

Geometric description of BTZ black hole thermodynamics

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We study the properties of the space of thermodynamic equilibrium states of the Bañados-Teitelboim-Zanelli (BTZ) black hole in $(2 + 1)$ gravity. We use the formalism of geometrothermodynamics to introduce in the space of equilibrium states a two-dimensional thermodynamic metric whose curvature is nonvanishing, indicating the presence of thermodynamic interaction, and free of singularities, indicating the absence of phase transitions. Similar results are obtained for generalizations of the BTZ black hole which include a Chern-Simons term and a dilatonic field. Small logarithmic corrections of the entropy turn out to be represented by small corrections of the thermodynamic curvature, reinforcing the idea that thermodynamic curvature is a measure of thermodynamic interaction.

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I. INTRODUCTION

The spacetime of a black hole in $(2 + 1)$ dimensions with negative cosmological constant provides an example of a lower-dimensional toy model which shares many of the important conceptual issues of general relativity in $(3 + 1)$ dimensions, but avoids some of the difficulties found in mathematical computations. This spacetime is known as the Bañados-Teitelboim-Zanelli (BTZ) black hole [1], and it warrants attention in its own right (for a review, see [2]). A key feature of this model lies in the simplicity of its construction. It is a spacetime with constant negative curvature and is obtained as a discrete quotient of three-dimensional anti-de Sitter space [3]. The BTZ spacetime is free of curvature singularities. Even so, all characteristic features of black holes such as the event horizon and Hawking radiation are present so that this model is a genuine black hole. Furthermore, despite its simplicity, the BTZ black hole plays an outstanding role in many of the recent developments in string theory, specially in the context of the AdS/CFT conjecture [4]. One of the most interesting aspects of black holes is related to their thermodynamic properties. In the case of the BTZ black hole, the extensive thermodynamic variables are the mass M , angular momentum J , and entropy S which is proportional to the horizon area. The intensive variables are the angular velocity at the horizon Ω and the Hawking temperature T . Although these quantities satisfy the laws of macroscopic thermodynamics, their microscopic origin remains obscure, and it is believed that it is related to the problem of quantization of gravity.

On the other hand, it is possible to introduce differential geometric concepts in ordinary thermodynamics. The most known structures were postulated by Weinhold [5] and Ruppeiner [6,7] who introduced Riemannian metrics in the space of equilibrium states of a thermodynamic system. These geometric structures can obviously be applied in black hole thermodynamics. For instance, the components of Weinhold's metric are simply defined as the second derivatives of the mass with respect to the extensive variables. The calculations are straightforward, but the geometric properties of the resulting manifolds are puzzling [8,9]. For instance, in the case of the BTZ black hole thermodynamics, where $M = M(S, J)$, the curvature of the equilibrium space turns out to be flat [10–12]. This flatness is usually interpreted as a consequence of the lack of thermodynamic interaction. However, if one applies a Legendre transformation $M \rightarrow \tilde{M} = M - J\Omega$, the resulting manifold is curved. This result is not in agreement with ordinary thermodynamics which is manifestly Legendre invariant. To overcome this inconsistency, the theory of geometrothermodynamics (GTD) was proposed recently [13–15]. It incorporates arbitrary Legendre transformations [16] into the geometric structure of the equilibrium space in an invariant manner. In this work, we study the equilibrium space of the BTZ black hole, and propose a thermodynamic metric whose curvature is nonzero and reproduces its main thermodynamic properties. In particular, we will see that the thermodynamic curvature is free of singularities. In GTD, this is interpreted as a consequence of the nonexistence of singular points at the level of the heat capacity, indicating that no (second-order) phase transitions occur. It turns out that these results coincide with the predictions of ordinary black hole thermodynamics as proposed by Davies [17]. We also include in our analysis the case of an additional dilatonic field and a Chern-

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Simons term at the level of the action. In all the cases presented in this work, it turns out that GTD correctly reproduces the thermodynamic properties of the corresponding system. Moreover, we analyze the leading logarithmic corrections to the entropy and show that they correspond to small perturbations at the level of the curvature of the equilibrium space. This can be considered as an additional indication that the thermodynamic curvature can be used to measure the thermodynamic interaction of a system.

This paper is organized as follows. In Sec. II we review the most important aspects of the BTZ black hole, emphasizing the thermodynamic interpretation of its physical parameters. In Sec. III we use the formalism of GTD to construct the thermodynamic phase space and the space of equilibrium states for the BTZ black hole. Section IV is devoted to study the GTD of generalizations of the BTZ black hole, including an additional dilatonic field and a Chern-Simons term. In all the cases we investigate the influence of small corrections of the entropy on the thermodynamic curvature. Finally, Sec. V contains discussions of our results and suggestions for further research. Throughout this paper we use units in which $c = k_B = \hbar = 8G = 1$.

II. THE BTZ BLACK HOLE

The BTZ black hole metric in spherical coordinates can be written as

$$ds^2 = -\left(-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}\right)dt^2 + r^2\left(d\varphi - \frac{J}{2r^2}dt\right)^2 + \frac{dr^2}{-M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}}, \quad (1)$$

where M and J are the mass and angular momentum, respectively. The BTZ metric is a classical solution of the field equations of $(2+1)$ gravity which follow from the action $I = 1/(16\pi) \int d^3x \sqrt{-g}(R - 2\Lambda)$, where $\Lambda = -1/l^2$ is the cosmological constant. The BTZ metric is characterized by a constant negative curvature and, therefore, can be obtained as a region of anti-de Sitter space with an appropriate identification of the boundaries [1]. The roots of the lapse function ($g_{tt} = 0$)

