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Superradiant instability of five-dimensional rotating charged AdS black holes

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We study the instability of small AdS black holes with two independent rotation parameters in minimal five-dimensional gauged supergravity to massless scalar perturbations. We analytically solve the Klein-Gordon equation for low-frequency perturbations in two regions of the spacetime of these black holes: namely, in the region close to the horizon and in the far-region. By matching the solutions in an intermediate region, we calculate the frequency spectrum of quasinormal modes. We show that in the regime of superradiance only the modes of *even* orbital quantum number undergo negative damping, resulting in exponential growth of the amplitude. That is, the black holes become unstable to these modes. Meanwhile, the modes of *odd* orbital quantum number do not undergo any damping, oscillating with frequency-shifts. This is in contrast with the case of four-dimensional small Kerr-AdS black holes which exhibit the instability to all modes of scalar perturbations in the regime of superradiance.

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I. INTRODUCTION

Nowadays the classical theory of black holes in four-dimensional asymptotically flat spacetime is thought of as elegant and well understood. General relativity provides a unique family of exact solutions for stationary black holes which, in the most general case, involves only three physical parameters: the mass, angular momentum and the electric charge. Since classically the stationary black holes are "dead" objects, it is of crucial importance to explore their characteristic responses to external perturbations of different sorts. *Superradiance* is one of such responses, namely, it is a phenomenon of amplification of scalar, electromagnetic and gravitational waves scattered by a rotating black hole.

Though the phenomenon of superradiance, as a Klein-paradox state of nongravitational quantum systems, has been known for a long time (see [1] and references therein), Zel'dovich was the first to suggest the idea of supperradiant amplification of waves when scattering by the rotating black hole [2]. In order to argue the idea he had explored a heuristic model of the scattering of a wave by a rotating and absorbing cylinder. It turned out that when the wave spectrum contains the frequency ω fulfilling the condition $\omega < m\Omega$, where m is the azimuthal number or magnetic quantum number of the wave and Ω is the angular velocity of the cylinder, the reflection of the wave occurs with amplification. In other words, the rotating cylinder effectively acts as an amplifier, transmitting its rotational energy to the reflected wave. Zel'dovich concluded that a similar phenomenon must occur with rotating black holes as well, where the horizon plays the role of an absorber. Similar arguments showing that certain modes of scalar waves must be amplified by a Kerr black hole were also given in [3]. A complete theory of the superradiance in the Kerr metric was developed by Starobinsky in [4]. (See also Ref. [5]). The appearance of the superradiance in string microscopic models of rotating black holes was studied in a recent paper [6].

Physically, the superradiant scattering is a process of stimulated radiation which emerges due to the excitations of negative energy modes in the ergosphere of the black hole. It is a wave analogue of the Penrose process [7], in which a particle entering the ergosphere decays into two particles, one of which has a negative energy relative to infinity and is absorbed by the black hole. This renders the other particle to leave the ergosphere with greater energy than the initial one, thereby extracting the rotational energy from the black hole. In a quantum-mechanical picture, the superradiance is of stimulated emission of quanta which must be accompanied by their spontaneous emission as well [2]. The spontaneous superradiance arises due to quantum instability of the vacuum in the Kerr metric, leading to a pair production of particles. When leaving the ergosphere these particles carry positive energy and angular momentum from the black hole to infinity, whereas inside the ergosphere they form negative energy and angular momentum flows into the black hole [8].

The phenomenon of superradiance, after all, has a deep conceptual significance for understanding the stability properties of the black holes. As early as 1971 Zel'dovich [2] noted that placing a reflecting mirror (a resonator) around a rotating black hole would result in reamplification of superradiant modes and eventually the system would develop instability. The effect of the instability was later studied in [9] and the system is now known as a "black hole bomb." This study has also created the motivation to answer general questions on the stability of rotating black holes against small external perturbations. Using analytical and numerical methods, it has been shown that the Kerr black holes are stable to massless scalar, electromagnetic, and gravitational perturbations [10]. However, the situation turned out to be different for perturbing massive bosonic fields. As is known, classical particles of energy E and mass m, obeying the condition $E < m$, perform a finite motion in the gravitational potential of the black hole. From quantum-mechanical point of view there exists a certain probability for tunneling such particles through the potential barrier into the black hole. In consequence of this, the bound states of the particles inside the potential well must become *quasistationary* or *quasinormal* (see, for instance [11] and references therein). Similarly, for fields with mass μ, the wave of frequency $\omega < \mu$ can be thought of as a "bound particle" and therefore must undergo repetitive reflections between the potential well and the horizon. In the regime of superradiance, this will cause exponential growth of the number of particles in the quasinormal states, developing the instability [12–15]. Thus, for massive bosonic fields the potential barrier of the black hole plays the role of a mirror in the heuristic model of the black hole bomb. There are also alternative models where a reflecting mirror leading to the instability arises due to an extra dimension which, from a Kaluza-Klein point of view, acts as a massive term (see for instance, Ref. [16]).

In recent years, the question of the stability of black holes to external perturbations has been the subject of extensive studies in four- and higher-dimensional spacetimes with a cosmological constant. In particular, analytical and numerical works have revealed the perturbative stability of nonrotating black holes in de Sitter or anti-de Sitter (dS/AdS) spacetimes of various dimensions [17,18]. Though the similar general analysis concerning the stability of rotating black holes in the cosmological spacetimes still remains an open question, significant progress has been achieved in understanding their superradiant instability [19–22]. The causal structure of the AdS spacetime shows that spatial infinity in it corresponds to a finite region with a timelike boundary. Because of this property, the spacetime exhibits a "boxlike" behavior, ensuring the repetitive reflections of massless bosonic waves between spatial infinity and a Kerr-AdS black hole. The authors of work [19] have shown that the Kerr-AdS black hole in five dimensions admits a corotating Killing vector which remains timelike everywhere outside the horizon, provided that the angular velocity of the boundary Einstein space does not exceed the speed of light. This means that there is no way to extract energy from the black hole. However, these authors have also given simple arguments showing that for over-rotating Kerr-AdS black holes whose typical size is constrained to $r_+ < l$, where l is a length scale determined by the negative cosmological constant, the superradiant instability may occur. That is, the small Kerr-AdS black holes may become unstable against external perturbations. The idea was further developed in [20,21]. In particular, it was found that there must exist a critical radius for the location of the mirror in the black hole bomb model. Below this radius the superradiant condition is violated and the system becomes classically stable. Extending this fact to the case of the small Kerr-AdS black holes in four dimensions, the authors proved that the black holes indeed exhibit a superradiant instability to massless scalar perturbations. Later on, it was shown that the small Kerr-AdS black holes are also unstable to gravitational perturbations [22]. In a recent work [23], it was argued that the instability properties of the Kerr-AdS black holes to gravitational perturbations are equivalent to those against massless scalar perturbations.

