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Ostrogradski approach for the Regge-Teitelboim type cosmology

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We present an alternative geometric inspired derivation of the quantum cosmology arising from a brane universe in the context of *geodetic gravity*. We set up the Regge-Teitelboim model to describe our universe, and we recover its original dynamics by thinking of such field theory as a second-order derivative theory. We refer to an Ostrogradski Hamiltonian formalism to prepare the system to its quantization. Our analysis highlights the second-order derivative nature of the RT model and the inherited geometrical aspect of the theory. A canonical transformation brings us to the internal physical geometry of the theory and induces its quantization straightforwardly. By using the Dirac canonical quantization method our approach comprises the management of both first- and second-class constraints where the counting of degrees of freedom follows accordingly. At the quantum level our Wheeler-De Witt equation agrees with previous results recently found. On these lines, we also comment upon the compatibility of our approach with the Hamiltonian approach proposed by Davidson and coworkers.

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I. INTRODUCTION

The concept of a relativistic extended object as a surface immersed in a bulk has increased the interest in physics due to its wide range of applications. One can model, for example, the smallest physical entities, like quarks, as vibrations of strings up to the entire universe as a relativistic extended object. Along a related line, with the advent of brane world universes, cosmology in the presence of extra dimensions has been the subject of intense research. In fact, the idea that our universe could be a 3 + 1 dimensional surface embedded in a higher dimensional spacetime was set up by Regge and Teiltelboim (RT) a long time ago [1] and pursued by many authors [2–5]. The scope of such a model is that gravitation can be described in a point- or stringlike fashion, as the world volume swept out by the motion of a three-dimensional spacelike brane evolving in a higher-dimensional bulk spacetime [1]. Recently, the RT brane model has been considered as one of the two main pillars of a unified branelike theory [6], where the Randall-Sundrum brane theory [7] is included. When one addresses this issue in a Minkowski spacetime, the model is named *geodetic gravity*, and it has been extensively studied by Davidson and coworkers [8–11]. Although the RT model is not the most popular theory for brane world universes (at the end of the last century, there was a revival of the idea that our universe could be a hypersurface; see, for example, [7,12]), it is very stimulating while thinking in the spirit of brane gravity *à la* string. The cosmology that arises from this model is interesting in its own right since it provides an alternative route to better understand classical cosmology in extra dimensions, and also it supplies a compelling model to apply the canonical quantization methods. Indeed, in the context of quantum brane cosmology [13] our universe can be explained through a tunneling process where the well-known problem of boundary conditions of four-dimensional cosmology is solved [14,15].

In most field theories the action depends usually on the fields and their first derivatives. By contrast, the RT model is a genuine second-order derivative model in the field variables, which are the embedding functions rather than the induced metric. Generally, one identifies and neglects a surface term associated with the linear dependence of the accelerations. Similarly, as in general relativity, it is a well-known fact that a "harmless" surface term can be neglected or removed at the beginning as occurs with the well-known Gibbons-Hawking-York term into the action. Whichever field context, the extremization of the corresponding action yields equations of motion of second order in derivatives in the field variables. Thus the RT model, like the Einstein-Hilbert action for general relativity, becomes transformed in an effective first-order field theory. However, by what formally appears to be a customary procedure, to follow such a strategy raises important limi-

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tations especially in the Hamiltonian framework for the RT model where it leads to certain troubles, as was noticed first by Regge and Teitelboim, due mainly to the fact that the scalar constraint is not written down in a closed form straightforwardly. In pursuing this endeavor, Davidson and coworkers tackled the problem successfully. They considered an extra nondynamical canonical field λ in the first-order Hamiltonian framework in order to get quadratic constraints of the phase space that recuperate the dynamics accordingly [9–11]. The explicit handling of the quantum RT model is made possible by extending the ordinary phase space, which in turn provides a wealth of information of the cosmology that this model possesses.

In the present paper we consider an alternative formulation for geodetic gravity which is strongly based in the Ostrogradski program for higher-order derivative theories [16,17]. For second-order theories this approach treats the velocities as independent fields. This is an unconventional viewpoint for the RT theory, and one might therefore wonder if such a description is viable at all since this does not necessarily represent a shortcoming: for this special case, the addition of more degrees of freedom is physically more accurate, but it means then that the first-order theory is incomplete in some sense. For this reason, it seems promising to start directly from the full RT model instead of omitting the surface term *a priori*. Hence, we pay close attention to a Hamiltonian approach for geodetic gravity constructed by applying the Ostrogradski scheme which in turn leads to the correct dynamics. In particular, it is of a great interest to use the full model straightforwardly for obtaining the quantum approach for brane cosmology. Our intention is to cope directly with the inherent second-order derivative nature of the RT model. As discussed below, we gain certain improvements of clarity by the use of this formalism in comparison with previous works. Contrary to the standard quadratic form of the constraints for ordinary first-order reparametrization invariant theories, in the Ostrogradski approach for RT field theory the constraints are projections of the momenta along the velocities as well as along the unit spacelike normal vector to the brane. To illustrate our development we specialize our considerations to a minisuperspace model where the inherent gauge invariance under the reparametrization of time is evident. We show that the canonical Dirac constraint quantization of this model casts into a satisfactory Wheeler-De Witt (WDW) equation on the wave function for a branelike universe. The handling of the quantum approach is made possible by a canonical transformation which results to be a Lorentz rotation in phase space. Such a transformation brings our constraints into a physically meaningful set which enable us to follow the standard Dirac constraint quantization programme. Our quantum treatment hence leads to a well-defined Wheeler-DeWitt equation which, even though it is technically complicated to solve, presents the right behavior for the quantum potential, estimating the accuracy of our approach.

The outline of the paper is as follows. In Sec. II we briefly review some geometrical aspects of the RT model for a general d-dimensional brane, which are of interest for the rest of the paper. This section will serve to explain our notation and to gain insight into our Hamiltonian approach. In Sec. III we adapt our approach to a minisuperspace model in which we specialize to the geometry generated by the Friedman-Robertson-Walker (FRW) model. We explicitly give the Lagrangian density associated with the RT model which includes the surface term. Section IV deals with the Ostrogradski approach for the model we are considering, and we develop the corresponding constraint analysis. In Sec. V we propose the gauge-fixing for the model in order to completely identify the structure of the reduced phase space. In Sec. VI, we study the quantization of our model within the scheme of Dirac quantization. Finally, in Sec. VII we draw some conclusions. As a general feature, our presentation avoids cumbersome notation and it is intended to be index-free where it is possible.