$$r_{\pm}^2 = \frac{l^2}{2} \left[M \pm \left(M^2 - \frac{J^2}{l^2} \right)^{1/2} \right] \quad (2)$$

define the horizons $r = r_{\pm}$ of the spacetime. In particular, the null hypersurface $r = r_+$ can be shown to correspond to an event horizon, which in this case is also a Killing horizon, whereas the inner horizon at r_- is a Cauchy horizon. From the expressions for the horizon radii, the following useful relations are obtained:

$$M = \frac{r_+^2 + r_-^2}{l^2}, \quad J = \frac{2r_+r_-}{l}. \quad (3)$$

From the area-entropy relationship, $S = 4\pi r_+$, we obtain an expression of the form $S = S(M, J)$ that can be rewritten as

$$M = \frac{S^2}{16\pi^2 l^2} + \frac{4\pi^2 J^2}{S^2}. \quad (4)$$

This equation relates all the thermodynamic variables entering the BTZ metric in the form $M = M(S, J)$ so that if we impose the first law of thermodynamics $dM = TdS + \Omega dJ$, the expressions for the temperature and the angular velocity can easily be computed as $T = \partial M / \partial S$, $\Omega = \partial M / \partial J$. It is convenient to write the final results in terms of the horizon radii by using the relations (3):

$$T = \frac{r_+^2 - r_-^2}{2\pi l^2 r_+}, \quad \Omega = \frac{r_-}{lr_+}. \quad (5)$$

The temperature is always positive and vanishes only in the case of an extremal black hole, i.e., when $r_+ = r_-$. The heat capacity at constant values of J is given as

$$C = T \frac{\partial T}{\partial S} = \frac{4\pi r_+(r_+^2 - r_-^2)}{r_+^2 + 3r_-^2}. \quad (6)$$

Following the fundamentals of black hole thermodynamics as formulated by Davies [17], the main thermodynamic properties of the BTZ black hole can be derived from the behavior of its thermodynamic variables M , T , Ω , and C in terms of the extensive variables S and J . We see that all thermodynamic variables are well behaved, except perhaps in the extremal limit $r_+ = r_-$, where the Hawking temperature and the heat capacity vanish. Since an absolute zero temperature is not allowed by the third law of thermodynamics, we conclude that the thermodynamic description breaks down in the extremal limit. The fact that the heat capacity C is always positive and free of singular points is usually interpreted as an indication that the BTZ black hole is a thermodynamically stable configuration where no phase transitions can occur. This is in contrast with black hole configurations in higher dimensions which, in general, are characterized by regions of high instabilities and a rich phase transitions structure.

It should be mentioned that Davies' formulation of phase transitions for black holes is not definitely settled and is still a subject of discussion. Alternative criteria for the existence of phase transitions of black holes have been proposed in different contexts [18–21]. A definite definition could be formulated only on the basis of a microscopic description that would lead to the ordinary macroscopic thermodynamics of black holes in the appropriate limit. Such a microscopic model for black holes must be related to a hypothetical model of quantum gravity which is still out of reach. We therefore use the intuitive definition of phase transitions as it is known from ordinary thermodynamics of black holes [17].

III. GEOMETROTHERMODYNAMICS OF THE BTZ BLACK HOLE

For the geometric description of the thermodynamics of the BTZ black hole in GTD, we first introduce the five-dimensional phase space \mathcal{T} with coordinates $\{M, S, J, T, \Omega\}$, a contact one-form $\Theta = dM - TdS - \Omega dJ$, and a metric

$$G = (dM - TdS - \Omega dJ)^2 + (TS + \Omega J)(-dTdS - d\Omega dJ), \quad (7)$$

which is Legendre invariant as described below. The triplet (\mathcal{T}, Θ, G) defines a contact Riemannian manifold that plays an auxiliary role in GTD. It is used to properly handle Legendre transformations. In fact, a Legendre transformation involves in general all the thermodynamic variables M, S, J, T , and Ω so that they must be independent from each other as they are in the phase space. Consider, for instance, the general Legendre transformation $\{M, S, J, T, \Omega\} \rightarrow \{\tilde{M}, \tilde{S}, \tilde{J}, \tilde{T}, \tilde{\Omega}\}$ with $M = \tilde{M} - \tilde{S}\tilde{T} - \tilde{J}\tilde{\Omega}$ and $S = -\tilde{T}$, $J = -\tilde{\Omega}$, $T = \tilde{S}$, $\Omega = \tilde{J}$. If we apply this transformation to the metric (7), it is easy to show that in the new coordinates it can be written as

$$\tilde{G} = (d\tilde{M} - \tilde{T}d\tilde{S} - \tilde{\Omega}d\tilde{J})^2 + (\tilde{T}\tilde{S} + \tilde{\Omega}\tilde{J})(-d\tilde{T}d\tilde{S} - d\tilde{\Omega}d\tilde{J}), \quad (8)$$

an expression which clearly indicates the Legendre invariance of the metric G . We will see below that the metric (7) is especially adapted for the analysis of the canonical ensemble in which the thermodynamical potential is usually given as the function $M = M(S, J)$ [22]. Since in ordinary thermodynamics different ensembles are related by Legendre transformations, the phase space \mathcal{T} turns out to be the appropriate arena to handle different statistical ensembles. For instance, the Legendre transformation mentioned above could be used to consider the geometric description of the corresponding grand-canonical ensemble.

