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Charged black holes on a Kaluza-Klein bubble

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We construct a solution of two black holes on a Kaluza-Klein bubble in Einstein-Maxwell-dilaton theory. We explore the consequences of the presence of charge for the properties of this solution, and obtain a generalized Smarr relation and first law.

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I. INTRODUCTION

In recent years the study of the higher-dimensional Einstein equations has led to interesting new types of solutions. Study of the generalized Weyl solutions by Emparan and Reall [1], for instance, has led to the construction of black ring solutions [2] and their generalizations (see e.g. [3] for a review).

The generalized Weyl solutions have been classified in terms of their rod structure [1]. They also include multi black hole solutions, analyzed in more detail by Tan and Teo [4], as well as sequences of Kaluza-Klein (KK) bubbles and black holes, analyzed by Elvang and Horowitz [5] and Elvang, Harmark, and Obers [6].

The presence of KK bubbles leads to interesting features for the black hole solutions. Black hole solutions with spherical horizon topology can, for instance, become arbitrarily large in the presence of a bubble [5]. In contrast, because of the finite size of the circle at infinity, without a bubble such KK black holes are limited in size, as demonstrated by Kudoh and Wiseman [7].

The bubbles also provide the means to hold the black holes apart, allowing for multi black hole spacetimes without conical singularites. Small pieces of bubble are sufficient to hold large black holes in equilibrium, i.e., they compensate the gravitational attraction of the black holes and at the same time prevent them from merging [5].

Recently, Kastor, Ray, and Traschen [8] also addressed the role of KK bubbles for the laws of black hole mechanics. Introducing the notion of bubble surface gravity, they obtained interesting relations, displaying an interchange symmetry between bubble and black hole properties.

Here we address the question as to how the properties of such Kaluza-Klein black hole-bubble spacetimes are affected by the presence of charge. In particular, we consider the five-dimensional Einstein-Maxwell-dilaton (EMD) gravity with arbitrary dilaton coupling parameter α. In the limit $\alpha \to 0$, Einstein-Maxwell theory is recovered. We here focus on the solution of two charged black holes on a Kaluza-Klein bubble.

In Sec. II we recall the action and equations of motion. We present the exact solution of two charged black holes on a Kaluza-Klein bubble in Sec. III. We analyze the horizon and global properties of this solution in Sec. IV. In Sec. V we derive Smarr-like formulas for the mass and tension and present the first law. We address the extremal limit in Sec. VI and compare with charged black strings in Sec. VII. We present our conclusions in Sec. VIII.

II. EMD GRAVITY

We consider the five-dimensional EMD gravity with action

$$S = \int d^5x\sqrt{-g}(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - e^{-2\alpha\varphi}F_{\mu\nu}F^{\mu\nu}) \tag{1}$$

and arbitrary dilaton coupling parameter α.

The field equations are given by

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi + 2e^{-2\alpha\varphi}\left(F_{\mu\sigma}F^{\sigma}_{\nu} - \frac{1}{6}g_{\mu\nu}F_{\lambda\sigma}F^{\lambda\sigma}\right),$$
$$\nabla_\mu(e^{-2\alpha\varphi}F^{\mu\nu}) = 0, \qquad \nabla_\mu\nabla^\mu\varphi = -\frac{\alpha}{2}e^{-2\alpha\varphi}F_{\sigma\lambda}F^{\sigma\lambda}. \tag{2}$$

The 5D EMD gravity has a dual description in terms of the 3-form field strength H defined by

$$H = e^{-2\alpha\varphi} \star F, \tag{3}$$

where $\star$ denotes the Hodge dual. In terms of the 3-form field strength H the action can be written as follows:

$$S = \int d^5x\sqrt{-g}(R - 2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi - \tfrac{1}{3}e^{2\alpha\varphi}H_{\mu\nu\sigma}H^{\mu\nu\sigma}). \tag{4}$$

The corresponding field equations of the dual theory are

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi + e^{2\alpha\varphi}\left(H_{\mu\sigma\lambda}H^{\sigma\lambda}_{\nu} - \frac{2}{9}g_{\mu\nu}H_{\rho\lambda\sigma}H^{\rho\lambda\sigma}\right),$$
$$\nabla_\mu(e^{2\alpha\varphi}H^{\mu\nu\lambda}) = 0, \qquad \nabla_\mu\nabla^\mu\varphi = \frac{\alpha}{6}e^{2\alpha\varphi}H_{\rho\sigma\lambda}H^{\rho\sigma\lambda}. \tag{5}$$

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III. EXACT SOLUTION

As an explicit example of the sequences of Kaluza-Klein bubbles and black holes [6], let us consider the solution describing two vacuum black holes on a Kaluza-Klein bubble discussed by Elvang and Horowitz [5]. This vacuum solution has metric

$$ds^2 = -e^{2U_t}dt^2 + e^{2U_\psi}d\psi^2 + e^{2U_\phi}d\phi^2 + e^{2\nu}(d\rho^2 + dz^2), \tag{6}$$

where

$$e^{2U_t} = \frac{(R_2 - \zeta_2)(R_4 - \zeta_4)}{(R_1 - \zeta_1)(R_3 - \zeta_3)}, \tag{7}$$

$$e^{2U_\psi} = (R_1 - \zeta_1)(R_4 + \zeta_4), \tag{8}$$

$$e^{2U_\phi} = \frac{R_3 - \zeta_3}{R_2 - \zeta_2}, \tag{9}$$

$$e^{2\nu} = \frac{Y_{14}Y_{23}}{4R_1R_2R_3R_4}\left(\frac{Y_{12}Y_{34}}{Y_{13}Y_{24}}\right)^{1/2}\frac{R_1 - \zeta_1}{R_4 - \zeta_4}, \tag{10}$$

and

$$R_i = \sqrt{\rho^2 + \zeta_i^2}, \tag{11}$$

$$Y_{ij} = R_iR_j + \zeta_i\zeta_j + \rho^2, \tag{12}$$

$$\begin{aligned} &\zeta_1 = z - a, \qquad \zeta_2 = z - b, \\ &\zeta_3 = z + b, \qquad \zeta_4 = z + c. \end{aligned} \tag{13}$$

The metric functions U_t, U_ϕ, and U_ψ may be regarded as Newtonian potentials produced by line masses along the z axis with densities $1/2$. These effective sources or rods characterize the solution [1]. The rod structure of this solution is shown in Fig. 1. The two finite rods $-c < z < -b$ and $b < z < a$ for the t coordinate correspond to event horizons of the space-time, while the finite rod $-b < z < b$ for the ϕ coordinate corresponds to the bubble. The two semi-infinite rods $z < -c$ and $z > a$ for the ψ coordinate