II. REGGE-TEITELBOIM MODEL

Consider a brane Σ of dimension d, evolving in a fixed Minkowski N dimensional background spacetime with metric $\eta_{\mu\nu}$. Its trajectory, or world volume m of dimension $d+1$, is described by the embedding $x^\mu = X^\mu(\xi^a)$, where x^μ are local coordinates for the background spacetime, ξ^a are local coordinates for m, and X^μ the embedding functions ($\mu, \nu = 0, 1, \ldots, N-1$; $a, b = 0, 1, \ldots, d$). We denote by $e^\mu{}_a = \partial_a X^\mu$ the tangent vectors to m. In this framework we introduce $N-d-1$ unit normal vectors to the world volume, denoted by $n^\mu{}_i$ ($i = 1, 2, \ldots, N-d-1$). These are defined implicitly by $n^i \cdot e_a = 0$, and we choose to normalize them as $n_i \cdot n_j = \delta_{ij}$.

The RT model for a d-dimensional brane Σ is defined by the action functional

$$S_{\mathrm{RT}}[X] = \frac{\alpha}{2}\int_m d^{d+1}\xi\sqrt{-g}\mathcal{R} - \int_m d^{d+1}\xi\sqrt{-g}\Lambda, \quad (1)$$

where the constant α has dimensions $[L]^{(1-d)}$, and g denotes the determinant of the induced metric $g_{ab} = \eta_{\mu\nu}e^\mu{}_a e^\nu{}_b = e_a \cdot e_b$. We have also included in this action a cosmological constant term, Λ. The extrinsic curvature of m is $K_{ab}{}^i = -n^i \cdot D_a e_b$, where $D_a = e^\mu{}_a D_\mu$ and D_μ is the covariant derivative in the bulk spacetime. The mean extrinsic curvature is given by the trace $K^i = g^{ab}K_{ab}{}^i$ where g^{ab} denotes the inverse of g_{ab}. The scalar curvature $\mathcal{R}$ of m can be obtained either directly from the induced metric g_{ab}, or, in terms of the extrinsic curvature, via the contracted Gauss-Codazzi equation, $\mathcal{R} = K^i K_i - K^i_{ab}K^{ab}_i$ [18,19].

The response of the action (1) to a deformation of the surface $X \to X + \delta X$ is characterized by a conserved stress tensor [20]

$$f^{a\mu} = -(\alpha \mathcal{G}^{ab} + \Lambda g^{ab}) e^{\mu}{}_{b}, \tag{2}$$

where $\mathcal{G}_{ab} = \mathcal{R}_{ab} - \frac{1}{2}\mathcal{R} g_{ab}$ is the world volume Einstein tensor, with $\mathcal{R}_{ab}$ being the Ricci tensor. This quantity will provide relevant physical information especially with consistent conservation laws. Following the line of reasoning of [20], the classical brane trajectories can be obtained from the covariant conservation law, $\nabla_a f^{a\mu} = 0$, where ∇_a is the covariant derivative compatible with the induced metric g_{ab} [21]. This yields [22]

$$T^{ab} K_{ab}{}^{i} = 0, \tag{3}$$

where $T^{ab} = \alpha \mathcal{G}^{ab} + \Lambda g^{ab}$. In fact, T^{ab} corresponds to the intrinsic stress tensor defined in the usual way by $-2/\sqrt{-g}(\delta S_{\rm RT}/\delta g_{ab})$. Its conservation is supported by the Bianchy identity. The equations of motion (3) are of second order in derivatives of the embedding functions because of the presence of the extrinsic curvature. This is so even though in the scalar curvature $\mathcal{R}$ we have the presence of the extrinsic curvature. Owing to the reparametrization invariance of the RT model, there are only $D - d - 1$ independent equations, along the normals; the remaining $d + 1$ tangential components are satisfied identically, as a consequence of the reparametrization invariance of the action (1).

An important quantity constructed with the conserved stress tensor is given by

$$\pi^{\mu} = \eta_a f^{a\mu} = -(\alpha \mathcal{G}^{ab} + \Lambda g^{ab}) \eta_a e^{\mu}{}_{b} \tag{4}$$

where η^a stands for the timelike unit normal vector to the brane Σ when it is viewed into m [23]. In fact, Eq. (4) is nothing but the conserved linear momentum associated with the Noether charge of the action (1) specialized to background translations [20]. The Σ basis, $\{\epsilon^{\mu}{}_{A}, \eta^{\mu}, n^{\mu}{}_{i}\}$, satisfies the completeness relation

$$\eta^{\mu\nu} = n^{\mu i} n^{\nu}{}_{i} - \eta^{\mu}\eta^{\nu} + h^{AB}\epsilon^{\mu}{}_{A}\epsilon^{\nu}{}_{B}, \tag{5}$$

where $h_{AB} = g_{ab}\epsilon^{a}{}_{A}\epsilon^{b}{}_{B}$ is the spatial metric on Σ and $\epsilon^{a}{}_{A}$ are the tangent vectors to Σ, $(A, B = 1, 2, \ldots, d)$. The vector η^{μ} stands for a timelike unit vector to Σ (see Refs. [21,24] for more details).

In the presence of other possible matter sources with stress tensor $T_m^{ab} = (-2/\sqrt{-g})\delta S_m/\delta g_{ab}$, where S_m is a matter action, we do not expect considerable modifications in our approach. The equations of motion (3) remain unchanged in form. It is sufficient to add the matter stress tensor to the original one described in (3). Similarly, the conserved linear momentum (4) is unaffected in form when another type of matter is included. It gets an additional contribution of the form $\pi_m^{\mu} = -T_m^{ab}\eta_a e^{\mu}{}_{b}$. These nice features allow us to develop straightforwardly a Hamiltonian analysis without substantial changes under the inclusion of matter fields. This fact was also noticed in [11].