The choice of the metric G as given in Eq. (7) follows from imposing Legendre invariance. Indeed, the terms entering this metric are of the form $E^a I^a$ and $dE^a dI^a$ ($a = 1, 2$), where $E^a = (S, J)$ are the extensive variables and $I^a = (T, \Omega)$ are the corresponding intensive variables, i.e., the variables and their differentials appear always in pairs in which each extensive variable is multiplied by its dual intensive variable. This choice guarantees invariance with respect to Legendre transformations which interchange extensive and intensive variables. We will use this criterion to construct analogous metrics in Sec. IV. Similar metrics were obtained in GTD in order to propose an invariant geometric description of the thermodynamics of higher dimensional black holes [23,24].

In GTD, we introduce also the geometric structure of the space of equilibrium states \mathcal{E} in the following manner: \mathcal{E} is a two-dimensional submanifold of \mathcal{T} that is defined by the

smooth embedding map $\varphi: \mathcal{E} \rightarrow \mathcal{T}$, satisfying the condition that the ‘‘projection’’ of the contact form Θ on \mathcal{E} vanishes, i.e., $\varphi^*(\Theta) = 0$, where φ^* is the pullback of φ , and that G induces a Legendre invariant metric g on \mathcal{E} by means of $g = \varphi^*(G)$. In principle, any two-dimensional subset of the set of coordinates of \mathcal{T} can be used to coordinate \mathcal{E} . For the sake of simplicity, we will use the set of extensive variables S and J which in ordinary thermodynamics corresponds to the energy representation. Then, the embedding map for this specific choice is $\varphi: \{S, J\} \mapsto \{M(S, J), S, J, T(S, J), \Omega(S, J)\}$. The condition $\varphi^*(\Theta) = 0$ is equivalent to the first law of thermodynamics and the conditions of thermodynamic equilibrium

$$dM = TdS + \Omega dJ, \quad T = \frac{\partial M}{\partial S}, \quad \Omega = \frac{\partial M}{\partial J}, \quad (9)$$

whereas the induced metric becomes

$$g = \left(S \frac{\partial M}{\partial S} + J \frac{\partial M}{\partial J} \right) \left(-\frac{\partial^2 M}{\partial S^2} dS^2 + \frac{\partial^2 M}{\partial J^2} dJ^2 \right). \quad (10)$$

This metric determines all the geometric properties of the equilibrium space \mathcal{E} . We see that in order to obtain the explicit form of the metric it is only necessary to specify the thermodynamic potential M as a function of S and J . In ordinary thermodynamics this function is usually referred to as the fundamental equation from which all the equations of state can be derived [25].

In general, it is possible to show that any metric in \mathcal{E} can be obtained as the pullback of a metric in \mathcal{T} . In particular, the Weinhold metric $g^W = (\partial^2 M / \partial E^a \partial E^b) dE^a dE^b$, with $E^a = \{S, J\}$, can be shown to be generated by a metric of the form $G^W = dM^2 - (TdS + \Omega dJ)^2 + dSdT + d\Omega dJ$ which can be shown to be noninvariant with respect to Legendre transformations [13]. This explains why Weinhold’s metric leads to contradictory results when different thermodynamic potentials are used in its definition.

As for the BTZ black hole, the fundamental equation $M = M(S, J)$ follows from the area-entropy relationship and is given in Eq. (4). Then, it is easy to compute the explicit form of the thermodynamic metric (10) which, using the expressions for S and J in terms of r_+ and r_- , can be written as

$$g = -\frac{r_+^2 + 3r_-^2}{4\pi^2 l^4} dS^2 + \frac{1}{l^2} dJ^2. \quad (11)$$

The corresponding thermodynamic curvature turns out to be nonzero and the scalar curvature can be expressed as

$$R = -\frac{3}{2} \frac{l^4}{(r_+^2 + 3r_-^2)^2}. \quad (12)$$

The general behavior of this curvature is illustrated in Fig. 1. The equilibrium manifold is a space of negative curvature for any values of the horizon radii and no singular points are present. This means that thermodynamic interaction is always present and that no phase transitions

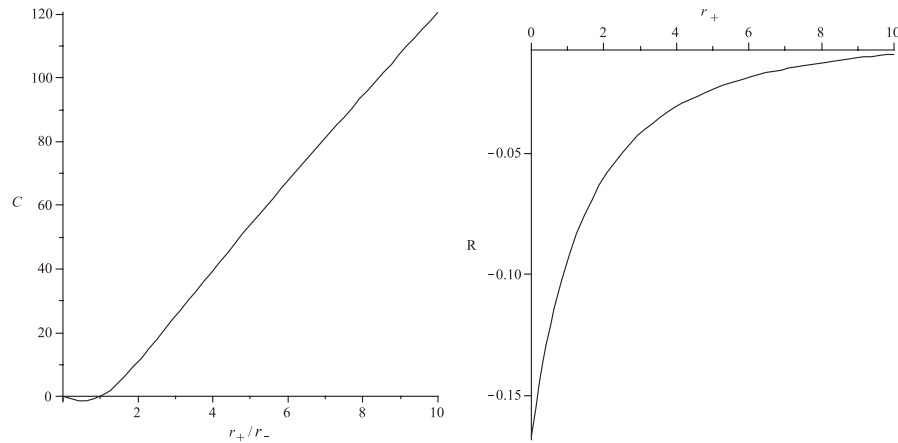


FIG. 1. Heat capacity and thermodynamic curvature of the BTZ black hole. A typical behavior is depicted for the specific values $l = 1$ and $r_- = 1$. The curvature is completely regular for the entire domain of the horizon radii.

can take place. Consequently, the BTZ black hole corresponds to a stable thermodynamic configuration. This interpretation coincides with the interpretation derived in Sec. II from the analysis of the thermodynamic variables. The heat capacity vanishes at the extremal limit $r_+ = r_-$ and becomes negative for $r_+ < r_-$. This region is, however, not allowed by the definition of the horizon radii. We conclude that the geometry of the equilibrium space correctly describes the thermodynamic behavior of the BTZ black hole. Indeed, one of the main goals of GTD is to interpret thermodynamic curvature as a measure of thermodynamic interaction and curvature singularities as points of phase transitions. We see that the thermodynamic metric proposed in GTD for the BTZ black hole meets these goals.

