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Challenging the generalized second law

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The generalized second law (GSL) of black hole thermodynamics states that the sum of changes in black hole entropy and the ordinary entropy of matter and fields outside the hole must be non-negative. In the classical limit, the GSL reduces to Hawking's area theorem. Neither law identifies the specific effects that make it work in particular situations. Motivated by Davies' recent *gedanken* experiment he used to infer a bound on the size of the fine structure constant from the GSL, we study a series of variants in which an electric test charge is lowered to a finite radius and then dropped into a Schwarzschild, a near-extremal magnetic Reissner-Nordström or a near-extremal Kerr black hole. For a classical charge, we demonstrate that a specific ''backreaction'' effect is responsible for protecting the area theorem in the near-extremal examples. For the magnetically charged Reissner-Nordström hole an area theorem violation is defused by taking into account a subtle source of repulsion of the charge: the spinning up of the black hole in the process of bringing the charge down to its dropping point. In Kerr hole case, the electric self-force on the charge is sufficient to right matters. However, in all experiments involving an elementary charge, the full GSL would apparently be violated were the fine structure constant greater than about order unity. We argue that in this case a quantum effect, the Unruh-Wald quantum buoyancy, may protect the GSL.

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I. INTRODUCTION

One of the most useful general results in black hole physics is Hawking's area theorem [1]: in the presence of matter and fields that obey the weak positive energy condition, the area of the event horizon of any black hole cannot decrease. This theorem is proved under the assumption that the congruence of the horizon's generators has no caustics in the future, in other words, that the generators do not run into a future singularity. The theorem does not identify specific effects that make it work in particular situations. A venerable application of this theorem is to demonstrate the necessity of superradiance by charged or rotating black holes [2]. The theorem is also closely related to black hole thermodynamics: the generalized second law (GSL) for black holes [3] is an extension of the area theorem that reduces back to it when Hawking's radiation has negligible effect.

Recently, Davies [4] has employed the GSL (and thus thermodynamics) to establish an upper bound on the fine structure constant, and to infer that a magnetic monopoles's spatial extent considerably exceeds its Compton length. He envisages a *gedanken* experiment in which a charged point particle is dropped from rest at infinity into a Schwarzschild black hole. By assuming that the GSL reduces to the area theorem, and consequently demanding that the area of the horizon grow upon ingestion of the charge, Davies concludes that the fine structure constant cannot be larger than unity. This derivation is interesting because of its thermodynamic origin. It immediately raises the question whether variations on the Davies gedanken experiment can yield additional insights into the possible bound and further conclusions of physical interest. To answer this question we examined a comprehensive set of variants designed to ''push the area theorem against the wall'': the dropping of either an extended or point charge *from rest at a finite distance* from a Schwarzschild, Reissner-Nordström (RN), and Kerr hole.

We start in Sec. II by reviewing Davies' argument, and showing that the key assumption is that the dropped particle is pointlike. Davies also assumes the Hawking emission is insignificant, but we point out that in the regime where the bound is obtained, the Hawking radiation entropy cannot actually be neglected in the GSL. To characterize this contribution we modify the experiment, imagining instead a series of point of charges are dropped into the hole. The result is a slightly weakened upper bound still close to Davies' value, provided we are allowed to make the timescale of the dropping appropriately small.

With this gedanken experiment in mind, we consider our new lowering-dropping process for a trio of black hole cases. First, in Sec. III, we allow the dropped object to be an extended classical charge. We start (Sec. III A) by enumerating the conditions that the object's charge, mass and radius must obey in relation to the hole's mass in order that various complications may be avoided. We then consider (Sec. III B) the dropping of the object from rest at a finite distance into a Schwarzschild black hole. We point out that one should take into account not only the gravitational force, but also the repulsive electromagnetic self-force the object is subject to. Analytic arguments show that no violation of the area theorem ensues in this case.

To further challenge the theorem we replace the Schwarzschild black hole by an almost extreme magnetically charged RN one (Sec. III C). Here, a violation of the area theorem would occur for classical charged objects satisfying the weak energy condition, even with the self-force accounted for. But, as made clear numerically, the theorem is saved by the intervention of an extra repulsion. The electric dipole moment induced in the hole by the angular momentum gradually transferred to it repels the charge as it is lowered. As a final example we look (Sec. III D) at the case of a charged object lowered along the symmetry axis of a nearly extreme (neutral) Kerr black hole down to a certain point, and then dropped into it. Here the only known force acting on the charge, apart from gravity, is the self-force. We compute this force anew, and show that if it is not properly accounted for, another inconsistency arises between the area theorem and the weak energy condition.

In Sec. IV, we consider lowering and dropping an elementary charge into our black holes. We allow for the classical "backreaction" effects discussed in the previous section, and employ a large black hole to minimize the effect of the Hawking emission. We find that the GSL, which reduces to the area theorem in this case, seems to imply an upper bound on the fine structure constant $\alpha \lesssim 2$, similar to Davies' bound. Based on these results, must one conclude that black hole thermodynamics truly requires a bound on the magnitude of the electromagnetic coupling in nature? We argue that in this case an entirely *quantum* effect, the average repulsive force on the elementary charge due to Compton scattering of photons from the black hole's "thermal atmosphere," an example of Unruh-Wald buoyancy, should not be neglected, and may be the protector of the GSL.

Section V summarizes our results and discusses possible future work. We also note that in the gedanken experiments where charges are dropped from infinity, strong self-field effects, which were neglected previously, are not negligible and could cause the process to break down. Throughout this paper we use units with $c = G = 1$, but display $\hbar$; hence $\sqrt{\hbar} = m_P$, the Planck mass. Numerical values and electromagnetic relations are stated with electrostatic units in mind.

II. CONSTRAINTS ON FINE STRUCTURE CONSTANT FROM THE GSL?

A. Davies' argument

We first review the argument of Davies [4] for setting constraints on the value of the fine structure constant and on the physical size of magnetic monopoles from the GSL. Davies considered a simple gedanken experiment where a *point* particle of electric (or magnetic) charge g with mass m is dropped from rest at infinity into a Schwarzschild black hole of mass of M. It is assumed that g, $m \ll M$ so that the in-falling test particle moves along the geodesics of the background Schwarzschild geometry. When the particle is absorbed, the black hole becomes an electric or magnetic RN solution. Therefore the change in the horizon cross-sectional area in the process is

$$\delta A = 4\pi(M + m + \sqrt{(M + m)^2 - g^2})^2 - 16\pi M^2. \quad (1)$$

Obviously $\delta A < 0$ whenever $M + m + \sqrt{(M + m)^2 - g^2} < 2M$, from which follows the *exact* criterion for the area to decrease, and for the GSL (more correctly the area theorem) to be violated:

$$g^2/m > 4M. \quad (2)$$

At this point Davies invokes quantum mechanics: the effective "radius" of the particle cannot be smaller than approximately its own Compton wavelength $\lambda_C = \hbar/m$; hence, the black hole diameter $4M$ must be greater than the last in order for the particle to be absorbed by, rather than scatter off the black hole. Therefore, the perceived violation of the area theorem entails

$$g^2/\hbar \gtrsim 1. \quad (3)$$

Accordingly, in Davies' view, the GSL requires that in nature

$$g^2/\hbar \lesssim 1. \quad (4)$$

Should we be incredulous that such fundamental bound on a coupling constant results from thermodynamics? With the electron's charge e in mind, Davies reminds us that a value of the fine structure constant as large as $\alpha = e^2/\hbar \gtrsim 1$ would induce spontaneous electron-positron pair creation in the vacuum [5], and is thus unviable.

Whenever g represents a magnetic monopole charge μ, the Dirac quantization condition leads to the basic constraint

$$\mu e \geq \hbar/2. \quad (5)$$

Now in terms of the observed fine structure constant $\alpha \approx 1/137$, this inequality becomes $\mu^2/\hbar \gtrsim (137/4)$, which clearly contradicts the aforementioned criterion $\mu^2/\hbar \lesssim 1$ for protecting the GSL. To circumvent this problem, Davies proposes that magnetic monopoles are never point particles, but rather extended objects with some effective proper radius $R \gg \hbar/m$. Since $R \lesssim 2M$ for the particle to be absorbed by the hole, he arrives at the new criterion

$$\mu^2 \lesssim mR \quad (6)$$

for the area theorem to hold. Combining this with (5) yields the additional constraint

$$\frac{e^2}{\hbar} \gtrsim \frac{\hbar/m}{R} \quad (7)$$

on the size of the fine structure constants in a world with monopoles.

