

Three-algebras and $\mathcal{N} = 6$ Chern-Simons gauge theories

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We derive the general form for a three-dimensional scale-invariant field theory with $\mathcal{N} = 6$ supersymmetry, $SU(4)$ R-symmetry and a $U(1)$ global symmetry. The results can be written in terms of a 3-algebra in which the triple product is not antisymmetric. For a specific choice of 3-algebra we obtain the $\mathcal{N} = 6$ theories that have been recently proposed as models for M2-branes in an $\mathbb{R}^8/\mathbb{Z}_k$ orbifold background.

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I. INTRODUCTION

M2-branes have recently been enjoying a period of considerable interest. One hopes that an understanding of the dynamics of multiple M2-branes will lead to a deeper and more microscopic understanding of M-theory. Motivated by the papers [1,2], in [3] we proposed a field-theory model of multiple M2-branes. This model was shown to admit $\mathcal{N} = 8$ supersymmetry (16 supercharges) in [4] and in [5], where the Lagrangian was also given. In this approach, the scalars and fermions take values in a 3-algebra \mathcal{A} .

A 3-algebra is a vector space with basis T^a , $a = 1, \dots, N$, endowed with a triple product [5],

$$[T^a, T^b, \bar{T}^c] = f^{abc} T^d. \quad (1)$$

Note that here we take the 3-algebra to be a *complex* vector space, and we have used a slightly different notation to keep track of the fact that, in this paper, $[\cdot, \cdot; \cdot]$ need only be antisymmetric in the first two indices. Furthermore, we require the f^{abc} to satisfy the following fundamental identity:

$$f^{ef\bar{g}}{}_b f^{cb\bar{a}}{}_d + f^{fe\bar{a}}{}_b f^{cb\bar{g}}{}_d + f^{*\bar{g}\bar{a}f}{}_b f^{ce\bar{b}}{}_d + f^{*\bar{a}\bar{g}e}{}_b f^{cf\bar{b}}{}_d = 0. \quad (2)$$

[We will give an alternative characterization of this condition in Eq. (40) below.] In [5] (and also [4]), we also required the f^{abc} to be real and antisymmetric in a, b, c . In that case, for any such triple product, one finds equations of motion that are invariant under 16 supersymmetries and $SO(8)$ R-symmetry.

To construct a Lagrangian we require a trace form on the 3-algebra that is linear in the second entry and complex antilinear in the first. This provides an inner product,

$$h^{\bar{a}b} = \text{Tr}(\bar{T}^{\bar{a}}, T^b). \quad (3)$$

For h^{ab} and $f^{abc}{}_d$ real, gauge invariance implies that $f^{abcd} = f^{abc}{}_e h^{ed}$ is totally antisymmetric. This leads to a Chern-Simons Lagrangian with 16 supersymmetries and $SO(8)$ R-symmetry [5]. When h^{ab} is also positive definite, it was recently shown that the one known example, in which $f^{abcd} \propto \varepsilon^{abcd}$, is essentially unique.¹ In [6,7] this maximally supersymmetric field theory was identified as describing two M2-branes in an $\mathbb{R}^8/\mathbb{Z}_2$ orbifold background.

Recently, there have been several attempts to relax these assumptions and construct additional models. In [8] it was suggested that $f^{abc}{}_d$ need not be totally antisymmetric, just antisymmetric a, b, c , and indeed this leads to an infinite number of models using the 3-algebra given in [9]. The equations of motion of [5] are still invariant under the 16 supersymmetries, but there is no gauge-invariant metric so it is not clear how to construct physical quantities such as energy.

More recently there have been proposals in which the metric h^{ab} has a Lorentzian signature [10–12]. This allows one to construct an associated 3-algebra for any Lie algebra, and the corresponding $\mathcal{N} = 8$ Lagrangian [5] has been proposed to describe M2-branes in flat \mathbb{R}^8 [10–12]. Although these models are built on a 3-algebra without a positive definite norm, the corresponding quantum theories have been argued to be unitary [10–12] and there are some encouraging features [13–17]. The current status of these models is unclear. In particular, one method for removing the negative norm states leads back to maximally supersymmetric Yang-Mills theory [18,19], although in a form that possesses both $SO(8)$ and spontaneously broken conformal symmetry.

Another option is to look for theories with a reduced number of supersymmetries. In [20–22] a class of Chern-Simons Lagrangians with $\mathcal{N} = 4$ supersymmetry (eight supercharges) was constructed. More recently, in [23] an

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¹All other 3-algebras are direct sums of the minimal four-dimensional 3-algebra [42–44].

infinite class of brane configurations was given whose low energy effective Lagrangian is a Chern-Simons theory with $SO(6)$ R-symmetry and $\mathcal{N} = 6$ supersymmetry (12 supercharges). These theories are related to N M2-branes in $\mathbb{R}^8/\mathbb{Z}_k$, including $k = 1$. The Lagrangians were studied in detail in [24–32]. More theories with $\mathcal{N} = 5$ and $\mathcal{N} = 6$ have also recently appeared in [33].

Thus it is of interest to generalize the construction of our model, based on 3-algebras, to the case of $\mathcal{N} = 6$ supersymmetry. We will see that this can be accomplished by relaxing the conditions on the triple product so that it is no longer real and antisymmetric in all three indices. Rather it is required to satisfy

$$f^{abc\bar{d}} = -f^{ba\bar{c}d} \quad \text{and} \quad f^{abc\bar{d}} = f^{*\bar{c}dab}. \quad (4)$$

The triple product is also required to satisfy the fundamental identity (2).

The rest of this paper is organized as follows. In Sec. II we revisit the analysis of [5], trying to be as general as possible. We will see that the model presented there is the most general with $\mathcal{N} = 8$ supersymmetry, scale invariance, and $SO(8)$ R-symmetry. Section III contains the main results of this paper. We follow the construction of [5], but only impose $\mathcal{N} = 6$ supersymmetry, scale invariance, $SU(4)$ R-symmetry, and a global $U(1)$. We find the supersymmetry transformations, the invariant Lagrangian, and the conditions on the structure constants $f^{abc\bar{d}}$. In Sec. IV we discuss the associated 3-algebra and show that a specific choice of triple product leads directly to the models in [23], as presented in [24]. As a result, we are able to provide the complete expressions for the Lagrangians in [23], including all the supersymmetry transformations, in a manifestly $SU(4)$ covariant form (see also [31,33]). Section V contains our conclusions. We collect our spinor conventions and some useful identities in an Appendix.