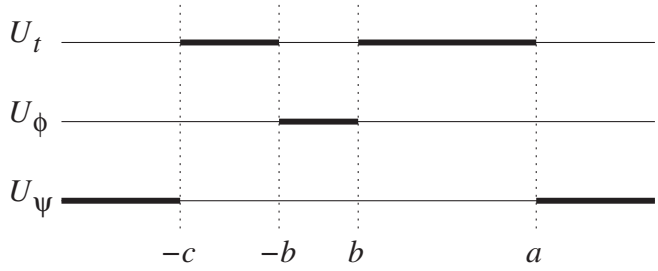

FIG. 1. The thin lines denote the z axis, and the thick lines denote the effective sources along this axis for the metric functions U_t, U_ϕ, and U_ψ.

then ensure that all sources add up to an infinite rod along the z axis, as required [1].

Using the method of [9] we generate a new EMD solution from the above vacuum solution [5]. This solution then describes two charged S^3-black holes sitting on a Kaluza-Klein bubble in the EMD gravity. For $\alpha = 0$ it reduces to a solution of Einstein-Maxwell gravity. The space-time metric of the solution is given by the line element

$$\begin{aligned} ds^2 = &-\frac{e^{2U_t}}{[\cosh^2\gamma - e^{2U_t}\sinh^2\gamma]^{2/1+\alpha_5^2}}dt^2 \\ &+ [\cosh^2\gamma - e^{2U_t}\sinh^2\gamma]^{1/1+\alpha_5^2} \\ &\times [e^{2U_\psi}d\psi^2 + e^{2U_\phi}d\phi^2 + e^{2\nu}(d\rho^2 + dz^2)]. \end{aligned} \tag{14}$$

The electric potential Φ and the dilaton field are given by

$$\Phi = \frac{\sqrt{3}\tanh\gamma}{2\sqrt{1+\alpha_5^2}}\frac{1 - e^{2U_t}}{1 - \tanh^2\gamma e^{2U_t}}, \tag{15}$$

$$e^{-\alpha\varphi} = [\cosh^2\gamma - e^{2U_t}\sinh^2\gamma]^{\alpha_5^2/1+\alpha_5^2}, \tag{16}$$

where $\alpha_5 = \frac{\sqrt{3}}{2}\alpha$ and γ is an arbitrary parameter. The Maxwell 2-form is, respectively,

$$F = -d\Phi \wedge dt. \tag{17}$$

VI. ANALYSIS OF THE SOLUTION

A. Asymptotic behavior

In order to study the asymptotic behavior of the solution we introduce the asymptotic coordinate r defined by

$$r = \sqrt{\rho^2 + z^2}. \tag{18}$$

Then in the asymptotic limit $r \to \infty$ we find

$$g_{tt} \approx -1 + \frac{a + c - 2b}{r}\left(1 + 2\frac{\sinh^2\gamma}{1+\alpha_5^2}\right), \tag{19}$$

$$g_{\phi\phi} \approx 1 + \frac{1}{r}\left[(a + c - 2b)\frac{\sinh^2\gamma}{1+\alpha_5^2} - 2b\right], \tag{20}$$

$$g_{\psi\psi} \approx \rho^2\left(1 - \frac{1}{r}\left[(a + c - 2b)\frac{\sinh^2\gamma}{1+\alpha_5^2} - (a + c)\right]\right), \tag{21}$$

$$\Phi \approx \frac{\sqrt{3}}{2}\frac{\cosh\gamma\sinh\gamma}{\sqrt{1+\alpha_5^2}}\frac{(a + c - 2b)}{r}, \tag{22}$$

$$e^{-\alpha\varphi} \approx 1 + \frac{\alpha_5^2\sinh^2\gamma}{1+\alpha_5^2}\frac{(a + c - 2b)}{r}. \tag{23}$$

The leading asymptotic form of the metric is

$$ds^2 \sim -dt^2 + \rho^2 d\psi^2 + d\rho^2 + dz^2 + d\phi^2. \quad (24)$$

In other words, asymptotically the space-time is $\mathcal{M}^4 \times S^1$. The compact dimension is parameterized by ϕ. In what follows we will consider spacetimes with a fixed length L of the Kaluza-Klein circle at infinity.

B. Regularity

The rod structure of the solution in general entails conical singularities. To cure such conical singularities at the location of the rods, the associated coordinate must be periodic with a particular period.

(i) For the semi-infinite rods $(-\infty, c]$ and $[a, +\infty)$ corresponding to the $\psi\psi$ part of the metric regularity requires the period

$$\Delta\psi = 2\pi \lim_{\rho\to 0} \sqrt{\frac{\rho^2 g_{\rho\rho}}{g_{\psi\psi}}} = 2\pi. \quad (25)$$

(ii) For the finite rod $[-b, b]$ corresponding to the $\phi\phi$ part of the metric regularity requires the period

$$\Delta\phi = 2\pi \lim_{\rho\to 0} \sqrt{\frac{\rho^2 g_{\rho\rho}}{g_{\phi\phi}}} = 8\pi \frac{b(a+c)}{\sqrt{(a+b)(b+c)}} = L, \quad (26)$$

where L is the length of the KK circle at infinity.

C. Horizons

1. First black hole horizon

The first black hole horizon is located at $\rho = 0$ for $-c < z < -b$. The metric of the spatial cross section of the horizon is given by

$$ds_h^2 = \cosh^{2/1+\alpha_5^2}\gamma \left[\frac{z+b}{z-b} d\phi^2 + 4(a-z)(z+c)d\psi^2 + \frac{(z-b)}{(z+b)(a-z)(z+c)} \frac{(a+c)^2(c-b)}{(c+b)} dz^2 \right].$$

The orbits of ϕ and ψ shrink to zero size at $z = -b$ and $z = -c$, respectively. This shows that the black hole horizon topology is S^3.

