Parallel-propagated frame along null geodesics in higher-dimensional black hole spacetimes

David Kubizňák,[1,2,*] Valeri P. Frolov,[1,†] Pavel Krtouš,[3,‡] and Patrick Connell[1,§]

[1]*Theoretical Physics Institute, University of Alberta, Edmonton, Alberta, Canada T6G 2G7*

[2]*DAMTP, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK*

[3]*Institute of Theoretical Physics, Faculty of Mathematics and Physics, Charles University in Prague, V Holešovičkách 2, Prague, Czech Republic*

(Received 31 October 2008; published 22 January 2009)

In [arXiv:0803.3259] the equations describing the parallel transport of orthonormal frames along timelike (spacelike) geodesics in a spacetime admitting a nondegenerate principal conformal Killing-Yano 2-form $\boldsymbol{h}$ were solved. The construction employed is based on studying the Darboux subspaces of the 2-form $\boldsymbol{F}$ obtained as a projection of $\boldsymbol{h}$ along the geodesic trajectory. In this paper we demonstrate that, although slightly modified, a similar construction is possible also in the case of null geodesics. In particular, we explicitly construct the parallel-transported frames along null geodesics in $D = 4, 5, 6$ Kerr-NUT-(A)dS spacetimes. We further discuss the parallel transport along principal null directions in these spacetimes. Such directions coincide with the eigenvectors of the principal conformal Killing-Yano tensor. Finally, we show how to obtain a parallel-transported frame along null geodesics in the background of the 4D Plebański-Demiański metric which admits only a conformal generalization of the Killing-Yano tensor.

DOI: 10.1103/PhysRevD.79.024018 PACS numbers: 04.50.−h, 02.40.−k, 04.20.Jb, 04.50.Gh

I. INTRODUCTION

Solving the parallel transport equations along null geodesics in a four-dimensional spacetime is a well-known problem with many physical applications. For example, in a geometric optics approximation linearly polarized photons and gravitons propagate along null geodesics while the corresponding polarization vectors are parallel-transported along the worldline [1] (see also Appendix A). This property was used to study the polarized radiation from black holes (see, e.g., [2–4] and references therein). The parallel-propagated frames are convenient for studying the form and shape of a thin ''pencil of light'' propagating in an external gravitational field. In the derivation of the equations for optical scalars the parallel-propagated frames play an important technical role (see, e.g., [5,6]). Another problem where such frames are useful is a so-called peeling-off property of the gravitational radiation in an asymptotically flat spacetime (see, e.g., [7–11] and references therein).

Recently, models of gravity with extra dimensions attracted a lot of attention. In order to rederive many of the important results of the four-dimensional optics in a curved spacetime in the higher-dimensional case it is necessary to consider first a problem of parallel transport along null geodesics. This is the main purpose of this paper.

We focus our attention on a special class of higher-dimensional spacetimes which admit a so-called *principal conformal Killing-Yano* (PCKY) tensor [12,13] (see also [14–16] for reviews). The most general metric element admitting such a tensor, the *canonical* (off shell) metric, was studied in [17–19]. Similar to 4D, when the vacuum Einstein equations with the cosmological constant are imposed the canonical metric becomes the Kerr-NUT-(A)dS spacetime [20].

Recently, a parallel-propagated frame along timelike (spacelike) geodesics in the canonical spacetime was constructed [21]. As it often happens, a limit when the velocity of the particle motion tends to the speed of light is singular, so that, the problem of parallel transport along null geodesics requires a special treatment. We deal with this problem in the present paper. The obtained results generalize the 4D results [22–24].

Consider a spacetime with a PCKY tensor $\boldsymbol{h}$, that is, a closed rank-2 nondegenerate conformal Killing-Yano (KY) tensor [25,26]. In such a spacetime the geodesic motion is completely integrable [27–29]. Let us concentrate on a *generic* null ray and denote its velocity vector $\boldsymbol{l}$. In our construction, starting with $\boldsymbol{l}$ we first generate two additional parallel-propagated vectors, one of which, say $\boldsymbol{n}$, is ''external'' to the null plane of $\boldsymbol{l}$ and can be made null. This means that the tangent space T at each point of $\boldsymbol{l}$ splits into a two-dimensional parallel-propagated subspace U spread by $\{\boldsymbol{l}, \boldsymbol{n}\}$ and a $(D-2)$-dimensional parallel-propagated subspace V orthogonal to U; $T = U \oplus V$.

The construction of the parallel-propagated frame is now similar to the timelike case. We consider the Darboux problem for the 2-form $\boldsymbol{F}$ obtained as a projection of the PCKY tensor $\boldsymbol{h}$ to a subspace V. Such a 2-form is automatically parallel-transported. In particular, each of the Darboux subspaces of $\boldsymbol{F}$ is independently parallel-transported. The zeroth value Darboux subspace is 3(4)-dimensional in an odd(even)-dimensional spacetime. The parallel-propagated vectors spanning it can be explicitly

*dk317@cam.ac.uk
†frolov@phys.ualberta.ca
‡Pavel.Krtous@utf.mff.cuni.cz
§pconnell@phys.ualberta.ca

constructed and we denote them $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}\}$ ($\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}, \boldsymbol{z}\}$). The Darboux subspaces, for nonzero eigenvalues of $\boldsymbol{F}$ are two-dimensional. One can obtain the parallel-propagated vectors spanning them by employing 2D (affine parameter)-dependent rotations.

For certain special geodesics the construction of a parallel-propagated frame has to be accordingly modified. Similar to the timelike (spacelike) case, the degeneracies occur for special geodesics for which the Darboux subspaces of $\boldsymbol{F}$ become more dimensional. In these cases the frame can be obtained by ''more involved'' (affine parameter)-dependent orthogonal transformations (see [21] for more details). An important example of different degeneracy is the case of parallel transport along principal null directions—discussed in Sec. VII.

The paper is organized as follows: In Sec. II we introduce the basic notations used in the paper and review the definition of the PCKY tensor. In Sec. III a generic construction of the parallel-propagated frames along null geodesics is outlined. Canonical metric element of a spacetime admitting the PCKY tensor and its basic properties are discussed in Sec. IV. In Sec. V the explicit form of the canonical metric is employed to concretize the generic construction. Examples of solutions in four, five, and six dimensional spacetimes are presented in Sec. VI. The parallel transport along principal null directions is described in Sec. VII. Section VIII is devoted to a discussion of obtained results and conclusions. More technical details, connected with the geometric optics in higher-dimensional spacetimes, principal null directions, and the parallel transport in the Plebański-Demiański metric are presented in Appendices A, B, and C, respectively.

II. BASIC NOTIONS AND NOTATIONS

In what follows we use the notations of [15,21]. We consider a D-dimensional spacetime M^D, equipped with the metric

$$\boldsymbol{g} = g_{ab}\boldsymbol{d}x^a\boldsymbol{d}x^b. \tag{1}$$

To treat both cases of even and odd dimensions simultaneously we denote

$$D = 2n + \varepsilon, \tag{2}$$

where $\varepsilon = 0$ and $\varepsilon = 1$ for even and odd number of dimensions, respectively. Operations $\flat$, $\sharp$ correspond to ''lowering''and ''rising'' of indices of vectors and forms, respectively. $\boldsymbol{\delta}$ denotes the coderivative. For a p-form $\boldsymbol{\alpha}_p$ one has $\boldsymbol{\delta}\boldsymbol{\alpha}_p = \boldsymbol{\epsilon} * \boldsymbol{d} * \boldsymbol{\alpha}_p$, where $\boldsymbol{d}$ denotes the exterior derivative, $*$ denotes the Hodge star operator, and $\boldsymbol{\epsilon} = (-1)^{p(D-p)+p-1}$. The ''hook'' operator $\lrcorner$ denotes ''contraction''. The scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is denoted by a dot; $\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{a}\lrcorner\boldsymbol{b}^\flat$.

Definition. A PCKY tensor $\boldsymbol{h}$ is a closed nondegenerate conformal Killing-Yano 2-form, $\boldsymbol{h} = \frac{1}{2}h_{ab}\boldsymbol{d}x^a \wedge \boldsymbol{d}x^b$. This implies that for all vector fields $\boldsymbol{X}$ there exists such a vector field $\boldsymbol{\xi}$ so that

$$\nabla_X \boldsymbol{h} = \boldsymbol{X}^\flat \wedge \boldsymbol{\xi}^\flat. \tag{3}$$

The condition of *nondegeneracy* means that in a generic point of the manifold the skew symmetric matrix h_{ab} has the (matrix) rank $2n$ and that the eigenvalues of $\boldsymbol{h}$ are functionally independent in some spacetime domain. In this domain, such eigenvalues may be used as ''natural'' coordinates (see [18,19] for more details). We exclude the possibility that $\boldsymbol{h}$ possesses constant eigenvalues, and, in particular, that it is covariantly constant; $\boldsymbol{\xi} \neq 0$. (For the discussion of cases when such degeneracies are admitted see [30,31].)

The Eq. (3) implies

$$\boldsymbol{d}\,\boldsymbol{h} = 0, \qquad \boldsymbol{\xi}^\flat = -\frac{1}{D-1}\boldsymbol{\delta}\boldsymbol{h}. \tag{4}$$

It can be shown [18], that for any spacetime admitting the PCKY tensor $\boldsymbol{h}$, $\boldsymbol{\xi}$ is a (*primary*) Killing vector. In tensor notations the definition (3) reads

$$\nabla_c h_{ab} = 2g_{c[a}\xi_{b]}, \qquad \xi_b = \frac{1}{D-1}\nabla_d h^d{}_b. \tag{5}$$

Let γ be a null geodesic and $l^a = dx^a/d\tau$ be a tangent vector to it; τ denotes the affine parameter. We denote the covariant derivative of a tensor $\boldsymbol{T}$ along γ by

$$\dot{\boldsymbol{T}} = \nabla_l \boldsymbol{T} = l^a\nabla_a \boldsymbol{T}. \tag{6}$$

In particular, $\dot{\boldsymbol{l}} = 0$.

III. GENERIC CONSTRUCTION OF A PARALLEL-PROPAGATED FRAME

In this section we outline the general construction of parallel-transported frames along *generic* geodesics.

A. Construction of parallel-transported vectors $\boldsymbol{m}$ and $\boldsymbol{n}$

Starting with $\boldsymbol{l}$ one can easily construct two additional parallel-propagated vectors. The construction is based on the following result: Let $\boldsymbol{h}$ be a PCKY tensor and $\boldsymbol{u}$ be a parallel-transported vector along a null geodesic $\boldsymbol{l}$, obeying $\boldsymbol{u}\cdot\boldsymbol{l} = 0$. Then the vector

$$\boldsymbol{w} = (\boldsymbol{u}\lrcorner\boldsymbol{h})^\sharp + \beta_{(u)}\boldsymbol{l} \tag{7}$$

is parallel-transported along $\boldsymbol{l}$, provided that

$$\dot{\beta}_{(u)} = \boldsymbol{u}\cdot\boldsymbol{\xi}. \tag{8}$$

Here $\boldsymbol{\xi}$ is the primary Killing vector (4).

To prove this statement we use Eq. (3) and the property of the hook operator. We find

$$\dot{\boldsymbol{w}} = (\boldsymbol{u} \lrcorner \dot{\boldsymbol{h}})^\sharp + \dot{\beta}_{(u)}\boldsymbol{l} = [\boldsymbol{u} \lrcorner (\boldsymbol{l}^\flat \wedge \boldsymbol{\xi}^\flat)]^\sharp + \dot{\beta}_{(u)}\boldsymbol{l}$$
$$= \boldsymbol{\xi}(\boldsymbol{u}\cdot\boldsymbol{l}) - \boldsymbol{l}(\boldsymbol{u}\cdot\boldsymbol{\xi}) + \dot{\beta}_{(u)}\boldsymbol{l} = \boldsymbol{l}[\dot{\beta}_{(u)} - \boldsymbol{u}\cdot\boldsymbol{\xi}].$$

Obviously, $\boldsymbol{u} = \boldsymbol{l}$ obeys the requirements and we may construct $[\beta = \beta_{(l)}]$

$$\boldsymbol{m} = \frac{1}{\sqrt{-\kappa_1}}[(\boldsymbol{l} \lrcorner \boldsymbol{h})^\sharp + \beta\boldsymbol{l}]. \tag{9}$$

Here, $\kappa_1 = Q_{ab}l^a l^b$ is a constant of geodesic motion corresponding to the conformal Killing tensor $Q_{ab} = h_{ac}h^c{}_b$. Since $\boldsymbol{\xi}$ is a primary Killing vector, $\dot{\beta} = \boldsymbol{l}\cdot\boldsymbol{\xi}$ is also a constant and β can be immediately integrated to get

$$\beta = (\boldsymbol{l}\cdot\boldsymbol{\xi})\tau. \tag{10}$$

We also find

$$\boldsymbol{m}\cdot\boldsymbol{l} = 0, \qquad \boldsymbol{m}\cdot\boldsymbol{m} = 1. \tag{11}$$

So, $\boldsymbol{m}$ given by (9) and (10) is the normalized spacelike vector which is orthogonal to $\boldsymbol{l}$ and parallel-propagated along γ.

Taking $\boldsymbol{u} = \boldsymbol{m}$ in (7) we may construct another parallel-transported vector

$$\boldsymbol{n} = \tilde{\boldsymbol{n}} + \tfrac{1}{2}(\tilde{\boldsymbol{n}}\cdot\tilde{\boldsymbol{n}})\boldsymbol{l}, \tag{12}$$

where

$$\tilde{\boldsymbol{n}} = \frac{1}{\sqrt{-\kappa_1}}[(\boldsymbol{m} \lrcorner \boldsymbol{h})^\sharp + \beta_{(m)}\boldsymbol{l}], \qquad \dot{\beta}_{(m)} = \boldsymbol{m}\cdot\boldsymbol{\xi}. \tag{13}$$

We find

$$\boldsymbol{n}\cdot\boldsymbol{l} = -1, \qquad \boldsymbol{n}\cdot\boldsymbol{m} = 0, \qquad \boldsymbol{n}\cdot\boldsymbol{n} = 0. \tag{14}$$

So, $\boldsymbol{n}$ is a null parallel-transported vector, external to $\boldsymbol{l}$ and orthogonal to $\boldsymbol{m}$. Moreover, we can easily show that it is independent of $\beta_{(m)}$. Indeed, using (13) we find that

$$\boldsymbol{n} = \frac{1}{\sqrt{-\kappa_1}}(\boldsymbol{m} \lrcorner \boldsymbol{h})^\sharp + C\boldsymbol{l},$$
$$C = \frac{1}{2\kappa_1^2}(Q^{(2)}_{ab}l^a l^b - \kappa_1\beta^2), \qquad Q^{(2)}_{ab} = Q_{ac}Q^c_b. \tag{15}$$

One might wonder whether it is not possible to generate more parallel-propagated vectors in this way. Unfortunately, the fact that $\boldsymbol{n}$ is external, $\boldsymbol{n}\cdot\boldsymbol{l} = -1$, shows that $\boldsymbol{n}$ does not obey the requirements of the lemma anymore and we have to proceed differently.[1]

[1]In fact, when $\boldsymbol{n}$ is used as a "seed" in (7) one obtains a new linear independent vector. This vector may be used as a seed again and so on—to produce the whole tower of linear independent vectors. These vectors are, however, not parallel-transported.