III. MINISUPERSPACE MODEL

We turn now to restrict the RT model itself (1) to the case of a minisuperspace model. Consider a 3-brane Σ, evolving in a 5-dimensional Minkowski spacetime, $ds^2 = -dt^2 + da^2 + a^2 d\Omega_3^2$, where $d\Omega_3^2$ stands for the metric of a unit 3-sphere, i.e., $d\Omega_3^2 = d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi \sin^2\theta d\phi^2$. For the sake of simplicity, we choose the function $\sin^2\chi$ in $d\Omega_3^2$ to consider a closed universe. If

$$x^{\mu} = X^{\mu}(\xi^a) = (t(\tau), a(\tau), \chi, \theta, \phi) \tag{6}$$

is a parametric representation of the trajectory of Σ, we assure that the geometry of the world volume generated is that of the FRW case. According to the cosmology jargon, $a(\tau)$ is known as the *scale factor*.

The basis adapted to the world volume is given by the four tangent vectors $e^{\mu}{}_{a}$ $(a = 0, 1, 2, 3)$ together with the unit spacelike normal vector

$$n_{\mu} = \frac{1}{N}(-\dot{a}, \dot{t}, 0, 0, 0), \tag{7}$$

where the dot stands for derivation with respect to τ. For short in the notation we have introduced the quantity, $N = \sqrt{\dot{t}^2 - \dot{a}^2}$, which coincides with the lapse function when we perform an ADM decomposition of the action (1) [21,24].

The metric induced on the world volume is given by

$$ds^2 = g_{ab} d\xi^a d\xi^b = -N^2 d\tau^2 + a^2 d\Omega_3^2. \tag{8}$$

The spatial components of this metric correspond to the metric associated with Σ when this is described by its embedding in the world volume itself. Furthermore, for this latter parametrization, we have $\eta^a = 1/N(1, 0, 0, 0)$ such that $g_{ab}\epsilon^{a}{}_{A}\eta^{b} = 0$.

The Ricci scalar associated with the metric (8) reads

$$\mathcal{R} = \frac{6\dot{t}}{a^2 N^4}(a\ddot{a}\dot{t} - a\dot{a}\ddot{t} + N^2\dot{t}). \tag{9}$$

The linear dependence that the Ricci scalar possesses in the accelerations of the variables $t(\tau)$ and $a(\tau)$ is particularly remarkable.

The Lagrangian density $\mathcal{L} = \sqrt{-g}(\frac{\alpha}{2}\mathcal{R} - \Lambda)$ thus becomes

$$\mathcal{L} = \frac{\Upsilon a \dot{t} \alpha}{N^3}(a\ddot{a}\dot{t} - a\dot{a}\ddot{t} + \dot{t}^3 - \dot{a}^2\dot{t}) - \frac{\Upsilon N a^3}{3}\Lambda, \tag{10}$$

where $\Upsilon = 3\sin\theta\sin^2\chi$. Thus, the RT action specialized to spherical configurations, in terms of an arbitrary parameter τ, is reduced to

$$S_{\rm RT} = 6\pi^2 \int d\tau L(a, \dot{a}, \ddot{a}, \dot{t}, \ddot{t}), \tag{11}$$

where the Lagrangian function is given by [25]

$$L(a, \dot{a}, \ddot{a}, \dot{t}, \ddot{t}) = \frac{a\dot{t}}{N^3}(a\ddot{a}\dot{t} - a\dot{a}\ddot{t} + N^2\dot{t}) - N a^3 H^2, \tag{12}$$

where we have introduced the constant quantity $H^2 := \Lambda/3\alpha$. Thus, we have only $a(\tau)$ and $t(\tau)$ as independent dynamical variables. Despite the acceleration dependence in the Lagrangian, as characterizes second-order derivative theories, the equations of motion remain second order in the field variables (see Eq. (3)).

We proceed now to evaluate both the Einstein and the extrinsic curvature tensors of the world volume as described by the metric (8). We have the nonvanishing explicit components

$$\mathcal{G}^{\tau}{}_{\tau} = -\frac{3\dot{t}^2}{a^2N^2},$$

$$\mathcal{G}^{\chi}{}_{\chi} = \mathcal{G}^{\theta}{}_{\theta} = \mathcal{G}^{\phi}{}_{\phi} = -\frac{\dot{t}^2}{a^2N^4}\left[2a\dot{t}\frac{d}{d\tau}\left(\frac{\dot{a}}{\dot{t}}\right) + N^2\right],$$

and

$$K^{\tau}{}_{\tau} = \frac{\dot{t}^2}{N^3}\frac{d}{d\tau}\left(\frac{\dot{a}}{\dot{t}}\right), \qquad K^{\chi}{}_{\chi} = K^{\theta}{}_{\theta} = K^{\phi}{}_{\phi} = \frac{\dot{t}}{aN}.$$

Clearly, we can read off immediately the spatial components of the extrinsic curvature tensor as well as its mean extrinsic curvature given by $\mathcal{K} = h^{AB}K_{AB} = 3\dot{t}/aN$.

As dictated by Eq. (3), there is only one equation of motion given by

$$\frac{d}{d\tau}\left(\frac{\dot{a}}{\dot{t}}\right) = -\frac{N^2}{a\dot{t}}\frac{\Theta}{\Phi}, \tag{13}$$

where we have introduced the functions

$$\Theta := \dot{t}^2 - 3N^2a^2H^2, \tag{14a}$$

$$\Phi := 3\dot{t}^2 - N^2a^2H^2, \tag{14b}$$

for simplicity. Equation (13) obviously only involves second derivatives of the field variables a and t. For any solution for $a(\tau)$ we have a gauge freedom to choose a function for $t(\tau)$ as we will see below.

IV. OSTROGRADSKI-HAMILTONIAN APPROACH

A deeper insight of the phase space structure of the theory defined by the Lagrangian (12) is achieved by an Ostrogradski procedure for higher-order derivative systems. (A complete description of the Hamiltonian formulation for branes whose action depends on the extrinsic curvature of their world volume is provided in [21,24].) The highest conjugate momenta to the velocities $\{\dot{t}, \dot{a}\}$ are, respectively,

$$P_t = \frac{\partial L}{\partial \ddot{t}} = -\frac{a^2\dot{a}\,\dot{t}}{N^3}, \tag{15}$$

$$P_a = \frac{\partial L}{\partial \ddot{a}} = \frac{a^2\dot{t}^2}{N^3}, \tag{16}$$

such that the highest momentum spacetime vector is

$$P_\mu = \frac{a^2\dot{t}}{N^3}(-\dot{a}, \dot{t}, 0, 0, 0) = \frac{a^2\dot{t}}{N^2}n_\mu. \tag{17}$$

Though this momentum has not a direct mechanical meaning it will become important to achieve a Legendre transformation in order to obtain the Hamiltonian function for our system (see Eq. (24) below). Note that the momentum P_μ is directed normal to the world volume.

