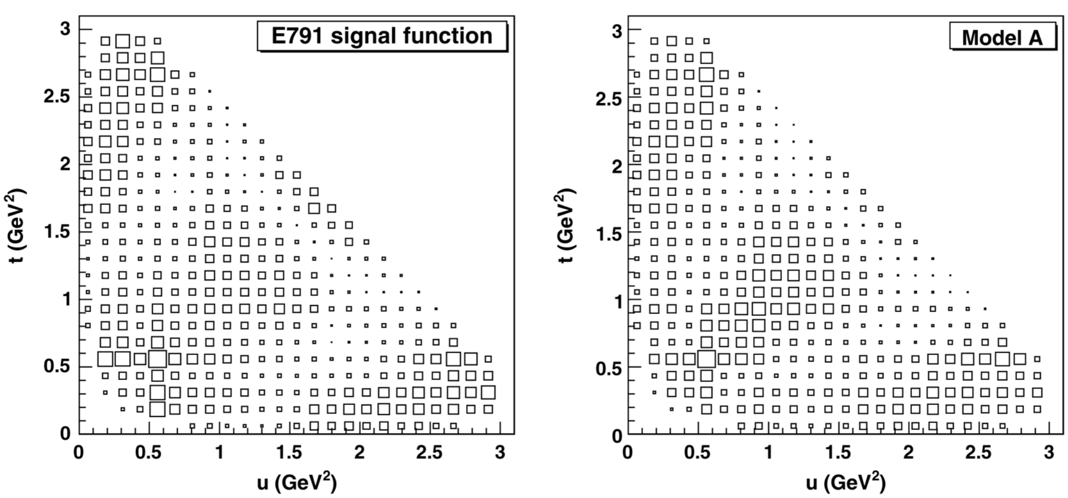

FIG. 1. Dalitz plot representation of the signal function employed by E791, Eq. (40), (left-hand side), same representation for the fitted amplitude given in Eq. (42) with the parameters of Table I for $\sqrt{x_{\rm cut}} = 1.2$ GeV (right-hand side).

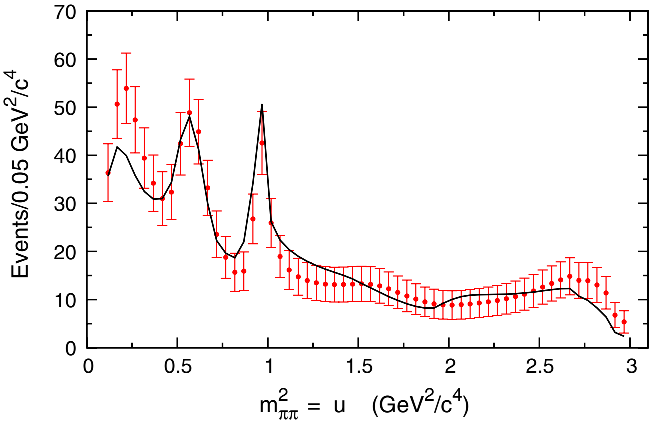

FIG. 2 (color online). Projection of the signal function of the E791 Collaboration (full dots) and the result of the fit (model A) for the parameters given in Table I with $\sqrt{x_{\rm cut}} = 1.2$ GeV.

description is reasonable, although not as good as in the ρ^0 and $f_0(980)$ region.

C. $D \to \sigma$ transition form factor

It is desirable to extract the form factors for the transition $D^+ \to \sigma\pi^+$ and $D^+ \to f_0(980)\pi^+$ from the fit. The continuous description of the S wave, however, renders this extraction difficult since the form factors are embedded in $\chi_{\rm eff}$. Nevertheless, we advance here a model which aims at determining $F_0^{D\to\sigma}(m_\pi^2)$ and $F_0^{D\to f_0}(m_\pi^2)$. We then need to separate the σ and $f_0(980)$ contributions in the S wave. The drawback is however a nonunified description of $(\pi^+\pi^-)_S$.