II. $\mathcal{N} = 8$

Before presenting the main results of this paper, we reexamine the closure of the $N = 8$ supersymmetry transformations given in [5] (see also [4]). In particular, we relax as many assumptions as possible to find the minimum requirements on f^{abcd} . We proceed by assuming scale invariance and an $SO(8)$ R-symmetry. The most general form for the supersymmetry transformations is then

$$\begin{aligned} \delta X_d^I &= i\bar{\epsilon}\Gamma^I\Psi_d \\ \delta\Psi_d &= D_\mu X_d^I\Gamma^\mu\Gamma^I\epsilon - \frac{1}{6}X_a^IX_b^JX_c^Kf^{abc\bar{d}}\Gamma^{IJK}\epsilon \\ &\quad + \frac{1}{2}X_a^IX_b^JX_c^Lg^{abc\bar{d}}\Gamma^I\epsilon \\ \delta\tilde{A}_\mu{}^c{}_d &= i\bar{\epsilon}\Gamma_\mu\Gamma_I X_a^I\Psi_b h^{abc\bar{d}}, \end{aligned} \quad (5)$$

where D_μ is a covariant derivative, and $g^{abc\bar{d}}$ and $h^{abc\bar{d}}$ define triple products on the algebra that are not antisymmetric (a possibility that was mentioned in [3]). Without loss of generality we may assume that $f^{abc\bar{d}}$ is antisym-

metric in a, b, c , while $g^{abc\bar{d}}$ is symmetric in a, b . All quantities are taken to be real.

To begin we consider the closure on X_d^I ,

$$[\delta_1, \delta_2]X_d^I = v^\mu D_\mu X_d^I + \tilde{\Lambda}^c{}_d X_c^I + \Omega^{IJc}{}_d X_c^I, \quad (6)$$

where

$$\begin{aligned} v^\mu &= -2i\bar{\epsilon}_2\Gamma^\mu\epsilon_1 & \tilde{\Lambda}^c{}_d &= -i\bar{\epsilon}_2\Gamma_{JK}\epsilon_1 X_a^J X_b^K f^{abc\bar{d}} \\ \Omega^{IJc}{}_d &= i\bar{\epsilon}_2\Gamma^{IJ}\epsilon_1 X_a^K X_b^K g^{abc\bar{d}}. \end{aligned} \quad (7)$$

The first two terms are familiar from [5]. The last transformation, however, mixes an internal symmetry with an R-symmetry, although we note it becomes a pure R-symmetry if $g^{abc\bar{d}}$ takes the form

$$g^{abc\bar{d}} = k^{ab}\delta^c{}_d. \quad (8)$$

This implies that R-symmetry must be gauged. A similar extension was successfully used in [34,35] except that the additional term was linear in X^I . As a result, the R-symmetry was not gauged, and the theory described a mass deformation that preserved all supersymmetries but broke the R-symmetry to $SO(4) \times SO(4)$.

R-symmetries cannot be gauged in rigid supersymmetry because the supercharges rotate into each other (by definition) and hence would have to become local symmetries. Thus we are forced to set $g^{abc\bar{d}} = 0$.

We now consider the fermions. Evaluating $[\delta_1, \delta_2]\Psi_d$ and using the Feirz identity (see the Appendix), we find four terms involving $\bar{\epsilon}_2\Gamma_\mu\Gamma_{LMNP}\epsilon_1$. After some manipulations, we reduce these terms to

$$\bar{\epsilon}_1\Gamma_\nu\Gamma_{LMNP}\epsilon_2\Gamma^\nu\Gamma^I\Gamma^{LMNP}\Gamma^J X_c^I X_a^J (f^{abc\bar{d}} - h^{abc\bar{d}})\Psi_b. \quad (9)$$

Closure implies that (9) must vanish and hence

$$h^{abc\bar{d}} = f^{abc\bar{d}}. \quad (10)$$

Thus we are left with just one tensor $f^{abc\bar{d}}$. As in [5], the algebra closes on the fermions using the on-shell condition

$$\Gamma^\mu D_\mu \Psi_d + \frac{1}{2}\Gamma_{IJ}X_a^I X_b^J \Psi_c f^{abc\bar{d}} = 0. \quad (11)$$

Next we turn to $[\delta_1, \delta_2]\tilde{A}_\mu{}^c{}_d$. Here we find a term that is fourth order in the scalars:

$$(\bar{\epsilon}_2\Gamma_\mu\Gamma_{IJKL}\epsilon_1)X_a^I X_e^J X_f^K X_g^L f^{efg\bar{b}} f^{abc\bar{d}}. \quad (12)$$

This term vanishes provided that

$$f^{[abc}{}_e f^{d]ef}{}_g = 0. \quad (13)$$

Given the antisymmetry of $f^{abc\bar{d}}$ in a, b, c , this is equivalent to the fundamental identity (2). Continuing, we find

$$\begin{aligned}
[\delta_1, \delta_2] \tilde{A}_\mu{}^c{}_d &= 2i(\tilde{\epsilon}_2 \Gamma^\nu \epsilon_1) \epsilon_{\mu\nu\lambda} \left(X_a^I D^\lambda X_b^I \right. \\
&\quad \left. + \frac{i}{2} \tilde{\Psi}_a \Gamma^\lambda \Psi_b \right) f^{abc}{}_d \\
&\quad - 2i(\epsilon_2 \Gamma_{IJ} \epsilon_1) X_a^I D_\mu X_b^J f^{abc}{}_d. \quad (14)
\end{aligned}$$

Gauge invariance requires that the last line be equal to $D_\mu \tilde{\Lambda}^c{}_d$. Writing $\tilde{A}_\mu{}^c{}_d = f^{abc}{}_d A_{\mu ab}$, this implies the condition

$$f^{abc}{}_e f^{fge}{}_d = f^{fga}{}_e f^{ebc}{}_d + f^{fgb}{}_e f^{aec}{}_d + f^{fgc}{}_e f^{abe}{}_d, \quad (15)$$

which ensures that the gauge symmetry acts as a derivation. Equation (15) is equivalent to (13) (e.g. see [8]), so we have recovered all the ingredients of [5].

III. $\mathcal{N} = 6$

In this section we relax the constraints on $f^{abc}{}_d$ to construct an infinite class of theories with fewer supersymmetries. We will construct a Lagrangian with 12 supercharges ($\mathcal{N} = 6$ supersymmetry), $SU(4)$ R-symmetry, and a $U(1)$ internal symmetry. We continue to assume that $h^{\bar{a}b}$ is positive definite, although no substantial changes arise if $h^{\bar{a}b}$ has a different signature.

We use a complex notation in which the $SO(8)$ R-symmetry of the $\mathcal{N} = 8$ theory is broken to $SU(4) \times U(1)$. The supercharges transform under the $SU(4)$ R-symmetry; the $U(1)$ provides an additional global symmetry. We introduce four complex 3-algebra valued scalar fields Z_a^A , $A = 1, 2, 3, 4$, as well as their complex conjugates $\bar{Z}_{A\bar{a}}$. Similarly, we denote the fermions by ψ_{Aa} and their complex conjugates by $\bar{\psi}_{\bar{a}A}$. A raised A index indicates that the field is in the $\mathbf{4}$ of $SU(4)$; a lowered index transforms in the $\bar{\mathbf{4}}$. We assign Z_a^A and ψ_{Aa} a $U(1)$ charge of 1. Complex conjugation raises or lowers the A index, flips the sign of the $U(1)$ charge, and interchanges $a \leftrightarrow \bar{a}$. The supersymmetry generators ϵ_{AB} are in the $\mathbf{6}$ of $SU(4)$ with vanishing $U(1)$ charge. They satisfy the reality condition $\epsilon^{AB} = \frac{1}{2} \epsilon^{ABCD} \epsilon_{CD}$.