The main purpose of the present paper is to address the superradiant instability of small rotating charged AdS black holes with two independent rotation parameters in minimal five-dimensional gauged supergravity. In Sec. II we discuss some properties of the spacetime metric given in the Boyer-Lindquist coordinates which are rotating at spatial infinity. In particular, we define a corotating Killing vector and calculate the angular velocities of the horizon as well as its electrostatic potential. We also discuss the "hidden" symmetries of the metric and demonstrate the separability of the Hamilton-Jacobi equation for massive charged particles.Section III is devoted to the study of the Klein-Gordon equation. We show that it is completely separable for massive charged particles and present the decoupled radial and angular equations in the most compact form. In Sec. IV we consider the near-horizon behavior of the radial equation and find the threshold frequency for the superradiance. In Sec. V we examine the instability of the small AdS black holes to low-frequency scalar perturbations. Here we construct the solution of the radial equation in the region close to the horizon and in the far-region. By matching these solutions in an intermediate region, we obtain the frequency spectrum for the quasinormal modes. We show that in the regime of superradiance the black hole exhibits instability to "selective" modes of the perturbations: Namely, only the modes of even orbital quantum number ℓ exponentially grow with time. We also show that the modes of odd ℓ do not exhibit any damping, but oscillate with frequency-shifts. In the Appendix we study the angular equation for AdS modified spheroidal harmonics in five dimensions.

II. THE METRIC AND ITS PROPERTIES

The general metric for rotating charged AdS black holes in the bosonic sector of minimal supergravity theory in five dimensions was recently found in [24]. The theory is described by the action

$$S = \int d^5x\sqrt{-g}\Big(R + \frac{12}{l^2} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} + \frac{1}{12\sqrt{3}}\epsilon^{\mu\nu\alpha\beta\lambda}F_{\mu\nu}F_{\alpha\beta}A_\lambda\Big), \quad (1)$$

leading to the coupled Einstein-Maxwell-Chern-Simons field equations

$$R_{\mu}{}^{\nu} = 2\left(F_{\mu\lambda}F^{\nu\lambda} - \frac{1}{6}\delta_{\mu}{}^{\nu}F_{\alpha\beta}F^{\alpha\beta}\right) - \frac{4}{l^2}\delta_{\mu}{}^{\nu}, \quad (2)$$

$$\nabla_{\nu}F^{\mu\nu} + \frac{1}{2\sqrt{3}\sqrt{-g}}\epsilon^{\mu\alpha\beta\lambda\tau}F_{\alpha\beta}F_{\lambda\tau} = 0. \quad (3)$$

The general black hole solution of [24] to these equations can be written in the form

$$\begin{aligned} ds^2 = & -\left(dt - \frac{a\sin^2\theta}{\Xi_a}d\phi - \frac{b\cos^2\theta}{\Xi_b}d\psi\right)\left[f\left(dt - \frac{a\sin^2\theta}{\Xi_a}d\phi - \frac{b\cos^2\theta}{\Xi_b}d\psi\right) + \frac{2Q}{\Sigma}\left(\frac{b\sin^2\theta}{\Xi_a}d\phi + \frac{a\cos^2\theta}{\Xi_b}d\psi\right)\right] \\ & + \Sigma\left(\frac{r^2dr^2}{\Delta_r} + \frac{d\theta^2}{\Delta_\theta}\right) + \frac{\Delta_\theta\sin^2\theta}{\Sigma}\left(adt - \frac{r^2+a^2}{\Xi_a}d\phi\right)^2 \\ & + \frac{\Delta_\theta\cos^2\theta}{\Sigma}\left(bdt - \frac{r^2+b^2}{\Xi_b}d\psi\right)^2 + \frac{1+r^2l^{-2}}{r^2\Sigma}\left(abdt - \frac{b(r^2+a^2)\sin^2\theta}{\Xi_a}d\phi - \frac{a(r^2+b^2)\cos^2\theta}{\Xi_b}d\psi\right)^2, \end{aligned} \quad (4)$$

where

$$\begin{aligned} f &= \frac{\Delta_r - 2abQ - Q^2}{r^2\Sigma} + \frac{Q^2}{\Sigma^2}, \\ \Xi_a &= 1 - \frac{a^2}{l^2}, \qquad \Xi_b = 1 - \frac{b^2}{l^2}, \\ \Delta_r &= (r^2+a^2)(r^2+b^2)(1+r^2l^{-2}) + 2abQ + Q^2 - 2Mr^2, \\ \Delta_\theta &= 1 - \frac{a^2}{l^2}\cos^2\theta - \frac{b^2}{l^2}\sin^2\theta, \\ \Sigma &= r^2 + a^2\cos^2\theta + b^2\sin^2\theta. \end{aligned} \quad (5)$$

We see that the metric is characterized by the parameters of mass M, electric charge Q as well as by two independent rotation parameters a and b. The cosmological constant is taken to be negative determining the cosmological length scale as $l^2 = -6/\Lambda$. Throughout this paper we suppose that the rotation parameters satisfy the relation $a^2, b^2 < l^2$.

For the potential one-form of the electromagnetic field, we have

$$A = -\frac{\sqrt{3}Q}{2\Sigma}\left(dt - \frac{a\sin^2\theta}{\Xi_a}d\phi - \frac{b\cos^2\theta}{\Xi_b}d\psi\right). \quad (6)$$

We recall that in their canonical forms, the Kerr-Newman-AdS metric in four dimensions as well as the Kerr-AdS metric in five dimensions are given in the Boyer-Lindquist coordinates which are rotating at spatial infinity. In order to be able to make an easy comparison of our description with those in four and five dimensions, we give the metric (4) in the asymptotically rotating Boyer-Lindquist coordinates $x^\mu = \{t, r, \theta, \phi, \psi\}$ with $\mu = 0, 1, 2, 3, 4$ (see Ref. [25]). It is easy to see that for $Q = 0$, it recovers the five-dimensional Kerr-AdS solution of [26].

The authors of [24] have calculated the physical parameters and examined the global structure and the supersymmetric properties of the solution in (4). In particular, they showed that for appropriate ranges of the parameters, the solution is free of closed timelike curves(CTCs) and naked singularities, describing a regular rotating charged black hole.

The determinant of the metric (4) does not involve the electric charge parameter Q and is given by

$$\sqrt{-g} = \frac{r\Sigma\sin\theta\cos\theta}{\Xi_a\Xi_b}, \quad (7)$$

whereas, the contravariant metric components have the form

$$\begin{aligned} g^{00} &= -\frac{1}{\Sigma}\left\{\frac{(r^2+a^2)(r^2+b^2)[r^2+l^2(1-\Xi_a\Xi_b)] + 2ab[(r^2+a^2+b^2)Q + abM]}{\Delta_r} - l^2\left(1 - \frac{\Xi_a\Xi_b}{\Delta_\theta}\right)\right\}, \\ g^{11} &= \frac{\Delta_r}{r^2\Sigma}, \qquad g^{22} = \frac{\Delta_\theta}{\Sigma}, \qquad g^{03} = \frac{\Xi_a}{\Sigma}\left\{\frac{a\Xi_b}{\Delta_\theta} - \frac{(r^2+b^2)[bQ + a\Xi_b(r^2+a^2)] + 2ab(aQ+bM)}{\Delta_r}\right\}, \\ g^{04} &= \frac{\Xi_b}{\Sigma}\left\{\frac{b\Xi_a}{\Delta_\theta} - \frac{(r^2+a^2)[aQ + b\Xi_a(r^2+b^2)] + 2ab(bQ+aM)}{\Delta_r}\right\}, \\ g^{33} &= \frac{\Xi_a^2}{\Sigma}\left\{\frac{\cot^2\theta + \Xi_b}{\Delta_\theta} + \frac{(r^2+b^2)[b^2 - a^2 + (r^2+a^2)(1-\Xi_b)] - 2b(aQ+bM)}{\Delta_r}\right\}, \\ g^{44} &= \frac{\Xi_b^2}{\Sigma}\left\{\frac{\tan^2\theta + \Xi_a}{\Delta_\theta} + \frac{(r^2+a^2)[a^2 - b^2 + (r^2+b^2)(1-\Xi_a)] - 2a(bQ+aM)}{\Delta_r}\right\}, \\ g^{34} &= -\frac{\Xi_a\Xi_b}{\Sigma}\left\{\frac{ab}{l^2}\left[\frac{1}{\Delta_\theta} - \frac{(r^2+a^2)(r^2+b^2)}{\Delta_r}\right] + \frac{2abM + (a^2+b^2)Q}{\Delta_r}\right\}. \end{aligned} \quad (8)$$