B. Projection formalism: operator $\boldsymbol{F}$

Let us consider the following 2-form $\boldsymbol{F}$:

$$F_{ab} = P^c_a P^d_b h_{cd}, \tag{16}$$

where

$$P_{ab} = g_{ab} + 2l_{(a}n_{b)} \tag{17}$$

is the projector to a $(D-2)$-dimensional space V, orthogonal to a 2-dimensional space U spanned by $\{\boldsymbol{l}, \boldsymbol{n}\}$. We have $P_{ab}l^b = 0$, $P_{ab}n^b = 0$.

Since vectors $\boldsymbol{l}$ and $\boldsymbol{n}$ are parallel-transported, so is P_{ab}; $\dot{P}_{ab} = 0$. Therefore we find

$$\dot{F}_{ab} = P^c_a P^d_b \dot{h}_{cd} = 2P^c_a P^d_b l_{[c}\xi_{d]} = 0, \tag{18}$$

where we have used Eq. (5). So, the 2-form $\boldsymbol{F}$ is parallel-transported. This implies that the eigenvalues of $\boldsymbol{F}$ as well as its Darboux subspaces are independently parallel-transported (see [21] for more details).

The problem of finding remaining parallel-transported vectors along null geodesic γ is now quite analogous to the problem of parallel transport along timelike (spacelike) geodesics. By solving the eigenvalue problem for the operator $\boldsymbol{F}^2$ one finds the eigenvectors spanning each of the Darboux subspaces of $\boldsymbol{F}$. In each Darboux subspace, the parallel-transported vectors are obtained by a τ-dependent orthogonal transformation of these eigenvectors.

The structure of the Darboux subspaces of $\boldsymbol{F}$ depends crucially on the chosen geodesic γ. As the number of dimensions is increased, the number of degenerate cases (corresponding to special geodesics) which require their own special treatment also increases. In what follows we concentrate on generic geodesics for which the discussion significantly simplifies.

C. Darboux subspace V_0

We denote the Darboux subspace corresponding to the zeroth eigenvalue of $\boldsymbol{F}$ by V_0. For a generic geodesic γ, V_0 is three-dimensional (four-dimensional) in the odd (even) number of spacetime dimensions. It is spanned by the parallel-transported vectors $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}\}$ and, in the even-dimensional case, $\boldsymbol{z}$ given by (20) below.

The fact that $\boldsymbol{l}$ and $\boldsymbol{n}$ are zero-value eigenvectors of $\boldsymbol{F}$ is trivial. To prove that $\boldsymbol{m}$ also belongs to V_0 we first notice that, $\boldsymbol{m}$ being orthogonal to $\{\boldsymbol{l}, \boldsymbol{n}\}$, is unaffected by projector $\boldsymbol{P}$, that is $P^b_a m^a = m^b$. Moreover, using (15) we realize that $s^a = h^a{}_b m^b$ is a linear combination of $\boldsymbol{l}$ and $\boldsymbol{n}$, $\boldsymbol{s} = \alpha\boldsymbol{l} + \beta\boldsymbol{n}$, which when projected again by $\boldsymbol{P}$ gives zero. So we have

$$F_{ad}m^d = P^b_a h_{bc}P^c_d m^d = P^b_a h_{bc}m^c = P^b_a s_b = 0. \tag{19}$$

In an even number of spacetime dimensions we consider an additional vector $\boldsymbol{z}$ given by

$$z = (l \lrcorner f)^{\sharp}, \tag{20}$$

where we have denoted

$$f = *h^{\wedge(n-1)} = * \underbrace{(h \wedge \ldots \wedge h)}_{\text{total of } (n-1) \text{ factors}}. \tag{21}$$

Tensor f is a Killing-Yano 2-form (see, e.g., [32]) and therefore vector z is automatically parallel-transported. Forms f and h are related as

$$h^a{}_c f^c{}_b \propto \delta^a_b. \tag{22}$$

It then follows that $h^a{}_b z^b \propto l^a$. Using this relation one can easily show that z is spacelike parallel-transported vector, orthogonal to $\{l, n, m\}$,

$$z \cdot l = 0, \qquad z \cdot m = 0, \qquad z \cdot n = 0. \tag{23}$$

It can be normalized, so that, $z \cdot z = 1$. Moreover, it is the last zero-value eigenvector of F spanning V_0. Indeed, we get

$$F^a{}_b z^b = P^a_d h^d{}_c P^c_b z^b = P^a_d h^d{}_c z^c \propto P^a_d l^d = 0.$$

To conclude, in the even-dimensional case, V_0 is spanned by explicitly constructed parallel-transported vectors $\{l, n, m, z\}$.

D. Parallel-transported vectors in remaining Darboux subspaces

We restrict ourselves by considering generic geodesics for which all the remaining Darboux subspaces of F are two-dimensional. Denote by V_i the Darboux subspace of F corresponding to the nonzero eigenvalue $-\lambda_i^2$ of the operator F^2,

$$F^2 v = -\lambda_i^2 v, \qquad v \in V_i, \tag{24}$$

and by $\{n_{\hat{i}}, \tilde{n}_{\hat{i}}\}$ the orthonormal basis in V_i. This basis is related to the parallel-propagated orthonormal basis $\{p_{\hat{i}}, \tilde{p}_{\hat{i}}\}$ spanning V_i by a 2D rotation

$$p_{\hat{i}} = \cos\gamma_i n_{\hat{i}} - \sin\gamma_i \tilde{n}_{\hat{i}} \qquad \tilde{p}_{\hat{i}} = \sin\gamma_i n_{\hat{i}} + \cos\gamma_i \tilde{n}_{\hat{i}},$$
$$\dot{\gamma}_i = \dot{n}_{\hat{i}} \cdot \tilde{n}_{\hat{i}} = -n_{\hat{i}} \cdot \dot{\tilde{n}}_{\hat{i}}. \tag{25}$$

If, at the initial point $\tau = 0$ bases $\{p_{\hat{i}}, \tilde{p}_{\hat{i}}\}$ and $\{n_{\hat{i}}, \tilde{n}_{\hat{i}}\}$ coincide, the initial conditions for Eqs. (25) are

$$\gamma_i(\tau = 0) = 0. \tag{26}$$

The construction of parallel-propagated vectors in a different Darboux subspace is exactly analogous, independent of the other constructions.

IV. KERR-NUT-(A)DS SPACETIMES AND THEIR PROPERTIES

A. Canonical metric element and Kerr-NUT-(A)dS spacetimes

The most general canonical metric element admitting the PCKY tensor reads [18,19]

$$g = \sum_{\mu=1}^{n-1} (\omega^{\hat{\mu}} \omega^{\hat{\mu}} + \tilde{\omega}^{\hat{\mu}} \tilde{\omega}^{\hat{\mu}}) + \omega^{\hat{n}} \omega^{\hat{n}} - \tilde{\omega}^{\hat{n}} \tilde{\omega}^{\hat{n}} + \varepsilon \omega^{\hat{\epsilon}} \omega^{\hat{\epsilon}}, \tag{27}$$

where the basis 1-forms are ($\mu = 1, \ldots, n-1$)

$$\omega^{\hat{n}} = \frac{dr}{\sqrt{Q_n}}, \qquad \omega^{\hat{\mu}} = \frac{dx_\mu}{\sqrt{Q_\mu}},$$
$$\tilde{\omega}^{\hat{n}} = \sqrt{Q_n} \sum_{j=0}^{n-1} A_n^{(j)} d\psi_j, \qquad \tilde{\omega}^{\hat{\mu}} = \sqrt{Q_\mu} \sum_{j=0}^{n-1} A_\mu^{(j)} d\psi_j,$$
$$\omega^{\hat{\epsilon}} = \sqrt{Q_\epsilon} \sum_{j=0}^{n} A^{(j)} d\psi_j, \qquad Q_\epsilon = \frac{-c}{A^{(n)}}. \tag{28}$$

We enumerate the basis $\{\omega\}$ so that $\tilde{\omega}^{\hat{n}}$ is a timelike 1-form (the only one). Here

$$A_\mu^{(j)} = \sum_{\substack{\nu_1 < \ldots < \nu_j \\ \nu_i \neq \mu}} x^2_{\nu_1} \ldots x^2_{\nu_j}, \qquad A^{(j)} = \sum_{\nu_1 < \ldots < \nu_j} x^2_{\nu_1} \ldots x^2_{\nu_j},$$
$$Q_\mu = \frac{X_\mu}{U_\mu}, \qquad U_\mu = \prod_{\substack{\nu=1 \\ \nu \neq \mu}}^{n} (x_\nu^2 - x_\mu^2), \qquad x_n^2 = -r^2, \tag{29}$$

and X_μ, X_n are arbitrary functions of x_μ, r, respectively. Time is denoted by ψ_0, azimuthal coordinates by ψ_j, $j = 1, \ldots, m = D - n - 1$, r is the Boyer-Lindquist type radial coordinate, and x_μ, $\mu = 1, \ldots, n-1$ stand for latitude coordinates.

The inverse metric reads

$$g^{-1} = \sum_{\mu=1}^{n-1} (e_{\hat{\mu}} e_{\hat{\mu}} + \tilde{e}_{\hat{\mu}} \tilde{e}_{\hat{\mu}}) + e_{\hat{n}} e_{\hat{n}} - \tilde{e}_{\hat{n}} \tilde{e}_{\hat{n}} + \varepsilon e_{\hat{\epsilon}} e_{\hat{\epsilon}}, \tag{30}$$

where

$$e_{\hat{n}} = \sqrt{Q_n} \partial_r, \qquad e_{\hat{\mu}} = \sqrt{Q_\mu} \partial_{x_\mu},$$
$$\tilde{e}_{\hat{n}} = \frac{1}{\sqrt{X_n U_n}} \sum_{j=0}^{m} r^{2(n-1-j)} \partial_{\psi_j},$$
$$\tilde{e}_{\hat{\mu}} = \frac{1}{\sqrt{Q_\mu} U_\mu} \sum_{j=0}^{m} (-x_\mu^2)^{n-1-j} \partial_{\psi_j},$$
$$e_{\hat{\epsilon}} = -\frac{\partial_{\psi_n}}{\sqrt{-cA^{(n)}}}. \tag{31}$$

The PCKY tensor for the canonical metric reads [12,32]

$$\boldsymbol{h} = \sum_{\mu=1}^{n-1} x_\mu \boldsymbol{\omega}^{\hat{\mu}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{\mu}} - r\boldsymbol{\omega}^{\hat{n}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{n}}. \tag{32}$$

Equations (27) and (32) mean that the basis $\{\boldsymbol{\omega}\}$ is an "orthogonal Darboux basis" of $\boldsymbol{h}$. We call such a basis a canonical one. The canonical basis is fixed uniquely by the PCKY tensor up to 2D rotations in each of the Killing-Yano 2-planes $\boldsymbol{\omega}^{\hat{n}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{n}}$, $\boldsymbol{\omega}^{\hat{\mu}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{\mu}}$. The basis $\{\boldsymbol{\omega}\}$ is a special canonical basis for which many of the Ricci coefficients of rotation vanish [18,19,33]. We call it a *principal canonical* basis.

The PCKY tensor $\boldsymbol{h}$ generates the whole towers of explicit and hidden symmetries [13]. Namely, it generates all the isometries $\boldsymbol{\partial}_{\psi_k}$, and, in particular, the primary Killing vector $\boldsymbol{\xi}$, (4)

$$\boldsymbol{\xi} = -\frac{1}{D-1}(\boldsymbol{\delta h})^\sharp = \boldsymbol{\partial}_{\psi_0}. \tag{33}$$

It also generates the set of the second-rank irreducible Killing tensors ($j = 1, \ldots, m$)

$$\begin{aligned}\boldsymbol{K}^{(j)} = &\sum_{\mu=1}^{n-1} A_\mu^{(j)}(\boldsymbol{\omega}^{\hat{\mu}}\boldsymbol{\omega}^{\hat{\mu}} + \tilde{\boldsymbol{\omega}}^{\hat{\mu}}\tilde{\boldsymbol{\omega}}^{\hat{\mu}}) \\ &+ A_n^{(j)}(\boldsymbol{\omega}^{\hat{n}}\boldsymbol{\omega}^{\hat{n}} - \tilde{\boldsymbol{\omega}}^{\hat{n}}\tilde{\boldsymbol{\omega}}^{\hat{n}}) + \varepsilon A^{(j)}\boldsymbol{\omega}^{\hat{\epsilon}}\boldsymbol{\omega}^{\hat{\epsilon}}.\end{aligned} \tag{34}$$

These objects are responsible for complete integrability of geodesic motion in the canonical spacetime.

When the vacuum Einstein equations with the cosmological constants are imposed

$$R_{ab} = (-1)^n(D-1)c_n g_{ab}, \tag{35}$$

metric functions $X_\mu(x_\mu)$ and $X_n(r)$ take the following specific form [33]:

$$\begin{aligned}X_n &= -\sum_{k=\varepsilon}^{n} c_k(-r^2)^k - 2Mr^{1-\varepsilon} + \frac{\varepsilon c}{r^2}, \\ X_\mu &= \sum_{k=\varepsilon}^{n} c_k x_\mu^{2k} - 2b_\mu x_\mu^{1-\varepsilon} + \frac{\varepsilon c}{x_\mu^2},\end{aligned} \tag{36}$$

and the canonical element becomes the general Kerr-NUT-(A)dS spacetime derived by Chen, Lü, and Pope [20]. The parameter c_n is proportional to the cosmological constant and the remaining constants c_k, $c > 0$, and b_μ are related to rotation parameters, mass, and NUT parameters.