We also see from Fig. 1 that the thermodynamic curvature is regular even at the extremal limit and below. This is probably a consequence of the fact there is no singularity inside the horizon. In fact, in the four-dimensional case the geometric description of black hole thermodynamics breaks down in the region $r_- > r_+$ as a consequence of the presence of a naked singularity [23].

To finish this section, it is worth mentioning that it is possible to consider the cosmological constant Λ as an additional extensive thermodynamic variable. In this case, the equilibrium space becomes three dimensional and it turns out that instead of Λ it is necessary to consider the radius of curvature l^2 as the additional variable. However, it would be necessary to perform a more detailed analysis in terms of statistical ensembles in order to understand the radius of curvature as a realistic thermodynamic variable [26].

IV. GEOMETROTHERMODYNAMICS OF BTZ GENERALIZATIONS

In this section we will investigate the geometry of the equilibrium space of certain generalizations of the BTZ

black hole. Our goal is to see whether these generalizations can also be interpreted as sources of thermodynamic interaction in the sense that they affect the thermodynamic curvature of the equilibrium space. We will focus our analysis on BTZ black holes with a dilatonic field, and an additional Chern-Simons (CS) term as well as thermal fluctuations of the BTZ black hole. We will see that in all the cases GTD correctly describes the thermodynamics of the corresponding system.

In order to investigate the effect of an additional dilatonic field it is necessary to consider Lagrangians of the form $\mathcal{L} = R - 2\Lambda + (\Delta\phi)^2 - V(\phi)$. Since the potential $V(\phi)$ can take, in principle, any desired form, the number of possible exact solutions to the corresponding field equations could be quite big. However, if we focus on black hole configurations the number of solutions reduces drastically. In fact, it turns out [27] that dilatonic black hole solutions are known only for very specific potentials which can be written as truncated series of the functions $\cot(\phi)$ and $\cot(\sqrt{2}\phi)$. Because of the uncommon form of these potentials, the physical interpretation of the dilatonic field is, at least, puzzling. In all the known cases [27], the found solutions are characterized by a dilatonic field which tends asymptotically to a constant value, say L , that is usually associated to the dilatonic charge. These solutions are generalizations of the BTZ spacetime in the sense that when the asymptotically constant dilaton vanishes, they all reduce to the nonrotating ($J = 0$) BTZ case. We performed a detailed analysis of the known dilatonic black hole configurations by considering the charge L as a thermodynamic variable. The area-entropy relationship is then used to obtain a thermodynamic fundamental equation of the form $S = S(M, L)$ from which all the relevant thermodynamic variables can be derived. It turns out that the general behavior of the Hawking temperature and the heat capacity is not significantly affected by the presence of the dilatonic field. This result is confirmed by the analysis of the corresponding thermodynamic curvature

of the metric of the equilibrium manifold which, as in the case of the BTZ spacetime, can easily be constructed from the explicit form of the fundamental equation $S = S(M, L)$. Indeed, the thermodynamic curvature is negative and regular in the entire region outside the outer horizon, and approaches a constant value in the limiting case of an extremal black hole. Our analysis shows that the presence of a dilatonic field essentially does not affect the thermodynamic behavior of the BTZ black hole. Perhaps the found solutions are too simple from a physical point of view due to the form of the potential which is chosen as a polynomial in a certain function with unspecified coefficients that can be adjusted to derive the mentioned exact solutions. Nevertheless, our results also show that GTD can handle correctly this kind of dilatonic generalization of the BTZ black hole.

A. The Chern-Simons charge

The inclusion of CS charges is important in the study of gravitational anomalies and for the case of (2 + 1) gravity it was performed in [28,29]. The addition of the gravitational CS term to the Einstein-Hilbert action with cosmological constant results in a new theory that is known as topologically massive gravity [30]. The BTZ solution turns out to be an exact solution of the corresponding field equations with a different mass and angular momentum [29]:

$$M = M_0 - \frac{k}{l^2} J_0, \quad J = J_0 - kM_0, \quad (13)$$

where M_0 and J_0 are the mass and angular momentum parameters of the original BTZ solution as given in Eq. (3), and k is the Chern-Simons coupling constant. Moreover, the expression for the entropy results modified into [29,31]

$$S = 4\pi \left(r_+ - \frac{k}{l} r_- \right). \quad (14)$$

It is then easy to show that, in terms of the new parameters M and J , the horizon radii can be expressed as

$$r_{\pm} = \frac{l}{2} \left[\left(\frac{lM + J}{l - k} \right)^{1/2} \pm \left(\frac{lM - J}{l + k} \right)^{1/2} \right], \quad (15)$$

which can then be introduced into the modified entropy (14) to obtain the fundamental equation in the entropy representation $S = S(M, J)$. The resulting equation can then be rewritten as

$$M = \frac{1}{8\pi^2 k^2} \left[S^2 + 8\pi^2 kJ + \frac{S}{l} \sqrt{(l^2 - k^2)(S^2 + 16\pi^2 kJ)} \right], \quad (16)$$

which is the fundamental equation in the mass representation. Introducing this fundamental equation into Eq. (10) and expressing the result in terms of the horizon radii, we obtain the thermodynamic metric,