The horizons of the black hole are governed by the equation $\Delta_r = 0$, which can be regarded as a cubic equation with respect to r^2. It has two real roots; $r_1 = r_+^2$ and $r_2 = r_0^2$. The largest of these roots, $r_+^2 > r_0^2$, represents the radius of the event horizon. However, when the equations

$$\Delta_r = 0, \qquad \frac{d\Delta_r}{dr} = 0 \tag{9}$$

are satisfied simultaneously, the two roots coincide, $r_+^2 = r_0^2 = r_e^2$, representing the event horizon of an extreme black hole. From these equations, it follows that the parameters of the extreme black hole must obey the relations

$$2M_e l^2 = 2(r_e^2 + a^2 + b^2 + l^2)r_e^2 + r_e^4 + a^2b^2 + (a^2 + b^2)l^2, \tag{10}$$

$$Q_e = \frac{r_e^2}{l}(2r_e^2 + a^2 + b^2 + l^2)^{1/2} - ab. \tag{11}$$

The time translational and rotational (bi-azimuthal) isometries of the spacetime (4) are defined by the Killing vector fields

$$\xi_{(t)} = \partial/\partial t, \qquad \xi_{(\phi)} = \partial/\partial\phi, \qquad \xi_{(\psi)} = \partial/\partial\psi. \tag{12}$$

Using these Killing vectors one can also introduce a corotating Killing vector

$$\chi = \xi_{(t)} + \Omega_a \xi_{(\phi)} + \Omega_b \xi_{(\psi)}, \tag{13}$$

where Ω_a and Ω_b are the angular velocities of the event horizon in two independent orthogonal 2-planes of rotation. We have

$$\begin{aligned}\Omega_a &= \frac{\Xi_a[a(r_+^2 + b^2) + bQ]}{(r_+^2 + a^2)(r_+^2 + b^2) + abQ},\\ \Omega_b &= \frac{\Xi_b[b(r_+^2 + a^2) + aQ]}{(r_+^2 + a^2)(r_+^2 + b^2) + abQ}.\end{aligned} \tag{14}$$

It is straightforward to show that the corotating Killing vector in (13) is null on the event horizon of the black hole, i.e. it is tangent to the null generators of the horizon, confirming that the quantities Ω_a and Ω_b are indeed the angular velocities of the horizon.

We also need the electrostatic potential of the horizon relative to an infinitely distant point. It is given by

$$\Phi_H = -A\cdot\chi = -(A_0 + \Omega_a A_\phi + \Omega_b A_\psi)|_{r=r_+}. \tag{15}$$

Substituting into this expression the components of the potential in (6), we find the explicit form for the electrostatic potential

$$\Phi_H = \frac{\sqrt{3}}{2}\frac{Qr_+^2}{(r_+^2 + a^2)(r_+^2 + b^2) + abQ}. \tag{16}$$

It is also important to note that, in addition to the global isometries, the spacetime (4) also possesses hidden symmetries generated by a second-rank Killing tensor. The existence of the Killing tensor ensures the complete separability of variables in the Hamilton-Jacobi equation for geodesic motion of uncharged particles [27]. Below, we describe the separation of variables in the Hamilton-Jacobi equation for charged particles.

A. The Hamilton-Jacobi equation for charged particles

The Hamilton-Jacobi equation for a particle of electric charge e moving in the spacetime under consideration is given by

$$\frac{\partial S}{\partial\lambda} + \frac{1}{2}g^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu} - eA_\mu\right)\left(\frac{\partial S}{\partial x^\nu} - eA_\nu\right) = 0, \tag{17}$$

where, λ is an affine parameter. Since the potential one-form (6) respects the Killing isometries (12) of the spacetime as well, $\mathcal{L}_\xi A^\mu = 0$, we assume that the action S can be written in the form

$$S = \frac{1}{2}m^2\lambda - Et + L_\phi\phi + L_\psi\psi + S_r(r) + S_\theta(\theta), \tag{18}$$

where the constants of motion represent the mass m, the total energy E and the angular momenta L_ϕ and L_ψ associated with the rotations in ϕ and ψ 2-planes. Substituting this action into Eq. (17) and using the metric components in (8) along with the contravariant components of the potential

$$\begin{aligned}A^0 &= \frac{\sqrt{3}Q}{2\Sigma}\frac{(r^2 + a^2)(r^2 + b^2) + abQ}{\Delta_r},\\ A^3 &= \frac{\sqrt{3}Q\Xi_a}{2\Sigma}\frac{a(r^2 + b^2) + bQ}{\Delta_r},\\ A^4 &= \frac{\sqrt{3}Q\Xi_b}{2\Sigma}\frac{b(r^2 + a^2) + aQ}{\Delta_r},\end{aligned} \tag{19}$$

and

$$A^\mu A_\mu = -\frac{3Q^2r^2}{4\Sigma\Delta_r}, \tag{20}$$

we obtain two independent ordinary differential equations for r and θ motions:

$$\begin{aligned}&\frac{\Delta_r}{r^2}\left(\frac{dS_r}{dr}\right)^2 + \frac{(abE - b\Xi_a L_\phi - a\Xi_b L_\psi)^2}{r^2}\\ &\quad - \frac{[(r^2 + a^2)(r^2 + b^2) + abQ]^2}{\Delta_r r^2}\\ &\quad \times\left\{E - \frac{L_\phi\Xi_a[a(r^2 + b^2) + bQ]}{(r^2 + a^2)(r^2 + b^2) + abQ}\right.\\ &\quad - \frac{L_\psi\Xi_b[b(r^2 + a^2) + aQ]}{(r^2 + a^2)(r^2 + b^2) + abQ}\\ &\quad \left. - \frac{\sqrt{3}}{2}\frac{eQr^2}{(r^2 + a^2)(r^2 + b^2) + abQ}\right\}^2 + m^2r^2 = -K,\end{aligned} \tag{21}$$

$$\Delta_\theta\left(\frac{dS_\theta}{d\theta}\right)^2 + \frac{L_\phi^2\Xi_a^2(\cot^2\theta + \Xi_b) + L_\psi^2\Xi_b^2(\tan^2\theta + \Xi_a) - 2abl^{-2}L_\phi L_\psi \Xi_a\Xi_b}{\Delta_\theta} + \frac{E^2l^2(\Delta_\theta - \Xi_a\Xi_b) - 2E(aL_\phi + bL_\psi)\Xi_a\Xi_b}{\Delta_\theta} + m^2(a^2\cos^2\theta + b^2\sin^2\theta) = K, \tag{22}$$