We postulate the following supersymmetry transformations (our spinor conventions are listed in the Appendix):

$$\begin{aligned}
\delta Z_d^A &= i\tilde{\epsilon}^{AB} \psi_{Bd} \\
\delta \psi_{Bd} &= \gamma^\mu D_\mu Z_d^A \epsilon_{AB} + f_1^{abc}{}_d Z_a^C \bar{Z}_{C\bar{b}} Z_c^A \epsilon_{AB} \\
&\quad + f_2^{abc}{}_d Z_a^C Z_b^D \bar{Z}_{B\bar{c}} \epsilon_{CD} \\
\delta \tilde{A}_\mu{}^c{}_d &= i\tilde{\epsilon}_{AB} \gamma_\mu Z_a^A \psi_b^B f_3^{abc}{}_d + i\tilde{\epsilon}^{AB} \gamma_\mu \bar{Z}_{A\bar{a}} \psi_{Bb} f_4^{abc}{}_d, \quad (16)
\end{aligned}$$

where $f_1^{abc}{}_d$, $f_2^{abc}{}_d$, $f_3^{abc}{}_d$, and $f_4^{abc}{}_d$ are tensors on the 3-algebra. Without loss of generality, we assume that $f_2^{(ab)\bar{c}}{}_d = 0$. The covariant derivative is defined by $D_\mu Z_d^A = \partial_\mu Z_d^A - \tilde{A}_\mu{}^c{}_d Z_c^A$. Therefore we require that

$D_\mu \bar{Z}_{A\bar{a}} = \partial_\mu \bar{Z}_{A\bar{a}} - \tilde{A}_\mu{}^{*\bar{c}}{}_{\bar{d}} \bar{Z}_{A\bar{c}}$. Supersymmetry then requires that $D_\mu \psi_d^A = \partial_\mu \psi_d^A - \tilde{A}_\mu{}^{*\bar{c}}{}_{\bar{d}} \psi_c^A$ and $D_\mu \psi_{Ad} = \partial_\mu \psi_{Ad} - \tilde{A}_\mu{}^c{}_d \psi_{Ac}$. These are the most general transformations that preserve the $SU(4)$, $U(1)$, and conformal symmetries.

In [3] the $\mathcal{N} = 8$ theory (without gauge fields) was written in terms of such a complex notation with manifest $SU(4) \times U(1)$ symmetry. However, the supersymmetries ϵ_{AB} were not considered in detail; the discussion focused on the other four supersymmetry generators ϵ that are $SU(4)$ singlets with $U(1)$ charge ± 2 . These supersymmetries have a natural $N = 2$ superspace interpretation; they require that f^{abcd} be real and totally antisymmetric. These supersymmetries will not, in general, be preserved in the models presented here. Indeed, imposing these as supersymmetries leads to the original $\mathcal{N} = 8$ theory (written in complex notation).

To begin, we first consider the closure of (16) on the scalars. Using the identities listed in the Appendix, we find that $[\delta_1, \delta_2] Z_d^A$ only closes onto translations and a gauge symmetry if

$$f_1^{abc}{}_d = f_2^{abc}{}_d. \quad (17)$$

In this case we find

$$[\delta_1, \delta_2] Z_d^A = v^\mu D_\mu Z_d^A + \Lambda_{\bar{c}b} f_2^{abc}{}_d Z_a^A, \quad (18)$$

where

$$\begin{aligned}
v^\mu &= \frac{i}{2} \tilde{\epsilon}_2^{CD} \gamma^\mu \epsilon_{1CD}, \\
\Lambda_{\bar{c}b} &= i(\tilde{\epsilon}_2^{DE} \epsilon_{1CE} - \tilde{\epsilon}_1^{DE} \epsilon_{2CE}) \bar{Z}_{D\bar{c}} Z_c^B. \quad (19)
\end{aligned}$$

The second term in (18) is a gauge transformation: $\delta_\Lambda Z_d^A = \Lambda_{\bar{c}b} f_2^{abc}{}_d Z_a^A$.

Next we examine the closure of the algebra on the fermions. After some work, we find that if

$$f_4^{abc}{}_d = -f_3^{abc}{}_d, \quad (20)$$

and

$$f_3^{abc}{}_d = f_2^{abc}{}_d, \quad (21)$$

then

$$\begin{aligned}
[\delta_1, \delta_2] \psi_{Dd} &= v^\mu D_\mu \psi_{Dd} + \Lambda_{\bar{b}a} f_2^{ca\bar{b}}{}_d \psi_{Dc} - \frac{i}{2} (\tilde{\epsilon}_1^{AC} \epsilon_{2AD} \\
&\quad - \tilde{\epsilon}_2^{AC} \epsilon_{1AD}) E_{Cd} + \frac{i}{4} (\tilde{\epsilon}_1^{AB} \gamma_\nu \epsilon_{2AB}) \gamma^\nu E_{Dd}, \quad (22)
\end{aligned}$$

where

$$\begin{aligned}
E_{Cd} &= \gamma^\mu D_\mu \psi_{Cd} + f_2^{abc}{}_d \psi_{Ca} Z_b^D \bar{Z}_{D\bar{c}} \\
&\quad - 2f_2^{abc}{}_d \psi_{Dd} Z_b^D \bar{Z}_{C\bar{c}} - \epsilon_{CDEF} f_2^{abc}{}_d \psi_c^D \psi_a^E Z_b^F. \quad (23)
\end{aligned}$$

Thus we see that the supersymmetry algebra closes if we impose the on-shell condition $E_{Cd} = 0$.

Finally we look at the gauge field $\tilde{A}_{\mu}{}^c{}_d$. In the closure there is a term that is fourth order in the scalars that vanishes when $f_2^{ab\bar{c}}{}_d$ satisfies the fundamental identity (2). At quadratic order in the fields, closure of the supersymmetry transformations gives

$$[\delta_1, \delta_2]\tilde{A}_{\mu}{}^c{}_d = -f_3^{ab\bar{c}}{}_d D_{\mu}(\Lambda_{\bar{b}a}) + \varepsilon_{\mu\nu\lambda} v^{\nu} (D^{\lambda} Z_a^A \bar{Z}_{A\bar{b}} - Z_a^A D^{\lambda} \bar{Z}_{A\bar{b}} - i\tilde{\psi}_{\bar{b}}^A \gamma^{\lambda} \psi_{Aa}) f_3^{ab\bar{c}}{}_d. \quad (24)$$

Thus if we impose the on-shell condition

$$\tilde{F}_{\mu\nu}{}^c{}_d = -\varepsilon_{\mu\nu\lambda} (D^{\lambda} Z_a^A \bar{Z}_{A\bar{b}} - Z_a^A D^{\lambda} \bar{Z}_{A\bar{b}} - i\tilde{\psi}_{\bar{b}}^A \gamma^{\lambda} \psi_{Aa}) f_2^{cab}{}_d, \quad (25)$$

we see that the supersymmetry algebra closes onto translations and gauge transformations

$$[\delta_1, \delta_2]\tilde{A}_{\mu}{}^c{}_d = v^{\nu} \tilde{F}_{\mu\nu}{}^c{}_d + D_{\mu}(\Lambda_{\bar{b}a} f_2^{cab}{}_d), \quad (26)$$

provided that $D_{\mu}(f_2^{cab}{}_d) = 0$. This is just the statement that $f_2^{cab}{}_d$ is an invariant tensor of the gauge algebra. In general it provides an additional condition on $f_2^{cab}{}_d$. However, we will see that it follows directly from the fundamental identity whenever there is a Lagrangian.