The conjugate momenta to the position variables $\{t, a\}$ are, respectively

$$p_t = \frac{\partial L}{\partial \dot{t}} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \ddot{t}}\right) = \frac{a\dot{t}}{N^3}[\dot{a}^2 + N^2(1 - a^2H^2)] =: -\Omega, \tag{18}$$

$$p_a = \frac{\partial L}{\partial \dot{a}} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \ddot{a}}\right) = -\frac{a\dot{a}}{N^3}[\dot{a}^2 + N^2(1 - a^2H^2)] = \left(\frac{\dot{a}}{\dot{t}}\right)\Omega. \tag{19}$$

Important to note is the fact that both momenta, p_t and p_a, are from a totally different nature. Indeed, while the momentum p_t is not influenced at all by the surface terms (as expected), the momentum p_a is obtained by two contributions: one coming from the ordinary theory and the other by a surface term. In this way, we can denote the momentum p_a as

$$p_a := \mathbf{p}_a + \mathfrak{p}_a, \tag{20}$$

where

$$\mathbf{p}_a = -\frac{a\dot{a}}{N^3}[\dot{a}^2 + N^2(3 - a^2H^2)], \tag{21}$$

$$\mathfrak{p}_a = \frac{2a\dot{a}}{N}. \tag{22}$$

It is crucial to recognize them as the canonical momentum worked out in [9] and as the momentum conjugated to the $a(\tau)$-variable when considering as the Lagrangian only the surface term $L_s = d/d\tau(a^2a/N)$, respectively.

To see the geometrical structure of this momentum, it is convenient to write p_μ as

$$p_\mu = \frac{\Omega}{\dot{t}}(-\dot{t}, \dot{a}, 0, 0, 0) = \frac{\Omega}{\dot{t}}\dot{X}_\mu. \tag{23}$$

We realize that this momentum is identical with the vector π_μ defined on Eq. (4), which is the projection of the conserved stress tensor along the unit timelike normal vector η^a to Σ.

The appropiate phase space of the system, $\Gamma := \{t, a, \dot{t}, \dot{a}; p_t, p_a, P_t, P_a\}$, has been identified explicitly. Thus in Γ, the Ostrogradski formalism yields the canonical Hamiltonian

$$H_0 = p \cdot \dot{X} + P \cdot \ddot{X} - L = p_a\dot{a} + p_t\dot{t} + J_{\mathcal{R}}, \tag{24}$$

where we have defined

$$J_{\mathcal{R}} = -\frac{a}{N}[\dot{a}^2 + N^2(1 - a^2H^2)] = \frac{N^2}{\dot{t}}\Omega. \tag{25}$$

This potential-like term results in an implicit function of the phase space variables in the combination N^3P^2. At first glance, this may look as an unnecessary complication to write the phase space quantities in terms of Ω, but this quantity results in a physical observable: It is nothing but the conserved bulk energy. Indeed, squaring the energy Eq. (20), results in the so-called evolution master equation [1,3–5]

$$N^2 + \dot{a}^2 = \gamma N^2 a^2 H^2, \tag{26}$$

where $\gamma = \gamma(a)$ satisfies the cubic equation $\gamma(\gamma - 1)^2 = \Omega^2/a^8H^6$.

A. Constraint analysis

Since we are dealing just with a second-order derivative theory linear in the accelerations, we have two primary constraints given by the definition of the momenta itself, and hence $\phi_\mu := P_\mu - \frac{a^2\dot{t}}{N^2}n_\mu = 0$. Instead of these constraints, here we will follow a different but convenient route. We choose to project the momentum (17) along the velocity vector as well as the normal vector to Σ, where in general $n^\mu = n^\mu(\dot{X}^\nu)$. This is supported by using the geometrical completeness relation (5) in $\phi_\mu = \eta_{\mu\nu}\phi^\nu$. Thus, we get the primary constraints

$$\mathcal{C}_1 = P \cdot \dot{X} = 0, \tag{27}$$

$$\mathcal{C}_2 = NP \cdot n - \frac{a^2\dot{t}}{N} = 0. \tag{28}$$

Therefore, the total Hamiltonian which generates time evolution of the fields is

$$H_T = H_0 + \lambda_1\mathcal{C}_1 + \lambda_2\mathcal{C}_2, \tag{29}$$

where λ_1 and λ_2 are Lagrange multipliers enforcing the primary constraints.

As customary, time-evolution for any canonical variable $z \in \Gamma$ reads

$$\dot{z} = \{z, H_T\}, \tag{30}$$

on the constraint surface, where the generalized Poisson bracket for any two functions $F(z)$ and $G(z)$ in Γ is appropriately defined as

$$\{F, G\} = \frac{\partial F}{\partial t}\frac{\partial G}{p_t} + \frac{\partial F}{\partial a}\frac{\partial G}{p_a} + \frac{\partial F}{\partial \dot{t}}\frac{\partial G}{P_t} + \frac{\partial F}{\partial \dot{a}}\frac{\partial G}{P_a} - (F \leftrightarrow G). \tag{31}$$

Important to mention is the fact that the total Hamiltonian (29) leads us directly to the right equations of motion (13) through the conventional Ostrogradski approach for higher-order derivative systems [17].

We also note that under the symplectic structure (31), the constraints (27) and (28) result to be in involution, $\{\mathcal{C}_1, \mathcal{C}_2\} = 0$. According to the Dirac program for constrained systems, both $\mathcal{C}_1$ and $\mathcal{C}_2$ must be preserved by the evolution which demands the existence of the secondary constraints

$$\mathcal{C}_3 = H_0 = p \cdot \dot{X} + N\left(a^3H^2 - \frac{1}{a^3}N^2P^2\right), \tag{32}$$

$$\mathcal{C}_4 = p \cdot n. \tag{33}$$

The vanishing of the canonical Hamiltonian is expected courtesy of the reparametrization invariance of the RT model. Hence the canonical Hamiltonian H_0 generates diffeomorphisms normal to the world volume. The secondary constraint (33) is characteristic for every brane model linear in accelerations. The process of generation of further constraints is stopped at this stage since $\mathcal{C}_3$ is preserved under evolution and the requirement of stationarity of $\mathcal{C}_4$ only determines a restriction on one of the Lagrange multipliers, namely, $\lambda_2 = N^3\Omega/a^2\Phi$. Thus, we are dealing with a wholly constrained theory with first- and second-class constraints, which is a consequence of the rich gauge symmetry of the RT model. The distinctive feature of the constraints (27) and (28) instead of ϕ_μ is that $\mathcal{C}_1$ and $\mathcal{C}_2$ are constraints that naturally generate the relationships (32) and (33) as befit a higher-order derivative brane theory [24].