where K is a constant of separation. The complete separability in the Hamilton-Jacobi equation (17) occurs due to the existence of a new quadratic integral of motion $K = K^{\mu\nu}p_\mu p_\nu$, which is associated with the hidden symmetries of the spacetime. Here $K^{\mu\nu}$ is an irreducible Killing tensor generating the hidden symmetries. Using Eq. (22) along with $-m^2 = g^{\mu\nu}p_\mu p_\nu$, we obtain that the Killing tensor is given by

$$\begin{aligned} K^{\mu\nu} &= l^2\left(1 - \frac{\Xi_a\Xi_b}{\Delta_\theta}\right)\delta_t^\mu\delta_t^\nu + \frac{\Xi_a\Xi_b}{\Delta_\theta}[a(\delta_t^\mu\delta_\phi^\nu + \delta_\phi^\mu\delta_t^\nu) \\ &\quad + b(\delta_t^\mu\delta_\psi^\nu + \delta_\psi^\mu\delta_t^\nu)] + \frac{1}{\Delta_\theta}\bigg[\Xi_a^2(\cot^2\theta + \Xi_b)\delta_\phi^\mu\delta_\phi^\nu \\ &\quad + \Xi_b^2(\tan^2\theta + \Xi_a)\delta_\psi^\mu\delta_\psi^\nu - \frac{ab\Xi_a\Xi_b}{l^2} \\ &\quad \times(\delta_\phi^\mu\delta_\psi^\nu + \delta_\psi^\mu\delta_\phi^\nu)\bigg] \\ &\quad - g^{\mu\nu}(a^2\cos\theta^2 + b^2\sin\theta^2) + \Delta_\theta\delta_\theta^\mu\delta_\theta^\nu. \end{aligned} \tag{23}$$

This expression agrees with that given in [27] up to terms involving symmetrized outer products of the Killing vectors. Similarly, for the vanishing cosmological constant, $l \to \infty$, it recovers the result of work [28].

III. THE KLEIN-GORDON EQUATION

We consider now the Klein-Gordon equation for a scalar field with charge e and mass μ in the background of the metric (4). It is given by

$$(D^\mu D_\mu - \mu^2)\Phi = 0, \tag{24}$$

where $D_\mu = \nabla_\mu - ieA_\mu$ and ∇_μ is a covariant derivative operator. Decomposing the indices as $\mu = \{1, 2, M\}$ in which $M = 0, 3, 4$, we can write down the above equation in the form

$$\begin{aligned} &\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\Delta_r}{r}\frac{\partial\Phi}{\partial r}\right) + \frac{1}{\sin 2\theta}\frac{\partial}{\partial\theta}\left(\sin 2\theta\Delta_\theta\frac{\partial\Phi}{\partial\theta}\right) \\ &\quad + \left(g^{MN}\frac{\partial^2\Phi}{\partial x^M\partial x^N} - 2ieA^M\frac{\partial\Phi}{\partial x^N} - e^2A_MA^M\right)\Sigma \\ &\quad = \mu^2\Sigma\Phi. \end{aligned} \tag{25}$$

It is easy to show that with the components of g^{MN} given in (8) and with Eqs. (19) and (20), this equation is manifestly separable in variables r and θ. That is, one can assume that its solution admits the ansatz

$$\Phi = e^{-i\omega t + im_\phi\phi + im_\psi\psi}S(\theta)R(r), \tag{26}$$

where m_ϕ and m_ψ are the "magnetic" quantum numbers related to ϕ and ψ 2-planes of rotation, so that they both must take integer values. In what follows, for the sake of certainty, we restrict ourselves to the case of positive frequency ($\omega > 0$) and positive m_ϕ and m_ψ.

The substitution of the expression (26) into Eq. (25) results in two decoupled ordinary differential equations for angular and radial functions. The angular equation is given by

$$\begin{aligned} &\frac{1}{\sin 2\theta}\frac{d}{d\theta}\left(\sin 2\theta\Delta_\theta\frac{dS}{d\theta}\right) + \frac{1}{\Delta_\theta}[\omega^2l^2(\Xi_a\Xi_b - \Delta_\theta) \\ &\quad - m_\phi^2\Xi_a^2(\cot^2\theta + \Xi_b) - m_\psi^2\Xi_b^2(\tan^2\theta + \Xi_a) \\ &\quad + 2\Xi_a\Xi_b\left(\omega am_\phi + \omega bm_\psi + \frac{ab}{l^2}m_\phi m_\psi\right) \\ &\quad - \Delta_\theta\mu^2(a^2\cos^2\theta + b^2\sin^2\theta)\bigg]S \\ &\quad = -\lambda S, \end{aligned} \tag{27}$$

where λ is a constant of separation. With regular boundary conditions at $\theta = 0$ and $\theta = \pi/2$, this equation describes a well-defined Sturm-Liouville problem with eigenvalues $\lambda_\ell(\omega)$, where ℓ is thought of as an "orbital" quantum number. The corresponding eigenfunctions are five-dimensional (AdS modified) spheroidal functions $S(\theta) = S_{\ell m_\phi m_\psi}(\theta|a\omega, b\omega)$. In some special cases of interest, the eigenvalues were calculated in the Appendix, see Eq. (A14).

The radial equation can be written in the form

$$\frac{\Delta_r}{r}\frac{d}{dr}\left(\frac{\Delta_r}{r}\frac{dR}{dr}\right) + U(r)R = 0, \tag{28}$$

where

$$\begin{aligned} U(r) &= -\Delta_r\left[\lambda + \mu^2r^2 + \frac{(ab\omega - b\Xi_am_\phi - a\Xi_bm_\psi)^2}{r^2}\right] \\ &\quad + \frac{[(r^2 + a^2)(r^2 + b^2) + abQ]^2}{r^2} \\ &\quad \times\left\{\omega - \frac{m_\phi\Xi_a[a(r^2 + b^2) + bQ]}{(r^2 + a^2)(r^2 + b^2) + abQ}\right. \\ &\quad - \frac{m_\psi\Xi_b[b(r^2 + a^2) + aQ]}{(r^2 + a^2)(r^2 + b^2) + abQ} \\ &\quad \left. - \frac{\sqrt{3}}{2}\frac{eQr^2}{(r^2 + a^2)(r^2 + b^2) + abQ}\right\}^2. \end{aligned} \tag{29}$$

When the cosmological constant vanishes, $l \to \infty$, the above expressions agree with those obtained in [29] for a five-dimensional Myers-Perry black hole.