By the direct calculation we find the horizon area

$$\mathcal{A}_h^{(1)} = 32\pi^2 \cosh^{3/1+\alpha_5^2}\gamma \frac{b(a+c)^2(c-b)}{(b+c)} \sqrt{\frac{c-b}{a+b}} = 4\pi \cosh^{3/1+\alpha_5^2}\gamma L(a+c)(c-b)\sqrt{\frac{c-b}{c+b}}. \quad (27)$$

The temperature of the horizon can be obtained from the surface gravity and is given by

$$T^{(1)} = \frac{1}{4\pi} \frac{\cosh^{-3/1+\alpha_5^2}\gamma}{a+c} \sqrt{\frac{c+b}{c-b}} = \cosh^{-3/1+\alpha_5^2}\gamma \frac{2b}{L\sqrt{(a+b)(c-b)}}. \quad (28)$$

2. First black hole near horizon metric

The near horizon geometry of the left black hole can be illustrated by performing the coordinate transformation [4]

$$\rho = \frac{1}{2}\sqrt{1 - \frac{2(c-b)}{R^2}} R^2 \sin 2\theta, \quad (29)$$

$$z = -\frac{1}{2}\left(1 - \frac{c-b}{R^2}\right) R^2 \cos 2\theta. \quad (30)$$

The near horizon metric then becomes

$$ds^2 = -\cosh^{-4/1+\alpha_5^2}\gamma \frac{f_b(\theta)}{f_a(\theta)} \left(1 - \frac{2(c-b)}{R^2}\right) dt^2 + \cosh^{2/1+\alpha_5^2}\gamma \frac{f_b(\theta)}{f_a(\theta)} \times \left\{ \frac{2(a+c)^2}{b+c} \left[\left(1 - \frac{2(c-b)}{R^2}\right)^{-1} dR^2 + R^2 d\theta^2 \right] + \frac{f_a(\theta)}{2f_b^2(\theta)} R^2 \cos^2\theta d\phi^2 + \frac{2f_a^2(\theta)}{f_b(\theta)} R^2 \sin^2\theta d\psi^2 \right\} \quad (31)$$

with

$$f_b(\theta) = 2b + (c-b)\cos^2\theta, \quad f_a(\theta) = a + b + (c-b)\cos^2\theta, \quad (32)$$

corresponding to a distorted black hole.

3. Second black hole horizon

The second black hole horizon is located at $\rho = 0$ for $b < z < a$. The metric of its spatial cross section is

$$ds_h^2 = \cosh^{2/1+\alpha_5^2}\gamma \left[\frac{z-b}{z+b} d\phi^2 + 4(a-z)(z+c)d\psi^2 + \frac{(z+b)}{(z-b)(a-z)(z+c)} \frac{(a+c)^2(a-b)}{(a+b)} dz^2 \right]. \quad (33)$$

The orbits of ϕ and ψ shrink to zero size at $z = b$ and $z = a$, respectively, showing that the horizon topology is S^3.

The horizon area is

$$\mathcal{A}_{(h)}^{(2)} = 32\pi^2 \cosh^{3/1+\alpha_5^2}\gamma \frac{b(a+c)^2(a-b)}{(a+b)} \sqrt{\frac{a-b}{b+c}} = 4\pi \cosh^{3/1+\alpha_5^2}\gamma L(a+c)(a-b)\sqrt{\frac{a-b}{a+b}}$$

and its temperature

$$T^{(2)} = \frac{1}{4\pi}\frac{\cosh^{-3/1+\alpha_5^2}\gamma}{a+c}\sqrt{\frac{a+b}{a-b}} = \cosh^{-3/1+\alpha_5^2}\gamma\frac{2b}{L\sqrt{(b+c)(a-b)}}. \quad (34)$$

The near horizon metric for the second black hole can be found from the near horizon metric of the first black hole by interchanging the parameters a and c. The near horizon metric shows that the second black holes is distorted by the interaction with the first black hole and the Kaluza-Klein bubble.

4. Bubble

The bubble is located at $\rho = 0$ for $-b < z < b$. The metric on the bubble for $t = \text{const}$ and $\rho = 0$ is given by

$$ds^2_{\text{bubble}} = \left[\cosh^2\gamma - \frac{(b^2-z^2)}{(a-z)(z+c)}\sinh^2\gamma\right]^{1/1+\alpha_5^2} \times\left[4(a-z)(z+c)d\psi^2 + \frac{4b^2(a+c)^2}{(a+b)(b+c)}\times\frac{dz^2}{b^2-z^2}\right]. \quad (35)$$

As it is clear from the rod structure diagram and as one sees from the bubble metric, the orbits of ψ do not vanish at $z = \pm b$. So the bubble has cylindrical topology.

5. Distance between black holes

The proper distance along a curve with $\psi = \text{const}$ between the black holes is given by

$$l_{12} = \frac{2b(a+c)\cosh^{1/1+\alpha_5^2}\gamma}{\sqrt{(a+b)(b+c)}} \times\int_{-b}^{b}\left[1-\frac{(b^2-z^2)}{(a-z)(z+c)}\tanh^2\gamma\right]^{1/2(1+\alpha_5^2)} \times\frac{dz}{\sqrt{b^2-z^2}}. \quad (36)$$

It can be shown that

$$\tfrac{1}{4}L \le l_{12} \le \tfrac{1}{4}L\cosh^{1/1+\alpha_5^2}\gamma. \quad (37)$$

D. Masses

1. ADM mass

In order to compute the ADM mass of the solution we use the general results of [10,11]. We then find

$$M_{\text{ADM}} = \frac{1}{2}L\left[a+c-b+\frac{3\sinh^2\gamma}{2(1+\alpha_5^2)}(a+c-2b)\right] = 4\pi\frac{b(a+c)}{\sqrt{(a+b)(b+c)}} \times\left[a+c-b+\frac{3\sinh^2\gamma}{2(1+\alpha_5^2)}(a+c-2b)\right]. \quad (38)$$

With the asymptotic form of the relevant metric components given by

$$g_{tt} \simeq -1 + \frac{c_t}{r}, \qquad g_{\phi\phi} \simeq 1 + \frac{c_\phi}{r}, \quad (39)$$

the mass (38) can be directly read off from the asymptotics of the metric potentials [12–14]

$$M_{\text{ADM}} = \tfrac{1}{4}L(2c_t - c_\phi). \quad (40)$$

The ADM mass can also be calculated via the generalized Komar integral introduced in [11]

$$M^K = -\frac{L}{16\pi}\oint_\infty dS^{\mu\nu\sigma}(2\eta_\sigma\nabla_\mu\xi_\nu + \xi_\nu\nabla_\mu\eta_\sigma), \quad (41)$$

where $dS^{\mu\nu\sigma}$ is the surface element and $\xi = \partial/\partial t$, $\eta = \partial/\partial\phi$ are the time translation and the Killing vector associated with the fifth dimension, respectively. Here the integration is performed over a two-dimensional sphere at spatial infinity of $\mathcal{M}^4$.