B. Critique

Davies assumes that the particle being dropped is a point particle, i.e., one whose localization scale is its own Compton length $\lambda_C = \hbar/m$. He further tacitly assumes that the black hole is so massive that production of entropy via the Hawking radiation [6] is completely negligible compared to the black hole entropy change in the process contemplated here. To see that these two assumptions are indispensable for his conclusions, let us imagine a purely classical variant of the argument.

Classically there is no GSL, but only the Hawking area theorem. One should not need to invoke quantum mechanics to prevent the theorem's violation when inequality (2) holds. Let us recall that g^2/m constitutes the "classical radius" r_c of our charged object or particle, the radius the object (regarded as spherical) would have were its rest mass m to be comprised exclusively of electromagnetic energy. Therefore, if we think of our charge as an extended classical object, its effective proper radius R must exceed r_c in order for the object's energy density to be positive everywhere (for an electron in the real world $r_c \approx \lambda_C/137$). If Eq. (2) holds so that $\delta A < 0$, and also $R \lesssim M$ so the object can indeed be absorbed, we find that $r_c \gtrsim R$. But this just means that the object contains regions with negative energy density. Thus, the violation of the area theorem is understandable: the theorem is always proved under the assumption that the weak positive energy condition is respected. By working entirely within classical physics we fail to come up with significant conclusions.

Let us now return to the quantum elementary particle. To be definite we take g to be the electric charge, e. We note that $\lambda_C = r_c/\alpha$. Thus, in a world where $\alpha > 1$, $\lambda_C < r_c$, and we can well ask, what is the actual size of our charge, λ_C as would be suggested by quantum considerations, or r_c as would be required by positivity of the internal energy density? In the former case the condition for the charge to be absorbed by the black hole is $\lambda_C < 4M$ and in the latter $r_c < 4M$. In the first case, we can rework the condition $\alpha > 1$ into the form $e \cdot (e/\lambda_C^2)\lambda_C > m$, which says that the electric field of our charge can impart to a like (virtual) particle in the vacuum an energy exceeding its own rest mass while that particle is a Compton length or so away. In the second case, $\alpha > 1$ is equivalent to $e \cdot (e/r_c^2) r_c > m$ so that, again, our charge can impart to a like virtual charge in the vacuum an energy greater than its rest mass while the virtual charge is a distance r_c away.

Thus, under both assumptions about the size of our charge, we would expect it to strongly polarize (if not outrightly break down) the vacuum adjacent to it. But then the expectation value of the stress-energy operator $\hat{T}_{\mu\nu}$, which appears as the source in the semiclassical Einstein equation,

$$G_{\mu\nu} = 8\pi\langle\hat{T}_{\mu\nu}\rangle, \tag{8}$$

need no longer satisfy the weak energy condition in the particle's neighborhood. This weakens the basis for the validity the area theorem in the situation we have in mind. Recall that in the Hawking radiation process, vacuum polarization opens a quantum loophole in the area theorem, and this is just what allows the black hole to evaporate and decrease its horizon area. Furthermore, in the first case the charge is smaller than its classical radius, which means, classically speaking, that its interior has negative energy density somewhere. For all these reasons one cannot, *a priori*, rely on the validity of the area theorem when $\alpha > 1$.

However, one can appeal to the presumably more reliable GSL,

$$\delta S_{\rm ext} + \tfrac{1}{4}\hbar^{-1}\delta A \geq 0, \tag{9}$$

namely, that if the black hole area decreases, the decrease must be more than compensated by generation of a suitable amount of radiation entropy. Davies tacitly assumes that the term $\delta S_{\rm ext}$ can be neglected here, so that the GSL reduces to the area theorem. One might think this would be so for a sufficiently massive black hole, in which case Hawking radiance would be suppressed. Let us check. By expanding Eq. (1) for e, $m \ll M$ we obtain the black hole entropy change contributed by the particle's in-fall alone:

$$\tfrac{1}{4}\hbar^{-1}\delta A \approx (2\pi/\hbar)(4Mm - e^2). \tag{10}$$

During the particle's disappearance, the hole radiates. Approximating this as emission of thermal radiation from a sphere of radius $2M$ at temperature $\hbar(8\pi M)^{-1}$, we find that the rate of thermal entropy emission is $\dot{S}_{\rm rad} \approx (1920M)^{-1}$. This is a contribution to $\delta S_{\rm ext}$. The emission induces a decrease in black hole entropy, which scales with M just as does the radiation entropy, and whose rate is known to be somewhat smaller in magnitude than $\dot{S}_{\rm rad}$; it makes a contribution to δA. Both together amount to

$$\dot{S}_{\rm BH} + \dot{S}_{\rm rad} = \gamma(1920M)^{-1}, \tag{11}$$

where the factor γ is dimensionless, positive, and of order unity. Putting together all above contributions we may write Eq. (9) as

$$\frac{\gamma\Delta t}{1920M} + \frac{2\pi(4Mm - e^2)}{\hbar} \geq 0, \tag{12}$$

where Δt is the global time that elapses as the charge goes down the hole.

What to take for Δt? For one charge it is a somewhat ambiguous quantity. So it helps to consider a series of like charges being dropped sequentially from rest at global time intervals Δt, with their starting points distributed around a sphere at large distance from the black hole. By the stationary character of Schwarzschild's spacetime, the charges will arrive at a specified radius r near the horizon also separated by time intervals Δt. Davies' bound comes from assuming that the charge is spatially almost as big as the black hole. Therefore, when one charge is just beginning to be absorbed by the black hole (i.e., its "center of mass" is about proper distance R from the horizon), its spatial separation from the subsequent charge must be at least R. This is the minimum criterion for the charges to avoid bumping into one another. In the Schwarzschild metric the rate of change of the proper distance from the horizon ℓ for a freely falling particle is

$$d\ell/dt = -\sqrt{(2M/r)(1 - 2M/r)}. \tag{13}$$

This formula can be used to calculate the time separation Δt. However, since r should be taken somewhat larger than $2M$, we can simply parameterize Δt as

$$\Delta t = \kappa R, \tag{14}$$

with κ at least a few times unity. We think of κ as fixed.

In our first case ($R = \hbar/m$), we have $\Delta t = \kappa\hbar/m$, so that Eq. (12) gives

$$\alpha \leq \frac{4Mm}{\hbar} + \frac{\gamma\kappa\hbar}{3840\pi Mm}. \tag{15}$$

It is immediately clear that large M is not most propitious for deriving the tightest bound on α. Let us instead minimize the r.h.s. of Eq. (15) with respect to M,while respecting Davies' requirement that $\hbar/m < 4M$. We get

$$\alpha \leq \max\left(1 + \frac{\gamma\kappa}{960\pi}, \sqrt{\frac{\gamma\kappa}{240\pi}}\right). \tag{16}$$

If indeed κ can be made just a few times unity, the upper bound on α is quite close to unity, as Davies contends. In the second case ($R = e^2/m = \alpha\hbar/m$), the appropriate parametrization is $\Delta t = \kappa\alpha\hbar/m$. This will transform Eq. (12) into

$$\left(1 - \frac{\gamma\kappa\hbar}{3840\pi Mm}\right)\alpha \leq \frac{4Mm}{\hbar}. \tag{17}$$

The additional demand that $e^2/m < 4M$, or $4Mm/\hbar > \alpha$, automatically causes Eq. (17) to be satisfied for all values of κ. So in this case the GSL sets no bound whatsoever on α.

We conclude that if, in a world with $\alpha > 1$, an elementary charge's spatial extent is its Compton length, then Davies' bound on α, or one very close to it, can indeed be derived from the GSL. This is contingent on existence of an appropriate charge dropping procedure with a κ not very large on scale unity, one without instabilities of the train of charges due to their Coulomb repulsion. It is also contingent on resolution of a quandary to be raised in Sec. V. If when $\alpha > 1$ an elementary charge is as large as its classical radius, no bounds on α can be derived from the GSL.