IV. THE SUPERRADIANCE THRESHOLD

The radial equation can be easily solved near the horizon. For this purpose, it is convenient to introduce a new radial function $\mathcal{R}$ defined by

$$R = \left[\frac{r}{(r^2 + a^2)(r^2 + b^2) + abQ}\right]^{1/2}\mathcal{R} \tag{30}$$

and a new radial, the so-called tortoise coordinate r_*, obeying the relation

$$\frac{dr_*}{dr} = \frac{(r^2 + a^2)(r^2 + b^2) + abQ}{\Delta_r}. \tag{31}$$

With these new definitions the radial Eq. (28) can be transformed into the form

$$\frac{d^2\mathcal{R}}{dr_*^2} + V(r)\mathcal{R} = 0, \tag{32}$$

where the effective potential is given by

$$\begin{aligned} V(r) = &-\frac{\Delta_r[r^2(\lambda + \mu^2 r^2) + (ab\omega - b\Xi_a m_\phi - a\Xi_b m_\psi)^2]}{[(r^2 + a^2)(r^2 + b^2) + abQ]^2} \\ &- \frac{\Delta_r}{2ru^{3/2}}\frac{d}{dr}\left(\frac{\Delta_r}{ru^{3/2}}\frac{du}{dr}\right) \\ &+ \left\{\omega - \frac{m_\phi \Xi_a[a(r^2 + b^2) + bQ]}{(r^2 + a^2)(r^2 + b^2) + abQ}\right. \\ &- \frac{m_\psi \Xi_b[b(r^2 + a^2) + aQ]}{(r^2 + a^2)(r^2 + b^2) + abQ} \\ &\left. - \frac{\sqrt{3}}{2}\frac{eQr^2}{(r^2 + a^2)(r^2 + b^2) + abQ}\right\}^2. \end{aligned} \tag{33}$$

For brevity, we have also introduced

$$u = \frac{(r^2 + a^2)(r^2 + b^2) + abQ}{r}. \tag{34}$$

In what follows, we consider a massless scalar field, $\mu = 0$. We see that at the horizon $r = r_+$ ($\Delta_r = 0$), the effective potential in Eq. (33) becomes

$$V(r_+) = (\omega - m_\phi \Omega_a - m_\psi \Omega_b - e\Phi_H)^2. \tag{35}$$

With this in mind, it is easy to verify that for an observer near the horizon, the asymptotic solution of the wave equation

$$\Phi = e^{-i\omega t + im_\phi \phi + im_\psi \psi} e^{-i(\omega - \omega_p)r_*} S(\theta), \tag{36}$$

corresponds to an ingoing wave at the horizon. The threshold frequency

$$\omega_p = m_\phi \Omega_a + m_\psi \Omega_b + e\Phi_H \tag{37}$$

determines the frequency range

$$0 < \omega < \omega_p \tag{38}$$

for which, the phase velocity of the wave changes its sign. As in the four-dimensional case [10], this fact is the signature of the superradiance. That is, when the condition (38) is fulfilled there must exist a superradiant outflow of energy from the black hole. From Eq. (37), it follows that the electric charge of the black hole changes the superradiance threshold frequency for charged particles.

Next, turning to the asymptotic behavior of the solution at spatial infinity, we recall that in this region the AdS spacetime reveals a boxlike behavior. In other words, at spatial infinity the spacetime effectively acts as a reflective barrier. Therefore, we require the vanishing field boundary condition

$$\Phi \to 0 \quad \text{as} \quad r \to \infty. \tag{39}$$

With the boundary conditions (36) and (39), namely, requiring a purely ingoing wave at the horizon and a purely damping wave at infinity, we arrive at a characteristic-value problem for complex frequencies of quasinormal modes of the massless scalar field, see [14]. The imaginary part of these frequencies describes the damping of the modes. A characteristic mode is stable if the imaginary part of its complex frequency is negative (*the positive damping*), while for the positive imaginary part, the mode undergoes exponential growth (*the negative damping*). In the latter case, the system will develop instability.

V. INSTABILITY

In this section we describe the instability for small-size five-dimensional AdS black holes, $r_+ \ll l$, in the regime of low-frequency perturbations. That is, we assume that the wavelength of the perturbations is much larger than the typical size of the horizon, $1/\omega \gg r_+$. In addition, we also assume slow rotation, i.e. we restrict ourselves to linear order terms in rotation parameters a and b. With these approximations, we can apply the similar method first developed by Starobinsky [4] and later on used by many authors (see, [21] and references therein) to construct the solutions of the radial Eq. (28) in the region near the horizon and in the far-region. It is remarkable that there exists an intermediate region where the two solutions overlap and matching these solutions enables us to calculate the frequency of quasinormal modes and explore the (in)stability of these modes.

A. Near-region solution

For small and slowly rotating black holes, in the region close to the horizon, $r - r_+ \ll 1/\omega$, and in the regime of low-frequency perturbations, $1/\omega \gg r_+$, the radial Eq. (28) can be approximated by the equation

$$4\Delta_x \frac{d}{dx}\left(\Delta_x \frac{dR}{dx}\right) + [x_+^3(\omega - \omega_p)^2 - \ell(\ell+2)\Delta_x]R = 0, \tag{40}$$

where we have used the new radial coordinate $x = r^2$ and

$$\Delta_x \simeq x^2 - 2Mx + Q^2 = (x - x_+)(x - x_-). \tag{41}$$

The superradiance threshold frequency is given by Eq. (37), in which one must now take

$$\Omega_a \simeq \frac{a}{x_+} + \frac{bQ}{x_+^2}, \qquad \Omega_b \simeq \frac{b}{x_+} + \frac{aQ}{x_+^2}, \qquad \Phi_H \simeq \frac{\sqrt{3}}{2}\frac{Q}{x_+}. \tag{42}$$

In obtaining the above equations we have neglected the term involving r_+^2/l^2 as well as all terms with square and higher orders in rotation parameters. With the approximation employed, the eigenvalues λ_ℓ are replaced by their five-dimensional flat spacetime value $\lambda_\ell \simeq \ell(\ell+2)$. (See the Appendix).

Next, it is convenient to define a new dimensionless variable

$$z = \frac{x - x_+}{x - x_-}, \tag{43}$$

which in the near-horizon region goes to zero, $z \to 0$. Then Eq. (40) can be put in the form

$$z(1-z)\frac{d^2R}{dz^2} + (1-z)\frac{dR}{dz} + \left[\frac{1-z}{z}\Omega^2 - \frac{\ell(\ell+2)}{4(1-z)}\right]R = 0, \tag{44}$$

where

$$\Omega = \frac{x_+^{3/2}}{2}\frac{\omega - \omega_p}{x_+ - x_-}. \tag{45}$$

It is straightforward to check that the ansatz

$$R(z) = z^{i\Omega}(1-z)^{1+\ell/2}F(z), \tag{46}$$

when substituting into the above equation, takes us to the hypergeometric equation of the form (A9) for the function $F(z) = F(\alpha, \beta, \gamma, z)$ with

$$\alpha = 1 + \ell/2 + 2i\Omega, \qquad \beta = 1 + \ell/2, \qquad \gamma = 1 + 2i\Omega. \tag{47}$$

The physical solution of this equation corresponding to the ingoing wave at the horizon, $z \to 0$, is given by

$$R(z) = Az^{-i\Omega}(1-z)^{1+\ell/2}F(1+\ell/2, 1+\ell/2-2i\Omega, 1 - 2i\Omega, z), \tag{48}$$

where A is a constant. For large enough values of the wavelength, this solution may overlap with the far-region solution. Therefore, we need to consider the large r ($z \to 1$) limit of this solution. For this purpose, we use the functional relation between the hypergeometric functions of the arguments z and $1 - z$ [30], which in our case has the form

$$\begin{aligned} F(1 + \ell/2, 1 + \ell/2 - 2i\Omega, 1 - 2i\Omega, z) &= \frac{\Gamma(-1-\ell)\Gamma(1-2i\Omega)}{\Gamma(-\ell/2)\Gamma(-\ell/2-2i\Omega)}F(1+\ell/2, 1+\ell/2 - 2i\Omega, 2+\ell, 1-z) \\ &+ \frac{\Gamma(1+\ell)\Gamma(1-2i\Omega)}{\Gamma(1+\ell/2)\Gamma(1+\ell/2-2i\Omega)}(1-z)^{-1-\ell} \\ &\times F(-\ell/2 - 2i\Omega, -\ell/2, -\ell, 1-z). \end{aligned} \tag{49}$$

Taking this into account in Eq. (48), we obtain that the large r behavior of the near-region solution is given by

$$R \sim A\Gamma(1-2i\Omega)\left[\frac{\Gamma(-1-\ell)(r_+^2 - r_-^2)^{1+\ell/2}}{\Gamma(-\ell/2)\Gamma(-\ell/2-2i\Omega)}r^{-2-\ell} + \frac{\Gamma(1+\ell)(r_+^2 - r_-^2)^{-\ell/2}}{\Gamma(1+\ell/2)\Gamma(1+\ell/2-2i\Omega)}r^{\ell}\right], \tag{50}$$

where we have also used the fact that $F(\alpha, \beta, \gamma, 0) = 1$.