2. Intrinsic masses

When dealing with multiobject configurations it is useful to properly define the intrinsic mass of each object in the configuration. As in four dimensions, the natural definition of the intrinsic mass of each black hole is the Komar integral evaluated on the horizon of the corresponding black hole

$$M_i^{\mathcal{H}} = -\frac{L}{16\pi}\oint_{\mathcal{H}_i} dS^{\mu\nu\sigma}(2\eta_\sigma\nabla_\mu\xi_\nu + \xi_\nu\nabla_\mu\eta_\sigma). \quad (42)$$

Here the integration is over a two-dimensional surface which is an intersection of the horizon with a constant t and ϕ hypersurface.

The intrinsic masses of the black holes for the solution under consideration are given by

$$M_1^{\mathcal{H}} = \frac{1}{2}L(c-b) = 4\pi\frac{b(a+c)(c-b)}{\sqrt{(a+b)(b+c)}} = \frac{1}{4\pi}\kappa_1\mathcal{A}_h^{(1)} = \frac{1}{2}T_h^{(1)}\mathcal{A}_h^{(1)}, \quad (43)$$

$$M_2^{\mathcal{H}} = \frac{1}{2}L(a-b) = 4\pi\frac{b(a+c)(a-b)}{\sqrt{(a+b)(b+c)}} = \frac{1}{4\pi}\kappa_2\mathcal{A}_h^{(2)} = \frac{1}{2}T_h^{(2)}\mathcal{A}_h^{(2)}, \quad (44)$$

where κ_i is the surface gravity of the ith black hole.

3. Bubble mass

In the same way we can define the intrinsic mass of the Kaluza-Klein bubble

$$M_{\text{bubble}} = -\frac{L}{16\pi}\oint_{\text{bubble}} dS^{\mu\nu\sigma}(2\eta_\sigma\nabla_\mu\xi_\nu + \xi_\nu\nabla_\mu\eta_\sigma) = 4\pi\frac{b^2(a+c)}{\sqrt{(a+b)(b+c)}} = \frac{1}{2}Lb. \tag{45}$$

The integration is over the two-dimensional boundary of a constant t and ϕ hypersurface at the bubble.

It is worth noting that the intrinsic mass of the bubble is given by

$$M_{\text{bubble}} = \frac{L}{8\pi}\kappa_b\mathcal{A}^{(b)} \tag{46}$$

where κ_b is the so-called bubble surface gravity and $\mathcal{A}^{(b)}$ is the bubble area. These quantities have been introduced in [8] and will be discussed in more detail in the next section. In our case the bubble is smooth, i.e., there are no conical singularities at the bubble. For such bubbles one obtains [8]

$$\kappa_b = \frac{2\pi}{L}, \tag{47}$$

and

$$\mathcal{A}^{(b)} = 2Lb, \tag{48}$$

where L is the length of the Kaluza-Klein circle at infinity. Note that κ_b and $\mathcal{A}^{(b)}$ are not affected by the presence of charge.

E. Electric charge

The electric charge of the ith black hole is defined by

$$Q_i = \frac{1}{2\pi}\oint_{\mathcal{H}_i} e^{-2\alpha\varphi}\star F = \frac{1}{2\pi}\oint_{\mathcal{H}_i} H. \tag{49}$$

Explicit calculations give

$$Q_i = \frac{\sqrt{3}\cosh\gamma\sinh\gamma}{\sqrt{1+\alpha_5^2}}M_i^{\mathcal{H}}. \tag{50}$$

Thus the individual black holes have the same charge to mass ratio.

Denoting by $\Phi_i^{\mathcal{H}}$ the electric potential evaluated on the horizon $\mathcal{H}_i$, we obtain explicitly

$$\Phi_1^{\mathcal{H}} = \Phi_2^{\mathcal{H}} = \frac{\sqrt{3}\tanh\gamma}{2\sqrt{1+\alpha_5^2}}. \tag{51}$$

F. Dilaton charge

Finally, we define the dilaton charge Σ via the asymptotic behavior of the dilaton field

$$\varphi \approx \frac{\Sigma}{r} = -\frac{\alpha_5^2\sinh^2\gamma(a+c-2b)}{\alpha(1+\alpha_5^2)r}. \tag{52}$$

Comparison with the electric charges and potential then yields the relation

$$Q_1\Phi_1^{\mathcal{H}} + Q_2\Phi_2^{\mathcal{H}} = -L\frac{\Sigma}{\alpha}. \tag{53}$$

The dilaton charge thus represents no independent parameter.

V. SMARR-LIKE RELATIONS AND FIRST LAW

A. Mass formula

It can be checked that our solution satisfies the following Smarr-like relation

$$M_{\text{ADM}} = M_1^{\mathcal{H}} + Q_1\Phi_1^{\mathcal{H}} + M_2^{\mathcal{H}} + Q_2\Phi_2^{\mathcal{H}} + M_{\text{bubble}}. \tag{54}$$

Here Eq. (53) can be used to replace the electromagnetic contributions by the dilaton contribution.

Below we give a general proof of the Smarr-like relation (54). Our starting point is the generalized Komar integral (41). Using the Gauss theorem we find that the ADM mass can be written as a sum of a bulk integral over a constant t and ϕ hypersurface and surface integrals on the black holes horizons and the bubble

$$\begin{aligned} M_{\text{ADM}} &= -\frac{L}{16\pi}\oint_\infty dS^{\mu\nu\sigma}(2\eta_\sigma\nabla_\mu\xi_\nu + \xi_\nu\nabla_\mu\eta_\sigma) \\ &= \frac{L}{8\pi}\int_\Sigma dS^{\mu\nu}[2\eta_\nu R_{\mu\lambda}\xi^\lambda - \xi_\nu R_{\mu\lambda}\eta^\lambda] \\ &\quad - \sum_i\frac{L}{16\pi}\oint_{\mathcal{H}_i} dS^{\mu\nu\sigma}(2\eta_\sigma\nabla_\mu\xi_\nu + \xi_\nu\nabla_\mu\eta_\sigma) \\ &\quad - \frac{L}{16\pi}\oint_{\text{bubble}} dS^{\mu\nu\sigma}(2\eta_\sigma\nabla_\mu\xi_\nu + \xi_\nu\nabla_\mu\eta_\sigma). \end{aligned} \tag{55}$$