Let us summarize our results so far. Henceforth we drop the subscript 2 on $f_2^{ab\bar{c}}{}_d$, which we take to be an invariant tensor of the gauge algebra that satisfies (2); the remaining tensors $f_1^{a\bar{c}b}{}_d$, $f_3^{a\bar{c}b}{}_d$, and $f_4^{c\bar{a}b}{}_d$ are related to $f^{ab\bar{c}}{}_d$ through (17), (20), and (21). The supersymmetry transformations are

$$\begin{aligned} \delta Z_d^A &= i\bar{\epsilon}^{AB} \psi_{Bd} \\ \delta \psi_{Bd} &= \gamma^{\mu} D_{\mu} Z_d^A \epsilon_{AB} + f^{ab\bar{c}}{}_d Z_a^C Z_b^A \bar{Z}_{C\bar{c}} \epsilon_{AB} \\ &\quad + f^{ab\bar{c}}{}_d Z_a^C Z_b^D \bar{Z}_{B\bar{c}} \epsilon_{CD} \\ \delta \tilde{A}_{\mu}{}^c{}_d &= -i\bar{\epsilon}_{AB} \gamma_{\mu} Z_a^A \psi_b^B f^{cab}{}_d + i\bar{\epsilon}^{AB} \gamma_{\mu} \bar{Z}_{A\bar{b}} \psi_{Ba} f^{cab}{}_d. \end{aligned} \quad (27)$$

In the case that f^{abcd} is real and antisymmetric in a, b, c , we recover the supersymmetry transformations of the $\mathcal{N} = 8$ theory.

Let us now construct an invariant Lagrangian. We have seen that the supersymmetry algebra closes into a translation plus a gauge transformation. On the field $\tilde{Z}_{A\bar{d}}$, we find

$$[\delta_1, \delta_2]\tilde{Z}_{A\bar{d}} = v^{\mu} D_{\mu} \tilde{Z}_{A\bar{d}} + \Lambda_{\bar{c}b}^* f^{*a\bar{b}c}{}_d \tilde{Z}_{A\bar{a}}, \quad (28)$$

with v and $\Lambda_{\bar{c}b}$ given in (19). The second term is a gauge transformation, $\delta_{\Lambda} \tilde{Z}_{A\bar{d}} = \Lambda_{\bar{c}b}^* f^{*a\bar{b}c}{}_d \tilde{Z}_{A\bar{a}} = -\Lambda_{\bar{b}c} f^{*a\bar{b}c}{}_d \tilde{Z}_{A\bar{a}}$. To construct a gauge-invariant Lagrangian (or, for that matter, any gauge-invariant observable), we need the metric to be gauge invariant, namely $\delta_{\Lambda}(h^{\bar{a}b} \tilde{Z}_{A\bar{a}} Z_b^A) = 0$. Therefore we must require

$$f^{ab\bar{c}d} = f^{*c\bar{d}ab}, \quad (29)$$

where $f^{ab\bar{c}d} = f^{ab\bar{c}}{}_e h^{\bar{d}e}$. This implies that $(\tilde{\Lambda}^{c\bar{d}})^* = -\tilde{\Lambda}^{\bar{d}c}$, where

$$\tilde{\Lambda}^{c\bar{d}} = \Lambda_{\bar{b}a} f^{cab\bar{d}}, \quad (30)$$

so the transformation parameters $\tilde{\Lambda}^c{}_d$ are elements of $u(N)$, although they are not in general all of $u(N)$.

The first term in (28) contains the translation. Note that it appears as part of a covariant derivative, $v^{\mu} D_{\mu} \tilde{Z}_{A\bar{d}} = v^{\mu} \partial_{\mu} \tilde{Z}_{A\bar{d}} - v^{\mu} \tilde{\Lambda}_{\mu}{}^{*c\bar{d}} \tilde{Z}_{A\bar{c}}$. The first part is the translation, while the second is another gauge transformation, with parameter $\tilde{\Lambda}^{*c\bar{d}} = -v^{\mu} \tilde{\Lambda}_{\mu}{}^{*c\bar{d}}$. This implies that the gauge field also takes values in $u(N)$.

With these results, it is not hard to show that an invariant Lagrangian (up to boundary terms) is given by

$$\begin{aligned} \mathcal{L} &= -D^{\mu} \tilde{Z}_A^a D_{\mu} Z_a^A - i\tilde{\psi}^{Aa} \gamma^{\mu} D_{\mu} \psi_{Aa} - V + \mathcal{L}_{CS} \\ &\quad - i f^{ab\bar{c}d} \tilde{\psi}_{\bar{d}}^A \psi_{Aa} Z_b^B \bar{Z}_{B\bar{c}} + 2i f^{ab\bar{c}d} \tilde{\psi}_{\bar{d}}^A \psi_{Ba} Z_b^B \bar{Z}_{B\bar{c}} \\ &\quad + \frac{i}{2} \varepsilon_{ABCD} f^{ab\bar{c}d} \tilde{\psi}_{\bar{d}}^A \psi_{\bar{c}}^B Z_a^C Z_b^D \\ &\quad - \frac{i}{2} \varepsilon^{ABCD} f^{cd\bar{a}b} \tilde{\psi}_{Ac} \psi_{Bd} \bar{Z}_{C\bar{a}} \bar{Z}_{D\bar{b}}, \end{aligned} \quad (31)$$

where the potential is

$$V = \frac{2}{3} \Upsilon_{Bd}^{CD} \tilde{\Upsilon}_{CD}^{Bd}, \quad (32)$$

where

$$\begin{aligned} \Upsilon_{Bd}^{CD} &= f^{ab\bar{c}}{}_d Z_a^C Z_b^D \bar{Z}_{B\bar{c}} - \frac{1}{2} \delta_{Bf}^C f^{ab\bar{c}}{}_d Z_a^E Z_b^D \bar{Z}_{E\bar{c}} \\ &\quad + \frac{1}{2} \delta_B^D f^{ab\bar{c}}{}_d Z_a^E Z_b^C \bar{Z}_{E\bar{c}}. \end{aligned} \quad (33)$$

The zero-energy solutions correspond to $\Upsilon_{Bd}^{CD} = 0$. This is equivalent to $\Upsilon_{Bd}^{CD} \epsilon_{CD} = 0$ for arbitrary ϵ_{CD} , which implies that the zero-energy solutions preserve all 12 supersymmetries.