B. Far-region solution

In this region $r_+ \gg M$ the effects of the black hole are suppressed and the radial Eq. (28) in this approximation is reduced to the form

$$\left(1 + \frac{r^2}{l^2}\right)\frac{d^2R}{dr^2} + \left(\frac{3}{r} + \frac{5r}{l^2}\right)\frac{dR}{dr} + \left[\frac{\omega^2}{1 + \frac{r^2}{l^2}} - \frac{\ell(\ell+2)}{r^2}\right]R = 0. \tag{51}$$

Defining a new variable

$$y = \left(1 + \frac{r^2}{l^2}\right), \tag{52}$$

we can also put the equation into the form

$$y(1-y)\frac{d^2R}{dy^2} + (1-3y)\frac{dR}{dy} - \frac{1}{4}\left[\frac{\omega^2 l^2}{y} - \frac{\ell(\ell+2)}{y-1}\right]R = 0. \tag{53}$$

We note that this is an equation in a pure AdS spacetime and therefore, we look for its solution satisfying the boundary conditions at infinity, $y \to \infty$, and at the origin of the AdS space, $y \to 1$.

Again, one can show that the ansatz

$$R = y^{\omega l/2}(1-y)^{\ell/2}F(y), \tag{54}$$

transforms Eq. (53) into the hypergeometric equation of the form (A9), where the parameters of the hypergeometric function $F(\alpha, \beta, \gamma, y)$ are given by

$$\alpha = 2 + \ell/2 + \omega l/2, \qquad \beta = \ell/2 + \omega l/2, \qquad \gamma = 1 + \omega l. \tag{55}$$

The solution of this equation vanishing at $y \to \infty$, i.e. obeying the boundary condition (39), is given by

$$R(y) = By^{-2-\ell/2}(1-y)^{\ell/2}F(2+\ell/2+\omega l/2, 2+\ell/2 - \omega l/2, 3, 1/y), \quad (56)$$

where B is a constant. We are also interested in knowing the small r ($y \to 1$) behavior of this solution. Using the expansion of the hypergeometric function in (56) in terms of the hypergeometric functions of the argument $1-y$ given by

$$\begin{aligned} F(2+\ell/2+\omega l/2, 2+\ell/2-\omega l/2, 3, 1/y) &= \frac{\Gamma(3)\Gamma(1+\ell)y^{2+\ell/2-\omega l/2}(y-1)^{-1-\ell}}{\Gamma(2+\ell/2+\omega l/2)\Gamma(2+\ell/2-\omega l/2)} \\ &\times F(1-\ell/2-\omega l/2, -1-\ell/2-\omega l/2, -\ell, 1-y) + \frac{\Gamma(3)\Gamma(-1-\ell)y^{2+\ell/2+\omega l/2}}{\Gamma(1-\ell/2+\omega l/2)\Gamma(1-\ell/2-\omega l/2)} \\ &\times F(2+\ell/2+\omega l/2, \ell/2+\omega l/2, 2+\ell, 1-y), \end{aligned} \quad (57)$$

we find that for small values of r the asymptotic solution has the form

$$R(r) \sim B\Gamma(3)(-1)^{\ell/2} \times \left[\frac{\Gamma(1+\ell)l^{2+\ell}r^{-2-\ell}}{\Gamma(2+\ell/2+\omega l/2)\Gamma(2+\ell/2-\omega l/2)} + \frac{\Gamma(-1-\ell)l^{-\ell}r^{\ell}}{\Gamma(1-\ell/2+\omega l/2)\Gamma(1-\ell/2-\omega l/2)}\right]. \quad (58)$$

Requiring the regularity of this solution at the origin of the AdS space ($r=0$), we obtain the quantization condition

$$2+\ell/2-\omega l/2 = -n, \quad (59)$$

where n is a non-negative integer being a "principal" quantum number. We recall that with this condition the gamma function $\Gamma(2+\ell/2-\omega l/2) = \infty$. Thus, we find that the discrete frequency spectrum for scalar perturbations in the five-dimensional AdS spacetime is given by

$$\omega_n = \frac{2n+\ell+4}{l}. \quad (60)$$

This formula generalizes the four-dimensional result of works in [31,32] to five dimensions. Since at infinity the causal structure of the AdS black hole is similar to that of the pure AdS background, it is natural to assume that Eq. (60) equally well governs the frequency spectrum at large distances from the black hole. However, the important difference is related to the inner boundaries which are different; for the AdS spacetime we have $r=0$, while for the black hole in this spacetime we have $r=r_+$. Therefore, to catch the effect of the black hole, the solution (58) must "respond" to the ingoing wave condition at the boundary $r=r_+$. Physically, this means that one must take into account the possibility for tunneling of the wave through the potential barrier into the black hole and scattering back. As we have described above, this would result in the quasinormal spectrum with the complex frequencies

$$\omega = \omega_n + i\sigma, \quad (61)$$

where σ is supposed to be a small quantity, describing the damping of the quasinormal modes. Taking this into account in Eq. (58), we first note that

$$\Gamma(2+\ell/2+\omega l/2)\Gamma(2+\ell/2-\omega l/2) = \Gamma(4+\ell+n+il\sigma/2)\Gamma(-n-il\sigma/2). \quad (62)$$

Next, applying to this expression the functional relations for the gamma functions [30]

$$\Gamma(k+z) = (k-1+z)(k-2+z)\dots(1+z)\Gamma(1+z), \qquad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}, \quad (63)$$

where k is a non-negative integer, it is easy to show that for $l\sigma \ll 1$

$$\Gamma(2+\ell/2+\omega l/2)\Gamma(2+\ell/2-\omega l/2) = -\frac{2i}{l\sigma}\frac{(\ell+3+n)!}{(-1)^{n+1}n!}. \quad (64)$$

Similarly, one can also show that

$$\Gamma(1-\ell/2+\omega l/2)\Gamma(1-\ell/2-\omega l/2) = \Gamma(-1-\ell-n)\Gamma(3+n). \quad (65)$$

Substituting now these expressions into Eq. (58), we obtain the desired form of the far-region solution at small values of r. It is given by

$$R = B\Gamma(3)(-1)^{\ell/2}\left[\frac{\Gamma(-1-\ell)l^{-\ell}r^{\ell}}{\Gamma(-1-\ell-n)\Gamma(3+n)} + i\sigma\frac{(-1)^{n+1}n!\Gamma(1+\ell)}{2(3+\ell+n)!}l^{3+\ell}r^{-2-\ell}\right]. \quad (66)$$

