Toy model for topology change transitions: Role of curvature corrections

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We consider properties of near-critical solutions describing a test static axisymmetric D-dimensional brane interacting with a bulk N-dimensional black hole ($N > D$). We focus our attention on the effects connected with curvature corrections to the brane action. Namely, we demonstrate that the second order phase transition in such a system is modified and becomes first order. We discuss possible consequences of these results for merger transitions between caged black holes and black strings.

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I. INTRODUCTION

Transitions with a change of Euclidean topology is a subject of wide physical interest. One interesting example is the phase transition connected with a nucleation of a black hole in a thermal bath. Consider a thermal field with temperature T in a flat spacetime. One can use Euclidean fields on a spacetime with the topology $R^{D-1} \times S^1$ to describe a canonical ensemble for such a field. The size of the compact dimension S^1 is $\beta = 1/T$. A nucleation of a black hole changes the Euclidean topology from $R^{D-1} \times S^1$ to $S^{D-2} \times R^2$. The corresponding Euclidean space after the black hole nucleation is the Gibbons-Hawking instanton [1].

Another important example of a similar phenomenon is the so-called merger phase transition which occurs in models with large extra dimensions when a black hole is localized in a spacetime which has additional k compact dimensions ($D = 4 + k$) (a caged black hole, for reviews see [2–6]). In the absence of the black hole such a spacetime has the topology $R^4 \times T^k$. Kol argued [7] that the black hole-black string phase transition includes a local topology change of the corresponding Euclidean manifold so that the singular geometry is a cone over $S^{D-3} \times S^2$ [8]. This topology change is similar to the conifold transition [9]. In the black hole phase the S^{D-3} is contractible while in the black string phase the S^2 is contractible. In order to achieve this topology change one has to pass a configuration which is singular at the tip of the cone. The "double-cone" over $S^{D-3} \times S^2$ is given by

$$ds^2 = d\rho^2 + \frac{\rho^2}{D-2}[d\chi^2 + \cos(\chi)^2 dt^2 + (D-4)d\Omega^2_{D-3}]. \quad (1)$$

For more details see [2,10].

Kol [11] proposed that there exists a relation between merger transitions and Choptuik's critical collapse [12,13]. This correspondence can be achieved by performing two analytic continuations. The physics of the critical collapse and merger transitions have some common features like a singular critical solution which turns out to be an attractor and a self-similar solution in the neighborhood of the singular point. A better understanding of one of the systems may shed light on the other.

It is interesting that there also exists a close similarity between the properties of *merger transitions* and a toy model proposed some time ago for study transitions during which the Euclidean topology is changed [14–16]. This model consists of an N-dimensional static bulk black hole and a D-dimensional brane ($D < N$) interacting with this black hole. The brane is assumed to be a test brane and infinitely thin. The former assumption means that one neglects the effects connected with the gravitational field of the brane, while the latter one implies that the effects of the brane thickness are neglected and its world sheet is a minimal surface which provides an extremum of the Dirac-Nambu-Goto (DNG) action. It is assumed that the brane is static and axisymmetric so that the induced geometry on the brane possesses the $O(D-1)$ group of isometry. It is also assumed that far from the black hole the brane surface is parallel to the equatorial plane of the bulk black hole. For such a brane there exists two qualitatively different configurations: One, which is called *subcritical*, is a brane which does not intersect the black hole event horizon, and the other, *supercritical*, is a brane crossing the horizon. In the latter configuration the induced geometry on the world sheet of the brane is the geometry of a D-dimensional black hole, which is called a *brane black hole*, or briefly *BBH*. Such a black hole is absent for a subcritical configuration. Thus by changing the position of the brane at infinity (*asymptotic data*), one generates a transition between BBH and no-BBH phases (see Fig. 1).

If this change is done adiabatically, then one deals with a one parameter set of quasistatic solutions. After Wick's rotation of time one gets a one parameter set of Euclidean induced metrics, with a change of the Euclidean topology of the induced metric at some critical value of the asymptotic data, which plays the role of an order parameter. The Euclidean topology changes from $S^1 \times R^{D-1}$ for the subcritical configuration to $R^2 \times S^{D-2}$ for the supercritical. It was demonstrated [14–16] that when the effects of the

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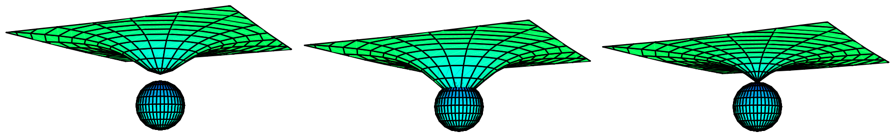

FIG. 1 (color online). Three possible types of configurations—the *subcritical* embedding (left) where the brane does not touch the black hole horizon, the *supercritical* (right) where we have an induced black hole on the brane (BBH), and the *critical* (center) which is singular at its tip.

brane stiffness are neglected the relation between the asymptotic data and the mass of the induced BBH for the transition between sub- and supercritical configurations is *universal*, that is it does not depend on the bulk black hole characteristics. Moreover, there is no mass gap for a creation of a BBH, so that the corresponding phase transition is of the second order. The near-critical solutions possess discrete (for $D \leq 6$) or continuous (for $D > 6$) self-similarity, which makes this transition formally similar both to the merger transitions and the near-critical collapse discovered by Choptuik [12]. (For a general review of the critical collapse see e.g. [13]. See also a discussion [17] of the critical collapse in a higher dimensional spacetime.). These properties and close similarity of near-critical solutions for both the BBH model and merger transitions make it interesting to consider in the framework of the BBH model some general problems which exist for this class of models.