The ‘‘twisted’’ Chern-Simons term \mathcal{L}_{CS} is given by

$$\begin{aligned} \mathcal{L}_{CS} &= \frac{1}{2} \varepsilon^{\mu\nu\lambda} (f^{ab\bar{c}d} A_{\mu\bar{c}b} \partial_{\nu} A_{\lambda\bar{d}a} \\ &\quad + \frac{2}{3} f^{ac\bar{d}}{}_g f^{g\bar{e}f}{}_b A_{\mu\bar{b}a} A_{\nu\bar{d}c} A_{\lambda\bar{f}e}). \end{aligned} \quad (34)$$

It satisfies

$$\frac{\delta \mathcal{L}_{CS}}{\delta \tilde{A}_{\lambda}{}^{ab}} f^{ac\bar{d}b} = \frac{1}{2} \varepsilon^{\lambda\mu\nu} \tilde{F}_{\mu\nu}{}^{c\bar{d}}, \quad (35)$$

up to integration by parts, where $\tilde{F}_{\mu\nu}{}^a{}_b = -\partial_{\mu} \tilde{A}_{\nu}{}^a{}_b + \partial_{\nu} \tilde{A}_{\mu}{}^a{}_b + \tilde{A}_{\nu}{}^a{}_e \tilde{A}_{\mu}{}^e{}_b - \tilde{A}_{\mu}{}^a{}_e \tilde{A}_{\nu}{}^e{}_b$. Just as in [5], this term can be viewed as a function of $\tilde{A}_{\mu}{}^c{}_d$ and not $A_{\mu\bar{c}d}$.

Note that the Lagrangian (31) is automatically gauge invariant since it is supersymmetric and supersymmetries close into gauge transformations. One can also confirm that the equations of motion give the on-shell conditions that we found above for closure of the supersymmetry algebra.

IV. THREE-ALGEBRAS AND THEIR CONSTRUCTION

A given tensor $f^{ab\bar{c}}_d$ defines a triple product on the algebra with (complex) generators T^a :

$$[T^a, T^b; \bar{T}^c] = f^{ab\bar{c}}_d T^d, \quad (36)$$

which is linear and antisymmetric in the first two entries and complex antilinear in the third. In a sense one may think of $[\cdot, \cdot; \cdot]$ as generating a map from the 3-algebra \mathcal{A} into the space of endomorphisms of \mathcal{A} , i.e. for a fixed pair $Y, \bar{Z} \in \mathcal{A}$, $[\cdot, Y; \bar{Z}]$ defines a linear map of \mathcal{A} into itself. We then obtain a triple product of any three elements $X, Y, \bar{Z} \in \mathcal{A}$ by evaluating the map $[\cdot, Y; \bar{Z}]$ on X .

For the case at hand, the triple product generates a gauge symmetry

$$\delta Z^A = \Lambda_{\bar{b}a} f^{ca\bar{b}}_d Z^c. \quad (37)$$

This is similar to the gauge symmetry in [5], but there are some important differences. Let us generalize the discussion of [36]. In what follows, we assume the existence of a gauge-invariant metric, so Λ^a_b extracted from (19) is an element of $u(N)$. The symmetries (4) imply that $\tilde{\Lambda}^c_d = f^{ca\bar{b}}_d \Lambda_{\bar{b}a}$ is also an element of $u(N)$ (where we assume for concreteness that the metric is positive definite). Thus $f^{ca\bar{b}}_d$ defines a map $f: u(N) \rightarrow u(N)$;

$$f(\Lambda)^c_d = \Lambda_{\bar{b}a} f^{ca\bar{b}}_d. \quad (38)$$

Let \mathcal{G} be the vector space generated by the image of f . The fundamental identity (2) implies that

$$[f(\Lambda_1), f(\Lambda_2)] = f(\Lambda_3), \quad (39)$$

where $\Lambda_{3\bar{a}b} = \Lambda_{1\bar{a}e} \Lambda_{2\bar{g}f} f^{ef\bar{g}}_b - \Lambda_{1\bar{b}e} \Lambda_{2\bar{g}f} f^{*e\bar{g}}_f \Lambda_{\bar{a}}$. In other words, the space \mathcal{G} of gauge transformations is closed under the ordinary matrix commutator and is therefore a Lie subalgebra of $u(N)$. In the special case that $f^{abcd} = -f^{acbd}$, we see that f^{abcd} is real and totally antisymmetric. In that case \mathcal{G} is generated by antisymmetric elements of $u(N)$. These are necessarily real and hence we recover the construction of [5] in which \mathcal{G} is a Lie subalgebra of $so(N)$.

Using the metric and the condition (30), we write the fundamental identity (2) as

$$f^{ab\bar{c}}_e f^{ef\bar{g}}_d = f^{af\bar{g}}_e f^{eb\bar{c}}_d + f^{bf\bar{g}}_e f^{ae\bar{c}}_d - f^{ef\bar{g}}_e f^{ab\bar{c}}_d, \quad (40)$$

which says that the gauge symmetry acts as a derivation. In particular, if we contract (40) with $\Lambda_{\bar{g}f}$ it is equivalent to the condition

$$\begin{aligned} \delta[Z^A, Z^B; \bar{Z}_C] &= [\delta Z^A, Z^B; \bar{Z}_C] + [Z^A, \delta Z^B; \bar{Z}_C] \\ &\quad + [Z^A, Z^B; \delta \bar{Z}_C], \end{aligned} \quad (41)$$

where $\delta Z^A = \tilde{\Lambda}^a_b Z^A T^b$. Thus we see that the gauge symmetry acts as a derivation.

To continue we give a characterization of tensors $f^{ab\bar{c}}_d$ that satisfy (2) and (4) by adapting a discussion from [37]. As we have noted, $f^{ab\bar{c}}_d$ generates the Lie algebra \mathcal{G} of gauge transformations. For any two generators T^a and T^b , we write

$$[X, T^a; \bar{T}^b]_d = \Gamma_A^{ab} (t^A)^c_d X_c, \quad (42)$$

where the Γ_A^{ab} are constants and the t^A are a matrix representation of \mathcal{G} inside $u(N)$. In particular, the t^A are anti-Hermitian. We note that

$$f^{ab\bar{c}}_d = \text{Tr}(\bar{T}^d, [T^a, T^b; \bar{T}^c]), \quad (43)$$

and thus

$$f^{ab\bar{c}}_d = \Gamma_A^{bc} (t^A)^{ad}, \quad (44)$$

where we have used the metric to raise an index. Since $f^{ab\bar{c}}_d = f^{*c\bar{d}ab}$, we also see that the Γ_A^{ad} must be such that