C. Overlapping

Comparing the large r behavior of the near-region solution in (50) with the small r behavior of the far-region solution in (66), we conclude that there exists an intermediate region $r_+ \ll r - r_+ \ll 1/\omega$ where these solutions overlap. In this region we can match them which allows us to obtain the damping factor in the form

$$\sigma = -2i\frac{(r_+^2-r_-^2)^{1+\ell}}{l^{3+2\ell}}\frac{(3+\ell+n)!(1+\ell+n)!}{(-1)^{\ell}n!(2+n)![\ell!(1+\ell)!]^2} \times \frac{\Gamma(1+\ell/2)\Gamma(1+\ell/2-2i\Omega)}{\Gamma(-\ell/2)\Gamma(-\ell/2-2i\Omega)}. \quad (67)$$

We note that in this expression the quantity Ω is given by

$$\Omega = \frac{r_+^3}{2}\frac{\omega_n - \omega_p}{r_+^2 - r_-^2}. \quad (68)$$

It is also important to note that, in contrast to the related expression in four dimensions [21], Eq. (67) involves the term $\ell/2$ in the arguments of the gamma functions. Therefore, its further evaluation requires us to consider the cases of even and odd values of ℓ separately.

1. Even ℓ

In this case using the functional relations [30]

$$\Gamma(k+iz)\Gamma(k-iz) = \Gamma(1+iz)\Gamma(1-iz)\prod_{j=1}^{k-1}(j^2+z^2), \tag{69}$$

$$\Gamma(1+iz)\Gamma(1-iz) = \frac{\pi z}{\sinh \pi z}, \tag{70}$$

one can show that

$$\frac{\Gamma(1+\ell/2)\Gamma(1+\ell/2-2i\Omega)}{\Gamma(-\ell/2)\Gamma(-\ell/2-2i\Omega)} = -2i\Omega[(\ell/2)!]^2\prod_{j=1}^{\ell/2}(j^2+4\Omega^2). \tag{71}$$

Substituting this expression into Eq. (67) we find that

$$\sigma = -(\omega_n-\omega_p)\frac{2(3+\ell+n)!(1+\ell+n)!}{(-1)^\ell n!(2+n)![\ell!(1+\ell)!]^2} \times \frac{r_+^3(r_+^2-r_-^2)^\ell}{l^{3+2\ell}}[(\ell/2)!]^2\prod_{j=1}^{\ell/2}(j^2+4\Omega^2). \tag{72}$$

We see that the sign of this expression crucially depends on the sign of the factor $(\omega_n - \omega_p)$ and in the superradiant regime $\omega_n < \omega_p$ it is positive. In other words, we have the negative damping effect, as we have discussed at the end of Sec. IV, resulting in exponential growth of the modes with characteristic time scale $\tau = 1/\sigma$. Thus, the small AdS black holes under consideration become unstable to the superradiant scattering of massless scalar perturbations of even ℓ or equivalently of even sum $m_\phi + m_\psi$. We recall that we consider the positive frequency modes and the positive magnetic quantum numbers m_ϕ and m_ψ.

2. Odd ℓ

In order to evaluate the combination of the gamma functions appearing in Eq. (67) for odd values of ℓ, we appeal to the relations [30]

$$\Gamma\left(k+\frac{1}{2}\right) = \pi^{1/2}2^{-k}(2k-1)!!,$$
$$\Gamma\left(\frac{1}{2}+iz\right)\Gamma\left(\frac{1}{2}-iz\right) = \frac{\pi}{\cosh \pi z}. \tag{73}$$

Using these relation along with those given in (63), after some algebra, we obtain that

$$\frac{\Gamma(1+\ell/2)\Gamma(1+\ell/2-2i\Omega)}{\Gamma(-\ell/2)\Gamma(-\ell/2-2i\Omega)} = \frac{(\ell!!)^2}{2^{1+\ell}}\prod_{j=1}^{(\ell+1)/2}\left[\left(j-\frac{1}{2}\right)^2+4\Omega^2\right]. \tag{74}$$

With this in mind, we put Eq. (67) in the form

$$\sigma = -i\frac{(r_+^2-r_-^2)^{1+\ell}}{l^{3+2\ell}}\frac{(3+\ell+n)!(1+\ell+n)!(\ell!!)^2}{(-1)^\ell 2^\ell n!(2+n)![\ell!(1+\ell)!]^2} \times \prod_{j=1}^{(\ell+1)/2}\left[\left(j-\frac{1}{2}\right)^2+4\Omega^2\right]. \tag{75}$$

We see that this expression is purely imaginary and it does not change the sign in the superradiant regime. In other words, these modes do not undergo any damping, but they do oscillate with frequency-shifts.

VI. CONCLUSION

In this paper, we have discussed the instability properties of small-size, $r_+ \ll l$, charged AdS black holes with two rotation parameters, which are described by the solution of minimal five-dimensional gauged supergravity recently found in [24]. The remarkable symmetries of this solution allow us to perform a complete separation of variables in the field equations governing scalar perturbations in the background of the AdS black holes.

We have begun with demonstrating the separability of variables in the Hamilton-Jacobi equation for massive charged particles as well as in the Klein-Gordon equation for a massive charged scalar field. In both cases, we have presented the decoupled radial and angular equations in their most compact form. Next, exploring the behavior of the radial equation near the horizon, we have found the threshold frequency for the superradiance of these black holes. Restricting ourselves to slow rotation and to low-frequency perturbations, when the characteristic wavelength scale is much larger than the typical size of the black hole, we have constructed the solutions of the radial equation in the region close to the horizon and in the far-region of the spacetime. Performing the matching of these solutions in an overlapping region of their validity, we have derived an analytical formula for the frequency spectrum of the quasinormal modes.

Analyzing the imaginary part of the spectrum for modes of even and odd ℓ separately, we have revealed a new feature: In the regime of superradiance only the modes of even ℓ undergo the negative damping, exponentially growing their amplitudes. On the other hand, the modes of odd ℓ turn out to be not sensitive to the regime of superradiance, oscillating without any damping, but with frequency-shifts. This new feature is inherent in the five-dimensional AdS black hole spacetime and absent in four dimensions where the small-size AdS black holes exhibit the instability to all modes of scalar perturbations in the regime of superradiance [21].

We emphasize once again that our result was obtained in the regime of low-frequency perturbations and for small-size, slowly rotating AdS black holes. Therefore, its validity is guaranteed for a certain range of the perturbation frequencies and parameters of the black holes within the approximation employed. The full analysis beyond this approximation requires a numerical work. Meanwhile, one should remember that the characteristic oscillating modes for the instability to occur (for superradiance) are governed by the radius of the AdS space. This means that the instability will not occur for an arbitrary range of the black hole parameters. We also emphasize that the different stability properties of even and odd modes of scalar perturbations arise only in the five-dimensional case with reflective boundary conditions. The physical reason for this is apparently related with the ‘‘fermionic constituents’’ of the five-dimensional AdS black hole. Therefore, it would be interesting to explore this effect in the spirit of work [33] using an effective string theory picture, where even ℓ refers to bosons and odd ℓ to fermions. This is a challenging project for future work.