On the horizons we have

$$dS_{\mu\nu\sigma} = \frac{3!}{\sqrt{g(\eta,\eta)}}\xi_{[\mu}N_\nu\eta_{\sigma]}d\omega, \tag{56}$$

where $d\omega$ is the magnitude of the two-dimensional surface element and N_ν is the second null normal to the horizon normalized such as $\xi^\nu N_\nu = -1$. Substituting into the horizon surface integrals and taking into account that ξ and η are commuting we obtain

$$-\frac{L}{16\pi}\oint_{\mathcal{H}_i} dS^{\mu\nu\sigma}(\xi_\nu\nabla_\mu\eta_\sigma) = 0 \tag{57}$$

and therefore

$$-\frac{L}{16\pi}\oint_{\mathcal{H}_i} dS^{\mu\nu\sigma}(2\eta_\sigma\nabla_\mu\xi_\nu+\xi_\nu\nabla_\mu\eta_\sigma)$$
$$=-\frac{L}{16\pi}\oint_{\mathcal{H}_i} dS^{\mu\nu\sigma}(2\eta_\sigma\nabla_\mu\xi_\nu)$$
$$=-\frac{L}{4\pi}\oint_{\mathcal{H}_i} d\omega\sqrt{g(\eta,\eta)}N_\nu\xi_\mu\nabla^\mu\xi^\nu. \quad (58)$$

By definition, on the horizons $\xi_\mu\nabla^\mu\xi^\nu=\kappa_i\xi^\nu$, where κ_i is the surface gravity of the ith horizon. Hence we obtain $N_\nu\xi_\mu\nabla^\mu\xi^\nu=\kappa_i N_\nu\xi^\nu=-\kappa_i$ and therefore

$$-\frac{L}{16\pi}\oint_{\mathcal{H}_i} dS^{\mu\nu\sigma}(2\eta_\sigma\nabla_\mu\xi_\nu+\xi_\nu\nabla_\mu\eta_\sigma)$$
$$=-\frac{L}{4\pi}\oint_{\mathcal{H}_i} d\omega\sqrt{g(\eta,\eta)}N_\nu\xi_\mu\nabla^\mu\xi^\nu$$
$$=\frac{\kappa_i L}{4\pi}\oint_{\mathcal{H}_i} d\omega\sqrt{g(\eta,\eta)}, \quad (60)$$

where we have made use of the fact that the surface gravity is constant on the horizon.

Further, taking into account that

$$L\oint_{\mathcal{H}_i} d\omega\sqrt{g(\eta,\eta)}=\oint_{\mathcal{H}_i} d\omega d\phi\sqrt{g(\eta,\eta)}=\mathcal{A}_h^{(i)} \quad (62)$$

is the area of the ith horizon we find that

$$-\frac{L}{16\pi}\oint_{\mathcal{H}_i} dS^{\mu\nu\sigma}(2\eta_\sigma\nabla_\mu\xi_\nu+\xi_\nu\nabla_\mu\eta_\sigma)$$
$$=\frac{1}{4\pi}\kappa_i\mathcal{A}_h^{(i)}$$
$$=\frac{1}{2}T^{(i)}\mathcal{A}_h^{(i)}=M_i^{\mathcal{H}}. \quad (63)$$

Let us denote by n^1_μ and n^2_μ the unit spacelike normals to the bubble surface. The bubble surface gravity is defined by the following equality on the bubble (see [8]):

$$\nabla_{[\mu}\eta_{\nu]}=\kappa_b(n^1_\mu n^2_\nu-n^1_\nu n^2_\mu). \quad (64)$$

As shown in [8], the bubble surface gravity κ_b is constant on the bubble surface. In our case this is obvious since $\kappa_b=2\pi/L$. For the bubble surface element we can write

$$dS_{\mu\nu\sigma}=\frac{3!}{\sqrt{|g(\xi,\xi)|}}\xi_{[\mu}n^1_\nu n^2_{\sigma]}d\omega. \quad (65)$$

Substituting in the bubble surface integral and taking into account that η vanishes on the bubble we have

$$\oint_{\text{bubble}} dS^{\mu\nu\sigma}(2\eta_\sigma\nabla_\mu\xi_\nu)=0, \quad (66)$$

and therefore we find

$$M_{\text{bubble}}=-\frac{L}{16\pi}\oint_{\text{bubble}} dS^{\mu\nu\sigma}(2\eta_\sigma\nabla_\mu\xi_\nu+\xi_\nu\nabla_\mu\eta_\sigma)$$
$$=-\frac{L}{16\pi}\oint_{\text{bubble}} dS^{\mu\nu\sigma}(\xi_\nu\nabla_\mu\eta_\sigma)$$
$$=\frac{L\kappa_b}{8\pi}\oint_{\text{bubble}}\sqrt{|g(\xi,\xi)|}d\omega=\frac{L}{8\pi}\kappa_b\mathcal{A}^{(b)}, \quad (67)$$

where

$$\mathcal{A}^{(b)}=\oint_{\text{bubble}}\sqrt{|g(\xi,\xi)|}d\omega \quad (68)$$

is the bubble area as defined in [8].

Let us now evaluate the bulk integral

$$I_{\text{bulk}}=\frac{L}{8\pi}\int_\Sigma dS^{\mu\nu}[2\eta_\nu R_{\mu\lambda}\xi^\lambda-\xi_\nu R_{\mu\lambda}\eta^\lambda]. \quad (69)$$

Using the field Eqs. (2) we obtain

$$I_{\text{bulk}}=\frac{L}{2\pi}\int_\Sigma dS^{\mu\nu}\Big[e^{-2\alpha\varphi}\Big(F_{\mu\sigma}\xi^\lambda F^\sigma_\lambda-\frac{1}{4}F^2\xi_\mu\Big)\eta_\nu$$
$$-\frac{1}{2}e^{-2\alpha\varphi}F_{\mu\sigma}\eta^\lambda F^\sigma_\lambda\xi_\nu\Big]. \quad (70)$$

Making use of the Lie symmetries $\mathcal{L}_\xi A=0$ and $\mathcal{L}_\eta A=0$ as well as the field equations one can show that the following relations are satisfied:

$$e^{-2\alpha\varphi}F^\mu_\sigma\xi^\lambda F^\sigma_\lambda=-\nabla_\sigma(e^{-2\alpha\varphi}F^{\mu\sigma}\xi^\lambda A_\lambda),$$
$$e^{-2\alpha\varphi}F^\mu_\sigma\eta^\lambda F^\sigma_\lambda=-\nabla_\sigma(e^{-2\alpha\varphi}F^{\mu\sigma}\eta^\lambda A_\lambda), \quad (71)$$
$$\tfrac{1}{4}e^{-2\alpha\varphi}F^2\xi^\mu=\nabla_\sigma(e^{-2\alpha\varphi}\xi^{[\mu}F^{\sigma]\lambda}A_\lambda).$$