III. LOWERING A CLASSICAL CHARGE AND THEN DROPPING IT

Given the above caveats on Davies' argument, can variants of the Davies' gedanken experiment yield stronger constraints on α, or even new conclusions? We first explore a variant that focuses on a classical charged object that is dropped from the vicinity of the black hole instead of from infinite distance. The new scenario is advantageous since the conserved Killing energy of the object, when ultimately dropped, is smaller than that pertaining to the object dropped from rest at spatial infinity. The consequent reduction of δA is more likely to challenge the area theorem. Furthermore, the use of a classical object allows us to escape the complications of the analysis of Sec. II B. Given that Davies' bound emerges when the black hole radius is almost as small as the elementary particle's, it would make little sense there to ignore Hawking's radiance, which is the more intense the smaller the black hole. Thus, since vacuum polarization issues force us to trade area theorem for GSL, we cannot avoid dealing with complications due to the radiation. By switching to a classical charged object, we no longer need to consider black holes with microscopic radius, and so we may hope the radiation plays a negligible role. We also explore whether use of RN or Kerr black holes, instead of Schwarzschild ones, can expose new things. An allied question we ask is whether there are any other forces, apart from gravitation and Coulomb ones, which are crucial for the outcome of the gedanken experiment.

A. Setup

We imagine that a classical object with charge q is first brought to rest at some finite radius from the horizon of a black hole, which we take of Schwarzschild, magnetic RN, or Kerr type, and then dropped freely. How is the charged object deposited at a finite radius? It could be lowered adiabatically from infinity with an appropriately designed apparatus. One possibility for this, discussed for the Schwarzschild case in Ref. [7], is a conical cable consisting of radial fibers filling the portion of space defined by some solid angle with vertex at the black hole. This cable can be modeled by the stress tensor $T^\mu{}_\nu = \mathrm{diag}[-\rho(r), S(r), 0, 0]$, where $\rho(r)$ is the proper density and $S(r)$ the tension. It turns out that this stress tensor respects the weak energy condition $|S|/\rho \leq 1$; therefore, the cable need not snap under tension at a finite radius from the hole [7] (see also our Appendix B for a demonstration that a properly constructed cable also will not snap in the Kerr spacetime). We consider the adiabatic lowering process to be on a time scale τ much longer than the inverse surface gravity κ^{-1} of the black hole. Thus, the change in horizon area in the course of the lowering process will be negligible [8].

In what follows, we treat the object as a uniformly charged sphere of proper radius R. We compute in the test particle approximation. In addition to the obvious requirement that m, $q \ll M$, this approximation implies three important additional conditions.

(1) The object's radius R should be greater than its classical radius, but much smaller than the black hole's radius (P shall denote magnetic monopole and a the specific angular momentum): $q^2/m < R \ll M + \sqrt{M^2 - P^2 - a^2}$.

(2) The magnitude of the electromagnetic energy-momentum tensor in the object's immediate vicinity, $|T_{\rm particle}| \sim q^2/R^4$, should be much less than the maximum background spacetime curvature $\sim 1/M^2$ due to the black hole. Thus, $R \gg \sqrt{qM}$.

(3) The magnitude of the object's interior energy-momentum tensor should be much less than the black hole's maximum curvature in its exterior. Thus, $R \gg m^{1/3}M^{2/3}$.

With condition 1 we insure that the object to be absorbed has positive energy density. The additional requirement $R \ll M$ makes the object "able to fit into the black hole" and smaller than the scale set by the curvature, so that finite size effects can be neglected. Conditions 2 and 3 keep the "backreaction" due to the distortion of the spacetime by the stress energy of the object itself negligible, so that the latter's equation of motion is, to good approximation, the same as in the fixed background geometry of the black hole.

B. Electric charge or magnetic monopole into Schwarzschild black hole

Our first example deals with an electrically charged object dropped from some distance into a Schwarzschild black hole. The first issue to resolve is the criterion that prevents the charge from breaking down the vacuum in its neighborhood (Schwinger effect) and thus neutralizing itself. For the breakdown to be suppressed, the object's electric field just outside it, $\mathcal{E}$, must be well below the critical field $m_e^2/(e\hbar)$, where m_e is the electron's mass and e is the elementary charge. Now by condition 2 of Sec. III A, $\mathcal{E} = q/R^2 < 1/M$. We must thus, in addition to the three conditions, require a minimum black hole mass: $M > e\hbar/m_e^2 = \alpha^{1/2}m_P^3/m_e^2 \approx 1.05 \cdot 10^{39}$ g (the numerical value assumes the real values of the fundamental constants). The Hawking radiation of such massive black hole is indeed negligible. And the requirement $m \ll M$ still permits the dropped object to contain many elementary particles and to be thoroughly classical.

The change in area during the entire process is given by Eq. (1) with m replaced by the conserved Killing energy E of the object,

$$\delta A = 4\pi(M + E + \sqrt{(M+E)^2 - g^2})^2 - 16\pi M^2. \quad (18)$$

During the assumed adiabatic lowering process, A shall be unchanged. In addition, we assume the relation $A_i = 16\pi M^2$ to remain valid as the particle is being lowered. This just means that M, the parameter governing the *background* metric, is taken as constant. Of course the active gravitational mass of the system changes during the lowering; this corresponds to a metric perturbation of relative order $O(m/M)$, which is then reflected in a commensurate correction to the object's conserved energy. But this last only leads to higher order corrections to δA. Therefore, we assume

$$E = -p_\mu \xi^\mu, \quad (19)$$

with canonical momentum p_μ derived from the effective Lagrangian of the object moving on the fixed curved background, and $\xi^\mu = (\partial/\partial t)^\mu$ is the timelike Killing vector. In our case

$$E = -mu_t - \tfrac{1}{2}qA_t^{\rm self}. \quad (20)$$

The first term is the fourth component of the four momentum of the particle; the second is the contribution of the object's electromagnetic vector potential to its own global energy [9,10].

The self-interaction of a charged object reflects the fact that Huygens' principle is not satisfied in curved spacetime. This means radiative fields propagate not only on characteristic surfaces, but also inside the light cone as backscattered "tails." This effect produces a nonlocal force term that depends on the metric and past history of the object. Even when the last is static, the background curvature distorts its electromagnetic field, which in turn "backreacts" on it. An alternative intuitive picture of the self-interaction is that it is due to a surface charge density induced on the black hole horizon, which acts like a conductor. However, this last intuition does not capture the situation entirely: the interaction turns out to be *repulsive* and not, as the analogy would suggest, attractive.

Since the object is initially nearly stationary at, say, $r = r_0$, its four velocity (which is normalized to unity) is $u^a = \xi^a/|\xi|$, where $|\xi| \equiv \sqrt{-\xi_\mu\xi^\mu|_{r=r_0}}$. $A_t^{\rm self}$ was calculated in Refs. [9,10] and others (see Appendix A for details) and found to be

$$A_t^{\rm self} = -\frac{Mq}{r_0^2}, \quad (21)$$

so that

$$E = m\sqrt{1 - 2M/r_0} + \frac{1}{2}Mq^2/r_0^2. \quad (22)$$

Now, according to criterion (2) with $m \to E$, the exact condition for a violation of the area theorem here would be

$$q^2 > 4mM\sqrt{1 - 2M/r_0} + 2M^2q^2/r_0^2. \quad (23)$$

To make this more transparent, let us express the dropping coordinate radius r_0 in terms of the proper distance ℓ_0 from the horizon,

$$\ell_0 = \int_{2M}^{2M+\epsilon}(1 - 2M/r)^{-1/2}dr. \quad (24)$$

If we assume $\epsilon = r_0 - 2M \ll 2M$, so that we are close to the horizon in coordinate sense, we find $\sqrt{1 - 2M/r_0} \approx \ell_0/4M$ so that criterion (23) for the area theorem violation becomes

$$q^2/m \gtrsim 2\ell_0. \tag{25}$$

Now the object can be lowered down to the hole only until its center of mass lies a proper distance $\ell_0 \approx R$ from the horizon, where it begins to be absorbed by the hole hole. Therefore, since we require $\ell_0 \gtrsim R$ for the lowering and dropping process, criterion (25) is inconsistent with condition 1 of Sec. III A. Were we to neglect the self-interaction contribution to the energy, the resulting criterion, $q^2/m \gtrsim \ell_0$, would still be inconsistent with condition 1. Hence, no violation of the area theorem can occur when a classical electrically charged sphere is lowered toward and then dropped into a Schwarzschild black hole. By duality, this conclusion is unchanged if we lower and then drop a magnetically charged sphere.