$$g = \frac{2lr_+^3 + kr_-(r_-^2 - 3r_+^2)}{2l^2 r_+^4 (l^2 - k^2)} \times \left[\frac{kr_-(r_-^2 + 3r_+^2) - lr_+(r_+^2 + 3r_-^2)}{4\pi^2 l^2} dS^2 + (lr_+ - kr_-) dJ^2 \right], \quad (17)$$

which describes the geometry of the equilibrium space and reduces to the thermodynamic metric for the BTZ black hole (11) in the limiting case $k \rightarrow 0$. Other important thermodynamic quantities can be computed from the fundamental equation (16). For instance, the Hawking temperature $T = \partial M / \partial S$ and the heat capacity $C = T(\partial^2 M / \partial S^2)^{-1}$ at constant J can be written in the form

$$T = \frac{r_+^2 - r_-^2}{2\pi l^2 r_+}, \quad (18)$$

$$C = \frac{4\pi(l^2 - k^2)r_+^2(r_+^2 - r_-^2)}{l[lr_+(r_+^2 + 3r_-^2) - kr_-(r_-^2 + 3r_+^2)]}.$$

The computation of the scalar curvature of the thermodynamic metric (17) is straightforward, but the resulting expression cannot be written in a compact form. We analyzed numerically the behavior of the thermodynamic curvature for different sets of values of the parameters l and k and the horizon radii r_+ and r_- . We found only one curvature singularity which is, however, always located at a distance r_+ that lies inside the inner horizon r_- . Clearly, this singularity is unphysical although it corresponds to a divergence which exists at the level of the heat capacity.

In the physically meaningful interval $r_+ \geq r_-$ with the additional condition $l > k$, which is necessary in order for the fundamental equation (16) to be well defined, it can be shown that the heat capacity is always positive, indicating that the BTZ-CS black hole is thermodynamically stable. The thermodynamic curvature corresponding to the metric (17) in this interval is as depicted in Fig. 2. The heat capacity and the thermodynamic curvature are indeed affected by the presence of the CS charge, but the general behavior remains the same. We only observe that the thermodynamic curvature has now a minimum value from which it grows as the outer horizon radius approaches the inner radius. Nevertheless, the behavior is completely regular in the entire physical interval. The fact that the curvature is free of singularities is interpreted in GTD as an indication of the absence of phase transitions structure. We conclude that the geometry of the equilibrium space, as described by the thermodynamic metric (17), correctly describes the thermodynamics of the BTZ-CS black hole.

An additional numerical analysis of the thermodynamic curvature shows that it diverges in the limit $k \rightarrow l$, indicating that at this point a phase transition occurs. This result seems to be relevant in the context of the recently proposed theory of chiral gravity [32]. Indeed, in an attempt to

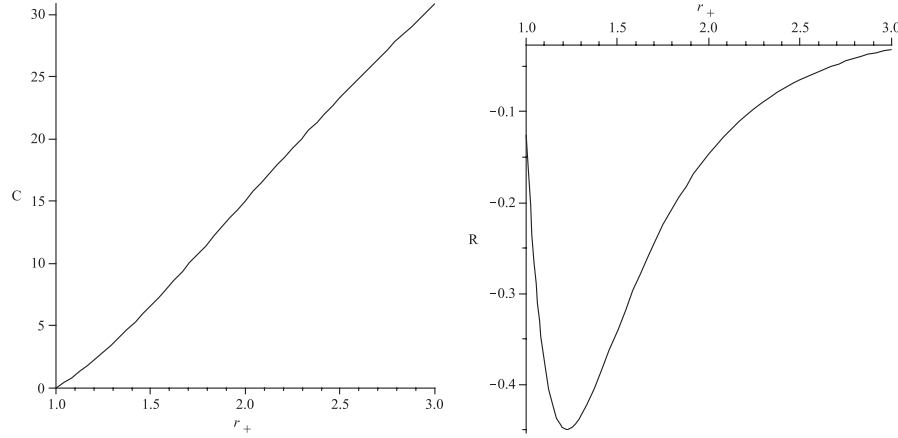


FIG. 2. The heat capacity and the thermodynamic curvature of the BTZ-CS black hole for the choice of values $l = 1$, $k = 1/2$, and $r_- = 1$ and the interval $r_+ > r_-$. The heat capacity is always positive and the thermodynamic curvature is free of singularities, ensuring thermodynamic stability.

determine the microscopic origin of the entropy of $(2 + 1)$ -dimensional black holes, the canonical quantization of topological massive gravity has been investigated with the result that the massive gravitons of the BTZ black holes possess negative energy, making the theory unstable and inconsistent. However, in the limiting case of chiral gravity, which corresponds to the limit $k = l$, the negative energy modes vanish, suggesting the possibility of a stable and consistent theory which is dual to a conformal field theory (CFT) on the boundary. If, furthermore, the zero-energy modes of the chiral limit could be discarded as pure gauge by choosing the boundary conditions in an appropriate form, the possibility would arise of obtaining a microscopic derivation of the black hole entropy by using the Cardy formula. This result has been intensively discussed in recent works [33–39]; it now seems to become clear that, if global issues are taken into account properly, in the chiral limit all the unpleasant modes of topological massive gravity vanish, and the dual CFT permits a microscopic determination of the black hole entropy.

The above result seems to be reinforced by the fact that in GTD the thermodynamic curvature, corresponding to the equilibrium space of the BTZ-CS black hole, diverges at the chiral limit. Indeed, the singularity at the level of the thermodynamic curvature indicates the existence of a point with a second-order phase transition where usually the associated system may undergo an abrupt change of state, accompanied by drastic modifications of its physical properties. It could be that the point where chiral gravity arises from topologically massive gravity corresponds to a phase transition in which all the states with negative energy vanish, giving rise to a more symmetric, stable and consistent theory dual to a CFT. Obviously, a more detailed analysis of the chiral point will be necessary in order to establish the conjectured relationship between the phase transition at the level of thermodynamic curvature and the vanishing of the negative modes. This, however, is beyond the scope of the present work.