$$f^{ab\bar{c}}_d = \sum_{AB} \Omega_{AB} (t^A)^{ad} (t^B)^{bc} \quad (45)$$

for some real and symmetric Ω_{AB} . If we now substitute this expression into the fundamental identity, we find

$$0 = \sum_{ABCDE} \Omega_{CD} (c^{CB}_E \Omega_{AB} + c^{CB}_A \Omega_{EB}) (t^A)^{ab} (t^E)^{cd} (t^D)^f_g, \quad (46)$$

where the c^{AB}_C are the structure constants of \mathcal{G} , i.e. $[t^A, t^B] = c^{AB}_C t^C$. Defining $(j^A)^B_C = c^{AB}_C$ to be the usual adjoint representation of \mathcal{G} , we see that the fundamental identity implies

$$[\Omega, j^C] = 0 \quad (47)$$

for all C , provided that Ω_{AB} is invertible. Thus by Schur's lemma, Ω_{AB} must be proportional to the identity in each simple component of \mathcal{G} . In particular, if the Lie algebra \mathcal{G} is of the form

$$\mathcal{G} = \oplus_\lambda \mathcal{G}_\lambda, \quad (48)$$

where \mathcal{G}_λ are commuting subalgebras of \mathcal{G} , then we find

$$f^{ab\bar{c}}_d = \sum_\lambda \omega_\lambda \sum_\alpha (t_\lambda^\alpha)^{ad} (t_\lambda^\alpha)^{bc}, \quad (49)$$

where the t_λ^α span a $u(N)$ representation of the generators of \mathcal{G}_λ and the ω_λ are arbitrary constants.

This would seem to furnish us with a very large class of $\mathcal{N} = 6$ Lagrangians. However, the $f^{ab\bar{c}}_d$ that we constructed in (49) do not necessarily satisfy $f^{ab\bar{c}}_d = -f^{ba\bar{c}}_d$. This condition must be imposed by hand as an additional constraint.

This form for $f^{ab\bar{c}}_d$ allows us to write the ‘‘twisted’’ Chern-Simons term as follows:

$$\begin{aligned} \mathcal{L}_{\text{CS}} &= \sum_\lambda \frac{1}{4d_\lambda \omega_\lambda} \text{Tr} \left(\tilde{A}_{(\lambda)} \wedge d\tilde{A}_{(\lambda)} \right. \\ &\quad \left. + \frac{2}{3} \tilde{A}_{(\lambda)} \wedge \tilde{A}_{(\lambda)} \wedge \tilde{A}_{(\lambda)} \right). \end{aligned} \quad (50)$$

Here $\tilde{A}_{(\lambda)}^c{}_d = \tilde{A}_{\mu\alpha}(t_\lambda^\alpha)^c{}_d dx^\mu$ is the projection of the gauge field onto the eigenspace \mathcal{G}_λ and d_λ is defined by the normalization $\text{Tr}(t_\lambda^\alpha t_\lambda^\beta) = d_\lambda \delta^{\alpha\beta}$. We are free to rescale the generators t_λ^α so that d_k agrees with the same quantity as calculated when the trace is taken to be in the fundamental representation of \mathcal{G}_λ . For the path integral to be well defined, the coefficient of a Chern-Simons term must be $k/4\pi$, where $k \in \mathbb{Z}$ [38]. This leads to a quantization condition of the form $\omega_\lambda = \pi/d_\lambda k$.

With these results, the Lagrangian can be written as

$$\begin{aligned} \mathcal{L} = & -\text{Tr}(D^\mu \bar{Z}_A, D_\mu Z^A) - i \text{Tr}(\bar{\psi}^A, \gamma^\mu D_\mu \psi_A) \\ & - V + \mathcal{L}_{\text{CS}} - i \text{Tr}(\bar{\psi}^A, [\psi_A, Z^B; \bar{Z}_B]) \\ & + 2i \text{Tr}(\bar{\psi}^A, [\psi_B, Z^B; \bar{Z}_A]) \\ & + \frac{i}{2} \varepsilon_{ABCD} \text{Tr}(\bar{\psi}^A, [Z^C, Z^D; \psi^B]) \\ & - \frac{i}{2} \varepsilon^{ABCD} \text{Tr}(\bar{Z}_D, [\bar{\psi}_A, \psi_B; \bar{Z}_C]), \end{aligned} \quad (51)$$

where

$$\begin{aligned} V = & \frac{2}{3} \text{Tr}(Y_B^{CD}, \bar{Y}_{CD}^B); \\ Y_B^{CD} = & [Z^C, Z^D; \bar{Z}_B] - \frac{1}{2} \delta_B^C [Z^E, Z^D; \bar{Z}_E] \\ & + \frac{1}{2} \delta_B^D [Z^E, Z^C; \bar{Z}_E], \end{aligned} \quad (52)$$

and \mathcal{L}_{CS} is given in (50).

We close this section by constructing an infinite class of examples. Let V_1 and V_2 be complex vector spaces with dimensions N_1 and N_2 , respectively. Consider the vector space \mathcal{A} of linear maps $X: V_1 \rightarrow V_2$. In general there is no natural notion of a product on \mathcal{A} , but there is a natural notion of a triple product:

$$[X, Y; \bar{Z}] = \lambda(XZ^\dagger Y - YZ^\dagger X). \quad (53)$$

Here \dagger denotes the transpose conjugate and λ is an arbitrary constant. If we introduce the inner product

$$\text{Tr}(\bar{X}, Y) = \text{tr}(X^\dagger Y), \quad (54)$$

where tr denotes the ordinary matrix trace, then one sees that $f^{abc\bar{d}}$ satisfies the correct symmetry properties as well as the fundamental identity.

From the Lie-algebra point of view, $V_1 \cong \mathbb{C}^{N_1}$ and $V_2 \cong \mathbb{C}^{N_2}$ can be regarded as the vector space of the fundamental representation of $U(N_1)$ and $U(N_2)$, respectively. The maps $X: V_1 \rightarrow V_2$ can then be viewed as states in the bifundamental representation $(\mathbf{N}_1, \bar{\mathbf{N}}_2)$. It is easy to see that the Lie algebra \mathcal{G} acts on X by

$$\delta X = XM_1 - M_2^\dagger X, \quad (55)$$

where M_1, M_2 are elements of $u(N_1)$ and $u(N_2)$, respec-

tively. Thus we see that $\mathcal{G} = u(N_1) \oplus u(N_2)$. Finally one can check that

$$\delta[X, Y; \bar{Z}] = [X, Y; \bar{Z}]M_1 - M_2^\dagger[X, Y; \bar{Z}], \quad (56)$$

which is a manifestation of the fundamental identity.