B. Geodesics

As we mentioned above, the geodesic motion in the canonical background (27)–(29) is completely integrable [27–29]. In particular, the null geodesic velocity takes the following form [13,15]:

$$\boldsymbol{l}^\flat = \sum_{\mu=1}^{n}(l_{\hat{\mu}}\boldsymbol{\omega}^{\hat{\mu}} + \tilde{l}_{\hat{\mu}}\tilde{\boldsymbol{\omega}}^{\hat{\mu}}) + \varepsilon l_{\hat{\epsilon}}\boldsymbol{\omega}^{\hat{\epsilon}}, \tag{37}$$

where

$$\begin{aligned}l_{\hat{n}} &= \frac{\sigma_n}{(X_n U_n)^{1/2}}(W_n^2 - X_n V_n)^{1/2}, \\ l_{\hat{\mu}} &= \frac{\sigma_\mu}{(X_\mu U_\mu)^{1/2}}(X_\mu V_\mu - W_\mu^2)^{1/2}, \qquad \tilde{l}_{\hat{n}} = \frac{W_n}{(X_n U_n)^{1/2}}, \\ \tilde{l}_{\hat{\mu}} &= \frac{1}{\sqrt{Q_\mu}}\frac{W_\mu}{U_\mu}, \qquad l_{\hat{\epsilon}} = -\frac{\Psi_n}{\sqrt{-cA^{(n)}}}.\end{aligned} \tag{38}$$

Here, the constants $\sigma_\mu = \pm 1$ ($\mu = 1, \ldots, n$) are independent of one another and we have defined

$$\begin{aligned}V_n &= -\sum_{j=1}^{m} r^{2(n-1-j)}\kappa_j, \qquad V_\mu = \sum_{j=1}^{m}(-x_\mu^2)^{n-1-j}\kappa_j, \\ W_n &= \sum_{j=0}^{m} r^{2(n-1-j)}\Psi_j, \qquad W_\mu = \sum_{j=0}^{m}(-x_\mu^2)^{n-1-j}\Psi_j.\end{aligned} \tag{39}$$

The quantities Ψ_j and κ_j are conserved and connected with the Killing vectors and the Killing tensors, respectively. We also have

$$\kappa_n = -\frac{\Psi_n^2}{c}. \tag{40}$$

The coordinate components of the velocity are

$$\begin{aligned}\dot{r} &= \frac{\sigma_n}{U_n}(W_n^2 - X_n V_n)^{1/2}, \\ \dot{x}_\mu &= \frac{\sigma_\mu}{|U_\mu|}(X_\mu V_\mu - W_\mu^2)^{1/2}, \\ \dot{\psi}_k &= \sum_{\mu=1}^{n-1}\frac{(-x_\mu^2)^{n-1-k}}{U_\mu X_\mu}W_\mu - \frac{r^{2(n-1-k)}}{U_n X_n}W_n - \varepsilon\frac{\Psi_n}{cA^{(n)}}\delta_{kn}.\end{aligned} \tag{41}$$

One can symbolically integrate equations for ψ_k. Let f be an arbitrary function of r and x_ν, obeying

$$\dot{f} = \frac{f_n(r)}{U_n} + \sum_{\nu=1}^{n-1}\frac{f_\nu(x_\nu)}{U_\nu}. \tag{42}$$

Then f can be written as (see Appendix C in [21])

$$f = \int \frac{\sigma_n f_n dr}{\sqrt{W_n^2 - X_n V_n}} + \sum_{\nu=1}^{n-1}\int \frac{\sigma_\nu \operatorname{sign}(U_\nu) f_\nu dx_\nu}{\sqrt{X_\nu V_\nu - W_\nu^2}}. \tag{43}$$

In particular, using the following identities (see, e.g., [34]):

$$\frac{1}{A^{(n)}} = -\frac{1}{r^2 U_n} + \sum_{\nu=1}^{n-1} \frac{1}{x_\nu^2 U_\nu},$$
$$1 = \frac{r^{2(n-1)}}{U_n} + \sum_{\mu=1}^{n-1} \frac{(-x_\mu^2)^{n-1}}{U_\mu}, \tag{44}$$

we find that

$$\psi_k = \int \frac{\sigma_n f_n^{(k)} dr}{\sqrt{W_n^2 - X_n V_n}} + \sum_{\mu=1}^{n} \int \frac{\sigma_\mu \,\mathrm{sign}(U_\mu) f_\mu^{(k)} dx_\mu}{\sqrt{X_\mu V_\mu - W_\mu^2}},$$
$$f_n^{(k)} = -\frac{W_n}{X_n} r^{2(n-1-k)} + \varepsilon \frac{\Psi_n}{c r^2} \delta_{kn},$$
$$f_\mu^{(k)} = \frac{W_\mu}{X_\mu} (-x_\mu^2)^{n-1-k} - \varepsilon \frac{\Psi_n}{c x_\mu^2} \delta_{kn}. \tag{45}$$

Similarly, we have

$$\tau = \int \frac{\sigma_n r^{2(n-1)} dr}{\sqrt{W_n^2 - X_n V_n}} + \sum_{\nu=1}^{n-1} \int \frac{\sigma_\nu \,\mathrm{sign}(U_\nu)(-x_\nu^2)^{n-1} dx_\nu}{\sqrt{X_\nu V_\nu - W_\nu^2}}. \tag{46}$$

V. PARALLEL TRANSPORT IN KERR-NUT-(A)DS SPACETIMES

A. Parallel-propagated frame

We shall construct the parallel-propagated frame for a geodesic motion in four steps. First, to simplify the calculations, we use the freedom of local 2D rotations in the KY 2-planes of $\boldsymbol{h}$ to introduce the *velocity adapted* canonical basis in which n components of the velocity vanish. Next, we generate parallel-transported vectors $\boldsymbol{m}$, $\boldsymbol{n}$, and possibly $\boldsymbol{z}$. In the third step, by studying the eigenvalue problem for the operator $\boldsymbol{F}^2$, we find the orthonormal 1-forms $\{\boldsymbol{\zeta}^{\hat{i}}, \tilde{\boldsymbol{\zeta}}^{\hat{i}}\}$ spanning each of the 2D Darboux subspaces V_i. Finally, in each V_i we rotate these 1-forms by an (affine parameter)-dependent rotation to obtain the (dual) parallel-transported frame.[2]

1. Velocity adapted canonical basis

In order to construct the velocity adapted canonical basis we perform the boost transformation in the $\{\tilde{\boldsymbol{\omega}}^{\hat{n}}, \boldsymbol{\omega}^{\hat{n}}\}$ 2-plane and the rotation transformations in each of the $\{\tilde{\boldsymbol{\omega}}^{\hat{\mu}}, \boldsymbol{\omega}^{\hat{\mu}}\}$ 2-planes

$$\tilde{\boldsymbol{o}}^{\hat{n}} = \cosh\alpha_n \tilde{\boldsymbol{\omega}}^{\hat{n}} + \sinh\alpha_n \boldsymbol{\omega}^{\hat{n}},$$
$$\boldsymbol{o}^{\hat{n}} = \sinh\alpha_n \tilde{\boldsymbol{\omega}}^{\hat{n}} + \cosh\alpha_n \boldsymbol{\omega}^{\hat{n}},$$
$$\tilde{\boldsymbol{o}}^{\hat{\mu}} = \cos\alpha_\mu \tilde{\boldsymbol{\omega}}^{\hat{\mu}} + \sin\alpha_\mu \boldsymbol{\omega}^{\hat{\mu}},$$
$$\boldsymbol{o}^{\hat{\mu}} = -\sin\alpha_\mu \tilde{\boldsymbol{\omega}}^{\hat{\mu}} + \cos\alpha_\mu \boldsymbol{\omega}^{\hat{\mu}}, \qquad \boldsymbol{o}^{\hat{\epsilon}} = \boldsymbol{\omega}^{\hat{\epsilon}}. \tag{47}$$

[2]In our setup it is somewhat more natural to work with 1-forms. One could, of course, similarly construct the parallel-transported frame of vectors.

Here, we choose

$$\cosh\alpha_n = \frac{\tilde{l}_{\hat{n}}}{\tilde{k}_{\hat{n}}}, \qquad \sinh\alpha_n = \frac{l_{\hat{n}}}{\tilde{k}_{\hat{n}}},$$
$$\cos\alpha_\mu = \frac{\tilde{l}_{\hat{\mu}}}{\tilde{k}_{\hat{\mu}}}, \qquad \sin\alpha_\mu = \frac{l_{\hat{\mu}}}{\tilde{k}_{\hat{\mu}}}, \tag{48}$$

and

$$\tilde{k}_{\hat{n}} = -\sqrt{\tilde{l}_{\hat{n}}^2 - l_{\hat{n}}^2} = -\sqrt{\frac{V_n}{U_n}}, \qquad \tilde{k}_{\hat{\mu}} = \sqrt{\tilde{l}_{\hat{\mu}}^2 + l_{\hat{\mu}}^2} = \sqrt{\frac{V_\mu}{U_\mu}}. \tag{49}$$

Such a transformation preserves the form of the metric as well as the form of the PCKY tensor.

$$\boldsymbol{g} = \sum_{\mu=1}^{n-1} (\boldsymbol{o}^{\hat{\mu}} \boldsymbol{o}^{\hat{\mu}} + \tilde{\boldsymbol{o}}^{\hat{\mu}} \tilde{\boldsymbol{o}}^{\hat{\mu}}) + \boldsymbol{o}^{\hat{n}} \boldsymbol{o}^{\hat{n}} - \tilde{\boldsymbol{o}}^{\hat{n}} \tilde{\boldsymbol{o}}^{\hat{n}} + \varepsilon \boldsymbol{o}^{\hat{\epsilon}} \boldsymbol{o}^{\hat{\epsilon}},$$
$$\boldsymbol{h} = \sum_{\mu=1}^{n-1} x_\mu \boldsymbol{o}^{\hat{\mu}} \wedge \tilde{\boldsymbol{o}}^{\hat{\mu}} - r \boldsymbol{o}^{\hat{n}} \wedge \tilde{\boldsymbol{o}}^{\hat{n}}. \tag{50}$$

Hence, the basis $\{\boldsymbol{o}\}$ is still canonical. Moreover, one obtains the following form of the velocity:

$$\boldsymbol{l}^\flat = \sum_{\mu=1}^{n} \tilde{k}_{\hat{\mu}} \tilde{\boldsymbol{o}}^{\hat{\mu}} + \varepsilon l_{\hat{\epsilon}} \boldsymbol{o}^{\hat{\epsilon}}. \tag{51}$$

This form simplifies considerably the subsequent calculations, especially the task of solving the eigenvalue problem for $\boldsymbol{F}^2$. We remark, that in the adapted basis $\{\boldsymbol{o}\}$ the components of the velocity depend on constants κ_j only; the constants Ψ_j and σ_μ are absorbed in the definition of the new frame.

2. Parallel-transported vectors in V_0

The eigenspace V_0 is spread by $\boldsymbol{l}$, the vectors $\boldsymbol{m}$ and $\boldsymbol{n}$ given by (9) and (12), and, in an even number of spacetime dimensions, by $\boldsymbol{z}$ (20). Let us express these vectors in the velocity adapted basis (47). The vector $\boldsymbol{l}$ is given by (49) and (51). Using (50) and (51), we find

$$\boldsymbol{m}^\flat = \frac{1}{\sqrt{-\kappa_1}} \Big[\sum_{\mu=1}^{n-1} (\tilde{k}_{\hat{\mu}} \beta \tilde{\boldsymbol{o}}^{\hat{\mu}} - \tilde{k}_{\hat{\mu}} x_\mu \boldsymbol{o}^{\hat{\mu}}) + \tilde{k}_{\hat{n}} \beta \tilde{\boldsymbol{o}}^{\hat{n}} - \tilde{k}_{\hat{n}} r \boldsymbol{o}^{\hat{n}} + \varepsilon \beta l_{\hat{\epsilon}} \boldsymbol{o}^{\hat{\epsilon}} \Big], \tag{52}$$

$$\boldsymbol{n}^\flat = \sum_{\mu=1}^{n-1} \Big[\tilde{k}_{\hat{\mu}} \Big(C + \frac{x_\mu^2}{\kappa_1} \Big) \tilde{\boldsymbol{o}}^{\hat{\mu}} + \tilde{k}_{\hat{\mu}} \frac{\beta x_\mu}{\kappa_1} \boldsymbol{o}^{\hat{\mu}} \Big] + \tilde{k}_{\hat{n}} \Big(C - \frac{r^2}{\kappa_1} \Big) \tilde{\boldsymbol{o}}^{\hat{n}} + \tilde{k}_{\hat{n}} \frac{\beta r}{\kappa_1} \boldsymbol{o}^{\hat{n}} + \varepsilon C l_{\hat{\epsilon}} \boldsymbol{o}^{\hat{\epsilon}}, \tag{53}$$

where

$$C = \frac{1}{2\kappa_1^2}\left(-r^4\tilde{k}_{\hat{n}}^2 + \sum_{\mu=1}^{n-1} x_\mu^4 \tilde{k}_{\hat{\mu}}^2 - \kappa_1\beta^2\right). \tag{54}$$

Moreover, using Eq. (33) we find

$$\dot{\beta} = \boldsymbol{l}\cdot\boldsymbol{\xi} = \boldsymbol{l}\cdot\boldsymbol{\partial}_{\psi_0} = \Psi_0,$$

and so

$$\beta = \Psi_0\tau. \tag{55}$$

Using Eq. (46) one can express this angle as a function of r and x_μ.

In an even number of spacetime dimensions we have an additional vector $\boldsymbol{z}$. We find

$$\boldsymbol{f} \propto x_1\ldots x_{n-1}\boldsymbol{o}^{\hat{n}}\wedge\tilde{\boldsymbol{o}}^{\hat{n}} + \sum_{\mu=1}^{n-1} x_1\ldots\check{x}_\mu\ldots x_{n-1}\boldsymbol{o}^{\hat{\mu}}\wedge\tilde{\boldsymbol{o}}^{\hat{\mu}}. \tag{56}$$

Here, the symbol $\propto$ means equality up to a constant factor, and $\check{x}_\mu$ denotes that in the sum over μ, x_μ is replaced by r. In consequence, we have the following expression for the (normalized) vector $\boldsymbol{z}$:

$$\boldsymbol{z}^\flat = \frac{1}{\sqrt{-\kappa_{n-1}}}\left(x_1\ldots x_{n-1}\tilde{k}_{\hat{n}}\boldsymbol{o}^{\hat{n}} - \sum_{\mu=1}^{n-1} x_1\ldots\check{x}_\mu\ldots x_{n-1}\tilde{k}_{\hat{\mu}}\boldsymbol{o}^{\hat{\mu}}\right). \tag{57}$$

Using the transformation inverse to (47),

$$\begin{aligned}
\tilde{\boldsymbol{\omega}}^{\hat{n}} &= \cosh\alpha_n\tilde{\boldsymbol{o}}^{\hat{n}} - \sinh\alpha_n\boldsymbol{o}^{\hat{n}},\\
\boldsymbol{\omega}^{\hat{n}} &= -\sinh\alpha_n\tilde{\boldsymbol{o}}^{\hat{n}} + \cosh\alpha_n\boldsymbol{o}^{\hat{n}},\\
\tilde{\boldsymbol{\omega}}^{\hat{\mu}} &= \cos\alpha_\mu\tilde{\boldsymbol{o}}^{\hat{\mu}} - \sin\alpha_\mu\boldsymbol{o}^{\hat{\mu}},\\
\boldsymbol{\omega}^{\hat{\mu}} &= \sin\alpha_\mu\tilde{\boldsymbol{o}}^{\hat{\mu}} + \cos\alpha_\mu\boldsymbol{o}^{\hat{\mu}}, \qquad \boldsymbol{\omega}^{\hat{\epsilon}} = \boldsymbol{o}^{\hat{\epsilon}},
\end{aligned} \tag{58}$$

one can easily obtain the above parallel-transported vectors in the principal basis $\{\boldsymbol{\omega}\}$.

3. Darboux subspaces V_i

Using (51) and (53) one can write down $\boldsymbol{F}$ in the velocity adapted basis. The general expression is quite involved and therefore we do not state it here. What is important is that one can show that $\boldsymbol{F}$ is independent of β. Let us denote $\{\boldsymbol{\zeta}\}$ the (dual) Darboux basis of $\boldsymbol{F}$. Then, for a generic geodesic, we have

$$\boldsymbol{F} = \sum_{i=1}^{n-2+\varepsilon} \lambda_i\boldsymbol{\zeta}^{\hat{i}}\wedge\tilde{\boldsymbol{\zeta}}^{\hat{i}}. \tag{59}$$

Here, $\{\boldsymbol{\zeta}^{\hat{i}}, \tilde{\boldsymbol{\zeta}}^{\hat{i}}\}$ are orthonormal vectors spanning the Darboux subspace V_i, $\lambda_i > 0$ are all different and correspond to the eigenvalues of $\boldsymbol{F}^2$; $\boldsymbol{F}^2\boldsymbol{v}_i = -\lambda_i^2\boldsymbol{v}_i$, $\boldsymbol{v}_i \in V_i$. Obtaining the general form of $\{\boldsymbol{\zeta}^{\hat{i}}, \tilde{\boldsymbol{\zeta}}^{\hat{i}}\}$ is the biggest obstacle in writing down a general formula for the parallel-propagated basis in an arbitrary number of dimensions. Concrete examples are in the next section.