Using these relations and the fact that the integrand has vanishing Lie derivative with respect to the Killing vector η the bulk integral can be presented as a sum of surface integrals by applying the Gauss theorem

$$I_{\text{bulk}}=\sum_i\frac{L}{4\pi}\oint_{\mathcal{H}_i} dS_{\mu\nu\sigma}\Big[e^{-2\alpha\varphi}(F^{\mu\sigma}\xi^\lambda A_\lambda+\xi^\mu F^{\sigma\lambda}A_\lambda)\eta^\nu-\frac{1}{2}e^{-2\alpha\varphi}F^{\mu\sigma}\eta^\lambda A_\lambda\xi^\nu\Big]$$
$$+\frac{L}{4\pi}\oint_{\text{bubble}} dS_{\mu\nu\sigma}\Big[e^{-2\alpha\varphi}(F^{\mu\sigma}\xi^\lambda A_\lambda+\xi^\mu F^{\sigma\lambda}A_\lambda)\eta^\nu-\frac{1}{2}e^{-2\alpha\varphi}F^{\mu\sigma}\eta^\lambda A_\lambda\xi^\nu\Big]$$
$$-\frac{L}{4\pi}\oint_{\infty} dS_{\mu\nu\sigma}\Big[e^{-2\alpha\varphi}(F^{\mu\sigma}\xi^\lambda A_\lambda+\xi^\mu F^{\sigma\lambda}A_\lambda)\eta^\nu-\frac{1}{2}e^{-2\alpha\varphi}F^{\mu\sigma}\eta^\lambda A_\lambda\xi^\nu\Big]. \quad (72)$$

The Killing vector η vanishes on the bubble and therefore the surface integral on the bubble vanishes too. The same holds for the surface integral at infinity since we consider a gauge in which A_λ vanishes at infinity. As a consequence we find that only the first term in (72) survives. Further, we consider pure electric solutions satisfying

$$F^{[\mu\sigma}\xi^{\nu]} = 0, \qquad \eta^\mu F_{\mu\nu} = 0. \tag{73}$$

Under these conditions and using the two-dimensional horizon surface element (56) we find that

$$\begin{aligned} I_{\text{bulk}} &= \sum_i \frac{L}{4\pi}\oint_{\mathcal{H}_i}(-\xi^\lambda A_\lambda)e^{-2\alpha\varphi}F^{\mu\sigma}(\xi_\mu N_\sigma - \xi_\sigma N_\mu) \\ &\quad\times\sqrt{g(\eta,\eta)}d\omega \\ &= \sum_i \frac{1}{4\pi}\oint_{\mathcal{H}_i}(-\xi^\lambda A_\lambda)e^{-2\alpha\varphi}F^{\mu\sigma}(\xi_\mu N_\sigma - \xi_\sigma N_\mu) \\ &\quad\times\sqrt{g(\eta,\eta)}d\phi d\omega \\ &= \sum_i \frac{1}{4\pi}\oint_{\mathcal{H}_i}(-\xi^\lambda A_\lambda)e^{-2\alpha\varphi}F^{\mu\sigma}dS_{\mu\sigma}, \end{aligned} \tag{74}$$

where we have taken into account that

$$\begin{aligned} &(\xi_\mu N_\sigma - \xi_\sigma N_\mu)g(\eta,\eta)d\phi d\omega = (\xi_\mu N_\sigma - \xi_\sigma N_\mu)d\mathcal{A} \\ &= dS_{\mu\nu} \end{aligned} \tag{75}$$

is the three-dimensional surface element on the horizon.

In the final step we use the fact that the potential $\Phi = -\xi^\lambda A_\lambda$ is constant[1] on the horizon and therefore we obtain

$$I_{\text{bulk}} = \sum_i \Phi_i^{\mathcal{H}}\frac{1}{4\pi}\oint_{\mathcal{H}_i} e^{-2\alpha\varphi}F^{\mu\sigma}dS_{\mu\sigma} = \sum_i \Phi_i^{\mathcal{H}}Q_i. \tag{76}$$

In this way we have proven the Smarr-like formula

$$M_{\text{ADM}} = \sum_i M_i^{\mathcal{H}} + \sum_i \Phi_i^{\mathcal{H}}Q_i + M_{\text{bubble}}. \tag{77}$$

Obviously the above relation can be generalized easily for the case when more than one bubble is present in the configuration.

B. Tension

A Smarr-like relation can also be found for the tension. The tension can be calculated by the following Komar integral [11]:

$$\mathcal{T}L = -\frac{1}{16\pi}\oint_\infty dS_{\mu\nu\sigma}(\eta^\sigma\nabla^\mu\xi^\nu + 2\xi^\nu\nabla^\mu\eta^\sigma). \tag{78}$$

With the asymptotic form of the relevant metric components given by Eq. (39), the tension can be directly read off from the asymptotics of the metric potentials [12–14]

$$\mathcal{T}L = \tfrac{1}{4}L(c_t - 2c_\phi). \tag{79}$$

Following the same method as for the derivation of the Smarr-like relation for ADM mass we find

$$\mathcal{T}L = \frac{1}{2}\sum_i M_i^{\mathcal{H}} + 2M_{\text{bubble}}. \tag{80}$$

Combining this relation with the relation for ADM mass one can easily show that the following relation holds

$$\mathcal{T}L = \frac{1}{2}M_{\text{ADM}} - \frac{1}{2}\sum_i \Phi_i^{\mathcal{H}}Q_i + \frac{3}{2}M_{\text{bubble}}.$$

C. First law

In the absence of bubbles the first law can be derived following the well-known method. In the presence of bubbles we have to find the contribution of the bubbles to the first law. It was, however, shown in [8] that the smooth bubbles (i.e. without conical singularities) do not give any contribution to the first law. In this paper we consider, namely, smooth bubbles. Therefore, in our case the first law conserves its usual form

$$\delta M_{\text{ADM}} = \tfrac{1}{4}T^{(1)}\delta A_h^{(1)} + \Phi_1\delta Q_1 + \tfrac{1}{4}T^{(2)}\delta A_h^{(2)} + \Phi_2\delta Q_2. \tag{81}$$

Let us note again that we keep L fixed.