C. Electric charge into magnetic Reissner-Nordstrom black hole

Now we calculate the horizon area change when an electrically charged spherical object is lowered toward and then dropped into a *near-extremal* RN black hole with mass M and magnetic charge P [11]. Our motivation for considering this case is that the spatial geometry becomes throatlike for near-extremal black holes, allowing the object to have smaller Killing energies at given proper distance from the horizon. And in contrast to the case of an electric charge assimilated by an electric RN, here there is no direct repulsive interaction between the electric charge and magnetic monopole to contribute to the conserved energy of the object.

We must still require $M > \alpha^{1/2} m_P^3/m_e^2 \approx 1.05 \cdot 10^{39}$ g so that the charge q shall cause no vacuum breakdown. Since the black hole is magnetically charged, it cannot cause vacuum breakdown, but it may polarize the neighboring vacuum with its near magnetic field P/M^2. The polarized vacuum would affect the electrodynamics and further complicate the already complex situation we analyze. Such polarization will be suppressed when $P/M^2 < m_e^2/(e\hbar)$. Since we are assuming $P \approx M$, we see that the lower bound we already required for M automatically suppresses vacuum polarization near the hole.

The exterior geometry is described by the line element and vector potential

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(\sin^2\theta d\theta + d\phi^2), \tag{26}$$

$$A_t = P/r, \tag{27}$$

where $f(r) = 1 - \frac{2M}{r} + \frac{P^2}{r^2}$. Now, as before, we imagine that by some mechanism the object with electric charge q and mass m is slowly lowered from spatial infinity to a fixed radius r_0 outside the black hole, but near the horizon radius, $r_+ = M + \sqrt{M^2 - P^2}$.

Without loss of generality we assume the object is lowered down the polar axis (z axis) to r_0. Since the configuration contains a magnetic monopole (in the magnetically charged black hole) and an electric charge, the electromagnetic and gravitational (in a curved spacetime) fields together possess a conserved angular momentum $L_z = qP$ along the z axis, which is independent of the distance between the object and black hole [12,13]. When the object is at spatial infinity, all this angular momentum is contained in the electromagnetic field, but during the lowering process, as the electromagnetic field lines penetrate the horizon, the angular momentum is transferred to the gravitational field, and the hole begins to rotate [12,13]. We again assume the lowering process is adiabatic, so that during this process the change in the horizon area will be negligible [14]. Once the particle reaches r_0, it is allowed to fall freely (radially) into the black hole.

The change in the area is approximately

$$\delta A \approx 4\pi\left(\left(\frac{qP}{M+E}\right)^2 + (M+E) + \sqrt{(M+E)^2 - P^2 - q^2 - \left(\frac{qP}{M+E}\right)^2}\right) - 4\pi(M + \sqrt{M^2 - P^2})^2, \tag{28}$$

where we have assumed the final state is a rotating, charged (Kerr-Newman) solution with angular momentum qP and mass $M + E$. The energy E contributed by the object is

$$E \approx m\sqrt{1 - 2M/r_0 + P^2/r_0^2} + \frac{Mq^2}{2r_0^2} + q\hat{A}_t, \tag{29}$$

where we ignore the $O(z^2)$ contributions of the hole's rotation to the redshift factor and self-energy.

The self-energy correction in a RN spacetime has the same form as in Schwarzschild spacetime [15,16]. The new term $\hat{A}_t$ originates from the repulsion of the charge q by the electric dipole induced by the rotation of the magnetically charged black hole. Since the charge is brought near the horizon, we may assume that the angular momentum parameter of the black hole is $a \approx qP/M$. For a magnetic Kerr-Newman black hole, the background A_t evaluated on the symmetry axis $\theta = 0$ is [17]

$$q\hat{A}_t = \frac{qPa}{r_0^2 + a^2} = \frac{q^2P^2}{M(r_0^2 + a^2)} \approx \frac{q^2P^2}{Mr_0^2}. \tag{30}$$

The proper distance to the horizon is given by integral

$$\ell_0 = \int_{r_+}^{r_0} (1 - 2M/r + P^2/r^2)^{-1/2} dr, \tag{31}$$

which has the analytic form

$$\ell_0 = \sqrt{r_0^2 - 2Mr_0 + P^2} + M\cosh^{-1}\left(\frac{r_0 - M}{\sqrt{M^2 - P^2}}\right). \tag{32}$$

If $r_0 - r_+ \ll r_+$, we can expand in a Taylor series as in Sec. III B. However, as the hole becomes more and more extremal ($\sqrt{M^2 - P^2} \to 0$), the case of most interest here, this approximation for the proper distance begins to break down.

Consequently, we calculated δA in Eq. (28) numerically for a wide range of values of P/M, m/M, q/M, and r_0/M. *In units with* $M = 1$ the range studied was $0.9 \leq P \leq 0.99\,999$, $10^{-12} \leq m \leq 10^{-3}$, $10^{-4}\sqrt{m} \leq q \leq \sqrt{m}$, and $1.00\,001 \leq r_0/r_+ \leq 1.05$. The upper end of the range of q is dictated by the condition that the classical radius of the object, q^2/m, be smaller than the black hole radius (approximately unity). For the smaller P this condition is enough to insure that the energy E monotonically increases with r_0 for $r_0 > r_+$, which means that the object is always attracted by the black hole, and can indeed be lowered all the way to the horizon. As P grows, the $E(r_0)$ curve develops a minimum near the horizon (with consequent repulsion inside this radius), but even as $P \to 1$, this does not happen for $q^2/m < 1/3$. Since the object cannot be entirely made up of electromagnetic energy (as it would then be unstable), its radius should well exceed q^2/m. Thus, for an object small enough to be dropped into the black hole, the possibility of its lowering being arrested by the said effect can be discounted.

The m range was sampled at ten values, each a power of 10 greater than the previous one; the ranges of the other variables were sampled 100 times each, with jumps of progressively growing (or decreasing) size in order to maximize coverage of interesting regions. For example, the region near $P = 1$ was more finely sampled than that near $P = 0.9$, and that near $r_0 = r_+$ was covered more finely than that near $r_0 = 1.1r_+$. For each point of the above grid we used Eq. (32) to calculate the proper distance from $r = r_0$ to the horizon. The overall range was $0.01165 \lesssim l_0 \lesssim 3.2499$. From the above grid of values we discarded points in conflict with conditions 1–3 of Sec. III A. Specifically, since we never specified R, the particle's radius, we just demanded that ℓ_0 exceed each of the quantities q^2/m, $\sqrt{q}$ and $m^{1/3}$. For all parameter combinations meeting these physical requirements for dropping a finite-sized charge into the black hole without disturbing it strongly, we found that $\delta A > 0$ and so the area theorem is respected.

We stress that for this to be true, the electric dipole-charge interaction in Eq. (29) *must* be included. For example, with it left out for $P = 0.99\,999$, $m = 10^{-7}$ and $q = 10^{-4}$, we find that $\delta A < 0$ for $r_0/r_+ < 1.058$. At this critical r_0, $\ell_0 \approx 7.919$ whereas $\{q^2/m, \sqrt{q}, m^{1/3}\} = \{0.1, 0.01, 0.0046\}$. Thus, with the object radius obeying $0.1 \ll R \ll 7.9$, conditions 1–3 are satisfied and we have a violation of the area theorem. (When only the self-force energy is included, the $E(r_0)$ curve has no minimum whatsoever for $q^2/m < 1$.)

We do not find evidence that the self-interaction term is necessary for satisfaction of the area theorem (this was also the case for the Schwarzschild black hole). Actually, when that term is neglected, and for P fairly close to M, there are formal violations of the area theorem for r_0 in the region where $E(r_0)$ rises with r_0, but these always occur for $l(r_0)$ just marginally larger than q^2/m (7% larger in the extreme case). In view of our above remarks about the relation between R and q^2/m, these cases cannot be counted as physical violations. We conclude that for RN black holes the area theorem is protected by the electric dipole-charge interaction.