B. Thermal fluctuations

If the canonical ensemble of a specific system is thermodynamically stable, it is well known that its entropy is subject to logarithmic and polynomial corrections, when thermal corrections are taken into account. If S_0 denotes the entropy calculated in the canonical ensemble of a thermodynamic system with temperature T and heat capacity C , the leading term of the thermal fluctuations leads to the entropy correction [40]

$$S = S_0 - \frac{1}{2} \ln(CT^2). \quad (19)$$

This and higher order corrections have been analyzed for many classes of black holes [41–43]. It has been pointed out that, up to an additive constant, the leading correction is logarithmic,

$$S = S_0 - \frac{3}{2} \ln(S_0), \quad (20)$$

and of quite general nature in the sense that it can be derived from a semiclassical approach as well as from completely different approaches to quantum gravity [44]. The BTZ black hole as well as the Chern-Simons and dilatonic generalizations analyzed above are characterized by heat capacities which are positive in the physically meaningful interval. This means that these structures are thermodynamically stable and their corrections to the entropy can be analyzed by using Eq. (20).

For the purposes of the present work, it is interesting to verify whether GTD is able to correctly handle entropy corrections in the sense that a small perturbation of the entropy would correspond to a small perturbation of the thermodynamic curvature. To investigate this question it is necessary to formulate GTD and the above results in the entropy representation. One of the advantages of GTD is indeed its flexibility in regard to different representations. We use as intuitive guidance the first law of thermodynam-

ics in the entropy representation, i.e., $dS = (1/T)dM - (\Omega/T)dJ$. Then, the fundamental equation should read $S = S(M, J)$ and the conditions for thermodynamic equilibrium are $\partial S/\partial M = 1/T$ and $\partial S/\partial J = -\Omega/T$ so that $1/T$ and $-\Omega/T$ are the variables dual to M and J , respectively. Consequently, the coordinates for the thermodynamic phase space \mathcal{T} of the BTZ black hole can be chosen as $\{S, M, J, 1/T, -\Omega/T\}$, and the fundamental form is $\Theta_S = dS - (1/T)dM + (\Omega/T)dJ$. Applying the same prescription used to construct the metric (7), we obtain the following metric for the phase space in the entropy representation:

$$G_S = \left(dS - \frac{1}{T}dM + \frac{\Omega}{T}dJ \right)^2 + \left(\frac{M}{T} - J\frac{\Omega}{T} \right) \left[-dM d\left(\frac{1}{T}\right) - dJ d\left(\frac{\Omega}{T}\right) \right]. \quad (21)$$

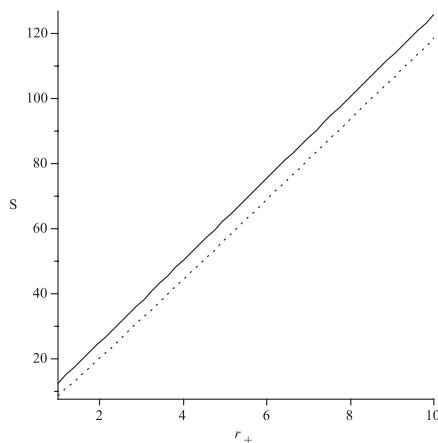
As for the coordinates of the space of equilibrium states \mathcal{E} , the natural choice is $\{M, J\}$ so that the smooth map $\varphi: \mathcal{E} \rightarrow \mathcal{T}$ with the condition $\varphi^*(\Theta_S) = 0$ implies the fundamental equation $S = S(M, J)$ and the equilibrium conditions given above. It is then easy to compute the metric $g_S = \varphi^*(G_S)$ which can be written as

$$g_S = \left(M \frac{\partial S}{\partial M} + J \frac{\partial S}{\partial J} \right) \left(-\frac{\partial^2 S}{\partial M^2} dM^2 + \frac{\partial^2 S}{\partial J^2} dJ^2 \right). \quad (22)$$

This metric represents the geometry of the equilibrium space for any thermodynamic system with fundamental equation $S = S(M, J)$. In the particular case of the BTZ black hole it reads

$$S_0 = 2\sqrt{2}\pi l \left[M + \left(M^2 - \frac{J^2}{l^2} \right)^{1/2} \right]^{1/2}. \quad (23)$$

Then, using the relationships (3), the corresponding thermodynamic metric can be expressed as



$$g_S = \frac{2\pi^2 l^4 r_+^2 (r_+^2 + 3r_-^2)}{(r_+^2 - r_-^2)^3} \left(dM^2 - \frac{1}{l^2} dJ^2 \right), \quad (24)$$

from which it can be shown that the thermodynamic curvature in the entropy representation becomes

$$R_S = \frac{(r_+^2 - r_-^2)^2 (5r_+^4 - 6r_+^2 r_-^2 + 9r_-^4)}{4\pi^2 r_+^4 (r_+^2 + 3r_-^2)^3}. \quad (25)$$

We notice that this curvature essentially reproduces the results obtained in Sec. III for the BTZ black hole geometrothermodynamics in the mass representation. In Fig. 3 the general behavior of the curvature (25) is shown for a specific choice of the parameters. It can be seen that the curvature is free of singularities in the entire interval, indicating that it corresponds to a stable thermodynamic system.