4. Parallel-transported basis

In order to construct parallel-transported vectors in each of the Darboux subspaces V_i we perform the rotation

$$\begin{aligned}
\boldsymbol{\pi}^{\hat{i}} &= \cos\gamma_i\boldsymbol{\zeta}^{\hat{i}} - \sin\gamma_i\tilde{\boldsymbol{\zeta}}^{\hat{i}} \qquad \tilde{\boldsymbol{\pi}}^{\hat{i}} = \sin\gamma_i\boldsymbol{\zeta}^{\hat{i}} + \cos\gamma_i\tilde{\boldsymbol{\zeta}}^{\hat{i}},\\
\dot{\gamma}_i &= \dot{\boldsymbol{\zeta}}^{\hat{i}}\cdot\tilde{\boldsymbol{\zeta}}^{\hat{i}} = -\boldsymbol{\zeta}^{\hat{i}}\cdot\dot{\tilde{\boldsymbol{\zeta}}}^{\hat{i}},
\end{aligned} \tag{60}$$

with the initial conditions $\gamma_i(\tau = 0) = 0$.

When $\dot{\gamma}_i$ given by the last equation can be written in the form

$$\dot{\gamma}_i = M_i\left[(r^2+\lambda_i^2)\prod_{\mu=1}^{n-1}(x_\mu^2-\lambda_i^2)\right]^{-1}, \tag{61}$$

where M_i is some constant, we can use the identity

$$\left[(r^2+\lambda^2)\prod_{\mu=1}^{n-1}(x_\mu^2-\lambda^2)\right]^{-1} = \frac{1}{(r^2+\lambda^2)U_n} - \sum_{\mu=1}^{n-1}\frac{1}{(x_\mu^2-\lambda^2)U_\mu}, \tag{62}$$

to symbolically integrate [cf. Eqs. (42) and (43)]

$$\begin{aligned}
\gamma_i &= \int\frac{\sigma_n\gamma_n^{(i)}dr}{\sqrt{W_n^2 - X_nV_n}} - \sum_{\nu=1}^{n-1}\int\frac{\sigma_\nu\,\mathrm{sign}(U_\nu)\gamma_\nu^{(i)}dx_\nu}{\sqrt{X_\nu V_\nu - W_\nu^2}},\\
\gamma_n^{(i)} &= \frac{M_i}{r^2+\lambda_i^2}, \qquad \gamma_\nu^{(i)} = \frac{M_i}{x_\nu^2-\lambda_i^2}.
\end{aligned} \tag{63}$$

We shall now give explicit examples of parallel-propagated frames in $D = 4, 5, 6$ canonical spacetimes.

VI. SPECIAL CASES

A. Parallel transport in 4D

The parallel transport along generic geodesics in the four-dimensional canonical spacetime, derived earlier in [23,24], is from the point of view of the above described theory trivial. We write it only for completeness and because it encapsulates the important subcase of parallel transport in the Carter's class of solutions [35,36]—describing, among others, a 4D rotating charged black hole in the cosmological background (see also Appendix C).

The metric reads

$$\boldsymbol{g} = -\tilde{\boldsymbol{\omega}}^{\hat{2}}\tilde{\boldsymbol{\omega}}^{\hat{2}} + \boldsymbol{\omega}^{\hat{2}}\boldsymbol{\omega}^{\hat{2}} + \tilde{\boldsymbol{\omega}}^{\hat{1}}\tilde{\boldsymbol{\omega}}^{\hat{1}} + \boldsymbol{\omega}^{\hat{1}}\boldsymbol{\omega}^{\hat{1}}, \tag{64}$$

where

$$\tilde{\boldsymbol{\omega}}^{\hat{2}} = \sqrt{\frac{X_2}{U_2}}(\boldsymbol{d}\psi_0 + x_1^2\boldsymbol{d}\psi_1), \qquad \boldsymbol{\omega}^{\hat{2}} = \sqrt{\frac{U_2}{X_2}}\boldsymbol{d}r,$$
$$\tilde{\boldsymbol{\omega}}^{\hat{1}} = \sqrt{\frac{X_1}{U_1}}(\boldsymbol{d}\psi_0 - r^2\boldsymbol{d}\psi_1), \qquad \boldsymbol{\omega}^{\hat{1}} = \sqrt{\frac{U_1}{X_1}}\boldsymbol{d}x_1, \tag{65}$$

and $U_2 = -U_1 = x_1^2 + r^2$. The PCKY tensor is

$$\boldsymbol{h} = x_1\boldsymbol{\omega}^{\hat{1}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{1}} - r\boldsymbol{\omega}^{\hat{2}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{2}}. \tag{66}$$

The components of the velocity are

$$\tilde{l}_{\hat{2}} = \frac{W_2}{\sqrt{X_2U_2}}, \qquad l_{\hat{2}} = \frac{\sigma_2}{\sqrt{X_2U_2}}\sqrt{W_2^2 - X_2V_2},$$
$$\tilde{l}_{\hat{1}} = \frac{-W_1}{\sqrt{X_1U_1}}, \qquad l_{\hat{1}} = \frac{\sigma_1}{\sqrt{X_1U_1}}\sqrt{X_1V_1 - W_1^2}, \tag{67}$$

where

$$W_2 = r^2\Psi_0 + \Psi_1, \qquad V_2 = -\kappa_1 > 0,$$
$$W_1 = -x_1^2\Psi_0 + \Psi_1, \qquad V_1 = \kappa_1. \tag{68}$$

In the velocity adapted frame $\{\boldsymbol{o}\}$, (47), the parallel-transported frame reads

$$\boldsymbol{l}^{\flat} = \tilde{k}_{\hat{1}}(-\tilde{\boldsymbol{o}}^{\hat{2}} + \tilde{\boldsymbol{o}}^{\hat{1}}), \qquad \tilde{k}_{\hat{1}} = -\tilde{k}_{\hat{2}} = \sqrt{\frac{V_2}{U_2}} = \sqrt{\frac{-\kappa_1}{U_2}},$$
$$\boldsymbol{m}^{\flat} = \frac{\tilde{k}_{\hat{1}}}{\sqrt{-\kappa_1}}(-\beta\tilde{\boldsymbol{o}}^{\hat{2}} + r\boldsymbol{o}^{\hat{2}} + \beta\tilde{\boldsymbol{o}}^{\hat{1}} - x_1\boldsymbol{o}^{\hat{1}}),$$
$$\boldsymbol{n}^{\flat} = \frac{\tilde{k}_{\hat{1}}}{\kappa_1}\left(\frac{U_2 + \beta^2}{2}\tilde{\boldsymbol{o}}^{\hat{2}} - \beta r\boldsymbol{o}^{\hat{2}} + \frac{U_2 - \beta^2}{2}\tilde{\boldsymbol{o}}^{\hat{1}} + \beta x_1\boldsymbol{o}^{\hat{1}}\right),$$
$$\boldsymbol{z}^{\flat} = \frac{\tilde{k}_{\hat{1}}}{\sqrt{-\kappa_1}}(x_1\boldsymbol{o}^{\hat{2}} + r\boldsymbol{o}^{\hat{1}}), \tag{69}$$

where $\beta = \Psi_0\tau$, or, in terms of r and x_1,

$$\beta = \int \frac{\sigma_2\Psi_0 r^2 dr}{\sqrt{W_2^2 - X_2V_2}} + \int \frac{\sigma_1\Psi_0 x_1^2 dx_1}{\sqrt{X_1V_1 - W_1^2}}. \tag{70}$$

B. Parallel transport in 5D

Next, we consider the 5D canonical spacetime. The metric reads

$$\boldsymbol{g} = -\tilde{\boldsymbol{\omega}}^{\hat{2}}\tilde{\boldsymbol{\omega}}^{\hat{2}} + \boldsymbol{\omega}^{\hat{2}}\boldsymbol{\omega}^{\hat{2}} + \tilde{\boldsymbol{\omega}}^{\hat{1}}\tilde{\boldsymbol{\omega}}^{\hat{1}} + \boldsymbol{\omega}^{\hat{1}}\boldsymbol{\omega}^{\hat{1}} + \boldsymbol{\omega}^{\hat{\epsilon}}\boldsymbol{\omega}^{\hat{\epsilon}}, \tag{71}$$

where

$$\tilde{\boldsymbol{\omega}}^{\hat{2}} = \sqrt{\frac{X_2}{U_2}}(\boldsymbol{d}\psi_0 + x_1^2\boldsymbol{d}\psi_1), \qquad \boldsymbol{\omega}^{\hat{2}} = \sqrt{\frac{U_2}{X_2}}\boldsymbol{d}r,$$
$$\tilde{\boldsymbol{\omega}}^{\hat{1}} = \sqrt{\frac{X_1}{U_1}}(\boldsymbol{d}\psi_0 - r^2\boldsymbol{d}\psi_1), \qquad \boldsymbol{\omega}^{\hat{1}} = \sqrt{\frac{U_1}{X_1}}\boldsymbol{d}x_1,$$
$$\boldsymbol{\omega}^{\hat{\epsilon}} = \frac{\sqrt{c}}{rx_1}[\boldsymbol{d}\psi_0 + (x_1^2 - r^2)\boldsymbol{d}\psi_1 - x_1^2r^2\boldsymbol{d}\psi_2], \tag{72}$$

and $U_2 = -U_1 = x_1^2 + r^2$. The PCKY tensor is

$$\boldsymbol{h} = x_1\boldsymbol{\omega}^{\hat{1}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{1}} - r\boldsymbol{\omega}^{\hat{2}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{2}}. \tag{73}$$

The components of the velocity are

$$\tilde{l}_{\hat{2}} = \frac{W_2}{\sqrt{X_2U_2}}, \qquad l_{\hat{2}} = \frac{\sigma_2}{\sqrt{X_2U_2}}\sqrt{W_2^2 - X_2V_2},$$
$$\tilde{l}_{\hat{1}} = \frac{-W_1}{\sqrt{X_1U_1}}, \qquad l_{\hat{1}} = \frac{\sigma_1}{\sqrt{X_1U_1}}\sqrt{X_1V_1 - W_1^2}, \tag{74}$$
$$l_{\hat{\epsilon}} = -\frac{\Psi_2}{\sqrt{c}x_1r},$$

where

$$W_1 = -x_1^2\Psi_0 + \Psi_1 - \frac{\Psi_2}{x_1^2}, \qquad V_1 = \kappa_1 + \frac{\Psi_2^2}{cx_1^2},$$
$$W_2 = r^2\Psi_0 + \Psi_1 + \frac{\Psi_2}{r^2}, \qquad V_2 = -\kappa_1 + \frac{\Psi_2^2}{cr^2}. \tag{75}$$

In the velocity adapted frame $\{\boldsymbol{o}\}$, (47), we have

$$\boldsymbol{l}^{\flat} = \tilde{k}_{\hat{2}}\tilde{\boldsymbol{o}}^{\hat{2}} + \tilde{k}_{\hat{1}}\tilde{\boldsymbol{o}}^{\hat{1}} + l_{\hat{\epsilon}}\boldsymbol{o}^{\hat{\epsilon}},$$
$$\boldsymbol{m}^{\flat} = \frac{1}{\sqrt{-\kappa_1}}(\beta\tilde{k}_{\hat{2}}\tilde{\boldsymbol{o}}^{\hat{2}} - r\tilde{k}_{\hat{2}}\boldsymbol{o}^{\hat{2}} + \beta\tilde{k}_{\hat{1}}\tilde{\boldsymbol{o}}^{\hat{1}} - x_1\tilde{k}_{\hat{1}}\boldsymbol{o}^{\hat{1}} + \beta l_{\hat{\epsilon}}\boldsymbol{o}^{\hat{\epsilon}}),$$
$$\boldsymbol{n}^{\flat} = \tilde{k}_{\hat{2}}\left(C - \frac{r^2}{\kappa_1}\right)\tilde{\boldsymbol{o}}^{\hat{2}} + \tilde{k}_{\hat{2}}\frac{\beta r}{\kappa_1}\boldsymbol{o}^{\hat{2}} + \tilde{k}_{\hat{1}}\left(C + \frac{x_1^2}{\kappa_1}\right)\tilde{\boldsymbol{o}}^{\hat{1}} + \tilde{k}_{\hat{1}}\frac{\beta x_1}{\kappa_1}\boldsymbol{o}^{\hat{1}} + Cl_{\hat{\epsilon}}\boldsymbol{o}^{\hat{\epsilon}}, \tag{76}$$

where

$$\tilde{k}_{\hat{2}} = -\sqrt{\frac{V_2}{U_2}}, \qquad \tilde{k}_{\hat{1}} = \sqrt{\frac{V_1}{U_1}},$$
$$C = \frac{1}{2\kappa_1^2}(-r^4\tilde{k}_{\hat{2}}^2 + x_1^4\tilde{k}_{\hat{1}}^2 - \kappa_1\beta^2), \tag{77}$$
$$\beta = \int \frac{\sigma_2\Psi_0 r^2 dr}{\sqrt{W_2^2 - X_2V_2}} + \int \frac{\sigma_1\Psi_0 x_1^2 dx_1}{\sqrt{X_1V_1 - W_1^2}}.$$

The 2-form $\boldsymbol{F}$ reads

$$\boldsymbol{F} = \lambda\boldsymbol{\zeta} \wedge \tilde{\boldsymbol{\zeta}}, \qquad \lambda = \frac{|\Psi_2|}{\sqrt{-c\kappa_1}}, \tag{78}$$

where

$$\boldsymbol{\zeta} = \frac{\sqrt{(r^2 + \lambda^2)(x_1^2 - \lambda^2)}}{rx_1}\left(-\frac{x_1F_r}{\sqrt{U_2}}\tilde{\boldsymbol{o}}^{\hat{2}} + \frac{rF_{x_1}}{\sqrt{U_2}}\tilde{\boldsymbol{o}}^{\hat{1}} + \boldsymbol{o}^{\hat{\epsilon}}\right),$$
$$\tilde{\boldsymbol{\zeta}} = \frac{\sqrt{r^2 + \lambda^2}}{\sqrt{U_2}}\left(-\frac{\lambda^2}{r^2 + \lambda^2}\frac{1}{F_rF_{x_1}}\boldsymbol{o}^{\hat{2}} + \boldsymbol{o}^{\hat{1}}\right). \tag{79}$$

Here we have introduced

$$F_r = \frac{x_1 l_{\hat{\epsilon}}}{\sqrt{V_2}}, \qquad F_{x_1} = -\frac{r l_{\hat{\epsilon}}}{\sqrt{-V_1}}, \tag{80}$$

which are functions of r, x_1, respectively. Using (60) we find

$$\begin{aligned} \dot{\gamma} &= \frac{M}{(x_1^2-\lambda^2)(r^2+\lambda^2)}, \\ M &= \frac{\lambda^2\Psi_2\Psi_0 - \Psi_2\Psi_1 - c\kappa_1}{\sqrt{-c\kappa_1}}, \end{aligned} \tag{81}$$

which is of the form (61). Therefore, the parallel-propagated forms $\{\boldsymbol{\pi}, \tilde{\boldsymbol{\pi}}\}$ are given by (60), where

$$\begin{aligned} \gamma &= \int \frac{\sigma_2 \gamma_r dr}{\sqrt{W_2^2 - X_2 V_2}} + \int \frac{\sigma_1 \gamma_{x_1} dx_1}{\sqrt{X_1 V_1 - W_1^2}}, \\ \gamma_r &= \frac{M}{r^2+\lambda^2}, \qquad \gamma_{x_1} = \frac{M}{x_1^2-\lambda^2}. \end{aligned} \tag{82}$$