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APPENDIX: ANGULAR EQUATION

The angular Eq. (27) can be transformed into a second-order Fuchsian equation by defining a new variable $z = \sin^2\theta$. Performing straightforward calculations, we obtain

$$\frac{d^2S}{dz^2} + \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-c}\right)\frac{dS}{dz} + \frac{1}{4z(1-z)\Delta_z^2} \times \Big[-\Delta_z\omega^2 l^2 - \frac{m_\phi^2\Xi_a^2}{z} - \frac{m_\psi^2\Xi_b^2}{1-z} + (\Xi_a - \Xi_b)(m_\phi^2\Xi_a - m_\psi^2\Xi_b) + \Xi_a\Xi_b l^2\Big(\omega + \frac{a}{l^2}m_\phi + \frac{b}{l^2}m_\psi\Big)^2 + \Delta_z(\lambda - \mu^2[b^2 z + a^2(1-z)])\Big]S = 0, \tag{A1}$$

where

$$\Delta_z = \Xi_a + (\Xi_b - \Xi_a)z, \qquad c = \frac{\Xi_a}{\Xi_a - \Xi_b}. \tag{A2}$$

This equation has four regular singular points $z = 0, 1, c, \infty$ and therefore can also be put in the form of the Heun equation [34]

$$\frac{d^2H}{dz^2} + \left(\frac{1+2\alpha}{z} + \frac{1+2\beta}{z-1} + \frac{1+2\gamma}{z-c}\right)\frac{dH}{dz} + \frac{\varepsilon\delta z + \eta}{z(z-1)(z-c)}H = 0, \tag{A3}$$

where the functions $H(z)$ and $S(z)$ are related by

$$S(z) = z^\alpha(1-z)^\beta(z-c)^\gamma H(z), \tag{A4}$$

and the associated parameters are given by

$$2\alpha = m_\phi, \qquad 2\beta = m_\psi, \qquad 2l\gamma = \omega l^2 + am_\phi + bm_\psi,$$
$$2\delta = m_\phi + m_\psi + 2(\gamma + 1 + \sqrt{1 + l^2\mu^2/4}),$$
$$2\varepsilon = m_\phi + m_\psi + 2(\gamma + 1 - \sqrt{1 + l^2\mu^2/4}),$$
$$2\eta = -(2\gamma + m_\phi)(1 + m_\phi) - c(m_\phi + m_\psi + m_\phi m_\psi) + \frac{c}{2\Xi_a}[\lambda - \mu^2a^2 - \omega^2l^2 + \Xi_b(4\gamma^2 - m_\phi^2 - m_\psi^2)]. \tag{A5}$$

We note that the regularity of the point at $z = \infty$ gives the following relation between the parameters

$$\delta + \varepsilon = 2(\alpha + \beta + \gamma + 1). \tag{A6}$$

We also note that the regularity of the solutions requires that the magnetic quantum numbers m_ϕ and m_ψ must take non-negative values. Therefore, below we imply only these values for m_ϕ and m_ψ. It turns out that for some special cases, namely, when the black hole has two equal rotation parameters ($a = b$, $\Xi_a = \Xi_b = \Xi$) or its rotation is slow enough, the above equation has only three regular singular points $z = 0, 1, \infty$. That is, the corresponding solution can be expressed in terms of the hypergeometric functions. We consider now these cases separately.

1. Equal rotation parameters

In this case Eq. (A1) reduces to the form

$$z(1-z)\frac{d^2S}{dz^2} + (1-2z)\frac{dS}{dz} + \frac{1}{4}\Big[\Big(\frac{\omega l^2 + a(m_\phi + m_\psi)}{l}\Big)^2 - \frac{m_\phi^2}{z} - \frac{m_\psi^2}{1-z} + \frac{\lambda - \mu^2a^2 - \omega^2l^2}{\Xi}\Big]S = 0. \tag{A7}$$

One can easily verify that with the substitution

$$S(z) = z^{m_\phi/2}(1-z)^{m_\psi/2}F(z). \tag{A8}$$

Equation (A7) goes over into the standard hypergeometric differential equation

$$z(1-z)\frac{d^2F}{dz^2} + [\gamma - (\alpha + \beta + 1)z]\frac{dF}{dz} - \alpha\beta F = 0, \tag{A9}$$

where the parameters are given by

$$2\alpha = 1 + \delta + m_\phi + m_\psi, \qquad 2\beta = 1 - \delta + m_\phi + m_\psi,$$
$$\gamma = 1 + m_\phi,$$
$$\delta^2 = 1 + \left(\frac{\omega l^2 + a(m_\phi + m_\psi)}{l}\right)^2 + \frac{\lambda - \mu^2 a^2 - \omega^2 l^2}{\Xi}. \tag{A10}$$

The general solution of this equation for $z \in (0, 1)$ has the form (see, for instance [30])

$$F(z) = A_1 z^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, z) + B_1 F(\alpha, \beta, \gamma, z), \tag{A11}$$

where A_1 and B_1 are constants. From the regularity of the solution at $z = 0$ and $z = 1$, we obtain $A_1 = 0$ and

$$\lambda = \Xi\bigg[(2j + m_\phi + m_\psi)(2j + m_\phi + m_\psi + 2) - 2\omega a(m_\phi + m_\psi) - \frac{a^2(m_\phi + m_\psi)^2}{l^2}\bigg] + a^2(\omega^2 + \mu^2), \tag{A12}$$

where j is a non-negative integer. Introducing now $\ell = 2j + m_\phi + m_\psi$, we can put this expression in the form

$$\lambda = \Xi\bigg[\ell(\ell + 2) - 2\omega a(m_\phi + m_\psi) - \frac{a^2(m_\phi + m_\psi)^2}{l^2}\bigg] + a^2(\omega^2 + \mu^2). \tag{A13}$$

We note that the new integer ℓ being the orbital quantum number must obey the condition $\ell \geq m_\phi + m_\psi$. Thus, it must take even (odd) values if the sum $m_\phi + m_\psi$ is even (odd). We also note that in the $a = b$ case, the eigenvalues are the same as in the absence of rotation (with redefinition of the separation constant). That is,

$$\lambda_\ell = \ell(\ell + 2). \tag{A14}$$

This is in agreement with the result obtained in [29].

2. Slow rotation

When the rotation of the black hole is slow enough, we can discard all terms of higher than linear order in rotation parameters a and b. In this case the angular Eq. (A1) takes the simple form

$$z(1 - z)\frac{d^2 S}{dz^2} + (1 - 2z)\frac{dS}{dz} - \frac{1}{4}\left(\frac{m_\phi^2}{z} + \frac{m_\psi^2}{1 - z} - \lambda_\ell\right)S = 0, \tag{A15}$$

where

$$\lambda_\ell = \lambda + 2\omega(am_\phi + bm_\psi). \tag{A16}$$

The similar substitution as in (A8) transforms this equation into Eq. (A9) with the parameters

$$2\alpha = 1 + \sqrt{1 + \lambda_\ell} + m_\phi + m_\psi,$$
$$2\beta = 1 - \sqrt{1 - \lambda_\ell} + m_\phi + m_\psi, \tag{A17}$$
$$\gamma = 1 + m_\phi.$$

It is easy to verify that its regular solution at $z = 0$ and $z = 1$ implies the eigenvalues given in (A14).

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