At the risk of repetitiousness we mention that analogous conclusions apply to the case where a *magnetically* charged sphere is lowered and then dropped into an *electrically* charged RN black hole. In that case, the area theorem is protected by the magnetic dipole-monopole interaction. Breakdown of the near vacuum by the black hole's Coulomb field and polarization of the vacuum by the sphere may both be avoided with the same lower bound on M stipulated earlier in this section.

D. Electric charge into Kerr black hole

Our third example concerns an object bearing charge q, which is lowered along the symmetry axis of a neutral, nearly extremal Kerr black hole down to a point where its Killing energy is E, and then dropped in. The extremal throatlike geometry permits one to challenge the area theorem to the utmost. Let M be the mass and $a = J/M$ the angular momentum per unit mass of the initial black hole. Again, problems stemming from vacuum breakdown by the object's Coulomb field may be obviated by requiring $M > \alpha^{1/2} m_P^3/m_e^2 \approx 1.05 \cdot 10^{39}$ g.

When necessary, we refer to the Boyer-Lindquist radial coordinate r. In the envisaged process the angular momentum of the hole is unchanged. Thus, the initial parameter a is transformed into $a(1 + E/M)^{-1}$ so that

$$\delta A = 4\pi(r_f^2 - r_+^2 + a^2(1 + E/M)^{-2} - a^2). \tag{33}$$

Here, $r_f = M + E + \sqrt{(M + E)^2 - q^2 - a^2(1 + E/M)^{-2}}$, and $r_+ \equiv M + \sqrt{M^2 - a^2}$ is the initial Kerr horizon radius. The conserved Killing energy for a particle on the symmetry axis is

$$E = m\sqrt{\frac{r_0^2 - 2Mr_0 + a^2}{r_0^2 + a^2}} - \frac{1}{2} q A_t^{\rm self}. \tag{34}$$

In Appendix A we calculate anew the self-interaction term for an object on the polar axis following the method of [9] and find

$$-\frac{1}{2} q A_t^{\rm self} = \frac{Mq^2}{2(r^2 + a^2)}. \tag{35}$$

Then we compare this with earlier results of Léauté and Linet [18], of Lohiya [16], and of Piazzese and Rizzi (PR) [19].

Just as in Sec. III C we here evaluate formula (33) numerically using specific values of m, q, a (in units of $M = 1$), and numerically integrating

$$\ell_0 = \int_{r_+}^{r_0} \sqrt{\frac{r^2 + a^2}{r^2 - 2Mr + a^2}} dr \tag{36}$$

(no analytic form is known) to find the proper distance from the horizon.

The range studied was $0.9 \le a \le 0.99\,999$, $10^{-12} \le m \le 10^{-3}$, $10^{-4}\sqrt{m} \le q \le \sqrt{m}$, and $1.000\,005 \le r_0/r_+ \le 1.05$. The upper end of the range of q was chosen by the consideration set forth in Sec. III C. We find numerically that for $q^2/m \le 1$ the energy $E(r_0)$ monotonically increases with r for $r > r_+$, so that the charge is always attracted to the black hole, and can be lowered all the way to the horizon.

The m range was sampled at ten values, each a power of 10 greater than the previous one; the ranges of the other variables were sampled 100 times each, with jumps of progressively growing (or decreasing) size so that the region near $a = 1$ was more finely sampled than that near $a = 0.9$, and that near $r_0 = r_+$ was covered more finely than that near $r_0 = 1.05r_+$. From the above grid we again discarded points for which ℓ_0 did not exceed each of the quantities q^2/m, $\sqrt{q}$ and $m^{1/3}$, thus securing compliance with conditions 1–3. We uncovered no violations of the theorem for the residual points.

When the self-force energy, Eq. (35), is not included, violations of the area theorem are found. For example, with $m = 3 \times 10^{-8}$, $q = 2 \times 10^{-5}$ and $a = 0.9999$, $\delta A < 0$ when $r_0 < 1.0145$, that is, when the object is dropped a proper length $\ell_0 < 0.3199$ from the horizon. Since $\{q^2/m, \sqrt{q}, m^{1/3}\} = \{0.0133, 0.00\,447, 0.00\,311\}$, if the size of the object obeys $0.0134 < R \ll 0.31$, conditions 1–3 are met, albeit marginally. The self-force is obviously an essential part of the workings of the area theorem.

We also examined the situation when the object's energy was calculated using the alternative form for the self-force (Eq. (A12) below) worked out by PR [19], to which corresponds the energy correction

$$E_{\rm PR} = \frac{q^2(M + r_0)}{2(r_0^2 + a^2)} + \frac{1}{2}\frac{q^2}{a}\arctan(r/a) - \frac{\pi q^2}{4a}, \tag{37}$$

instead of that in Eq. (35). We easily found violations of the area theorem. For the same parameters a, e, and m as above, we find that $\delta A < 0$ when $r_0 < 1.0144$, that is, when the object is dropped a proper length $\ell_0 < 0.2718$ from the horizon. Obviously conditions 1–3 can be satisfied in this case for $0.0134 < R \ll 0.271$. These and similar violations of the area theorem constitute independent evidence that the PR formula for the self-force is incorrect.

IV. LOWERING AN ELEMENTARY CHARGE

In this section we suppose that the above gedanken experiments of Sec. III are carried out with an elementary particle of mass m and charge e. To be concrete we work in the Schwarzschild case, where we can make use of the analytic results in Sec. III B.

The criterion (25) for $\delta A < 0$ translates into

$$e^2/m = \alpha \cdot \hbar/m \gtrsim 2\ell_0. \tag{38}$$

As discussed earlier, in a world where $\alpha > 1$ the spatial extent of the particle might be given by its classical radius e^2/m. In such eventuality the last criterion could not be satisfied since it requires the still suspended particle's center to be closer to the horizon than its own radius. The conclusion might be that no violation of the area theorem occurs, but clearly one cannot draw any bound on α.

If instead the particle's effective size is about a Compton wavelength, its center cannot be deposited closer to the horizon than that distance, so $\ell_0 \gtrsim \hbar/m$. Thus, violation of the area law would entail $\alpha \gtrsim 2$. Assuming Hawking radiation effects can be neglected, we may conclude that the area theorem implies the bound

$$\alpha \lesssim 2. \tag{39}$$

(In the RN and Kerr cases this bound should also be of order unity when we include the dipole repulsion and self-forces.) Apart from the factor 2, this is bound (4) inferred by Davies from the freely falling charge experiment. The weaker constraint here results from our inclusion of the self-force energy. Note that Davies would also have obtained $\alpha \lesssim 2$ had he demanded that the black hole *radius* exceeds the Compton length.

Now as we stressed in Sec. II B, in a world where $\alpha \gtrsim 1$, the area theorem is unreliable because the charge will polarize the vacuum, so that the expectation value of the stress-energy tensor operator need not then satisfy the weak energy condition. In an attempt to settle the issue here, we again appeal to the more reliable GSL and use Eq. (11) for the net contribution of the Hawking radiation to the change in total entropy (black hole plus exterior). We repeat the derivation of Eq. (12), but this time with m replaced by the E of Eq. (22) with $\sqrt{1 - 2M/r_0} \approx \ell_0/4M$ and $r_0 \approx 2M$. We find that the GSL requires

$$\frac{\gamma \Delta t}{1960M} + \frac{\pi}{\hbar}(2\ell_0 m - e^2) \ge 0. \tag{40}$$

What to take for the disappearance time Δt? We are allowed to lower down the charge to within roughly a Compton wavelength from the horizon, so that $\hbar/m$ is a lower bound on Δt. And because M sets the scale of the gravitational field, a geneous upper bound should be a few times M. We conclude that the first term in Eq. (40) is small compared to unity, and thus negligible compared to $2\pi\ell_0 m/\hbar$. Hence, the GSL gives $e^2/\hbar \lesssim 2\ell_0/\lambda_C$; since we can arrange for ℓ_0 to approach λ_C, we recover inequality (39).