C. Parallel transport in 6D

Finally we consider the 6D canonical spacetime. The metric reads

$$\begin{aligned} \boldsymbol{g} &= -\tilde{\boldsymbol{\omega}}^{\hat{3}}\tilde{\boldsymbol{\omega}}^{\hat{3}} + \boldsymbol{\omega}^{\hat{3}}\boldsymbol{\omega}^{\hat{3}} + \sum_{\mu=1}^{2}(\boldsymbol{\omega}^{\hat{\mu}}\boldsymbol{\omega}^{\hat{\mu}} + \tilde{\boldsymbol{\omega}}^{\hat{\mu}}\tilde{\boldsymbol{\omega}}^{\hat{\mu}}), \\ \boldsymbol{\omega}^{\hat{3}} &= \sqrt{\frac{U_3}{X_3}}\boldsymbol{d}r, \qquad \boldsymbol{\omega}^{\hat{2}} = \sqrt{\frac{U_2}{X_2}}\boldsymbol{d}x_2, \qquad \boldsymbol{\omega}^{\hat{1}} = \sqrt{\frac{U_1}{X_1}}\boldsymbol{d}x_1, \\ \tilde{\boldsymbol{\omega}}^{\hat{3}} &= \sqrt{\frac{X_3}{U_3}}(\boldsymbol{d}\psi_0 + A_r\boldsymbol{d}\psi_1 + x_1^2x_2^2\boldsymbol{d}\psi_2), \\ \tilde{\boldsymbol{\omega}}^{\hat{2}} &= \sqrt{\frac{X_2}{U_2}}(\boldsymbol{d}\psi_0 + A_{x_2}\boldsymbol{d}\psi_1 - x_1^2r^2\boldsymbol{d}\psi_2), \\ \tilde{\boldsymbol{\omega}}^{\hat{1}} &= \sqrt{\frac{X_1}{U_1}}(\boldsymbol{d}\psi_0 + A_{x_1}\boldsymbol{d}\psi_1 - x_2^2r^2\boldsymbol{d}\psi_2). \end{aligned} \tag{83}$$

Here

$$\begin{aligned} A_r &= x_1^2 + x_2^2, \qquad A_{x_1} = x_2^2 - r^2, \\ A_{x_2} &= x_1^2 - r^2, \qquad U_3 = (x_1^2+r^2)(x_2^2+r^2), \\ U_2 &= -(x_2^2+r^2)(x_1^2-x_2^2), \qquad U_1 = (x_1^2+r^2)(x_1^2-x_2^2). \end{aligned}$$

The PCKY tensor is

$$\boldsymbol{h} = x_1\boldsymbol{\omega}^{\hat{1}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{1}} + x_2\boldsymbol{\omega}^{\hat{2}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{2}} - r\boldsymbol{\omega}^{\hat{3}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{3}}. \tag{84}$$

The components of the velocity are

$$\begin{aligned} \tilde{l}_{\hat{3}} &= \frac{W_3}{\sqrt{X_3U_3}}, \qquad l_{\hat{3}} = \frac{\sigma_3}{\sqrt{X_3U_3}}\sqrt{W_3^2 - X_3V_3}, \\ \tilde{l}_{\hat{2}} &= -\frac{W_2}{\sqrt{X_2U_2}}, \qquad l_{\hat{2}} = \frac{\sigma_2}{\sqrt{X_2U_2}}\sqrt{X_2V_2 - W_2^2}, \\ \tilde{l}_{\hat{1}} &= \frac{W_1}{\sqrt{X_1U_1}}, \qquad l_{\hat{1}} = \frac{\sigma_1}{\sqrt{X_1U_1}}\sqrt{X_1V_1 - W_1^2}, \end{aligned} \tag{85}$$

where

$$\begin{aligned} W_3 &= r^4\Psi_0 + r^2\Psi_1 + \Psi_2, \qquad V_3 = -r^2\kappa_1 - \kappa_2, \\ W_2 &= x_2^4\Psi_0 - x_2^2\Psi_1 + \Psi_2, \qquad V_2 = -x_2^2\kappa_1 + \kappa_2, \\ W_1 &= x_1^4\Psi_0 - x_1^2\Psi_1 + \Psi_2, \qquad V_1 = -x_1^2\kappa_1 + \kappa_2. \end{aligned} \tag{86}$$

The parallel-transported vectors spanning V_0 are

$$\begin{aligned} \boldsymbol{l}^{\flat} &= \tilde{k}_{\hat{3}}\tilde{\boldsymbol{o}}^{\hat{3}} + \tilde{k}_{\hat{2}}\tilde{\boldsymbol{o}}^{\hat{2}} + \tilde{k}_{\hat{1}}\tilde{\boldsymbol{o}}^{\hat{1}}, \\ \boldsymbol{m}^{\flat} &= \frac{1}{\sqrt{-\kappa_1}}(\beta\tilde{k}_{\hat{3}}\tilde{\boldsymbol{o}}^{\hat{3}} - r\tilde{k}_{\hat{3}}\boldsymbol{o}^{\hat{3}} + \beta\tilde{k}_{\hat{2}}\tilde{\boldsymbol{o}}^{\hat{2}} - x_2\tilde{k}_{\hat{2}}\boldsymbol{o}^{\hat{2}} \\ &\quad + \beta\tilde{k}_{\hat{1}}\tilde{\boldsymbol{o}}^{\hat{1}} - x_1\tilde{k}_{\hat{1}}\boldsymbol{o}^{\hat{1}}), \\ \boldsymbol{n}^{\flat} &= \tilde{k}_{\hat{3}}\left(C - \frac{r^2}{\kappa_1}\right)\tilde{\boldsymbol{o}}^{\hat{3}} + \tilde{k}_{\hat{3}}\frac{\beta r}{\kappa_1}\boldsymbol{o}^{\hat{3}} \\ &\quad + \sum_{\mu=1}^{2}\left[\tilde{k}_{\hat{\mu}}\left(C + \frac{x_\mu^2}{\kappa_1}\right)\tilde{\boldsymbol{o}}^{\hat{\mu}} + \tilde{k}_{\hat{\mu}}\frac{\beta x_\mu}{\kappa_1}\boldsymbol{o}^{\hat{\mu}}\right], \\ \boldsymbol{z}^{\flat} &= \frac{1}{\sqrt{-\kappa_2}}(x_1x_2\tilde{k}_{\hat{3}}\boldsymbol{o}^{\hat{3}} - rx_1\tilde{k}_{\hat{2}}\boldsymbol{o}^{\hat{2}} - rx_2\tilde{k}_{\hat{1}}\boldsymbol{o}^{\hat{1}}), \end{aligned} \tag{87}$$

where

$$\begin{aligned} \tilde{k}_{\hat{3}} &= -\sqrt{\frac{V_3}{U_3}}, \qquad \tilde{k}_{\hat{2}} = \sqrt{\frac{V_2}{U_2}}, \qquad \tilde{k}_{\hat{1}} = \sqrt{\frac{V_1}{U_1}}, \\ C &= \frac{1}{2\kappa_1^2}(-r^4\tilde{k}_{\hat{n}}^2 + x_2^4\tilde{k}_{\hat{2}}^2 + x_1^4\tilde{k}_{\hat{1}}^2 - \kappa_1\beta^2), \\ \beta &= \int \frac{\Psi_0\sigma_3 r^4 dr}{\sqrt{W_3^2 - X_3V_3}} - \int \frac{\Psi_0\sigma_2 x_2^4 dx_2}{\sqrt{X_2V_2 - W_2^2}} \\ &\quad + \int \frac{\Psi_0\sigma_1 x_1^4 dx_1}{\sqrt{X_1V_1 - W_1^2}}. \end{aligned}$$

The 2-form $\boldsymbol{F}$ reads

$$\boldsymbol{F} = \lambda\boldsymbol{\zeta} \wedge \tilde{\boldsymbol{\zeta}}, \qquad \lambda = \sqrt{\frac{\kappa_2}{\kappa_1}}, \tag{88}$$

where

$$\boldsymbol{\zeta} = \sqrt{\frac{(r^2+\lambda^2)(\lambda^2-x_2^2)}{U_1}}(F_1\tilde{\boldsymbol{o}}^{\hat{3}} + F_2\tilde{\boldsymbol{o}}^{\hat{2}} + \tilde{\boldsymbol{o}}^{\hat{1}}),$$
$$\tilde{\boldsymbol{\zeta}} = \frac{1}{\lambda}\sqrt{\frac{(r^2+\lambda^2)(\lambda^2-x_2^2)}{U_1}}(rF_1\boldsymbol{o}^{\hat{3}} + x_2F_2\boldsymbol{o}^{\hat{2}} + x_1\boldsymbol{o}^{\hat{1}}). \tag{89}$$

Here we have introduced

$$F_1 = \frac{U_1\tilde{k}_{\hat{1}}\tilde{k}_{\hat{3}}}{\kappa_1(r^2+\lambda^2)}, \qquad F_2 = \frac{U_1\tilde{k}_{\hat{2}}\tilde{k}_{\hat{1}}}{\kappa_1(\lambda^2-x_2^2)}. \tag{90}$$

Using Eq. (60) we find

$$\dot{\gamma} = \frac{-\lambda(\Psi_2 - \lambda^2\Psi_1 + \lambda^4\Psi_0)}{(x_1^2-\lambda^2)(x_2^2-\lambda^2)(r^2+\lambda^2)}. \tag{91}$$

This means that also in 6D the angle γ can be symbolically integrated—it is given by (63)—and the parallel-transported forms $\{\boldsymbol{\pi}, \tilde{\boldsymbol{\pi}}\}$, (60), explicitly constructed.

VII. PRINCIPAL NULL DIRECTIONS

So far we have described the construction of a parallel-propagated frame along generic null geodesics. Similar to the timelike case, with the increasing number of spacetime dimensions, the number of degenerate cases increases for which this construction has to be modified. One type of degeneracy occurs for special geodesics for which the spectrum of the operator $\boldsymbol{F}$ is degenerate. These geodesics are characterized by a special choice of constants of motion and the parallel transport along them was partly described in [21]. In this section, we concentrate on a more fundamental degeneracy which happens when the very construction of the external vector $\boldsymbol{n}$, (12), fails. Such a degeneracy occurs for the velocity vector $\boldsymbol{l}$ which is the eigenvector of the PCKY tensor. In this case one cannot proceed with constructing the 2-form $\boldsymbol{F}$ and the whole procedure described in Sec. II breaks down. In the following subsection we show that such a situation occurs for an important class of geodesics called the *principal null directions*. The parallel-propagated frame along these directions is described in the next subsection.

A. Principal null directions as the eigenvectors of the PCKY tensor

The principal null directions (the Weyl aligned null geodesics, see, e.g., [37]) play an important role in many physical situations. In a spacetime admitting the PCKY tensor the principal null directions, $\boldsymbol{l}_\pm$, coincide with the (real) eigenvectors of the PCKY tensor

$$\boldsymbol{l}_\pm \lrcorner \boldsymbol{h} = \pm\lambda\boldsymbol{l}_\pm^\flat, \tag{92}$$

see [38] or Appendix C.1 in [16].

Namely, for the canonical metric element the affine-parametrized principal null directions are [33,39]

$$\boldsymbol{l}_\pm = \frac{1}{\sqrt{Q_n}}(\tilde{\boldsymbol{e}}_{\hat{n}} \pm \boldsymbol{e}_{\hat{n}}) = \pm\boldsymbol{\partial}_r + \frac{1}{X_n}\sum_{j=0}^{m} r^{2(n-1-j)}\boldsymbol{\partial}_{\psi_j}. \tag{93}$$

These directions are characterized by the following constants of motion:

$$\kappa_j = 0, \qquad \Psi_j = -A_n^{(j)}, \qquad \Psi_n = 0, \tag{94}$$

where $j = 0, \ldots, n-1$ and the constant Ψ_n is relevant only in odd dimensions. Equation (93) gives also coordinate components of $\boldsymbol{l}_\pm$, which lead to the equations for geodesics

$$\dot{r} = \pm 1, \qquad \dot{x}_\mu = 0, \qquad \dot{\psi}_j = \frac{r^{2(n-1-j)}}{X_n}, \tag{95}$$

where $\mu = 1, \ldots, n-1$ and $j = 0, \ldots, m$. These can be integrated to get

$$r = \pm\tau, \qquad \psi_j = \pm\int_0^{\pm\tau}\frac{r^{2(n-1-j)}}{X_n}dr + \psi_j^{(0)}. \tag{96}$$

B. Parallel transport

Now, we turn to the task of parallel transport along the principal null direction $\boldsymbol{l}_+$.[3] The parallel-propagated frame can be obtained by a sequence of local Lorentz transformations of the canonical basis $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{e}_{\hat{\mu}}, \tilde{\boldsymbol{e}}_{\hat{\mu}}, \boldsymbol{e}_{\hat{\epsilon}}\}$,

$$\boldsymbol{l} = \frac{1}{\sqrt{2}}(\tilde{\boldsymbol{e}}_{\hat{n}} + \boldsymbol{e}_{\hat{n}}), \qquad \boldsymbol{n} = \frac{1}{\sqrt{2}}(\tilde{\boldsymbol{e}}_{\hat{n}} - \boldsymbol{e}_{\hat{n}}), \tag{97}$$

along the geodesic (see Appendix B). Here, $\mu = 1, \ldots, n-1$, and $\boldsymbol{e}_{\hat{\epsilon}}$ is relevant only in an odd number of spacetime dimensions. The resulting parallel-propagated frame is $\{{}^\parallel\boldsymbol{l}, {}^\parallel\boldsymbol{n}, {}^\parallel\boldsymbol{e}_{\hat{\mu}}, {}^\parallel\tilde{\boldsymbol{e}}_{\hat{\mu}}, {}^\parallel\boldsymbol{e}_{\hat{\epsilon}}\}$, where

$$\begin{aligned}
{}^\parallel\boldsymbol{l} &= \frac{1}{\sqrt{Q_n}}\boldsymbol{l},\\
{}^\parallel\boldsymbol{n} &= \sqrt{Q_n}\boldsymbol{n} + \sum_{\mu=1}^{n-1}\sqrt{2}\sqrt{Q_\mu}\tilde{\boldsymbol{e}}_{\hat{\mu}} + \varepsilon\sqrt{2}\sqrt{Q_\epsilon}\boldsymbol{e}_{\hat{\epsilon}}\\
&\quad + \Big(\sum_{\mu=1}^{n-1}Q_\mu + \varepsilon Q_\epsilon\Big)\frac{1}{\sqrt{Q_n}}\boldsymbol{l},\\
{}^\parallel\boldsymbol{e}_{\hat{\mu}} &= \frac{x_\mu}{\sqrt{x_\mu^2+r^2}}\boldsymbol{e}_{\hat{\mu}} - \frac{r}{\sqrt{x_\mu^2+r^2}}\Big(\tilde{\boldsymbol{e}}_{\hat{\mu}} + \sqrt{2}\frac{\sqrt{Q_\mu}}{\sqrt{Q_n}}\boldsymbol{l}\Big),\\
{}^\parallel\tilde{\boldsymbol{e}}_{\hat{\mu}} &= \frac{r}{\sqrt{x_\mu^2+r^2}}\boldsymbol{e}_{\hat{\mu}} + \frac{x_\mu}{\sqrt{x_\mu^2+r^2}}\Big(\tilde{\boldsymbol{e}}_{\hat{\mu}} + \sqrt{2}\frac{\sqrt{Q_\mu}}{\sqrt{Q_n}}\boldsymbol{l}\Big),\\
{}^\parallel\boldsymbol{e}_{\hat{\epsilon}} &= \boldsymbol{e}_{\hat{\epsilon}} + \sqrt{2}\frac{\sqrt{Q_\epsilon}}{\sqrt{Q_n}}\boldsymbol{l}.
\end{aligned} \tag{98}$$

[3]The parallel transport along $\boldsymbol{l}_-$ is analogous.