With this choice of 3-algebra, the action (51) becomes

$$\begin{aligned} \mathcal{L} = & -\text{tr}(D^\mu Z_A^\dagger D_\mu Z^A) - i \text{tr}(\bar{\psi}^{A\dagger} \gamma^\mu D_\mu \psi_A) \\ & - V + \mathcal{L}_{\text{CS}} - i \lambda \text{tr}(\bar{\psi}^{A\dagger} \psi_A Z_B^\dagger Z^B - \bar{\psi}^{A\dagger} Z^B Z_B^\dagger \psi_A) \\ & + 2i \lambda \text{tr}(\bar{\psi}^{A\dagger} \psi_B Z_A^\dagger Z^B - \bar{\psi}^{A\dagger} Z^B Z_A^\dagger \psi_B) \\ & + i \lambda \varepsilon_{ABCD} \text{tr}(\bar{\psi}^{A\dagger} Z^C \psi^{B\dagger} Z^D) \\ & - i \lambda \varepsilon^{ABCD} \text{tr}(Z_D^\dagger \bar{\psi}_A Z_C^\dagger \psi_B). \end{aligned} \quad (57)$$

For $N_1 = N_2$ this is the $\mathcal{N} = 6$ action of [23], as written in component form in [24]. For $N_1 \neq N_2$ we obtain the $U(N_1) \times U(N_2)$ models proposed in [39].

V. CONCLUSIONS

In this paper we have studied the general form of three-dimensional Lagrangians with $\mathcal{N} = 6$ supersymmetry, $SU(4)$ R-symmetry, and a $U(1)$ global symmetry. The resulting Lagrangians are of Chern-Simons form, with interacting scalars and vectors that take values in a so-called 3-algebra. As with the $\mathcal{N} = 8$ model previously constructed, the Lagrangian is entirely determined by specifying a triple product on a 3-algebra that satisfies the fundamental identity. For $\mathcal{N} = 6$, the tensor $f^{abc\bar{d}}$ that defines triple product need not be real or totally antisymmetric.²

We believe that the $\mathcal{N} = 6$ theories relevant for multiple M2-branes are classified by tensors $f^{abc\bar{d}}$ that satisfy the fundamental identity (2) and possess the symmetry properties (4). There is at least one very natural form for the triple product that leads to the models of [23] with gauge group $U(N) \times U(N)$. It would certainly be interesting to see if there are any other examples and hence other models. For example, $\mathcal{N} = 6$ models with gauge group $Sp(2N) \times O(2)$ have appeared in [33]. In addition, perhaps there is a connection to the embedding tensor approach studied in [40], or to the work of [14,41] that classifies totally antisymmetric 3-algebras.

We note that we have emphasized the role of triple products and 3-algebras even though the resulting Lagrangians can be viewed as relatively familiar Chern-Simons gauge theories based on Lie algebras with matter fields. From our point of view, the dynamical fields have interactions that are most naturally defined in terms of a triple product. Thus even though the 3-algebra may not be

²In the special case that f^{abcd} is totally antisymmetric, it is also real and the Lagrangian becomes that of the $\mathcal{N} = 8$ theory.

an independent structure apart from a Lie algebra, we believe the triple product is the central concept behind the M2-brane dynamics. For example in [6], the light states on the Coulomb branch of the $\mathcal{N} = 8$ theory were found to have masses, at least in the classical theory, that are proportional to the area of a triangle whose vertices end on the M2-branes. This is a consequence of the appearance of the triple product in the dynamics and hints to underlying M-theory degrees of freedom analogous to the open strings that arise in D-branes.

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APPENDIX

In this paper all spinor quantities in lower case letters are those of three-dimensional Minkowski space with real two-component spinors. Spinor quantities with capitol letters refer to 11-dimensional Minkowski space with 32 component spinors (although the supersymmetry generators are always denoted by a lower case ϵ). In both cases γ_μ , $\mu = 0, 1, 2$ and Γ_m , $m = 0, 1, 2, \dots, 10$ are sets of real γ matrices with $\gamma_{012} = 1$ (respectively $\Gamma_{012345678910} = 1$) and $\bar{\epsilon} = \epsilon^T \gamma_0$ (respectively $\bar{\epsilon} = \epsilon^T \Gamma_0$). The eight transverse directions are labeled by the scalars X^I , $I, J = 1, \dots, 8$ or in terms of four complex scalars Z^A , $A, B = 1, 2, 3, 4$, with complex conjugates \bar{Z}_A .

In three dimensions the Fierz transformation is

$$(\bar{\lambda}\chi)\psi = -\frac{1}{2}(\bar{\lambda}\psi)\chi - \frac{1}{2}(\bar{\lambda}\gamma_\nu\psi)\gamma^\nu\chi. \quad (\text{A1})$$

Furthermore, we note the following useful identities:

$$\begin{aligned} \frac{1}{2}\bar{\epsilon}_1^{CD}\gamma_\nu\epsilon_{2CD}\delta_B^A &= \bar{\epsilon}_1^{AC}\gamma_\nu\epsilon_{2BC} - \bar{\epsilon}_2^{AC}\gamma_\nu\epsilon_{1BC} \\ 2\bar{\epsilon}_1^{AC}\epsilon_{2BD} - 2\bar{\epsilon}_2^{AC}\epsilon_{1BD} &= \bar{\epsilon}_1^{CE}\epsilon_{2DE}\delta_B^A - \bar{\epsilon}_2^{CE}\epsilon_{1DE}\delta_B^A \\ &\quad - \bar{\epsilon}_1^{AE}\epsilon_{2DE}\delta_B^C + \bar{\epsilon}_2^{AE}\epsilon_{1DE}\delta_B^C \\ &\quad + \bar{\epsilon}_1^{AE}\epsilon_{2BE}\delta_D^C - \bar{\epsilon}_2^{AE}\epsilon_{1BE}\delta_D^C \\ &\quad - \bar{\epsilon}_1^{CE}\epsilon_{2BE}\delta_D^A + \bar{\epsilon}_2^{CE}\epsilon_{1BE}\delta_D^A \end{aligned} \quad (\text{A2})$$

$$\begin{aligned} \frac{1}{2}\epsilon_{ABCD}\bar{\epsilon}_1^{EF}\gamma_\mu\epsilon_{2EF} &= \bar{\epsilon}_{1AB}\gamma_\mu\epsilon_{2CD} - \bar{\epsilon}_{2AB}\gamma_\mu\epsilon_{1CD} \\ &\quad + \bar{\epsilon}_{1AD}\gamma_\mu\epsilon_{2BC} - \bar{\epsilon}_{2AD}\gamma_\mu\epsilon_{1BC} \\ &\quad - \bar{\epsilon}_{1BD}\gamma_\mu\epsilon_{2AC} + \bar{\epsilon}_{2BD}\gamma_\mu\epsilon_{1AC}. \end{aligned} \quad (\text{A3})$$

In 11 dimensions the Fierz transformation is

$$\begin{aligned} (\bar{\epsilon}_2\chi)\epsilon_1 - (\bar{\epsilon}_1\chi)\epsilon_2 &= -\frac{1}{16}\left(2(\bar{\epsilon}_2\Gamma_\mu\epsilon_1)\Gamma^\mu\chi \right. \\ &\quad - (\bar{\epsilon}_2\Gamma_{IJ}\epsilon_1)\Gamma^{IJ}\chi \\ &\quad \left. + \frac{1}{4!}(\bar{\epsilon}_2\Gamma_\mu\Gamma_{IJKL}\epsilon_1)\Gamma^\mu\Gamma^{IJKL}\chi\right), \end{aligned} \quad (\text{A4})$$

where ϵ_1 , ϵ_2 , and χ have the same chirality with respect to Γ_{012} .