Following Dirac's program, the set of constraints should be separated into subsets of first- and second-class constraints [26]. It is quite well known that for each pair of second-class constraints there is one degree of freedom which is not physically important and has to be removed from the theory, and for each first-class constraint one degree of freedom is removed. For our system we have two first-class phase space constraints,

$$\mathcal{F}_1 = \mathcal{C}_1, \tag{34}$$

$$\mathcal{F}_2 = \frac{N^3\Omega}{a^2\Phi}\mathcal{C}_2 + \mathcal{C}_3, \tag{35}$$

and two second-class constraints. The selection of the second-class constraints is straightforward (see the Appendix). We choose them as

$$\mathcal{S}_1 = \mathcal{C}_2, \tag{36}$$

$$\mathcal{S}_2 = \mathcal{C}_4. \tag{37}$$

Note that as we have two linear independent first-class constraints, we have the presence of two gauge transformations in the RT model. In the Appendix we discuss more thoroughly the Poisson brackets among the phase space constraints. The counting of degrees of freedom is as follows: dof = [(Total number of canonical variables) −

$2 \times (\text{first} - \text{class constraints}) - (\text{number of second} - \text{class constraints})]/2 = [8 - (2 \times 2) - 2]/2 = 1,$ which agrees with the number of normals to the world volume [24]. Such a single degree of freedom can be identified as the scale factor $a(\tau)$.

V. GAUGE-FIXING

According to the conventional Dirac scheme, in order to extract the physical meaningful phase space for a constrained system we need a gauge-fixing prescription which entails the introduction of extra constraints, avoiding in this way the gauge freedom generated by constraints (34) and (35). To achieve this we will consider the conventional cosmic gauge condition

$$\varphi_1 = N - 1 = \sqrt{\dot{t}^2 - \dot{a}^2} - 1 \approx 0, \tag{38}$$

and the generalized evolution Eq. (26)

$$\varphi_2 = N^2 + \dot{a}^2 - \gamma N^2 H^2 a^2 \approx 0, \tag{39}$$

where the $\approx$ symbol stands for weak equality in the Dirac approach for constrained systems [26]. From the geometric point of view, this set of gauge conditions is good enough since the matrix $(\{\mathcal{F}, \varphi_{1,2}\})$ is nondegenerate in the constraint surface. Indeed, under the Poisson bracket structure (31), it is straightforward to show that gauges φ_1 and φ_2 form a second-class algebra with the constraints $\mathcal{F}_1$ and $\mathcal{F}_2$

$$\begin{aligned} &\{\varphi_1, \mathcal{F}_1\} = \varphi_1 + 1, \qquad \{\varphi_1, \mathcal{F}_2\} = 0, \\ &\{\varphi_2, \mathcal{F}_1\} = 2\varphi_2 + 2\gamma H^2 a^2, \qquad \{\varphi_2, \mathcal{F}_2\} = F(a, \dot{a}, \dot{t}), \end{aligned} \tag{40}$$

where $F(a, \dot{a}, \dot{t})$ is a complicated function [27]. Consequently, velocities $\dot{t}$ and $\dot{a}$ must be discarded as dynamical degrees of freedom.

The use of the completeness relation (5) turns out to be very useful at this level. It allows us to express the quantity P^2 by the equivalent expression $-(P \cdot \eta)^2 + (P \cdot n)^2$. This suggests the implementation of the following canonical transformation to a new set of phase space variables:

$$\begin{aligned} &N := \sqrt{\dot{t}^2 - \dot{a}^2}, \qquad \Pi_N := \frac{1}{N}(P \cdot \dot{X}), \\ &\upsilon := \operatorname{arctanh}\left(-\frac{p_a}{p_t}\right), \qquad \Pi_\upsilon := N(P \cdot n), \end{aligned} \tag{41}$$

together with the coordinate transformation $Z^\mu := X^\mu - \{X^\mu, \upsilon\}\Pi_\upsilon$, while the momenta p_μ remains unaltered. Such transformation can be physically interpreted as a Lorentz rotation in phase space which, straightforwardly, preserves the structure of the canonical Poisson brackets

$$\{N, \Pi_N\} = 1 = \{\upsilon, \Pi_\upsilon\}, \qquad \{Z^\mu, p_\nu\} = \delta^\mu{}_\nu, \tag{42}$$

as expected. A transformation of the same kind was considered in [28] for a quantum treatment of a kink model. To understand in a more transparent way the geometric content of this transformation let us go through the intrinsic angular momentum density of the RT model calculation carefully. In terms of the new phase space variables the intrinsic angular momentum is given by $\mathcal{M}_i^{\mu\nu} = \eta_a \mathcal{M}_i^{a\mu\nu} = NP^{[\mu}\dot{X}^{\nu]}$ [20]. Such momentum is conserved in the sense that $\nabla_a \mathcal{M}_i^{a\mu\nu} = 0$. This contribution shows up as an effect of the finite width of the brane in comparison with flimsy branes described by Dirac-Nambu-Goto action. The only nonvanishing contribution to the intrinsic angular momentum reads

$$\mathcal{M}_{i\eta n} = \frac{1}{2} N \Pi_\upsilon, \tag{43}$$

which in turn tell us that our new variable Π_υ can be thought of as an angular momentum component.

In attempting to take into account the new phase space variables, the first- and second-class constraints (34)–(37) become

$$\mathcal{F}_1 = N\Pi_N, \tag{44}$$

$$\mathcal{F}_2 = p \cdot \dot{X} + N\left(a^3 H^2 + \frac{1}{a^3} N^2 \Pi_N^2 - \frac{1}{a^3}\Pi_\upsilon^2\right), \tag{45}$$

and

$$\mathcal{S}_1 = \Pi_\upsilon - \frac{a^2 \dot{t}}{N} = 0, \tag{46}$$

$$\mathcal{S}_2 = p_a \dot{t} + p_t \dot{a} = \mathfrak{p}_a \dot{t} - \frac{2a\dot{a}}{N}\dot{t} = 0, \tag{47}$$

respectively. Note that in the second-class constraint (47) we split the momentum conjugated to a according to relation (20), and hence, the second-class constraint (47) results in identity (22). Furthermore, second-class identities (46) and (47) will become auspicious at the quantum level since they enclose important operator identities.