Similar to 4D, it may be convenient to find a parallel-propagated complex null frame. This is done in Appendix B.

VIII. CONCLUSIONS

In this paper we have studied the equations describing the parallel-transport along null geodesics in spacetimes which possess a (nondegenerate) principal conformal Killing-Yano (PCKY) tensor. When the vacuum Einstein equations with the cosmological constant are imposed, this class of metrics coincides with the Kerr-NUT-(A)dS spacetimes, describing the higher-dimensional rotating black holes with NUT parameters, in an asymptotically flat or (A)dS background. In particular, a solution of this problem gives effective tools for studying the polarization of light beams in backgrounds of considered black hole metrics.

A tangent vector to the null ray, which is evidently parallel-propagated, determines a $(D-1)$-dimensional null plane to which it is orthogonal. Our main observation is, that using the PCKY tensor one can obtain a parallel-propagated vector along the null geodesic which does not belong to this null plane. We used these two parallel-propagated vectors to construct a projection operator on a $(D-2)$-dimensional subspace. By using the eigenvectors of the PCKY tensor projected to this subspace we found two-dimensional Darboux planes invariant under the parallel transport, and by proper rotations in these planes we constructed the required parallel-propagated basis. Though the idea of this construction is rather simple, concrete calculations in higher-dimensional spacetimes are quite involved. We performed them concretely in spacetimes with $D \leq 6$. In each of these cases the final first order ordinary differential equations specifying rotations in the 2D Darboux planes were solved by the separation of variables. We expect that this remains true for any number of dimensions.

The class of the Kerr-NUT-(A)dS spacetimes we considered in this paper belongs to the algebraic type D. These spacetimes possess two special congruences of null geodesics called the principal null directions. Tangent vectors to these geodesics are ''eigenvectors'' of the Weyl tensor. At the same time they are null eigenvectors of the PCKY tensor. This property implies that for this subclass of null geodesics the problem of the parallel transport becomes degenerate and requires special consideration. We have studied this degenerate case and directly solved the corresponding equations of parallel transport. This result might be useful for studying the peeling-off property in the Kerr-NUT-(A)dS spacetimes.

ACKNOWLEDGMENTS

D. K. acknowledges support from the University of Cambridge and the University of Alberta. V. F. thanks the Natural Sciences and Engineering Research Council of Canada and the Killam Trust for their financial support. P. K. is supported by Grant No. GAČR 202/06/0041 and the Czech Ministry of Education under Project No. MSM0021610860. P. C. is grateful to the Department of Physics at the University of Alberta for continued financial support.

APPENDIX A: GEOMETRIC OPTICS IN HIGHER-DIMENSIONAL SPACETIMES

It is well known that if the wave length of massless field radiation is much smaller than a characteristic scale on which the gravitational field changes one can use the geometric optics approximation. In this approximation a normal to a surface of constant phase is a null vector which is tangent to a null geodesic describing a motion of a massless quantum. We collect here useful relations of the geometric optics in a higher-dimensional curved spacetime. To make the presentation concrete we discuss the electromagnetic field propagation. We closely follow the nice presentation of the Misner, Thorne, and Wheeler book [1], which requires only tiny changes connected with the number of dimensions D which is now not four but arbitrary. Maxwell equations in a spacetime with the metric g_{ab}, $(a, b = 0, \ldots, D-1)$ in the Lorentz gauge have the form

$$\nabla^b \nabla_b A^a - R^a_b A^b = 0, \tag{A1}$$

$$\nabla_a A^a = 0. \tag{A2}$$

We write the potential A_a in the form

$$A_a = \Re\{[\mathcal{A} + O(\epsilon)]_a e^{iS/\epsilon}\}. \tag{A3}$$

Here ϵ is a small parameter.

Substituting (A3) into (A2) and keeping the term of the leading order ϵ^{-1} one obtains

$$l^a \mathcal{A}_a = 0, \qquad l_a = \nabla_a S. \tag{A4}$$

Similarly, substituting (A3) into (A1) and keeping the terms of order ϵ^{-2} and ϵ^{-1} one gets

$$l_a l^a = 0, \tag{A5}$$

$$l^b \nabla_b \mathcal{A}_a = -\frac{1}{2} \mathcal{A}_a \nabla_b l^b. \tag{A6}$$

Since $\nabla_b l_a = \nabla_b \nabla_a S = \nabla_a \nabla_b S = \nabla_a l_b$ the equation (A5) implies that

$$l^b \nabla_b l^a = \nabla_l l^a = 0. \tag{A7}$$

Hence integral lines of l^a

$$\frac{dx^a}{d\tau} = l^a \tag{A8}$$

are null geodesics and τ is an affine parameter.

We call *l plane* a $(D-1)$-dimensional null plane formed by the vectors $\boldsymbol{v}$ orthogonal to $\boldsymbol{l}$, $\boldsymbol{v} \cdot \boldsymbol{l} = 0$. Relation (A4)

shows that the vector $\mathcal{A}$ lies in the l plane. Consider a gauge transformation of the potential $A_a \to A_a + \nabla_a \alpha$, where $\alpha = \Re[\epsilon\gamma \exp(iS/\epsilon)]$. This transformation generates the following map $\mathcal{A}_a \to \mathcal{A}_a + \gamma l_a$. This means that the vector $\mathcal{A}$ is determined up to the transformation

$$\mathcal{A} \to \mathcal{A} + \gamma \boldsymbol{l}. \tag{A9}$$

Thus for a nontrivial electromagnetic field the vector $\mathcal{A}$ is spacelike. Let us write $\mathcal{A} = \mathcal{A}\boldsymbol{e}$, where $\boldsymbol{e}\cdot\boldsymbol{e} = 1$. We call $\mathcal{A}$ the amplitude and $\boldsymbol{e}$ the polarization vector.

Since $e^a \nabla_l e_a = 0$, the equation (A6) implies

$$\nabla_l \boldsymbol{e} = 0, \tag{A10}$$

$$\nabla_l \mathcal{A} + \tfrac{1}{2}\mathcal{A}\nabla_b l^b = 0. \tag{A11}$$

The first equation, (A10), shows that the vector of polarization $\boldsymbol{e}$ is parallel-transported along the null geodesic, while the second equation implies

$$\nabla_a(\mathcal{A}^2 l^a) = 0. \tag{A12}$$

This conserved current gives the conservation law for the ''number of photon''

$$N = \int d\Sigma_a \mathcal{A}^2 l^a, \tag{A13}$$

where $d\Sigma$ is a volume element of a $(D-1)$-dimensional spacelike Cauchy surface.

Denoted by $\boldsymbol{e}_i$, $i = 1, \ldots, D-2$ a set of $(D-2)$ parallel-propagated mutually orthogonal unit vectors. An arbitrary vector of the linear polarization $\boldsymbol{e}$ can be decomposed in this basis as follows

$$\boldsymbol{e} = \sum_{i=1}^{D-2} b_i \boldsymbol{e}_i, \tag{A14}$$

where b_i are constant coefficients.

APPENDIX B: PARALLEL TRANSPORT ALONG PRINCIPAL NULL DIRECTIONS

In this appendix we present the details of the construction of a parallel-transported frame along the principal null directions. Besides the basis $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{e}_{\hat\mu}, \tilde{\boldsymbol{e}}_{\hat\mu}, \boldsymbol{e}_{\hat\epsilon}\}$, it is useful to consider also the complex null Darboux basis $\{\boldsymbol{l}, \boldsymbol{n}, \boldsymbol{m}_{\hat\mu}, \bar{\boldsymbol{m}}_{\hat\mu}, \boldsymbol{e}_{\hat\epsilon}\}$,

$$\boldsymbol{m}_{\hat\mu} = \frac{1}{\sqrt{2}}(\tilde{\boldsymbol{e}}_{\hat\mu} + i\boldsymbol{e}_{\hat\mu}), \qquad \bar{\boldsymbol{m}}_{\hat\mu} = \frac{1}{\sqrt{2}}(\tilde{\boldsymbol{e}}_{\hat\mu} - i\boldsymbol{e}_{\hat\mu}), \tag{B1}$$

$\mu = 1, \ldots, n-1$, with the only nonvanishing scalar products

$$\boldsymbol{l}\cdot\boldsymbol{n} = -1, \qquad \boldsymbol{m}_{\hat\mu}\cdot\bar{\boldsymbol{m}}_{\hat\mu} = 1, \tag{B2}$$

in which the PCKY tensor $\boldsymbol{h}$, (32), takes the form

$$\boldsymbol{h} = r\boldsymbol{l}^\flat \wedge \boldsymbol{n}^\flat + i\sum_\mu x_\mu \boldsymbol{m}_\mu{}^\flat \wedge \bar{\boldsymbol{m}}_\mu{}^\flat. \tag{B3}$$

The covariant derivatives of the Darboux basis along the principal null direction $\boldsymbol{l}_+ \propto \boldsymbol{l}$ are

$$\begin{aligned}
\nabla_{l_+}\boldsymbol{l} &= \frac{\sqrt{Q_{n,r}}}{\sqrt{Q_n}}\boldsymbol{l},\\
\nabla_{l_+}\boldsymbol{n} &= -\frac{\sqrt{Q_{n,r}}}{\sqrt{Q_n}}\boldsymbol{n} + \varepsilon\sqrt{2}\frac{\sqrt{Q_\epsilon}}{\sqrt{Q_n}}\boldsymbol{e}_{\hat\epsilon} + \sqrt{2}\sum_{\mu=1}^{n-1}\frac{\sqrt{Q_\mu}}{\sqrt{Q_n}}\\
&\quad\times\left(\frac{x_\mu}{x_\mu^2+r^2}\boldsymbol{e}_{\hat\mu} + \frac{r}{x_\mu^2+r^2}\tilde{\boldsymbol{e}}_{\hat\mu}\right),\\
\nabla_{l_+}\boldsymbol{e}_{\hat\mu} &= \frac{x_\mu}{x_\mu^2+r^2}\tilde{\boldsymbol{e}}_{\hat\mu} + \frac{\sqrt{2}x_\mu}{x_\mu^2+r^2}\frac{\sqrt{Q_\mu}}{\sqrt{Q_n}}\boldsymbol{l},\\
\nabla_{l_+}\tilde{\boldsymbol{e}}_{\hat\mu} &= -\frac{x_\mu}{x_\mu^2+r^2}\boldsymbol{e}_{\hat\mu} + \frac{\sqrt{2}r}{x_\mu^2+r^2}\frac{\sqrt{Q_\mu}}{\sqrt{Q_n}}\boldsymbol{l},\\
\nabla_{l_+}\boldsymbol{e}_{\hat\epsilon} &= \frac{\sqrt{2}}{r}\frac{\sqrt{Q_\epsilon}}{\sqrt{Q_n}}\boldsymbol{l}.
\end{aligned} \tag{B4}$$

For the null basis these are equivalent to

$$\begin{aligned}
\nabla_{l_+}\boldsymbol{m}_{\hat\mu} &= \frac{ix_\mu}{x_\mu^2+r^2}\boldsymbol{m}_{\hat\mu} + \frac{i}{x_\mu+ir}\frac{\sqrt{Q_\mu}}{\sqrt{Q_n}}\boldsymbol{l},\\
\nabla_{l_+}\bar{\boldsymbol{m}}_{\hat\mu} &= \frac{-ix_\mu}{x_\mu^2+r^2}\bar{\boldsymbol{m}}_{\hat\mu} - \frac{i}{x_\mu-ir}\frac{\sqrt{Q_\mu}}{\sqrt{Q_n}}\boldsymbol{l},\\
\nabla_{l_+}\boldsymbol{n} &= -\frac{\sqrt{Q_{n,r}}}{\sqrt{Q_n}}\boldsymbol{n} + \varepsilon\sqrt{2}\frac{\sqrt{Q_\epsilon}}{\sqrt{Q_n}}\boldsymbol{e}_{\hat\epsilon}\\
&\quad+\sum_{\mu=1}^{n-1}\frac{\sqrt{Q_\mu}}{\sqrt{Q_n}}\left(\frac{-i}{x_\mu-ir}\boldsymbol{m}_{\hat\mu} + \frac{i}{x_\mu+ir}\bar{\boldsymbol{m}}_{\hat\mu}\right),
\end{aligned} \tag{B5}$$

with the equations for $\boldsymbol{l}$ and $\boldsymbol{e}_{\hat\epsilon}$ unchanged.