Are there any relevant effects that might change this conclusion? One possibly important effect is Unruh-Wald or quantum buoyancy [20]. This buoyancy comes about because a suspended, hence accelerated, object perceives the quantum vacuum outside a black hole as an "atmosphere" of thermal radiation. Since the local temperature at the bottom of the object is higher than that at its top (due to the varying redshift factor), the associated pressure gradient exerts an outwardly directed force on the object. Consequently, the conserved energy $E(r_0)$ receives an extra positive contribution from the work done by the buoyancy on the object. In the classical object examples in Sec. III this contribution is negligible, but in any case makes it more difficult to violate the area theorem.

For an elementary point charge in the thermal atmosphere the fluid description of radiation that underlies Unruh and Wald's original calculation of buoyancy [20] must be replaced by one based on the momentum transfer to the charge due to Compton scattering of atmosphere photons. Thus, it is not obvious whether the neglect of this effect is also similarly justified for the point particle case. In Ref. [21] one of us calculated, in Schwarzschild spacetime, the average repulsive force measured at infinity on a suspended elementary charge due to Compton scattering finding

$$\mathbf{f} = \frac{\hbar\hat{\mathbf{z}}}{270\pi M}\frac{\ell_0^2 r_c^3}{(\ell_0^2 - r_c^2)^3}, \tag{41}$$

where $\hat{\mathbf{z}}$ is a unit vector pointing away from the horizon in an orthonormal frame centered at the particle. For $\alpha \ll 1$, $\ell_0 > \lambda_C \gg r_c$ and one can rewrite (41) as

$$\mathbf{f} \approx (2/135\pi)\alpha^3(\lambda_C/\ell_0)^4\mathbf{f}_{\text{grav}}, \tag{42}$$

where $\mathbf{f}_{\text{grav}} = \hat{\mathbf{z}}m(4M)^{-1}$ is the gravitational force as measured at infinity. Thus, for the observed $\alpha \approx 1/137$, the repulsive force is clearly negligible compared to the gravitational attraction since $\ell_0 > \lambda_C$ during the lowering.

However, for our case of interest, $\alpha \gtrsim 1$, Eq. (41) is unreliable because it is based on a leading order (tree level in α) computation of the differential scattering cross section, a procedure not justified when QED is strongly coupled. It would be useful to determine the momentum transfer due to Compton scattering to higher order, but this calculation is beyond the scope of this paper. Instead, let us try gain some insight by naively extrapolating Eqs. (41) and (42) into the strongly coupled regime $\alpha \sim 1$. Equation (42) suggests the repulsion is *not* strongly suppressed compared to the gravitational attraction when $\ell_0 \sim \lambda_C$, while Eq. (41) suggests that the repulsive force actually diverges as the dropping distance ℓ_0 approaches a Compton length from the horizon.

Though not definitive, this argument suggests that the interaction of the elementary particle with the thermal atmosphere makes a significant contribution to the energetics of the lowering process when $\alpha \gtrsim 1$. Because of this contribution the charge will probably reach a floating point near the horizon [20,21]. At any rate, the change in energetics will weaken the implied constraint on the fine structure constant to an extent we cannot determine with the facts available. It is even possible that by preventing GSL violation, the buoyancy may, in fact, remove the constraint completely.

V. SUMMARY AND DISCUSSION

We reassessed the gedanken experiment used by Davies [4] to set constraints on the value of the fine structure constant $\alpha = e^2/\hbar$ and on the spatial extent of magnetic monopoles. We showed that in order for constraints to follow from the area theorem, one must assume the object dropped from infinity is a quantum "point" particle whose localization scale is its Compton wavelength $\lambda_C = \hbar/m$. If instead an extended classical charged object is dropped, violation of the area theorem transpires only when the object violates the weak energy condition. Hence, the only thing we learn in this case is the obvious requirement that realistic classical objects must obey the classical energy conditions. However, for an intrinsically quantum elementary charge, the violation of Davies' constraint $e^2 < \hbar$ implies that the charge strongly polarizes the surrounding vacuum, raising doubts as to whether the weak energy condition is maintained.

Therefore, we are forced to fall back on the full GSL. To analyze the effect of the Hawking contribution to the entropy change, δS_{ext}, we imagined the gedanken experiment to be performed by dropping a series of particles into the hole at a certain rate. If the (strong) assumption is made that no instabilities arise when the dropping rate is tuned to the maximum value required by the gedanken experiment, Davies' bound on α is only moderately weakened.

With the Davies experiment in mind we then examined a new class of gedanken experiments in which classical charged test objects are lowered toward and then dropped into black holes of Schwarzschild, RN, or Kerr type. It turns out that this apparently simpler type of experiment does tell us something interesting about black hole physics. When an electrically (magnetically) charged object is dropped into a near-extremal magnetic (electric) RN black hole, the subtle correction to the object's Killing energy due to the electric (magnetic) dipole repulsion turns out to be crucial for consistency between the area theorem and the weak energy condition. We show numerically that when this effect is included, the black hole area decreases only in cases where the object is more compact than its classical radius, under which circumstance the area theorem is expected to fail anyway. Similarly, when a charged object is lowered along the symmetry axis of a Kerr black hole and then dropped in, it is the electric self-force on the particle that prevents violation of the area theorem.

Thus the classical area theorem is only valid for near-extremal black holes when backreaction effects like self-forces and the dipole repulsion in the RN case are properly taken into account. A naive analysis that only considers the gravitational and Coulomb forces can disclose apparent violations. In this respect it is interesting to note that similar backreaction effects also seem to play an important role in maintaining the cosmic censorship hypothesis when one attempts to destroy the horizon by overcharging or over-spinning a near-extremal black hole. In the test particle limit cosmic censorship can apparently be violated in some cases by overcharging [22], but it appears that higher order backreaction effects prevent the horizon's destruction, and the concomitant violation of the GSL [23].

In Sec. IV we considered gedanken experiments where an elementary particle is lowered to nearly a Compton wavelength away from the horizon and then dropped in. We obtain what seems to be a very robust bound on α. On the other hand, we showed that a neglected quantum effect, the Unruh-Wald quantum buoyancy felt by the charge, as it is held at rest in the "thermal atmosphere" near the horizon, is likely to make a significant contribution to its conserved energy. This effect could protect the GSL at large α, and make the bound unnecessary. Again, it is interesting to note that the Unruh-Wald buoyancy has also been shown to be key in upholding cosmic censorship and preventing the destruction of the black hole horizon in a similar experiment where an electric charge is lowered to an electric RN hole [24].

In light of the above, let us reconsider the bound on α that follows from Davies' original experiment. There the particle is dropped from rest at infinity and would seem to freely fall into the hole, so that the thermal atmosphere should have no effect on it. Note however that the radiation reaction on the charge due to self-field effects was neglected by Davies (who correctly notes that any radiated energy that escapes entering the hole will only make the bound on α the stronger). This neglect is actually justified only when the classical radius of the particle is much smaller than the radius of curvature of the background spacetime: $e^2/m \ll M$ [22]. But the Davies argument requires $e^2/m = \alpha \cdot \hbar/m \sim 4\alpha M$. Thus, if we want to allow *a priori* the possibility that $\alpha > 1$, radiation reaction may be important. One effect of it would be to make the charge move in a radial nongeodesic trajectory. Because the charge would then be accelerating, some variant of quantum buoyancy might act on it (although not exactly Unruh-Wald buoyancy from thermal radiation). Near the horizon the new buoyancy may be strong, so it may actually stop the charge and make it rebound outward. It is beyond the scope of this paper to clarify this issue by calculations.

Thus, in both methods for dropping the elementary charge, subtle effects—quantum buoyancy in one case and radiation reaction together with quantum buoyancy in the second—"muddy the waters," and it is not at all clear whether any bound on the fine structure constant is required by the GSL. In view of the lessons of Sec. III in the classical regime, it is not unreasonable to speculate that when all effects are properly considered, the full GSL will be found to be respected without the need for a fundamental bound on α. As in its early history, the black hole area theorem may help us to understand new physical effects.