VI. EXTREMAL LIMIT

The extremal solutions are obtained in the limit $\gamma \to \infty$, keeping the charges Q_1 and Q_2 fixed. The explicit form of the extremal solutions is the following:

$$\begin{aligned} ds^2 &= -\left(1 + \frac{q_2}{R_2} + \frac{q_1}{R_3}\right)^{-(2/1+\alpha_5^2)}dt^2 \\ &\quad + \left(1 + \frac{q_2}{R_2} + \frac{q_1}{R_3}\right)^{1/1+\alpha_5^2}\Bigg[(R_2 - \zeta_2)(R_3 + \zeta_3)d\psi^2 \\ &\quad + \frac{R_3 - \zeta_3}{R_2 - \zeta_2}d\phi^2 + \frac{Y_{23}}{2R_2R_3}\frac{R_2 - \zeta_2}{R_3 - \zeta_3}(d\rho^2 + dz^2)\Bigg], \\ \Phi &= \frac{\sqrt{3}}{2\sqrt{1+\alpha_5^2}}\frac{\frac{q_2}{R_2} + \frac{q_1}{R_3}}{1 + \frac{q_2}{R_2} + \frac{q_1}{R_3}}, \\ e^{-\alpha\varphi} &= \left(1 + \frac{q_2}{R_2} + \frac{q_1}{R_3}\right)^{\alpha_5^2/1+\alpha_5^2}, \end{aligned} \tag{82}$$

where the parameters q_1 and q_2 are given by

[1]The fact that the potential Φ is constant on the horizon can be shown by the well-known manner. On the horizon we have $R_{\mu\nu}\xi^\mu\xi^\nu = 0$ which, in view of the field equations, implies that $\partial_\mu\Phi$ is null on the horizon. Since ξ_μ is null on the horizon and orthogonal to $\partial_\mu\Phi$ it follows that on the horizon $\partial_\mu\Phi$ is proportional to ξ_μ, $\partial_\mu\Phi = \sigma\xi_\mu$. For an arbitrary vector ζ^μ tangential to the horizon we have $\mathcal{L}_\zeta\Phi = \sigma\zeta^\mu\xi_\mu = 0$ (on the horizon) which shows that Φ is constant on the horizon.

$$q_i = 2\sqrt{\frac{1+\alpha_5^2}{3}}\frac{Q_i}{L}. \tag{83}$$

Strictly speaking, the extremal solutions are singular and they cannot be interpreted as extremal black holes. As one can see the dilaton field is divergent on the candidate horizons $R_2 = 0$ and $R_3 = 0$ for $\alpha \neq 0$. As we shall show below, only in the Einstein-Maxwell case ($\alpha = 0$) the solutions describe two extremal black holes sitting on a Kaluza-Klein bubble.

The extremal solutions can be casted in new coordinates r and ϑ by performing the transformation

$$\rho = \sqrt{r^2 - 2rb}\sin\vartheta, \qquad z = (r-b)\cos\vartheta, \tag{84}$$

which gives

$$\begin{aligned}
ds^2 &= -\left[1 + \frac{q_2}{r-b-b\cos\vartheta} + \frac{q_1}{r-b+b\cos\vartheta}\right]^{-(2/1+\alpha_5^2)} dt^2 \\
&\quad + \left[1 + \frac{q_2}{r-b-b\cos\vartheta} + \frac{q_1}{r-b+b\cos\vartheta}\right]^{1/1+\alpha_5^2}\left[\frac{dr^2}{1-\frac{2b}{r}} + r^2 d\vartheta^2 + r^2\sin^2\vartheta d\psi^2 + \left(1-\frac{2b}{r}\right)d\phi^2\right], \\
\Phi &= \frac{\sqrt{3}}{2\sqrt{1+\alpha_5^2}}\left[1 - \frac{1}{1+\frac{q_2}{r-b-b\cos\vartheta} + \frac{q_1}{r-b+b\cos\vartheta}}\right], \\
e^{-\alpha\varphi} &= \left[1 + \frac{q_2}{r-b-b\cos\vartheta} + \frac{q_1}{r-b+b\cos\vartheta}\right]^{\alpha_5^2/1+\alpha_5^2}.
\end{aligned} \tag{85}$$

The new coordinates r and ϑ appear to be more suitable from a physical point of view and the extremal solutions look more tractable. In the particular case $q_2 = q_1 = 0$ we obtain the 5D solution describing a static KK bubble in standard spherical coordinates. The extremal solutions can be interpreted physically as solutions describing two charged objects whose gravitational attraction is balanced by the electric repulsion in the presence of a Kaluza-Klein bubble.

Let us now show that, in the Einstein-Maxwell case, the extremal solutions describe two extremal black holes on a Kaluza-Klein bubble. For this purpose we consider the metric in the vicinity of the candidate-horizon $r = 2b$, $\vartheta = 0$ and put

$$r = 2b + y_1^2, \qquad \vartheta = y_2, \tag{86}$$

where y_1 and y_2 are small. Expanding the metric around $r = 2b$ and $\vartheta = 0$ and performing the coordinate transformation

$$dt = d\tau - \sqrt{2bq_2^3}\,d\left(\frac{1}{R}\right) \tag{87}$$

where $R = y_1^2 + \frac{b}{2}y_2^2$, we obtain

$$ds^2 = -\frac{R^2}{q_2^2}d\tau^2 - \sqrt{\frac{2b}{q_2}}dRd\tau + 8bq_2 d\Omega_3^2. \tag{88}$$

Here $d\Omega_3^2$ is the metric on the unit three-dimensional sphere, i.e.,

$$d\Omega_3^2 = d\theta^2 + \sin^2\theta d\psi^2 + \cos^2\theta d\phi_1^2, \tag{89}$$

where $\phi_1 = \phi/4b$ and $\theta = \arctan(\sqrt{\frac{b}{2}}\frac{y_2}{y_1})$. Since $L = 8\pi b$ in the extremal limit the angle ϕ_1 has the canonical period $\Delta\phi_1 = 2\pi$.

It is seen from (88) that the metric remains regular as $r \to 2b$ and $\vartheta \to 0$, and the same is true for the electromagnetic field. Therefore $r = 2b$, $\vartheta = 0$ is a (degenerate) event horizon with S^3 topology as follows from (88). The radius of the horizon is $\sqrt{8bq_2}$ and hence the area of the horizon is

$$\mathcal{A}_h^{(2)E}(\alpha = 0) = 2\sqrt{\pi}\left(\frac{2Q_2}{\sqrt{3}}\right)^{3/2}. \tag{90}$$

The area is also found from (34) by taking the extremal limit.