For energies within the elastic domain, $x < 4m_K^2$, the form factor of Eq. (33) is proportional to $\cos\delta_{\pi\pi}(x)$ [22]. Thus, it has a zero at the point x_0 where $\delta_{\pi\pi}(x_0) = \pi/2$. Numerically, from the analysis of Ref. [5], we have $x_0 \approx (0.828\ {\rm GeV})^2$. We split our S-wave amplitude at this point and consider that for energies $x < x_0$ the $(\pi^+\pi^-)_S$ is dominated by the transition $D \to \sigma$ and hence described by Eq. (28), whereas for $x > x_0$ the dominant transition is $D \to f_0(980)$ given by Eq. (29). In practice, we substitute Eq. (30) by Eq. (28) when $x < x_0$ and by Eq. (29) when $x > x_0$. We shall refer to this modified version of the model as "model B."

TABLE II. Fit of model B with $\sqrt{x_{\rm cut}} = 1.2$ GeV to E791 signal function Eq. (40). Uncertainties include all the possible sources (see text).

$\chi_\sigma \tilde{F}_0^{D\to\sigma}(m_\pi^2)$	$(8.4 \pm 1.4)\ {\rm GeV}^{-1}$
$\chi_{f_0} \tilde{F}_0^{D\to f_0}(m_\pi^2)$	$(5.6 \pm 1.9)\ {\rm GeV}^{-1}$
$\alpha_{f_2} \times 10^5$	5.3 ± 0.8
ϕ_{f_2} (rd)	-6.5 ± 0.4
$\alpha_{\rho'} \times 10^6$	2.1 ± 0.9
$\phi_{\rho'}$ (rd)	-0.2 ± 0.6
χ^2/d.o.f.	0.21

With this modification, we can now consider the products $\chi_\sigma \tilde{F}_0^{D\to\sigma}(m_\pi^2)$ and $\chi_{f_0} \tilde{F}_0^{D\to f_0}(m_\pi^2)$ from Eqs. (28) and (29) as free parameters of the fit. The results of the fit are shown in Table II. The calculated uncertainties take into account the three possible sources: statistics, changing the set of $R_i^n(x)$ and varying $x_{\rm cut}$. The dominant error in the case of $\chi_\sigma \tilde{F}_0^{D\to\sigma}(m_\pi^2)$ and $\chi_{f_0} \tilde{F}_0^{D\to f_0}(m_\pi^2)$ comes from the two different sets of $R_i^n(x)$ functions that we have at our disposal from Refs. [36,38]. This procedure is conservative, since it probably yields an overestimation of the theoretical error. The central values are obtained from the most recent determination of Ref. [38]. In this fit, statistical uncertainties are larger due to the additional parameter of the model indicating that, within the present experimental constraints, model B is somehow overparametrized as compared to model A. Since the plots for this model are similar to Figs. 1 and 2 we refrain from displaying them here.

Let us now use the results of Table II to obtain the transition form factor for $D \to \sigma$. In order to disentangle the product $\chi_\sigma \tilde{F}_0^{D\to\sigma}(m_\pi^2)$ we calculate χ_σ from Eq. (26). Gardner and Meißner gave $\chi_\sigma = 20\ {\rm GeV}^{-1}$ in Ref. [20] while employing the production functions $R_i^n(x)$ from Ref. [38], we get $\chi_\sigma \approx 22\ {\rm GeV}^{-1}$. From Eq. (26) one sees that χ_σ depends on quantities that are not well known such as Γ_σ and m_σ and we shall take these values as an indication. Nevertheless, with our value one obtains

$$\tilde{F}_0^{D\to\sigma}(m_\pi^2) = 0.38 \pm 0.06. \tag{45}$$

This result is to be compared to 0.79 ± 0.15 from Ref. [31], 0.57 ± 0.09 from Ref. [43] and 0.42 ± 0.05 from Ref. [44] which is however evaluated at $q^2 = 0$. Our value is compatible with the last one, but smaller than those of Refs. [31,43]. More detailed discussions on the form factor $\tilde{F}_0^{D\to\sigma}(m_\pi^2)$ are beyond the scope of this work.