One can develop further the constraint $\mathcal{F}_2$ (45) by expressing the velocities in terms of the momenta, $\dot{a} = -(N/a)[\mathbf{p}_a/((\gamma-1)a^2H^2 + 2)]$ and $\dot{t} = -\Omega N/(\gamma - 1) \times a^3 H^2$, by using the second gauge (39). Thus, a direct calculation on the first term in (45) yields

$$\begin{aligned} p \cdot \dot{X} = &-\left(\frac{N}{a[(\gamma-1)a^2H^2 + 2]}\right)\mathbf{p}_a^2 - \left(\frac{N\Omega}{(\gamma-1)a^3H^2}\right)p_t \\ &+ \mathfrak{p}_a \dot{a}. \end{aligned} \tag{48}$$

Finally, after a lengthy but straightforward computation the constraint $\mathcal{F}_2$ can be cast as

$$\begin{aligned} \mathcal{F}_2 = N\Bigg\{\mathbf{p}_a^2 - a\Bigg[&-\left(\frac{\Omega}{(\gamma-1)a^3H^2}\right)p_t + \frac{\mathfrak{p}_a \dot{a}}{N} + a^3H^2 \\ &+ \frac{1}{a^3}N^2\Pi_N^2 - \frac{1}{a^3}\Pi_\upsilon^2\Bigg][(\gamma-1)a^2H^2 + 2]\Bigg\}. \end{aligned} \tag{49}$$

We could also use Eq. (18) and the second-class condition (47), which impose the identities $p_t = -\Omega$ and $\mathfrak{p}_a = 2a\dot{a}/N$, respectively, to conclude that the constraint $\mathcal{F}_2 \approx 0$ gives rise to a quadratic expression for the involved momenta. That is, the second gauge condition (39) shifts the problem from the linear dependence in the momenta to a convenient quadratic expression for the physical momenta. To close this section we must mention that the constraints $\mathcal{F}_1$ and $\mathcal{F}_2$ form an algebra isomorphic to the Lie algebra associated with the lower triangular subgroup of $SL(2,\mathbb{R})$ as argued in the Appendix. This is the starting point to achieve an algebraic quantization as we will sketch below.

VI. QUANTIZATION

In this section we study the canonical quantization of our system. Also, we will sketch an alternative different quantum theory for our model which emerges from the corresponding first-class symmetries. To this end, we emphasize the totally dissimilar nature which first- and second-class constraints play in the quantum theory, and also, we explore the different senses in which the physical states of our theory can be defined.

We start in the conventional way by promoting the classical constraints into operators, densely defined on a common domain in a proper Hilbert space. As it is well known, we can only achieve a consistent classical theory by implementation of the Dirac bracket. Once this is done, the second-class constraints are eliminated off the theory by converting them into strong identities. At the quantum level this is mirrored by defining the quantum commutator of two quantum operators as

$$[\hat{A}, \hat{B}] := i\widehat{\{A, B\}^*}, \tag{50}$$

where the Dirac bracket $\{\cdot,\cdot\}^*$ is defined in Eq. (A7). Thus, with this prescription the operators corresponding to second-class constraints are also enforced as operator identities [26]. For our system, this yields the quantum operator expressions

$$\hat{\mathcal{S}}_1 = \hat{\Pi}_v - \frac{a^2\hat{\dot{t}}}{N} = 0, \tag{51}$$

$$\hat{\mathcal{S}}_2 = \hat{\mathfrak{p}}_a - \frac{2a\hat{\dot{a}}}{N} = 0, \tag{52}$$

which, in particular, tell us the character of the quantum operators $\hat{\Pi}_v$ and $\hat{\mathfrak{p}}_a$. For the rest of the variables, we choose to work on the "position" representation, where we consider the position operators by multiplication and their associated momenta operators by $-i$ times the corresponding derivative operator when applied on states defined on a suitable Hilbert space.

By defining the quantum first-class constraints as

$$\hat{\mathcal{F}}_1 := -iN\frac{\partial}{\partial N}, \tag{53}$$

$$\hat{\mathcal{F}}_2 := N\Big\{-\frac{\partial^2}{\partial a^2} - \Big[\frac{i\Omega}{(\gamma-1)a^3H^2}\frac{\partial}{\partial t} + 2a(\gamma H^2a^2 - 1) + a(1-\gamma)H^2a^2 - \frac{1}{a^3}\Big(N\frac{\partial}{\partial N}\Big)^2\Big]a[(\gamma-1)H^2a^2 + 2]\Big\}, \tag{54}$$

we will work on the assumption that the commutators of these quantum constraints form a closed Lie algebra which will be also isomorphic to the algebra $\mathfrak{g}$. In fact, the classical first-class constraints are isomorphic to the algebra $\mathfrak{g}$ associated with the lower triangular subgroup G of $SL(2,\mathbb{R})$ (See the Appendix). Quantization of the lower triangular subgroup of $SL(2,\mathbb{R})$ by algebraic methods was extensively studied in [29] (see also [30] for comparison). Now we explore the rather different senses in which the quantum constraints can be used to define appropriate physical states.