In the same manner one can show that $r = 2b$, $\vartheta = \pi$ is a (degenerate) event horizon with S^3 topology and area

$$\mathcal{A}_h^{(1)E}(\alpha = 0) = 2\sqrt{\pi}\left(\frac{2Q_1}{\sqrt{3}}\right)^{3/2}. \tag{91}$$

The extremal limit of (27) gives the same expression for the area of the first extremal black hole. For the two extremal Einstein-Maxwell black holes sitting on a Kaluza-Klein bubble we have

$$\begin{aligned}
M^E_{\text{bubble}} &= 4\pi b^2 = \frac{L^2}{16\pi}, \\
M^E_{\text{ADM}} &= \frac{L^2}{16\pi} + \frac{\sqrt{3}}{2}(Q_1 + Q_2).
\end{aligned} \tag{92}$$

When $\alpha \neq 0$ we consider near extremal (NE) solutions. We here find that the area of the NE black holes tends to zero

$$\mathcal{A}_h^{(1)\text{NE}}(\alpha \neq 0) \to 0, \qquad \mathcal{A}_h^{(2)\text{NE}}(\alpha \neq 0) \to 0. \tag{93}$$

The temperature of the NE black holes is

$$T^{(i)}(\alpha_5^2<2)\to 0,\qquad T^{(i)}(\alpha_5^2=2)\to\frac{1}{2\sqrt{2\pi Q_i}},\qquad T^{(i)}(\alpha_5^2>2)\to\infty. \tag{94}$$

The mass of the bubble and the ADM mass are, respectively,

$$M^{\rm NE}_{\rm bubble}\approx 4\pi b^2=\frac{L^2}{16\pi},\qquad M^{\rm NE}_{\rm ADM}\approx\frac{L^2}{16\pi}+\frac{\sqrt{3}}{2\sqrt{1+\alpha_5^2}}(Q_1+Q_2). \tag{95}$$

The proper distance l_{12} between the black holes becomes arbitrarily large when extremality is approached.

VII. BLACK STRINGS

For the vacuum solution the question was addressed whether the total area of the two black holes $\mathcal{A}_{\rm 2BH}$ is larger or smaller than the area of a black string $\mathcal{A}^s_h$ with the same mass [5]. It turned out, that

$$\mathcal{A}_{\rm 2BH}<\mathcal{A}^s_h$$

showing that the black string is entropically favored. Here, we address this question for charged black holes.

In order to generate the solution describing a five-dimensional black string in Einstein-Maxwell-dilaton gravity we consider the four-dimensional Schwarzschild solution trivially embedded in five dimensions

$$ds^2_{\rm Sch}=-\left(1-\frac{2m}{r}\right)dt^2+\frac{dr^2}{1-\frac{2m}{r}}+r^2d\Omega^2+dx_5^2. \tag{96}$$

Using the method of Ref. [9] and (96) as a seed solution one obtains the following EMD solution

$$\begin{aligned}ds^2=&-\frac{1-\frac{2m}{r}}{[1+\frac{2m}{r}\sinh^2\gamma_1]^{2/1+\alpha_5^2}}dt^2\\&+\left[1+\frac{2m}{r}\sinh^2\gamma_1\right]^{1/1+\alpha_5^2}\\&\times\left[\frac{dr^2}{1-\frac{2m}{r}}+r^2d\Omega^2+dx_5^2\right],\\ e^{-2\alpha\varphi}=&\left[1+\frac{2m}{r}\sinh^2\gamma_1\right]^{2\alpha_5^2/1+\alpha_5^2},\\ \Phi=&\frac{\sqrt{3}}{2}\frac{\cosh\gamma_1\sinh\gamma_1}{\sqrt{1+\alpha_5^2}}\frac{2m}{r+2m\sinh^2\gamma_1}.\end{aligned} \tag{97}$$

An alternative derivation of this solution has been given in [15].

The ADM mass, the charge, the horizon area, and the temperature of the EMD black string are given by

$$\begin{aligned}M^s_{\rm ADM}&=mL\left[1+\frac{3}{2(1+\alpha_5^2)}\sinh^2\gamma_1\right],\\ Q^s&=\frac{\sqrt{3}\cosh\gamma_1\sinh\gamma_1}{\sqrt{1+\alpha_5^2}}mL,\\ \mathcal{A}^s_h&=16\pi\cosh^{3/1+\alpha_5^2}\gamma_1 m^2L,\\ T^s&=\frac{\cosh^{-(3/1+\alpha_5^2)}\gamma_1}{8\pi m}\end{aligned} \tag{98}$$

and the following Smarr-like relation is satisfied:

$$M^s_{\rm ADM}=\tfrac{1}{2}\mathcal{A}^s_hT^s+\Phi_hQ^s. \tag{99}$$

The extremal limit of the black string solution is obtained for $\gamma_1\to\infty$ and $m\to 0$ keeping the charge Q^s finite. The extremal solution is singular with zero horizon area for all coupling parameters α_5. The temperature of the near extremal solution tends to zero for $\alpha_5^2<1/2$, to infinity for $\alpha_5^2>1/2$, and the temperature is finite for $\alpha_5^2=1/2$.

Comparing now the area of two charged black holes $\mathcal{A}_{\rm 2BH}$ with the area of a charged string of the same mass, charge, and the length L of the Kaluza-Klein circle, we find

$$\mathcal{A}_{\rm 2BH}<\mathcal{A}^s_h. \tag{100}$$

This shows that the solution is globally unstable as all solutions of this type.

VIII. CONCLUSION

We have constructed a solution of two black holes on a Kaluza-Klein bubble in Einstein-Maxwell-dilaton theory and studied the consequences of the presence of charge for the properties of this solution. Furthermore, we have obtained a generalized Smarr relation and first law for this solution, extending recent vacuum results [8].

While we have focussed here on a particular solution [5], the calculations are easily extended to general sequences of smooth KK bubbles and black holes [6]. For these we expect the Smarr relation

$$M=\sum_i M^{\mathcal{H}}_i+\sum_i\Phi^{\mathcal{H}}_iQ_i+\sum_i M^i_{\rm bubble}$$

and first law

$$\delta M=\frac{1}{4}\sum_i T^i\delta\mathcal{A}^i_h+\sum_i\Phi^{\mathcal{H}}_i\delta Q_i.$$

As for all solutions of the considered type, the question of their classical stability remains open.

Future work may involve the inclusion of rotation [16]. Considering the solutions from the point of view of string theory, on the other hand, may necessitate the inclusion of further fields for consistency.

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