Concerning the $f_0(980)$, we use the recent determination of $F_0^{D\to f_0}(m_\pi^2)$ from the phenomenological analysis of Ref. [32] in order to estimate χ_{f_0}. The latter can be compared with the values from Refs. [22,23] to check the consistency of both approaches. From Ref. [32] we have $\tilde{F}_0^{D\to f_0}(m_\pi^2) = 0.215$, and, with the result for the product $\chi_{f_0} \tilde{F}_0^{D\to f_0}(m_\pi^2)$ obtained in the fit, this yields

$$\chi_{f_0} = 26 \pm 9\ {\rm GeV}^{-1}. \tag{46}$$

This result agrees within uncertainties with the ones employed in B decays: $\chi_{f_0} = 23.5\ {\rm GeV}^{-1}$ and $\chi_{f_0} = 33.5\ {\rm GeV}^{-1}$ in the two models of Ref. [22] and $\chi_{f_0} = 28.9\ {\rm GeV}^{-1}$ in Ref. [23]. This comparison is to be done with care since one always has the product of the normalization χ_{f_0} and the transition form factors. Therefore, smaller values for χ_{f_0} can be compensated by larger form factors.

TABLE III. Fit fractions (in %) from E791 [1], FOCUS [13], CLEO (isobar model) [14], Oller [18], PDG [26] and models A and B. The uncertainties have been summed quadratically. Results marked with an asterisk are the sum of all S-wave contributions and are extrapolated from the original works. From Ref. [18] we included the 6% of $K\bar{K} \to (\pi\pi)_S$ in the $(\pi\pi)_S$ fit fraction.

	E791	FOCUS	CLEO	Oller	PDG	Mod. A	Mod. B
σ	46.3 ± 9.2	⋯	41.8 ± 2.9	⋯	42.2 ± 2.7	⋯	⋯
NR	7.8 ± 7.8	⋯	<3.5	17	<3.5	⋯	⋯
$f_0(980)$	6.2 ± 1.4	⋯	4.1 ± 0.9	⋯	4.8 ± 1.0	⋯	⋯
$f_0(1370)$	2.3 ± 1.7	⋯	2.6 ± 1.9	3	2.4 ± 1.3	⋯	⋯
$f_0(1500)$	⋯	⋯	3.4 ± 1.3	⋯	3.4 ± 1.3	⋯	⋯
$(\pi^+\pi^-)_S$	$(63 \pm 12)^*$	56.0 ± 3.9	$(51.9 \pm 3.8)^*$	108	56 ± 4	50.4 ± 2.7	56.6 ± 5.2
$\rho(770)$	33.6 ± 3.9	30.8 ± 3.9	20.0 ± 2.5	36	25 ± 4	40 ± 4	34.4 ± 0.4
$f_2(1270)$	19.4 ± 2.5	11.7 ± 1.9	18.2 ± 2.7	21	15.4 ± 2.5	8.6 ± 2.1	11.8 ± 0.8
$\rho(1450)$	0.7 ± 0.8	⋯	<2.4	1	<2.4	1.5 ± 0.6	0.8 ± 0.4
$\sum_i f_i$	116.3	98.5	90.1	186	⋯	100.2	103.6

V. SUMMARY AND DISCUSSIONS

It is elucidative to quantify the weight of the different contributions in our model and compare them with other results in the literature. This is usually done through the fit fractions, defined for a given contribution R as

$$f_R = \frac{\int_{\mathcal{D}} dudt|\mathcal{M}_R(u,t)|^2}{\int_{\mathcal{D}} dudt|\sum_i \mathcal{M}_i(u,t)|^2}, \quad (47)$$

where $\mathcal{D}$ indicates that the integrals are to be performed over the whole Dalitz plot. The fit fractions give an idea of how important a given resonance or partial wave is to the total decay amplitude. The sum of the fit fractions does not necessarily add up to one due to interference effects. However, one expects that this sum should not deviate widely from unity. In Table III we compare the fit fractions of models A and B with those of other works as well as with the values quoted by the PDG [26]. In our fit fractions we included the uncertainty from $\sqrt{x_{\rm cut}}$ as well as that arising from the use of different sets of $R_i^n(x)$ from Refs. [36,38].