A. Naïve Dirac constraints

First, we explore the Wheeler-DeWitt equation emerging by considering the physical states Ψ of the theory as those defined by naïve Dirac conditions

$$\hat{\mathcal{F}}_1\Psi = 0, \tag{55}$$

$$\hat{\mathcal{F}}_2\Psi = 0. \tag{56}$$

Equation (55) simply tells us that our physical states Ψ are not explicitly depending on the phase space variable N. However, due mainly to the complexity of our WDW Eq. (56), we have not succeed in finding explicit solutions for the physically admissible quantum states. We note that the last term in the operator (54) will bring a vanishing contribution to the WDW equation, and also we see that the t-dependence can be avoided by assuming $\Psi(a,t) := e^{-i\Omega t}\psi$, where $\psi := \psi(a)$ satisfies the WDW equation

$$\Big[-\frac{\partial^2}{\partial a^2} + U(a)\Big]\psi(a) = 0, \tag{57}$$

where the potential $U(a)$ is given by

$$U(a) = a^2[(\gamma-1)H^2a^2 + 2]^2(1 - \gamma H^2a^2), \tag{58}$$

which is recognized as the potential function found in [9] by repeated use of the master evolution constraint (26) in Eq. (56). The behavior of this potential is drawn in Fig. 1, where we can see the characteristic potential barrier. As discussed by Davidson and coworkers, it can be shown that after considering appropriate boundary conditions the big-bang singularity in our quantum theory can be neutralized

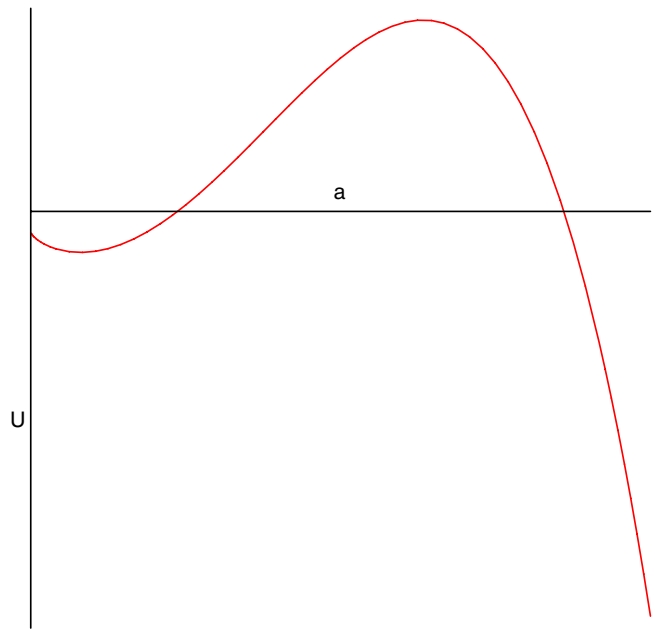

FIG. 1 (color online). WDW potential for geodetic gravity with naïve Dirac constraints.

by properly choosing the origin as inaccessible to wave packets. For further details on the behavior of the potential $U(a)$, the reader is referred to [9].

B. Modified Dirac constraints

As discussed in Refs. [31,32], there exists a procedure which allows us to reduce nonunimodular groups to unimodular ones and this in turn brings a remarkable alteration for systems amenable to geometric or algebraic quantization which comprises a modification for the Dirac conditions on physical states. Let $\{\hat{C}_a\}$ be a set of quantum constraints operators that generate a nonunimodular gauge group with the commutators $[\hat{C}_a, \hat{C}_b] = i f^c_{ab} \hat{C}_c$, where f^a_{bc} are the structure constants of the corresponding Lie algebra. Thus, the "unimodularization" procedure for nonunimodular groups dictates the consideration of the physical states $|\Psi\rangle$ as those satisfying $\hat{C}_a|\Psi\rangle = -(i/2) f^b_{ab}|\Psi\rangle$. Such modified Dirac conditions agree with the naïve Dirac constraints if, and only if, the group is unimodular.

Accordingly, for our theory the modified Dirac conditions for the gauge group invariant quantization of the system can be shown to be equivalent to

$$\left[\hat{\mathcal{F}}_1 - \frac{i}{2}\right]|\Psi\rangle = 0, \tag{59}$$

$$\hat{\mathcal{F}}_2|\Psi\rangle = 0, \tag{60}$$

which consequently define physical states $|\Psi\rangle$. Equation (59) is equivalent to the homogeneity condition $|\Psi(rN)\rangle = r^{-1/2}|\Psi(N)\rangle$ for $r > 0$ [29]. Further, (59) can be explicitly solved by taking $|\Psi\rangle = \frac{A}{N^{1/2}}|\psi\rangle$, where A is a constant, and $|\psi\rangle$ is a function of the variables a and t. Once more we also do not have control on the explicit solutions for the physically admissible quantum states. The t-dependence can be avoided by assuming $|\psi(a, t)\rangle := e^{-i\Omega t}|\varphi\rangle$, where $|\varphi\rangle$ is thought of as a function of the scale factor a only, which satisfies the WDW equation

$$\left[-\frac{\partial^2}{\partial a^2} + U(a) + \frac{(\gamma - 1)a^2H^2 + 2}{4a^2}\right]|\varphi(a)\rangle = 0, \tag{61}$$

where the potential $U(a)$ was described in the previous subsection. Hence, we see that our modified quantum theory brings out an extra potential term into our WDW equation, which succinctly differs from the one found with the naïve Dirac procedure.

We note that the extra term is purely emerging from the modified quantum Dirac Eqs. (59) and (60), and it is completely absent while considering the naïve Dirac procedure. This term will be nonvanishing even in the Einstein limit ($\gamma \to 1$), where it goes as a^{-2}. Further studies about the possible physical implications of this term could be carried out. The behavior of the modified potential is drawn in Fig. 2, where we can notably see that the central barrier potential present in Fig. 1 is almost vanishing while an infinite barrier emerges at the origin. To the best of our knowledge, the resulting unbounded potential is not realistic despite the fact that first-class constraints suggest this modified description. Nevertheless, one cannot resist the speculation of such possible quantum behavior. Thus, rather than a nice potential, this time it is a more compli-

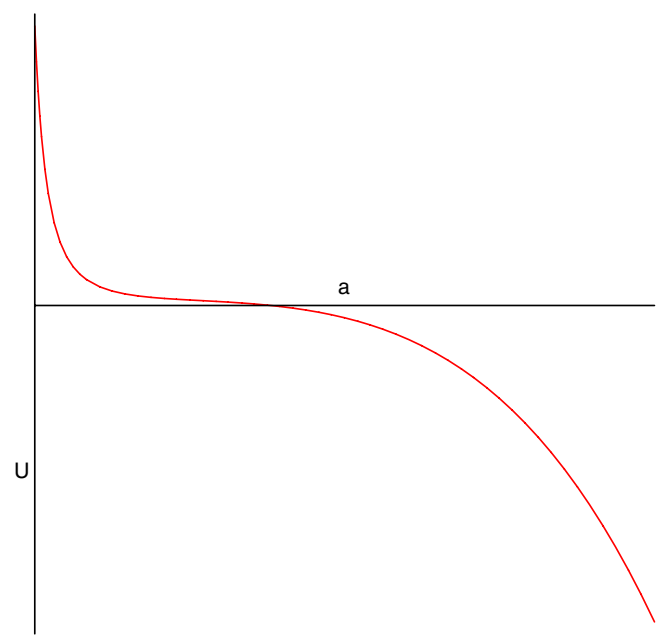

FIG. 2 (color online). WDW potential for geodetic gravity with modified Dirac constraints.

cated function with distinct features notwithstanding the internal constraint symmetries that demand an unimodularization procedure.