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APPENDIX A: CALCULATION OF THE SELF-ENERGY IN KERR GEOMETRY

Here, we extend the method of [9] to calculate the self-interaction correction

$$E_{\rm corr} = -\tfrac{1}{2} q A_t^{\rm self} \tag{A1}$$

to the conserved energy of a stationary test object on the symmetry axis of the Kerr geometry. The procedure starts with Maxwell's equations for the vector potential A_μ with a point charge source at $r = r_0$ on the axis. Fortunately, an analytic solution was found by Léauté [25], who was able to extend the earlier solution of Copson and Linet [26,27] in the Schwarzschild background to the Kerr case. In Boyer-Lindquist coordinates (t, r, θ, ϕ),

$$\begin{aligned} A_t(r,\theta) = -\frac{q}{\Sigma_0 \Sigma}\Big[&(r_0 r + a^2\cos\theta) \\ &\times\Big(M + \frac{(r-M)(r_0-M) - (M^2-a^2)\cos\theta}{R}\Big) \\ &+ a^2(r - r_0\cos\theta)\frac{(r-M)-(r_0-M)\cos\theta}{R}\Big], \end{aligned} \tag{A2}$$

where $\Sigma_0 = r_0^2 + a^2$, $\Sigma = r^2 + a^2$, and $R = (r-M)^2 + (r_0-M)^2 - 2(r-M)(r_0-M)\cos\theta - (M^2-a^2)\sin^2\theta$.

We first re-express this solution in isotropic coordinates. In Schwarzschild geometry one can define a new radial coordinate ϱ, which makes the spatial 3-metric conformally flat and isotropic. While no such possibility exists for the Kerr spacetime, the coordinate change $r = \bar{\varrho} + M + \frac{(M^2-a^2)}{4\bar{\varrho}}$ does put the metric into the form

$$\begin{aligned} ds^2 = &-A(\bar{\varrho},\theta)dt^2 + B(\bar{\varrho},\theta)[d\bar{\varrho}^2 + \bar{\varrho}^2 d\theta^2] \\ &- C(\bar{\varrho},\theta)dt d\phi + D(\bar{\varrho},\theta)d\phi^2, \end{aligned} \tag{A3}$$

which is isotropic in the $(\bar{\varrho}, \theta)$ plane. Since the solution in question has only A_t, A_ϕ components, we may transform A_t

to the new coordinates by just re-expressing the function $A_t(r, \theta)$ as a function of $\bar{\varrho}$ and θ, which we shall denote $A_t(\bar{\varrho}, \theta)$, and then use the substitutions

$$\bar{\varrho}\cos\theta \rightarrow \bar{\varrho}_0 + \varrho\cos\vartheta, \tag{A4}$$

$$\bar{\varrho} \rightarrow \sqrt{\bar{\varrho}_0^2 + \varrho^2 + 2\bar{\varrho}_0\varrho\cos\vartheta} \tag{A5}$$

to transform to a new set of isotropic coordinates $(\varrho, \vartheta, \phi)$ centered on the location of the object's center of mass $\bar{\varrho}_0$. Expanding $A_t(\varrho, \vartheta)$ in powers of ϱ, we find

$$A_t = -q|\xi|\epsilon^{-1} - \mathcal{A} + \mathcal{B}\cos\vartheta + O(\varrho) + \cdots, \tag{A6}$$

where $|\xi| = \sqrt{\frac{r_0^2 - 2Mr_0 + a^2}{r_0^2 + a^2}}$, $\epsilon = |\xi|^{-1}\varrho$ is proper distance from the center of mass, and $\mathcal{A}$ and $\mathcal{B}$ are constants depending on M, a, q, and ϱ_0. The first term in (A6) is the Coulomb potential of a point charge (t component) redshifted to the location r_0 in the gravitational field of the hole. Assuming the object is spherical, in the limit $\varrho \rightarrow 0$ this last term renormalizes the rest mass of the object

$$m_{\mathrm{rem}} = m + \lim_{\epsilon\rightarrow 0} q^2|\xi|/2\epsilon, \tag{A7}$$

while first and higher order contributions in ϱ vanish. The third term $\mathcal{B}\cos\vartheta$ vanishes when averaged over the angular direction ϑ in our isotropic space. Converting the complicated expression for $\mathcal{A}$ in terms of $\bar{\varrho}_0$ back to Boyer-Lindquist r_0, one is left with just

$$E_{\mathrm{corr}} = \frac{Mq^2}{2(r_0^2 + a^2)}. \tag{A8}$$

This result is consistent with the literature on the subject. Léauté and Linet [18] and Lohiya [16] found a repulsive force along the polar axis

$$\bar{F} = \frac{Mq^2 r_0}{(r_0^2 + a^2)^2}. \tag{A9}$$

Their method was to re-express the vector potential (A2) in coordinates where the gravitational field in the vicinity of the particle is locally homogeneous (a local Rindler frame) to first order. In these coordinates the potential has the form of that for an accelerated point charge in flat spacetime plus an additional term due to the background curvature. Using the formula for the electric field $E_i = \partial_i A_t$ and averaging over all directions yields the result above. In contrast, Lohiya calculated the electric field due to the point charge along the symmetry axis of Kerr in Boyer-Lindquist coordinates. In the limit $r \rightarrow r_0$, the result is again a Coulomb piece whose corresponding energy renormalizes the mass of the particle, the self-force term mentioned earlier, and terms depending on the sign of $r - r_0$, which average out to zero if we imagine the charge is assembled in a spherically symmetric manner.

The contribution that Léauté, Linet, and Lohiya's self-force makes to E can be found as follows. The work done in a local orthonormal frame at $r = r_0$ when the charge is displaced a proper distance $d\ell$ is

$$d\bar{W} = \overline{F(r_0')}d\ell. \tag{A10}$$

At infinity this work is measured as $dE = |\xi|d\bar{W}$ because the with redshift factor is $|\xi|$. Thus,

$$E = \int_{r_0}^{\infty} |\xi|\bar{F}d\ell, \tag{A11}$$

where $d\ell = |\xi|^{-1}dr_0'$ along the axis. Since the redshift factors cancel, this is just an integral of the self-force from r_0 to ∞, which agrees with our result Eq. (A8).

Piazzese and Rizzi [19] re-examined the Kerr self-force problem and came up with a different self-force

$$\bar{F} = \frac{q^2(Mr_0 - a^2)}{(r_0^2 + a^2)^2}, \tag{A12}$$

again by differentiating A_t to find an electric field in coordinates where the gravitational field is locally homogeneous. Using (A11) one now finds the energy correction to be given by Eq. (37) instead of Eq. (A8). As we remarked in Sec. III D, this alternative energy correction is inconsistent with the area theorem.

Independently of the above, we believe PR's analysis is in error. They define the proper displacement $d\ell$ in the formula

$$\bar{F} = \frac{1}{|\xi|}\frac{dE}{d\ell} \tag{A13}$$

in terms of the Boyer-Lindquist coordinate difference $r - r_0$. In this formulation an extra term proportional to $|r - r_0|$ in the expansion for A_t appears to contribute to the force. However, the correct physical picture of the force involves a displacement of the charge location r_0 itself. In other words, one should take the limit as $r \rightarrow r_0$ in A_t first, and only then differentiate with respect to r_0 to find the self-force.

APPENDIX B: MUST A CABLE ALONG THE KERR POLAR AXIS ALWAYS SNAP?

One possible inconsistency in our gedanken experiments is an instability in the lowering and dropping process. Maybe the cable always snaps before we can place the object at rest at the desired radius r_0 near the horizon? In Sec. III A, we briefly described the cable model of Ref. [7], which indicates it is possible, at least in principle, to complete a lowering and dropping process in the Schwarzschild geometry. There the cable was considered to be conical, made up of thin radial fibers filling the portion of space defined by some solid angle with vertex near the black hole. With this cable design the enormous tension needed to hold a load stationary near the horizon can be distributed over a steadily increasing cross-sectional area.

However, the situation is more complicated in the Kerr spacetime. We must also consider the effect of the hole's rotation on the cable system, the so-called "dragging of inertial frames." When the cable is placed along the symmetry axis ($\theta = 0$), the rigid fiber elements will begin to rotate with the black hole in the ϕ direction at an angular velocity that depends on the distance from the horizon

$$\omega(r) = \frac{-g_{t\phi}}{g_{\phi\phi}} = \frac{2Mra}{(r^2 + a^2)^2} \tag{B1}$$

(Boyer-Lindquist coordinates). As the fibers twist around each other, the stress tensor of the cable will come to include not only the radial tension and mass density, but also shearing, torsional stresses ($\phi\phi$ component), and a flow of mass-energy in the ϕ direction.