To clarify the content of Table III, one should remark that the E791 Collaboration employed the isobar model [1], as already commented. Note that from the CLEO analyses [14], we only quote the isobar model results. FOCUS has performed an analysis where the S wave is described continuously by means of a K matrix previously determined from $\pi\pi$ scattering data [13]. This model, therefore, is the closest to ours since both are based on $\pi\pi$ scattering analyses and share the on-shell approximation. Oller's results [18], based on a description of FSIs obtained in the context of unitarized ChPT [19], also rely on previous analysis of $\pi\pi$ scattering and include off-shell effects. In his model, the nonresonant amplitude is kept and, although the resemblance to E791 data is striking, the price to pay is a huge interference that makes the sum of fit fractions to be 186% with 108% of $\pi\pi$ S wave in the final state. Finally, we do not sum the fit fractions from the PDG [26] since they come from many different analyses. Our results are in general agreement with the others. The S wave is within the experimental results and the sum of the fit fractions of our models is very reasonable.

The CLEO Collaboration has performed an analysis where two models based on the $\pi\pi$ scattering T matrix were applied [14]. The model of Schechter (nonisobar model, and chiral Lagrangian with 7 parameters) has similar fit fractions to our results although the coupling to the $K\bar{K}$ channel is not included in the model. The model of Achasov (isobar model with 12 parameters) produces results closest to [18] with large interference effects. The $(\pi^+\pi^-)_S$ has a fit fraction of about 70% and the total sum is roughly 140%.

Since we have performed an analysis that starts from the weak vertex, we can calculate the total branching ratio for the decay $D^+ \to \pi^+\pi^-\pi^+$ using the fit results for the parameters of the models. This gives

$$\text{model A: } \mathcal{B}(D^+ \to \pi^+\pi^-\pi^+) = (2.2^{+0.7}_{-0.5}) \times 10^{-3},$$
$$\text{model B: } \mathcal{B}(D^+ \to \pi^+\pi^-\pi^+) = (2.5^{+0.4}_{-0.3}) \times 10^{-3}, \quad (48)$$

whose central values are smaller than the experimental average [26]

$$\mathcal{B}^{\rm PDG}(D^+ \to \pi^+\pi^-\pi^+) = (3.21 \pm 0.19) \times 10^{-3}. \quad (49)$$

The results are, however, coherent with the fact that we miss a part of the contribution in the σ region. Concerning this discrepancy, one can advance some hypothesis. As already stated, this is likely due to the off-shell effects that are not considered in our form factor. Another possible cause for the less prominent σ peak and, consequently, for the smaller branching ratio are three-body final state interactions. There are indications of three-body effects in $D^+ \to K^-\pi^+\pi^-$ coming from a new analysis technique where the S wave is treated bin by bin in an almost model-independent way [45,46]. The $K\pi$ S-wave phase thus obtained is considerably different from the $K\pi$ scattering results and this has been interpreted as an indication of genuine three-body effects [47]. To our knowledge, this type of analysis has never been done for $D^+ \to \pi^+\pi^-\pi^+$ but, in principle, three-body effects could also be important in this case.

In conclusion, given the relative simplicity of our model, the small number of parameters and the fixed phases between the S wave and the ρ^0, the model is able to provide a fair description of the experimental data for $D^+ \to \pi^+\pi^-\pi^+$ within a unitary approach for the $\pi\pi$ final state interactions. Among many possible improvements to this work, one can think of introducing P-wave form factors to describe the $\rho(770)^0$ and the $\rho(1450)^0$.

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