We also found the following identities useful:

$$\begin{aligned} \Gamma_M\Gamma^{IJ}\Gamma^M &= 4\Gamma^{IJ} \\ \Gamma_M\Gamma^{IJKL}\Gamma^M &= 0 \\ \Gamma^{IJP}\Gamma^{KLMN}\Gamma_P &= -\Gamma^I\Gamma^{KLMN}\Gamma^J + \Gamma^J\Gamma^{KLMN}\Gamma^I \\ \Gamma^I\Gamma^{KL}\Gamma^J - \Gamma^J\Gamma^{KL}\Gamma^I &= 2\Gamma^{KL}\Gamma^{IJ} - 2\Gamma^{KJ}\delta^{IL} + 2\Gamma^{KI}\delta^{JL} \\ &\quad - 2\Gamma^{LI}\delta^{JK} + 2\Gamma^{LJ}\delta^{IK} - 4\delta^{KJ}\delta^{IL} \\ &\quad + 4\delta^{KI}\delta^{JL} \\ \Gamma^{IJM}\Gamma^{KL}\Gamma_M &= 2\Gamma^{KL}\Gamma^{IJ} - 6\Gamma^{KJ}\delta^{IL} + 6\Gamma^{KI}\delta^{JL} \\ &\quad - 6\Gamma^{LI}\delta^{JK} + 6\Gamma^{LJ}\delta^{IK} + 4\delta^{KJ}\delta^{IL} \\ &\quad - 4\delta^{KI}\delta^{JL}. \end{aligned} \quad (\text{A5})$$

[1] A. Basu and J. A. Harvey, Nucl. Phys. **B713**, 136 (2005).
[2] J. H. Schwarz, J. High Energy Phys. **11** (2004) 078.
[3] J. Bagger and N. Lambert, Phys. Rev. D **75**, 045020 (2007).
[4] A. Gustavsson, arXiv:0709.1260.
[5] J. Bagger and N. Lambert, Phys. Rev. D **77**, 065008 (2008).
[6] N. Lambert and D. Tong, Phys. Rev. Lett. **101**, 041602

(2008).
[7] J. Distler, S. Mukhi, C. Papageorgakis, and M. Van Raamsdonk, J. High Energy Phys. **05** (2008) 038.
[8] U. Gran, B. E. W. Nilsson, and C. Petersson, J. High Energy Phys. **10** (2008) 067.
[9] H. Awata, M. Li, D. Minic, and T. Yoneya, J. High Energy Phys. **02** (2001) 013.
[10] J. Gomis, G. Milanesi, and J. G. Russo, J. High Energy

- Phys. 06 (2008) 075.
- [11] S. Benvenuti, D. Rodriguez-Gomez, E. Tonni, and H. Verlinde, arXiv:0805.1087.
- [12] P.M. Ho, Y. Imamura, and Y. Matsuo, J. High Energy Phys. 07 (2008) 003.
- [13] H. Lin, J. High Energy Phys. 07 (2008) 136.
- [14] J. Figueroa-O'Farrill, P. de Medeiros, and E. Mendez-Escobar, J. High Energy Phys. 07 (2008) 111.
- [15] J. Gomis, D. Rodriguez-Gomez, M. Van Raamsdonk, and H. Verlinde, J. High Energy Phys. 08 (2008) 094.
- [16] S. Banerjee and A. Sen, arXiv:0805.3930.
- [17] S. Cecotti and A. Sen, arXiv:0806.1990.
- [18] M.A. Bandres, A.E. Lipstein, and J.H. Schwarz, J. High Energy Phys. 07 (2008) 117.
- [19] B. Ezhuthachan, S. Mukhi, and C. Papageorgakis, J. High Energy Phys. 07 (2008) 041.
- [20] D. Gaiotto and E. Witten, arXiv:0804.2907.
- [21] H. Fuji, S. Terashima, and M. Yamazaki, arXiv:0805.1997.
- [22] K. Hosomichi, K.M. Lee, S. Lee, S. Lee, and J. Park, J. High Energy Phys. 07 (2008) 091.
- [23] O. Aharony, O. Bergman, D.L. Jafferis, and J. Maldacena, J. High Energy Phys. 10 (2008) 091.
- [24] M. Benna, I. Klebanov, T. Klose, and M. Smedback, J. High Energy Phys. 09 (2008) 072.
- [25] T. Nishioka and T. Takayanagi, J. High Energy Phys. 08 (2008) 001.
- [26] Y. Honma, S. Iso, Y. Sumitomo, and S. Zhang, Phys. Rev. D **78**, 105011 (2008).
- [27] J.A. Minahan and K. Zarembo, J. High Energy Phys. 09 (2008) 040.
- [28] J. Bhattacharya and S. Minwalla, arXiv:0806.3251.
- [29] Y. Imamura and K. Kimura, Prog. Theor. Phys. **120**, 509 (2008).
- [30] A. Armoni and A. Naqvi, J. High Energy Phys. 09 (2008) 119.
- [31] D. Gaiotto, S. Giombi, and X. Yin, arXiv:0806.4589.
- [32] G. Grignani, T. Harmark, and M. Orselli, arXiv:0806.4959.
- [33] K. Hosomichi, K.M. Lee, S. Lee, S. Lee, and J. Park, J. High Energy Phys. 09 (2008) 002.
- [34] J. Gomis, A.J. Salim, and F. Passerini, J. High Energy Phys. 08 (2008) 002.
- [35] K. Hosomichi, K.M. Lee, and S. Lee, Phys. Rev. D **78**, 066015 (2008).
- [36] J. Bagger and N. Lambert, J. High Energy Phys. 02 (2008) 105.
- [37] A. Gustavsson, J. High Energy Phys. 04 (2008) 083.
- [38] S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982).
- [39] O. Aharony, O. Bergman, and D.L. Jafferis, J. High Energy Phys. 11 (2008) 043.
- [40] E.A. Bergshoeff, M. de Roo, and O. Hohm, Classical Quantum Gravity **25**, 142001 (2008); E.A. Bergshoeff, M. de Roo, O. Hohm, and D. Roest, J. High Energy Phys. 08 (2008) 091.
- [41] P. de Medeiros, J. Figueroa-O'Farrill, and E. Mendez-Escobar, J. High Energy Phys. 08 (2008) 045.
- [42] P-A. Nagy, arXiv:0712.1398v2.
- [43] J.P. Gauntlett and J.B. Gutowski, arXiv:0804.3078.
- [44] G. Papadopoulos, J. High Energy Phys. 05 (2008) 054.