In order to investigate in GTD the entropy correction for the BTZ black hole, we must introduce the entropy (23) into the general expression for the logarithmic correction (20), and the resulting corrected entropy must be inserted into the general form of the thermodynamic metric (22) from which the corresponding curvature can be derived in the standard manner. The resulting expressions cannot be written in a compact form. Therefore, we perform a graphical analysis. The results are presented in Fig. 3. The graphics show clearly that a small correction of the entropy leads to a small perturbation of the thermodynamic curvature. We performed similar analysis for the BTZ-CS and BTZ dilatonic black holes. The thermodynamic metrics of the corresponding equilibrium spaces in the entropy representation can be derived as described above for the BTZ black hole. In general, we conclude that a small perturbation at the level of the entropy corresponds to a small perturbation at the level of the thermodynamic curvature. We interpret this result as a further indication that in GTD the thermodynamic curvature can be used as a measure of thermodynamic interaction.

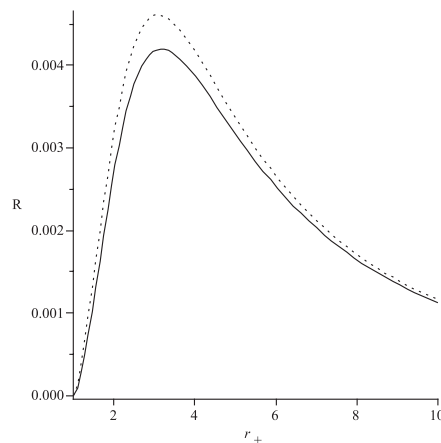


FIG. 3. Behavior of the entropy and thermodynamic curvature for the BTZ black hole (solid curves) and the corresponding corrections (dashed curves). Our choice of parameter values is $l = 1$, $r_- = 1$, and $r_+ > 1$.

V. CONCLUSIONS

In this work we used the formalism of GTD to construct a thermodynamic metric for the space of equilibrium states of the BTZ black hole and its generalizations which include an additional Chern-Simons term and a dilatonic field. In all these cases we showed that the thermodynamic curvature (scalar) is in general different from zero, indicating the presence of thermodynamic interaction, and free of singularities, indicating the absence of phase transitions. This result is in accordance with the goals of GTD and allows us to investigate the thermodynamic properties of the BTZ black holes in terms of the geometric properties of the corresponding space of equilibrium states.

The thermodynamic metric proposed in this work has been applied to the case of black hole configurations in four and higher dimensions with and without cosmological constant [23,24]. In general, it has been shown that this thermodynamic metric correctly describes the thermodynamic behavior of the corresponding black hole configurations. One additional advantage of this thermodynamic metric is its invariance with respect to total Legendre transformations. This means that the results are independent of the thermodynamic potential used to generate the thermodynamic metric. Also, the generality of the method of GTD allows us to easily implement different thermodynamic representations. In particular, in the case of BTZ black holes we presented in Sec. III the mass representation and in Sec. IV B the entropy representation.

For all BTZ black holes analyzed in this work, we showed that small perturbations at the level of the thermodynamic potential lead to small perturbations at the level of the thermodynamic curvature. This is not a trivial result that contrasts with the results obtained by using other metric structures. In fact, Weinhold's metric leads to a zero thermodynamic curvature for the BTZ black hole and to big perturbations of the curvature when small perturbations of the entropy are taken into account [11]. This is not in agreement with the idea of describing a thermodynamic interaction in terms of curvature, which is one of the aims of applying geometric concepts in thermodynamics. In the case of GTD, we have shown that the thermodynamic curvature vanishes in the case of an ideal gas [45] where no thermodynamic interaction is present. In all the cases investigated so far in which thermodynamic interaction is present, the thermodynamic curvature turns out to be nonvanishing. Moreover, if the thermodynamic system possesses a nontrivial phase transition structure at the level of the heat capacity, as in the case of the van der Waals gas and higher dimensional black hole configurations, the corresponding thermodynamic curvature turns out to diverge

at exactly those points where phase transitions occur [13,23,24]. Here we have shown that the thermodynamic metric proposed for the equilibrium space not only leads to a nonzero thermodynamic curvature for the BTZ black hole, but also induces small perturbations on the thermodynamic curvature when small perturbations of the thermodynamic potential are taken into account. We interpret this result as an additional indication that the thermodynamic curvature proposed in GTD could be used as a realistic measure for thermodynamic interaction. Nevertheless, a deeper and more general analysis will be necessary to establish the definite significance of the thermodynamic curvature.

It would be interesting to further analyze the manifold of equilibrium states of the BTZ black hole in the context of the variational principles proposed in GTD [45]. If it turns out that the thermodynamic metric investigated in the present work for the BTZ black holes satisfies the Nambu-Goto-like equations [46], an additional interpretation of BTZ configurations would emerge in terms of bosonic strings. Recently, we analyzed the set of geodesics in the equilibrium space of thermodynamic systems in the specific case of an ideal gas [47] and found a very rich geometric structure. It would be interesting to study the geometric structure of the space of geodesics in the case of the BTZ black hole. Moreover, in analogy to the investigation reported in [48], an additional test of the applicability of GTD could be performed in the case of two-dimensional gravity theories in which simpler black hole configurations exist.

In our geometric construction of GTD as presented above, the thermodynamic phase space plays only an auxiliary role in the sense that it is used only to correctly apply Legendre transformations and to guarantee Legendre invariance of metric structures. Nevertheless, it would be interesting to analyze its geometric properties. We know, for instance, that its curvature must be nonzero since a flat metric in the phase space is not Legendre invariant. It would be interesting to classify the phase space in terms of the properties of its curvature. Another open question is the significance of the geodesics of the phase space and their relation to the geodesics of the equilibrium space which, at least in the case of the ideal gas, represent quasistatic thermodynamic processes [47].