Now, we change these Darboux bases to the parallel-transported ones by a sequence of the local Lorentz transformations. (Such transformations preserve the orthogonality and the normalization of the frame.) Guided by Eq. (93), our first transformation is the boost in the $\{\boldsymbol{l}, \boldsymbol{n}\}$ plane

$$\begin{aligned}
&{}^{\mathrm{B}}\boldsymbol{l} = \frac{1}{\sqrt{Q_n}}\boldsymbol{l}, \qquad {}^{\mathrm{B}}\boldsymbol{n} = \sqrt{Q_n}\boldsymbol{n}, \qquad {}^{\mathrm{B}}\boldsymbol{e}_{\hat\mu} = \boldsymbol{e}_{\hat\mu},\\
&{}^{\mathrm{B}}\tilde{\boldsymbol{e}}_{\hat\mu} = \tilde{\boldsymbol{e}}_{\hat\mu}, \qquad {}^{\mathrm{B}}\boldsymbol{e}_{\hat\epsilon} = \boldsymbol{e}_{\hat\epsilon}, \qquad {}^{\mathrm{B}}\boldsymbol{m}_{\hat\mu} = \boldsymbol{m}_{\hat\mu}, \qquad {}^{\mathrm{B}}\bar{\boldsymbol{m}}_{\hat\mu} = \bar{\boldsymbol{m}}_{\hat\mu}.
\end{aligned} \tag{B6}$$

The covariant derivatives along $\boldsymbol{l}_+$ changes to

$$
\begin{aligned}
&\nabla_{l_+}{}^{\mathrm{B}}\boldsymbol{l} = 0,\\
&\nabla_{l_+}{}^{\mathrm{B}}\boldsymbol{n} = \sqrt{2}\sum_{\mu=1}^{n-1}\sqrt{Q_\mu}\Big(\frac{x_\mu}{x_\mu^2+r^2}{}^{\mathrm{B}}\boldsymbol{e}_{\hat{\mu}} + \frac{r}{x_\mu^2+r^2}{}^{\mathrm{B}}\tilde{\boldsymbol{e}}_{\hat{\mu}}\Big)\\
&\qquad\qquad + \varepsilon\sqrt{2}\sqrt{Q_\epsilon}{}^{\mathrm{B}}\boldsymbol{e}_{\hat{\epsilon}},\\
&\nabla_{l_+}{}^{\mathrm{B}}\boldsymbol{e}_{\hat{\mu}} = \frac{x_\mu}{x_\mu^2+r^2}{}^{\mathrm{B}}\tilde{\boldsymbol{e}}_{\hat{\mu}} + \frac{\sqrt{2}x_\mu\sqrt{Q_\mu}}{x_\mu^2+r^2}{}^{\mathrm{B}}\boldsymbol{l},\\
&\nabla_{l_+}{}^{\mathrm{B}}\tilde{\boldsymbol{e}}_{\hat{\mu}} = -\frac{x_\mu}{x_\mu^2+r^2}{}^{\mathrm{B}}\boldsymbol{e}_{\hat{\mu}} + \frac{\sqrt{2}r\sqrt{Q_\mu}}{x_\mu^2+r^2}{}^{\mathrm{B}}\boldsymbol{l},\\
&\nabla_{l_+}{}^{\mathrm{B}}\boldsymbol{e}_{\hat{\epsilon}} = \sqrt{2}\frac{\sqrt{Q_\epsilon}}{r}{}^{\mathrm{B}}\boldsymbol{l},
\end{aligned}
\tag{B7}
$$

or, for the null frame

$$
\begin{aligned}
&\nabla_{l_+}{}^{\mathrm{B}}\boldsymbol{m}_{\hat{\mu}} = \frac{ix_\mu}{x_\mu^2+r^2}{}^{\mathrm{B}}\boldsymbol{m}_{\hat{\mu}} + \frac{i\sqrt{Q_\mu}}{x_\mu+ir}{}^{\mathrm{B}}\boldsymbol{l},\\
&\nabla_{l_+}{}^{\mathrm{B}}\bar{\boldsymbol{m}}_{\hat{\mu}} = \frac{-ix_\mu}{x_\mu^2+r^2}{}^{\mathrm{B}}\bar{\boldsymbol{m}}_{\hat{\mu}} - \frac{i\sqrt{Q_\mu}}{x_\mu-ir}{}^{\mathrm{B}}\boldsymbol{l},\\
&\nabla_{l_+}{}^{\mathrm{B}}\boldsymbol{n} = \sum_{\mu=1}^{n-1}\sqrt{Q_\mu}\Big(\frac{-i}{x_\mu-ir}{}^{\mathrm{B}}\boldsymbol{m}_{\hat{\mu}} + \frac{i}{x_\mu+ir}{}^{\mathrm{B}}\bar{\boldsymbol{m}}_{\hat{\mu}}\Big)\\
&\qquad\qquad + \varepsilon\sqrt{2}\sqrt{Q_\epsilon}{}^{\mathrm{B}}\boldsymbol{e}_{\hat{\mu}}.
\end{aligned}
\tag{B8}
$$

Next transformation is a multinull rotation, leaving ${}^{\mathrm{B}}\boldsymbol{l}$ fixed. Actually, this transformation can be decomposed into a sequence of commuting null rotations each of which combines only vectors $\boldsymbol{l}$, $\boldsymbol{n}$ and vectors $\boldsymbol{m}_{\hat{\mu}}$, $\bar{\boldsymbol{m}}_{\hat{\mu}}$ from one KY 2-plane. Namely, for each $\mu = 1, \ldots, n-1$ we perform the null rotation characterized by the parameter $\sqrt{Q_\mu}$,

$$
\begin{aligned}
&{}^{\mathrm{N}}\boldsymbol{l} = {}^{\mathrm{B}}\boldsymbol{l}, \qquad {}^{\mathrm{N}}\boldsymbol{m}_{\hat{\mu}} = {}^{\mathrm{B}}\boldsymbol{m}_{\hat{\mu}} + \sqrt{Q_\mu}{}^{\mathrm{B}}\boldsymbol{l},\\
&{}^{\mathrm{N}}\bar{\boldsymbol{m}}_{\hat{\mu}} = {}^{\mathrm{B}}\bar{\boldsymbol{m}}_{\hat{\mu}} + \sqrt{Q_\mu}{}^{\mathrm{B}}\boldsymbol{l}, \qquad {}^{\mathrm{N}}\boldsymbol{e}_{\hat{\mu}} = {}^{\mathrm{B}}\boldsymbol{e}_{\hat{\mu}},\\
&{}^{\mathrm{N}}\tilde{\boldsymbol{e}}_{\hat{\mu}} = {}^{\mathrm{B}}\tilde{\boldsymbol{e}}_{\hat{\mu}} + \sqrt{2}\sqrt{Q_\mu}{}^{\mathrm{B}}\boldsymbol{l},
\end{aligned}
\tag{B9}
$$

in odd dimension accompanied by

$$
{}^{\mathrm{N}}\boldsymbol{e}_{\hat{\epsilon}} = {}^{\mathrm{B}}\boldsymbol{e}_{\hat{\epsilon}} + \sqrt{2}\sqrt{Q_\epsilon}{}^{\mathrm{B}}\boldsymbol{l}. \tag{B10}
$$

The transformed vectors are orthogonal and have to be completed by properly transformed vector ${}^{\mathrm{B}}\boldsymbol{n}$,

$$
\begin{aligned}
{}^{\mathrm{N}}\boldsymbol{n} &= {}^{\mathrm{B}}\boldsymbol{n} + \sum_{\mu=1}^{n-1}\sqrt{Q_\mu}({}^{\mathrm{B}}\boldsymbol{m}_{\hat{\mu}} + {}^{\mathrm{B}}\bar{\boldsymbol{m}}_{\hat{\mu}}) + \varepsilon\sqrt{2}\sqrt{Q_\epsilon}{}^{\mathrm{B}}\boldsymbol{e}_{\hat{\epsilon}}\\
&\quad + \Big(\sum_{\mu=1}^{n-1}Q_\mu + \varepsilon Q_\epsilon\Big){}^{\mathrm{B}}\boldsymbol{l}\\
&= {}^{\mathrm{B}}\boldsymbol{n} + \sum_{\mu=1}^{n-1}\sqrt{2}\sqrt{Q_\mu}{}^{\mathrm{B}}\tilde{\boldsymbol{e}}_{\hat{\mu}} + \varepsilon\sqrt{2}\sqrt{Q_\epsilon}{}^{\mathrm{B}}\boldsymbol{e}_{\hat{\epsilon}}\\
&\quad + \Big(\sum_{\mu=1}^{n-1}Q_\mu + \varepsilon Q_\epsilon\Big){}^{\mathrm{B}}\boldsymbol{l}.
\end{aligned}
\tag{B11}
$$

Let us remark here, that since the parameters of the null rotations are real, each of them actually mix only three directions ${}^{\mathrm{B}}\boldsymbol{l}$, ${}^{\mathrm{B}}\boldsymbol{n}$, ${}^{\mathrm{B}}\tilde{\boldsymbol{e}}_{\hat{\mu}}$; the direction ${}^{\mathrm{B}}\boldsymbol{e}_{\hat{\mu}}$ remains fixed. The covariant derivatives of the null rotated frame are simple

$$
\begin{aligned}
&\nabla_{l_+}{}^{\mathrm{N}}\boldsymbol{l} = 0, \qquad \nabla_{l_+}{}^{\mathrm{N}}\boldsymbol{n} = 0, \qquad \nabla_{l_+}{}^{\mathrm{N}}\boldsymbol{e}_{\hat{\epsilon}} = 0,\\
&\nabla_{l_+}{}^{\mathrm{B}}\boldsymbol{e}_{\hat{\mu}} = \frac{x_\mu}{x_\mu^2+r^2}{}^{\mathrm{N}}\tilde{\boldsymbol{e}}_{\hat{\mu}}, \qquad \nabla_{l_+}{}^{\mathrm{B}}\tilde{\boldsymbol{e}}_{\hat{\mu}} = -\frac{x_\mu}{x_\mu^2+r^2}{}^{\mathrm{N}}\boldsymbol{e}_{\hat{\mu}},\\
&\nabla_{l_+}{}^{\mathrm{B}}\bar{\boldsymbol{m}}_{\hat{\mu}} = \frac{-ix_\mu}{x_\mu^2+r^2}{}^{\mathrm{N}}\bar{\boldsymbol{m}}_{\hat{\mu}}, \qquad \nabla_{l_+}{}^{\mathrm{B}}\boldsymbol{m}_{\hat{\mu}} = \frac{ix_\mu}{x_\mu^2+r^2}{}^{\mathrm{N}}\boldsymbol{m}_{\hat{\mu}}.
\end{aligned}
\tag{B12}
$$

Finally, we perform the spatial rotation in each spatial KY 2-plane, by the angle $\varphi_\mu = \arctan\frac{r}{x_\mu}$,

$$
\begin{aligned}
&{}^{\|}\boldsymbol{l} = {}^{\mathrm{N}}\boldsymbol{l}, \qquad {}^{\|}\boldsymbol{n} = {}^{\mathrm{N}}\boldsymbol{n}, \qquad {}^{\|}\boldsymbol{e}_{\hat{\epsilon}} = {}^{\mathrm{N}}\boldsymbol{e}_{\hat{\epsilon}},\\
&{}^{\|}\boldsymbol{e}_{\hat{\mu}} = \frac{x_\mu}{\sqrt{x_\mu^2+r^2}}{}^{\mathrm{N}}\boldsymbol{e}_{\hat{\mu}} - \frac{r}{\sqrt{x_\mu^2+r^2}}{}^{\mathrm{N}}\tilde{\boldsymbol{e}}_{\hat{\mu}},\\
&{}^{\|}\tilde{\boldsymbol{e}}_{\hat{\mu}} = \frac{r}{\sqrt{x_\mu^2+r^2}}{}^{\mathrm{N}}\boldsymbol{e}_{\hat{\mu}} + \frac{x_\mu}{\sqrt{x_\mu^2+r^2}}{}^{\mathrm{N}}\tilde{\boldsymbol{e}}_{\hat{\mu}},\\
&{}^{\|}\boldsymbol{m}_{\hat{\mu}} = \sqrt{\frac{x_\mu-ir}{x_\mu+ir}}{}^{\mathrm{N}}\boldsymbol{m}_{\hat{\mu}}, \qquad {}^{\|}\bar{\boldsymbol{m}}_{\hat{\mu}} = \sqrt{\frac{x_\mu+ir}{x_\mu-ir}}{}^{\mathrm{N}}\bar{\boldsymbol{m}}_{\hat{\mu}}.
\end{aligned}
\tag{B13}
$$

The resulting frame $\{{}^{\|}\boldsymbol{l}, {}^{\|}\boldsymbol{n}, {}^{\|}\boldsymbol{e}_{\hat{\mu}}, {}^{\|}\tilde{\boldsymbol{e}}_{\hat{\mu}}, {}^{\|}\boldsymbol{e}_{\hat{\epsilon}}\}$ (or, alternatively, $\{{}^{\|}\boldsymbol{l}, {}^{\|}\boldsymbol{n}, {}^{\|}\boldsymbol{m}_{\hat{\mu}}, {}^{\|}\bar{\boldsymbol{m}}_{\hat{\mu}}, {}^{\|}\boldsymbol{e}_{\hat{\epsilon}}\}$) is parallel-transported

$$
\begin{aligned}
&\nabla_{l_+}{}^{\|}\boldsymbol{l} = 0, \quad \nabla_{l_+}{}^{\|}\boldsymbol{n} = 0, \quad \nabla_{l_+}{}^{\|}\boldsymbol{e}_{\hat{\epsilon}} = 0, \quad \nabla_{l_+}{}^{\|}\boldsymbol{e}_{\hat{\mu}} = 0,\\
&\nabla_{l_+}{}^{\|}\tilde{\boldsymbol{e}}_{\hat{\mu}} = 0, \quad \nabla_{l_+}{}^{\|}\boldsymbol{m}_{\hat{\mu}} = 0, \quad \nabla_{l_+}{}^{\|}\bar{\boldsymbol{m}}_{\hat{\mu}} = 0.
\end{aligned}
\tag{B14}
$$

Combining all the transformations together we arrive at the result (98), or, for the complex null frame

$$
\begin{aligned}
{}^{\shortparallel}\boldsymbol{l} &= \frac{1}{\sqrt{Q_n}}\boldsymbol{l},\\
{}^{\shortparallel}\boldsymbol{n} &= \sqrt{Q_n}\boldsymbol{n} + \Big(\sum_{\mu=1}^{n-1} Q_\mu + \varepsilon Q_\epsilon\Big)\frac{1}{\sqrt{Q_n}}\boldsymbol{l}\\
&\quad + \sum_{\mu=1}^{n-1}\sqrt{Q_\mu}(\boldsymbol{m}_{\hat{\mu}} + \bar{\boldsymbol{m}}_{\hat{\mu}}) + \varepsilon\sqrt{2}\sqrt{Q_\epsilon}\boldsymbol{e}_{\hat{\epsilon}}\\
{}^{\shortparallel}\boldsymbol{m}_{\hat{\mu}} &= \sqrt{\frac{x_\mu - ir}{x_\mu + ir}}\Big(\boldsymbol{m}_{\hat{\mu}} + \frac{\sqrt{Q_\mu}}{\sqrt{Q_n}}\boldsymbol{l}\Big)\\
{}^{\shortparallel}\bar{\boldsymbol{m}}_{\hat{\mu}} &= \sqrt{\frac{x_\mu + ir}{x_\mu - ir}}\Big(\bar{\boldsymbol{m}}_{\hat{\mu}} + \frac{\sqrt{Q_\mu}}{\sqrt{Q_n}}\boldsymbol{l}\Big)\\
{}^{\shortparallel}\boldsymbol{e}_{\hat{\epsilon}} &= \boldsymbol{e}_{\hat{\epsilon}} + \sqrt{2}\frac{\sqrt{Q_\epsilon}}{\sqrt{Q_n}}\boldsymbol{l}.
\end{aligned}
\tag{B15}
$$

APPENDIX C: PARALLEL TRANSPORT IN THE PLEBAŃSKI-DEMIAŃSKI FAMILY OF SOLUTIONS

In the main text we have demonstrated how to construct a parallel-propagated frame along null geodesics in spacetimes admitting the PCKY tensor. In this appendix we show how to modify this construction for an important family of 4D spacetimes described by the Plebański-Demiański metric. Such a family generally admits only a nonclosed generalization of the PCKY tensor—the (nondegenerate) conformal Killing-Yano (CKY) tensor. Our construction generalizes the results presented in [23,24].

1. Plebański-demiański metric

The Plebański-Demiański metric [40] describes a large class of four-dimensional type D spacetimes. The concrete form of physical metrics is obtained by a due limiting procedure (see, e.g., [41] for a recent review). In this way one can obtain, for example, the metric of an accelerated rotating charged black hole in the cosmological background.