To circumvent this problem we propose a new, less rigid, suspension in which the fibers are allowed to rotate freely, as dictated by the geometry, without twisting. The object of mass m is attached to the lowest of a system of separate light rigid disks, which are orthogonal to the symmetry axis and concentric with it; the uppermost disk is attached to some distant fixed structure. Each disk has a number of fibers affixed to its top surface, and each such fiber is suspended by a bearing from a circular groove in the next disk up in such way that the suspended end can slide without friction. This arrangement prevents the shearing between fibers; at each tier the fibers will rotate with respect to infinity with the local $\omega(r)$. We suppose the number of fibers grows from tier to tier, as does the number of grooves per disk, replicating the conical structure already mentioned. We shall require that the full width of the suspension lie within a solid angle of small opening θ with the apex at the suspended object. In this way we can proceed to the limit $\theta \to 0$ at the end of our calculations.

In the Kerr spacetime it is convenient to work with the field of orthonormal tetrad frames corresponding to locally nonrotating observers; these will rotate with the fibers at the angular velocity $\omega(r)$. The basis vectors for this orthonormal frame are, in terms of Boyer-Lindquist coordinates, [28]

$$\begin{aligned} \mathbf{e}_{\hat{t}} &= \left(\frac{\mathcal{A}}{\Sigma\Delta}\right)^{1/2} \frac{\partial}{\partial t} + \frac{2Mar}{(\mathcal{A}\Sigma\Delta)^{1/2}} \frac{\partial}{\partial\phi}, \\ \mathbf{e}_{\hat{r}} &= \left(\frac{\Delta}{\Sigma}\right)^{1/2} \frac{\partial}{\partial r}, \qquad \mathbf{e}_{\hat{\theta}} = \left(\frac{1}{\Sigma}\right)^{1/2} \frac{\partial}{\partial\theta}, \\ \mathbf{e}_{\hat{\phi}} &= \left(\frac{\Sigma}{\mathcal{A}}\right)^{1/2} \frac{1}{\sin\theta} \frac{\partial}{\partial\phi}, \end{aligned} \tag{B2}$$

where $\Delta = r^2 + a^2 - 2Mr$, $\mathcal{A} = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta$, and $\Sigma = r^2 + a^2\cos^2\theta$. In this frame there is no energy flux in the ϕ direction, and the only nonzero components of the stress tensor, averaged over many fibers, are the mass density ρ and the tension per unit cross section S, both of which we naturally take to depend only on r, since the fibers run parallel to the symmetry axis:

$$T^{\hat{t}\hat{t}} = \rho(r), \qquad T^{\hat{r}\hat{r}} = S(r). \tag{B3}$$

From these we may calculate $T^{\mu\nu}$ in the Boyer-Lindquist coordinate frame.

A stationary configuration of matter outside the hole obeys the conservation law

$$T^{\nu}_{\mu;\nu} = \frac{(\sqrt{-g}T^{\nu}_{\mu})_{,\nu}}{\sqrt{-g}} - \frac{1}{2} g_{\alpha\beta,\mu} T^{\alpha\beta} = 0. \tag{B4}$$

Along the symmetry axis only the $\hat{r}$ component of this gives a nontrivial condition

$$\frac{dS}{dr} = \frac{M\rho(r)(a^2 - r^2) + S(r)(a^2M + 3r^2M - 2r^2a - 2r^3)}{(r^2 - 2Mr + a^2)(r^2 + a^2)}. \tag{B5}$$

By assuming that the suspension flares upward so as to fill a small solid angle around the axis, we may continue to use this equation slightly off axis. In this case, S signifies stress along the fibers. Our goal is to show that there is a (nonsingular) solution to this equation that satisfies the appropriate boundary conditions and the weak energy condition, $|S|/\rho \le 1$.

Reference [7] assumed a constant (average) proper density $\rho = \rho_0$; this implies infinite sound speed in the suspension apparatus. We avoid this unphysical assumption. Assuming that the stretching is adiabatic, the constitutive relation should be $S = -\mathcal{S}(\rho)$, where $\mathcal{S}(\rho_0) = 0$ for the density ρ_0 of the unstressed apparatus and $\mathcal{S}(\rho) > 0$ for $\rho > \rho_0$. (Recall that $S < 0$ for a suspension under tension; in this state the density should exceed ρ_0, since elastic energy adds to ρ.) The squared speed of sound along the apparatus axis is $\mathcal{S}'(\rho)$; we thus assume $\mathcal{S}'(\rho) < 1$ for all $\rho \ge \rho_0$. It is immediately clear that $\mathcal{S}(\rho) < \rho$. Thus, $|S| < \rho$ so that the weak energy condition is satisfied as a result of the causality assumption.

To make calculations tractable we shall specialize to a linear $\mathcal{S}$, that is,

$$S = K \cdot (\rho_0 - \rho), \tag{B6}$$

where K is the squared speed of sound. Consequently, we assume $K < 1$ to preserve causality; again this immediately implies that $|S|/\rho < 1$. Therefore, the weak energy condition will be automatically satisfied, provided there exists a nonsingular solution for the stress.

Using Eq. (B6) we eliminate $\rho(r)$ in favor of $S(r)$ in Eq. (B5). The general solution to the resulting differential equation for $S(r)$, with C an integration constant, is

$$\begin{aligned} S(r) = (r^2 - 2Mr + a^2)^{-\gamma}(r^2 + a^2)^{\gamma-1}\bigg[\int_{r_0}^{r} M\rho_0(a^2 - r'^2) \\ \times (r'^2 + a^2)^{-\gamma}(r'^2 - 2Mr' + a^2)^{\gamma-1}dr' + C\bigg], \end{aligned} \tag{B7}$$

where $r = r_0$ marks the lower end of the suspension and $\gamma = \frac{K-1}{2K} < 0$ given our causality restriction. The term involving the integral describes the average stress (tension per unit area) due to the weight of fibers and disks, $S_{\rm susp}$. The second term is due to the load on the end of the suspension. Using Eq. (B7), we can solve for the integration constant C in terms of $S(r_0)$

$$C = S(r_0)(r_0^2 - 2Mr_0 + a^2)^\gamma (r_0^2 + a^2)^{\gamma-1}. \tag{B8}$$

For a suspended point mass m, $S(r_0) = -ma^{\hat{r}}/\alpha$, where $a^{\hat{r}}$ is the radial acceleration in the orthonormal frame at r_0 and α is the cross section of the suspension at the bottom. We thus find that

$$S_{\rm load} = \frac{mM(a^2 - r_0^2)(r_0^2 - 2Mr_0 + a^2)^{\gamma-(1/2)}}{\alpha(r_0^2 + a^2)^{\gamma+(1/2)}(r^2 - 2Mr + a^2)^{1-\gamma}(r^2 + a^2)^\gamma}. \tag{B9}$$

Note that this stress is negative (as befits a tension) and, for sufficiently large r, its magnitude decreases monotonically as r^{-2} because the suspension's cross section increases.

Generically $S_{\rm load}$ becomes very large as r_0 approaches the horizon radius r_+. In (B6) this corresponds to a cable density $\rho \gg \rho_0$. In this regime the validity of the linear constitutive relation is doubtful. However, by taking the suspended mass m sufficiently small on scale M, and by employing a suspension with an already large cross section α at its bottom, the large stresses can be controlled.

Hence, our remaining task to check that there are nonsingular solutions for the stress. Inspection of the integral piece of (B7) indicates no obvious problems; for example, for large r, $S(r) \sim r^{-1}$. We have numerically integrated in the cases where $K < 1$ and for various values of ρ_0 to deduce $S_{\rm susp}(r)$ near the horizon. We scanned through a series of dropping points $r_0 > r_+$ near the horizon and found well-behaved solutions throughout. Thus, it seems the weak energy condition can be satisfied during the lowering process, so there is no reason *of principle* that will force our suspension to break. We conclude there is no inconsistency in the lowering and dropping process envisaged in Sec. III D.

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