VII. CONCLUDING REMARKS

By making use of the Ostrogradski formalism we have developed an alternative Hamiltonian description of the RT brane model. Unlike the Hamiltonian treatment by Davidson and coworkers for this model [10,11], our analysis above keeps the original variables without the necessity of introducing nondynamical variables. At first sight, this may look like an unnecessary complication since the configuration space is initially increased only to be reduced again at a later stage by imposing the constraints and fixing the gauge. Nevertheless, it is hoped that despite computational complications we have provided an improvement of physical clarity, in particular, concerning the geometrical meaning of the constraints of the theory and the physical content of the achieved canonical transformation. Although the Ostrogradski approach has a price to pay, since neither the momentum P has the meaning of mechanical momentum nor H_0 has to do with the energy of the system, as it is customary, these quantities are adequate for providing a set of canonical equations which correctly describe the evolution of the system. Also, an important point to mention is that the formalism is rich enough to demonstrate the real role of the extra terms coming from the surface: the phase space constraints of the system impose identities for these quantities which are valid at both classical and quantum levels, hence eliminating the unphysical degrees of freedom.

In spite of the fact that this model is a second-order derivative theory and since it is well known that at quantum level the energy for higher-order derivative Lagrangians is almost always unbounded below, the RT model results in an exception due to its linear dependence on the accelerations which in turn contain important physical information commonly absent in higher-order derivative theories [33]. Like it or not, until now a Hamiltonian approach for the RT field theory demands the use of extra unphysical degrees of freedom at the beginning which by means of the phase space constraints are frozen out. Our adopted treatment renders the passage to a full quantization for the system which can be achieved by means of an inspired canonical transformation. We conclude further that the Ostrogradski quantum approach has exactly the same unique degree of freedom as the Davidson and coworkers approach. Although our Wheeler-DeWitt equation for the scale factor is not analytically manageable, it is good enough to substract from it some interesting features. In particular, the potential we found is exactly the same as the one extensively discussed by Davidson *et al.* [9]. Furthermore, our Hamiltonian approach makes feasible the quantum treatment of Lagrangians with linear higher-order derivative dependence in the fields.

It is suggesting that relativistic theories linear in the accelerations, for which characteristic surface terms are commonly neglected, are, as a generic feature, reluctant to quantization. To present day, quantization for these kinds of systems have been mainly studied by considering some extra degrees of freedom by several other methods. From this point of view, our intention has also been to introduce the Hamiltonian-Ostrogradski approach as a geometrical powerful method to beset this sort of system. An specific example of this would be to apply our treatment to the almost forgotten idea concerning the rigid bubble electron, for which a linear correction in the extrinsic curvature of the electron surface is added to the Dirac-Nambu-Goto action in contrast to the conventional first-order method where a surface term is omitted [34]. It will be worked elsewhere.

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APPENDIX A: ALGEBRAIC PROPERTIES OF THE CONSTRAINTS

We can construct the matrix $\mathcal{C}_{AB}$ whose elements are the Poisson brackets of all the constraints $\mathcal{C}_A$ where $A, B = 1, 2, 3, 4$. Hence,

$$(\mathcal{C}_{AB}) = \frac{1}{aN}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a^2\Phi \\ 0 & 0 & 0 & N^3\Omega \\ 0 & a^2\Phi & -N^3\Omega & 0 \end{pmatrix}, \tag{A1}$$

in the constraint surface. This matrix has rank 2, which is a signal that we have two first-class constraints [26]. It is also important to mention that constraints $\mathcal{C}_1$ and $\mathcal{C}_3$ form an algebra, namely,

$$\{\mathcal{C}_1, \mathcal{C}_1\} = 0, \qquad \{\mathcal{C}_1, \mathcal{C}_3\} = -\mathcal{C}_3, \qquad \{\mathcal{C}_3, \mathcal{C}_3\} = 0, \tag{A2}$$

which reflects the invariance under reparametrizations of the RT field theory as a fundamental gauge symmetry. Indeed, this algebra results in an isomorphism of the Lie algebra $\mathfrak{g}$ associated with the lower triangular subgroup of $SL(2, \mathbb{R})$ with positive diagonal elements, G. Such Lie algebra $\mathfrak{g}$ is spanned by the matrices [35]

$$h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad e^- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{A3}$$

whose commutator is

$$[h, e^-] = -2e^-, \tag{A4}$$

and hence we realize the isomorphism through the identification $\mathcal{C}_1 \mapsto h/2$ and $\mathcal{C}_3 \mapsto e^-$. Among the relevant properties of the subgroup G we refer to the fact that G is two-dimensional, non-Abelian, connected, and nonunimodular. This last property plays an important role in our quantum theory, as developed in Sec. VI.

Also, among the second-class constraints, (36) and (37), we can construct the matrix $\mathcal{S}_{IJ} = \{\mathcal{S}_I, \mathcal{S}_J\}$, given by

$$(\mathcal{S}_{IJ}) = \frac{a\Phi}{N}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{A5}$$

and its inverse

$$(\mathcal{S}^{IJ}) = \frac{N}{a\Phi}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{A6}$$

where $I, J = 1, 2$. The matrix $\mathcal{S}^{IJ}$ will help us to construct a Dirac bracket in the standard way: Let f and g be two arbitrary functions then

$$\{f, g\}^* := \{f, g\} - \sum \{f, \mathcal{S}_I\}\mathcal{S}^{IJ}\{\mathcal{S}_J, g\}, \tag{A7}$$

where $\{\cdot, \cdot\}$ stands for the Poisson bracket defined in (31). As it is well known, classically, the Dirac bracket is essential to eliminate the second-class constraint of the theory by converting them into simple functional identities. The need for the Dirac bracket is also very relevant at the quantum theory since we can only reach a consistent quantization procedure through the implementation of this bracket [26].

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