The whole Plebański-Demiański class of solutions possesses the CKY tensor [42]. Such a tensor is responsible for complete integrability of a null geodesic motion. When the acceleration parameter is removed the corresponding subclass of solutions obtained earlier by Carter [35,36] allows the PCKY tensor [43] and the solution of parallel transport was already described in Sec. VI A (see also [21] for the timelike case). In order to see the impact of the presence of a nontrivial acceleration on parallel transport, we write the Plebański-Demiański metric in the notations of Sec. VI A. So we have

$$
\boldsymbol{g} = -\tilde{\boldsymbol{\omega}}^{\hat{2}}\tilde{\boldsymbol{\omega}}^{\hat{2}} + \boldsymbol{\omega}^{\hat{2}}\boldsymbol{\omega}^{\hat{2}} + \tilde{\boldsymbol{\omega}}^{\hat{1}}\tilde{\boldsymbol{\omega}}^{\hat{1}} + \boldsymbol{\omega}^{\hat{1}}\boldsymbol{\omega}^{\hat{1}}, \tag{C1}
$$

where

$$
\begin{aligned}
\tilde{\boldsymbol{\omega}}^{\hat{2}} &= \Omega\sqrt{\frac{X_2}{U_2}}(\boldsymbol{d}\psi_0 + x_1^2\boldsymbol{d}\psi_1), \qquad \boldsymbol{\omega}^{\hat{2}} = \Omega\sqrt{\frac{U_2}{X_2}}\boldsymbol{d}r,\\
\tilde{\boldsymbol{\omega}}^{\hat{1}} &= \Omega\sqrt{\frac{X_1}{U_1}}(\boldsymbol{d}\psi_0 - r^2\boldsymbol{d}\psi_1), \qquad \boldsymbol{\omega}^{\hat{1}} = \Omega\sqrt{\frac{U_1}{X_1}}\boldsymbol{d}x_1.
\end{aligned}
\tag{C2}
$$

Here, $U_2 = -U_1 = x_1^2 + r^2$ and $\Omega = (1 - x_1 r)^{-1}$. For $X_1 = X_1(x_1)$ and $X_2 = X_2(r)$ we refer to the metric as the *off shell* metric. Such a metric possesses two (Hodge dual) nondegenerate CKY 2-forms [42]

$$
\boldsymbol{h} = \Omega(-r\boldsymbol{\omega}^{\hat{2}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{2}} + x_1\boldsymbol{\omega}^{\hat{1}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{1}}), \tag{C3}
$$

$$
\boldsymbol{k} = -\Omega(x_1\boldsymbol{\omega}^{\hat{2}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{2}} + r\boldsymbol{\omega}^{\hat{1}} \wedge \tilde{\boldsymbol{\omega}}^{\hat{1}}), \tag{C4}
$$

which are connected with the background isometries as

$$
\boldsymbol{\xi}_{(h)} = -\frac{1}{3}\boldsymbol{\delta h} = \boldsymbol{\partial}_{\psi_0}, \qquad \boldsymbol{\xi}_{(k)} = -\frac{1}{3}\boldsymbol{\delta k} = \boldsymbol{\partial}_{\psi_1}. \tag{C5}
$$

These isometries together with the hidden symmetry of the CKY tensor make the null geodesic motion in the off shell background (C1) and (C2) completely integrable. The tetrad components of the velocity are

$$
\begin{aligned}
\tilde{l}_{\hat{2}} &= \frac{W_2}{\Omega\sqrt{X_2U_2}}, \qquad l_{\hat{2}} = \frac{\sigma_2}{\Omega\sqrt{X_2U_2}}\sqrt{W_2^2 - V_2X_2},\\
\tilde{l}_{\hat{1}} &= \frac{-W_1}{\Omega\sqrt{X_1U_1}}, \qquad l_{\hat{1}} = \frac{\sigma_1}{\Omega\sqrt{X_1U_2}}\sqrt{V_1X_1 - W_1^2},
\end{aligned}
\tag{C6}
$$

where

$$
\begin{aligned}
W_2 &= r^2\Psi_0 + \Psi_1, \qquad V_2 = -\kappa_1 > 0,\\
W_1 &= -x_1^2\Psi_0 + \Psi_1, \qquad V_1 = \kappa_1.
\end{aligned}
\tag{C7}
$$

The constant κ_1 corresponds to the conformal Killing tensor $Q_{ab} = k_{ac}k_b{}^c$, whereas the constants Ψ_0 and Ψ_1 are associated with the Killing vectors $\boldsymbol{\partial}_{\psi_0}$ and $\boldsymbol{\partial}_{\psi_1}$, respectively.[4]

For the special choice of metric functions

$$
\begin{aligned}
X_1 &= -k - 2nx_1 + \epsilon x_1^2 - 2mx_1^3\\
&\quad + (k + e^2 + g^2 + \Lambda/3)x_1^4,\\
X_2 &= k + e^2 + g^2 - 2mr + \epsilon r^2 - 2nr^3 - (k + \Lambda/3)r^4,
\end{aligned}
\tag{C8}
$$

and the vector potential

$$
\boldsymbol{A} = -\frac{1}{\Omega}\Big(\frac{er}{\sqrt{U_2X_2}}\tilde{\boldsymbol{\omega}}^{\hat{2}} + \frac{gx_1}{\sqrt{U_1X_1}}\tilde{\boldsymbol{\omega}}^{\hat{1}}\Big), \tag{C9}
$$

[4]One can formally recover the "nonaccelerating" class of solutions [35,36] by taking the conformal factor $\Omega = 1$. After that, the CKY tensor $\boldsymbol{h}$ becomes the PCKY tensor discussed earlier and $\boldsymbol{k}$ becomes the Killing-Yano tensor. At the same time $\boldsymbol{\xi}_{(h)}$ still coincides with $\boldsymbol{\partial}_{\psi_0}$ and $\boldsymbol{\xi}_{(k)}$ vanishes.

the Plebański-Demiański metric obeys the Einstein-Maxwell equations with e and g the electric and magnetic charges and the cosmological constant Λ.

2. Construction of a parallel-propagated frame

Let us now construct a parallel-propagated frame in the off shell background (C1) and (C2). We start by observing that having a CKY 2-form $\boldsymbol{\omega}$, that is a 2-form obeying the CKY equations

$$\nabla_X \boldsymbol{\omega} = \frac{1}{3} \boldsymbol{X} \lrcorner \boldsymbol{d\omega} + \boldsymbol{X}^\flat \wedge \boldsymbol{\xi}^\flat, \qquad \boldsymbol{\xi}^\flat = -\frac{1}{D-1} \boldsymbol{\delta\omega}, \tag{C10}$$

and a null geodesic velocity vector $\boldsymbol{l}$, one can construct the following parallel-transported vector:

$$\boldsymbol{w} = (\boldsymbol{l} \lrcorner \boldsymbol{\omega})^\sharp + \beta \boldsymbol{l}, \qquad \dot{\beta} = \boldsymbol{l} \cdot \boldsymbol{\xi}. \tag{C11}$$

Indeed, using the defining property (C10) we obtain

$$\begin{aligned} \dot{\boldsymbol{w}} &= (\boldsymbol{l} \lrcorner \dot{\boldsymbol{\omega}})^\sharp + \dot{\beta} \boldsymbol{l} = \frac{1}{3} \boldsymbol{l} \lrcorner (\boldsymbol{l} \lrcorner \boldsymbol{d\omega}) \\ &\quad + [\boldsymbol{l} \lrcorner (\boldsymbol{l}^\flat \wedge \boldsymbol{\xi}^\flat)]^\sharp + \dot{\beta} \boldsymbol{l} = \boldsymbol{l}(\dot{\beta} - \boldsymbol{l} \cdot \boldsymbol{\xi}). \end{aligned} \tag{C12}$$

Here we have used the obvious fact that the first term in the second line is zero and then proceeded in the same way as in Sec. III A.

In particular, this means that for the off shell Plebański-Demiański metric we may construct the following two parallel-transported vectors:

$$\boldsymbol{m} = \frac{1}{\sqrt{-\kappa_1}} [(\boldsymbol{l} \lrcorner \boldsymbol{h})^\sharp + \beta_h \boldsymbol{l}], \tag{C13}$$

$$\boldsymbol{z} = \frac{1}{\sqrt{-\kappa_1}} [(\boldsymbol{l} \lrcorner \boldsymbol{k})^\sharp + \beta_k \boldsymbol{l}]. \tag{C14}$$

Moreover, using (C5) we find[5]

$$\beta_h = \Psi_0 \tau, \qquad \beta_k = \Psi_1 \tau. \tag{C15}$$

The last parallel-transported vector $\boldsymbol{n}$ is simply determined by the normalization conditions

$$\begin{aligned} \boldsymbol{n} \cdot \boldsymbol{l} &= -1, \qquad \boldsymbol{n} \cdot \boldsymbol{m} = 0, \\ \boldsymbol{n} \cdot \boldsymbol{z} &= 0, \qquad \boldsymbol{n} \cdot \boldsymbol{n} = 0. \end{aligned} \tag{C16}$$

Let us explicitly write down the form of the (off shell) parallel-transported frame in the velocity adapted basis $\{\boldsymbol{o}\}$, (47). We have

$$\begin{aligned} \boldsymbol{l}^\flat &= \tilde{k}_{\hat{1}} (-\tilde{\boldsymbol{o}}^{\hat{2}} + \tilde{\boldsymbol{o}}^{\hat{1}}), \qquad \tilde{k}_{\hat{1}} = -\tilde{k}_{\hat{2}} = \frac{1}{\Omega} \frac{\sqrt{V_2}}{\sqrt{U_2}}, \\ \boldsymbol{m}^\flat &= \frac{\tilde{k}_{\hat{1}}}{\sqrt{-\kappa_1}} (-\beta_h \tilde{\boldsymbol{o}}^{\hat{2}} + r\Omega \boldsymbol{o}^{\hat{2}} + \beta_h \tilde{\boldsymbol{o}}^{\hat{1}} - x_1 \Omega \boldsymbol{o}^{\hat{1}}), \\ \boldsymbol{n}^\flat &= \frac{\tilde{k}_{\hat{1}}}{\kappa_1} \Big[\frac{\Omega^2 U_2 + \beta_h^2 + \beta_k^2}{2} \tilde{\boldsymbol{o}}^{\hat{2}} - \Omega(\beta_h r + \beta_k x_1) \boldsymbol{o}^{\hat{2}} \\ &\quad + \frac{\Omega^2 U_2 - \beta_h^2 - \beta_k^2}{2} \tilde{\boldsymbol{o}}^{\hat{1}} + \Omega(\beta_h x_1 - \beta_k r) \boldsymbol{o}^{\hat{1}} \Big], \\ \boldsymbol{z}^\flat &= \frac{\tilde{k}_{\hat{1}}}{\sqrt{-\kappa_1}} (-\beta_k \tilde{\boldsymbol{o}}^{\hat{2}} + x_1 \Omega \boldsymbol{o}^{\hat{2}} + \beta_k \tilde{\boldsymbol{o}}^{\hat{1}} + r\Omega \boldsymbol{o}^{\hat{1}}). \end{aligned} \tag{C17}$$

It is easy to see, that one can formally recover the "non-accelerating" limit of the PCKY tensor, Eq. (69), by setting $\Omega = 1$ ($\beta_k = 0$).

[5] Let us remark here, that contrary to Sec. IV B one cannot, for the Plebański-Demiański metric, "separate" the affine parameter τ as in (46). Formally, this is due to the presence of a nontrivial (nonseparable) conformal factor Ω.

[1] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973).
[2] R. F. Stark and P. A. Connors, Nature (London) **266**, 429 (1977).
[3] P. A. Connors and R. F. Stark, Nature (London) **269**, 128 (1977).
[4] P. A. Connors, T. Piran, and R. F. Stark, Astrophys. J. **235**, 224 (1980).
[5] F. A. E. Pirani, in *Brandeis Lectures on General Relativity*, edited by S. Deser and K. W. Ford (Prentice-Hall, Englewood Cliffs, NJ, 1965), pp. 249–372.
[6] V. P. Frolov, in *Problems in the General Theory of Relativity and the Theory of Group Representation, The Proceedings of the Lebedev Physics Institute of the Academy of Sciences of the USSR*, edited by N. G. Basov , (Consultants Bureau, New York, 1979), p. 72.
[7] R. Sachs, Proc. R. Soc. A **264**, 309 (1961).
[8] R. K. Sachs, Proc. R. Soc. A **270**, 103 (1962).
[9] E. T. Newman and R. Penrose, J. Math. Phys. (N.Y.) **3**, 566 (1962).
[10] R. Penrose, Proc. R. Soc. A **284**, 159 (1965).
[11] P. Krtouš and J. Podolský, Classical Quantum Gravity **21**, R233 (2004).
[12] V. P. Frolov and D. Kubizňák, Phys. Rev. Lett. **98**, 011101 (2007).
[13] P. Krtouš, D. Kubizňák, D. N. Page, and V. P. Frolov, J. High Energy Phys. 02 (2007) 004.
[14] V. P. Frolov, Prog. Theor. Phys. Suppl. **172**, 210 (2008).
[15] V. P. Frolov and D. Kubizňák, Classical Quantum Gravity **25**, 154005 (2008).
[16] D. Kubizňák, Ph.D. thesis, University of Alberta, Edmonton, Alberta, Canada, 2008.

[17] T. Houri, T. Oota, and Y. Yasui, Phys. Lett. B **656**, 214 (2007).
[18] P. Krtouš, V. P. Frolov, and D. Kubizňák, Phys. Rev. D **78**, 064022 (2008).
[19] P. Krtouš, V. P. Frolov, and D. Kubizňák (unpublished).
[20] W. Chen, H. Lü, and C. N. Pope, Classical Quantum Gravity **23**, 5323 (2006).
[21] P. Connell, V. P. Frolov, and D. Kubizňák, Phys. Rev. D **78**, 024042 (2008).
[22] J. A. Marck, Proc. R. Soc. A **385**, 431 (1983).
[23] J. A. Marck, Phys. Lett. A **97**, 140 (1983).
[24] N. Kamran and J. A. Marck, J. Math. Phys. (N.Y.) **27**, 1589 (1986).
[25] S. Tachibana, Tôhoku Math. J. **21**, 56 (1969).
[26] T. Kashiwada, Nat. Sci. Rep. Ochanomizu Univ. **19**, 67 (1968).
[27] D. N. Page, D. Kubizňák, M. Vasudevan, and P. Krtouš, Phys. Rev. Lett. **98**, 061102 (2007).
[28] P. Krtouš, D. Kubizňák, D. N. Page, and M. Vasudevan, Phys. Rev. D **76**, 084034 (2007).
[29] T. Houri, T. Oota, and Y. Yasui, J. Phys. A **41**, 025204 (2008).
[30] T. Houri, T. Oota, and Y. Yasui, Phys. Lett. B **666**, 391 (2008).
[31] T. Houri, T. Oota, and Y. Yasui, arXiv:0805.3877.
[32] D. Kubizňák and V. P. Frolov, Classical Quantum Gravity **24**, F1 (2007).
[33] N. Hamamoto, T. Houri, T. Oota, and Y. Yasui, J. Phys. A **40**, F177 (2007).
[34] V. P. Frolov, P. Krtouš, and D. Kubizňák, J. High Energy Phys. 02 (2007) 005.
[35] B. Carter, Commun. Math. Phys. **10**, 280 (1968).
[36] B. Carter, Phys. Lett. A **26**, 399 (1968).
[37] A. A. Coley, Classical Quantum Gravity **25**, 033001 (2008).
[38] L. Mason and A. Taghavi-Chabert, arXiv:0805.3756.
[39] V. Pravda, A. Pravdová, and M. Ortaggio, Classical Quantum Gravity **24**, 4407 (2007).
[40] J. F. Plebański and M. Demiański, Ann. Phys. (N.Y.) **98**, 98 (1976).
[41] J. B. Griffiths and J. Podolský, Int. J. Mod. Phys. D **15**, 335 (2006).
[42] D. Kubizňák and P. Krtouš, Phys. Rev. D **76**, 084036 (2007).
[43] B. Carter, J. Math. Phys. (N.Y.) **28**, 1535 (1987).