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- [1] M. Bañados, C. Teitelboim, and J. Zanelli, *Phys. Rev. Lett.* **69**, 1849 (1992).
- [2] S. Carlip, *Quantum Gravity in 2 + 1 Dimensions* (Cambridge University Press, Cambridge, UK, 1998).
- [3] M. Bañados, M. Henneaux, C. Teitelboim, and J. Zanelli, *Phys. Rev. D* **48**, 1506 (1993).
- [4] E. Witten, *Adv. Theor. Math. Phys.* **2**, 505 (1998); arXiv: hep-th/9803131.
- [5] F. Weinhold, *J. Chem. Phys.* **63**, 2479 (1975); **63**, 2484 (1975); **63**, 2488 (1975); **63**, 2496 (1975); **65**, 559 (1976).
- [6] G. Ruppeiner, *Phys. Rev. A* **20**, 1608 (1979).
- [7] G. Ruppeiner, *Rev. Mod. Phys.* **67**, 605 (1995); **68**, 313 (1996).
- [8] J. Åman, I. Bengtsson, and N. Pidokrajt, *Gen. Relativ. Gravit.* **35**, 1733 (2003).
- [9] J. Åman, I. Bengtsson, and N. Pidokrajt, *Gen. Relativ. Gravit.* **38**, 1305 (2006).
- [10] R. Cai and J. Cho, *Phys. Rev. D* **60**, 067502 (1999).
- [11] T. Sarkar, G. Sengupta, and B. N. Tiwari, *J. High Energy Phys.* **11** (2006) 015.
- [12] A. J. M. Medved, *Mod. Phys. Lett. A* **23**, 2149 (2008).
- [13] H. Quevedo, *J. Math. Phys. (N.Y.)* **48**, 013506 (2007).
- [14] H. Quevedo, *Gen. Relativ. Gravit.* **40**, 971 (2008).
- [15] H. Quevedo and A. Vazquez, *AIP Conf. Proc.* **977**, 165 (2008); arXiv:math-ph/0712.0868.
- [16] V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer Verlag, New York, 1980).
- [17] P. C. W. Davies, *Rep. Prog. Phys.* **41**, 1313 (1978).
- [18] A. Curir, *Gen. Relativ. Gravit.* **13**, 417 (1981).
- [19] D. Pavon, *Phys. Rev. D* **43**, 2495 (1991).
- [20] J. Katz, I. Okamoto, and O. Kaburaki, *Classical Quantum Gravity* **10**, 1323 (1993).
- [21] O. Kaburaki, I. Okamoto, and J. Katz, *Phys. Rev. D* **47**, 2234 (1993).
- [22] M. M. Caldarelli, G. Cognola, and D. Klemm, *Classical Quantum Gravity* **17**, 399 (2000).
- [23] J. L. Alvarez, H. Quevedo, and A. Sanchez, *Phys. Rev. D* **77**, 084004 (2008).
- [24] H. Quevedo and A. Sanchez, *J. High Energy Phys.* **09** (2008) 034.
- [25] H. B. Callen, *Thermodynamics and an Introduction to Thermostatistics* (John Wiley & Sons, Inc., New York, 1985).
- [26] S. Cruz (unpublished).
- [27] K. C. K. Chan, *Phys. Rev. D* **55**, 3564 (1997).
- [28] P. Kraus and F. Larsen, *J. High Energy Phys.* **01** (2006) 022.
- [29] S. N. Solodukhin, *Phys. Rev. D* **74**, 024015 (2006).
- [30] S. Deser, R. Jackiw, and S. Templeton, *Phys. Rev. Lett.* **48**, 975 (1982).
- [31] M. Park, *Phys. Rev. D* **77**, 026011 (2008).
- [32] W. Li, W. Song, and A. Strominger, *J. High Energy Phys.* **04** (2008) 082.
- [33] S. Carlip, S. Deser, A. Waldron, and D. K. Wise, arXiv: hep-th/0803.3998.
- [34] W. Li, W. Song, and A. Strominger, arXiv:hep-th/0805.3101.
- [35] D. Grumiller and N. Johansson, *J. High Energy Phys.* **07** (2008) 134.
- [36] G. Giribet, M. Kleban, and M. Porrati, *J. High Energy Phys.* **10** (2008) 045.
- [37] D. Grumiller, R. Jackiw, and N. Johansson, arXiv:hep-th/0806.4185.
- [38] S. Carlip, S. Deser, A. Waldron, and D. K. Wise, *Phys. Lett. B* **666**, 272 (2008).
- [39] S. Carlip, *J. High Energy Phys.* **10** (2008) 078.
- [40] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon Press, London, UK, 1980).
- [41] S. Das, P. Majumdar, and R. K. Bhaduri, *Classical Quantum Gravity* **19**, 2355 (2002).
- [42] S. S. More, *Classical Quantum Gravity* **22**, 4129 (2005).
- [43] M. Park, *J. High Energy Phys.* **12** (2004) 041.
- [44] S. Carlip, *Classical Quantum Gravity* **17**, 4175 (2000).
- [45] A. Vazquez, H. Quevedo, and A. Sanchez, arXiv:hep-th/0805.4819.
- [46] J. Polchinski, *String Theory: An Introduction to the Bosonic String* (Cambridge University Press, Cambridge, UK, 2001).
- [47] H. Quevedo, A. Sanchez, and A. Vazquez, arXiv:math-ph/0811.0222.
- [48] J. Åman, J. Bedford, D. Grumiller, and N. Pidokrajt, *J. Phys. Conf. Ser.* **66**, 012007 (2007); arXiv:gr-qc/0611119.