Before discussing these problems we mention that the universality of the near-critical behavior in the BBH model is a consequence of the following fact: only near-horizon properties of these solutions are important . In this near-horizon domain there is no dimensionful parameter which determines the behavior of the system. As a result of this the system has scaling properties and the related phase transition is of the second order. This is why the model captures many universal features of various physical systems [18]. One important example of such a system provides a holographic description of the meson melting phase transition of matter in the fundamental representation [19–23]. The configuration consists of N_c color Dp-branes and N_f flavor Dq-branes when $p < q$. The addition of the flavor Dq-branes is dual to the addition of matter in the fundamental representation in the gauge theory. In the limit $N_c \gg N_f$ the Dp-branes are described by a p-brane supergravity action (“black Dp-branes”) while the Dq-branes are described by the DNG action. In other words, we can say that there are Dq-branes in the background of Dp-branes. In this system, the point where the brane touches the black hole horizon corresponds to a certain temperature where the mesons melt [24]. The holographic description of the melting corresponds to the transition from the subcritical embedding to the supercritical one.

Let us consider merger transitions in more detail. It is evident that the double-cone solution (1) is smooth everywhere except for the tip $\rho = 0$ where the curvature becomes singular. Near the tip the Kretchsman curvature scalar $\mathcal{R}^2 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ infinitely grows

$$\mathcal{R}^2 = \frac{4(D-3)^2(D-2)}{(D-4)\rho^4}. \quad (2)$$

The existence of the infinite curvature indicates that the solution obtained in the framework of classical Einstein gravity should be modified by quantum corrections. In other words, the naked singularity that is formed during the merger transition in its classical description might be resolved by the inclusion of quantum corrections into the classical action [25]. This conclusion is important. It means that if the transition between a black hole and black string phases occurs through the merger transition, one can expect the formation of a region with very high (up to the Planckian) curvature in a system characterized by macroscopic parameters (size of extra dimensions). An important question is how quantum gravity effects modify an adopted picture of classical merger transition.

Trying to answer this question one inevitably meets two difficult problems. One is of technical origin, namely, how the near-critical solutions for the merger problem are modified by quantum gravity corrections, for example, by adding to the Einstein-Hilbert action quadratic curvature corrections which arise in one loop computations. One can expect that the corresponding corrections become important when the curvature near the tip reaches the Planckian scale. The other more difficult problem is the following. At the corresponding Planckian scale the higher loop quantum gravity terms might also become important. If this happens, it would indicate that a complete solution of the problem requires the summation of all quantum loops or the use of a more fundamental theory of gravity, such as string theory. All of the above makes the problem very complicated for analysis.

For this reason it is interesting to analyze a much simpler BBH model which has qualitatively the same behavior as merger transitions. Its critical solutions also have curvature singularity at the cone tip where it touches the horizon. One can expect that adding terms quadratic in the extrinsic curvature to the classical DNG Lagrangian, which are analogous to local one loop corrections in quantum gravity, may “cure” this “disease.” Such curvature corrections naturally arise as a result of the stiffness effect [26]. In the case of strings they were suggested by Polyakov [27]. We can think about such terms as corrections that come from the finite thickness of the brane which is ignored in the DNG action. The DNG action can be considered as the zeroth order in the expansion in the width over a typical length in the system [28,29]. Usually the small parameter in such an expansion is the ratio of the thickness of the brane to the characteristic radius of the brane bending. It is instructive first to study effects connected with the leading order terms in this expansion. In this analogy the thickness of the brane plays the role similar to the Planck scale. Moreover, if one describes the brane as a special topologically stable solution of some nonlinear field theory, one may, in principle, answer not only the question of how the quadratic curvature corrections modify the near-critical solutions, but also investigate the complete field theoretical object behavior in the near-critical regime. Effectively this corresponds to the summation of all the stiffness corrections.

In the present paper we focus on the first problem, namely, we will analyze how the lowest order stiffness corrections modify the phase transition in the BBH systems.

A good analogy that helps to understand the effect of stiffness terms is a stiff bar. Consider the bending of a stiff bar. When we take into account the effect of the stiffness of the bar, its bending costs energy. Hence we cannot bend the bar as much as we want and it would eventually break long before a sharp tip is created. The sharp tip corresponds to the singular critical solution of the DNG action. One might expect that inclusion of higher derivative terms to the Lagrangian would prevent the creation of such a singular solution and will form a first order phase transition long before. Indeed, as we will see this is what happens in the BBH system for the subcritical configuration.

The higher derivative corrections to the BBH system can serve as a toy model for the singularity resolution of ‘‘small BHs.’’ Small BHs are singular limits of BH parameters in which the horizon becomes singular (see for example [30,31]). If we look at the induced BH on the brane, the critical solution has a singularity exactly of this type.

The paper is organized as follows. In Sec. II we review the main results concerning the near-critical branes obtained in the absence of stiffness in the BBH model. In Sec. III we discuss the curvature corrections for a stiff brane. The stiff brane equations are presented in Sec. IV. Section V contains the analysis of near-critical solutions in the linear approximation. In Secs. VI and VII the numerical results for near-critical branes are presented. Section VIII contains a summary of the obtained results and their discussion.

II. NONSTIFF BRANES

In this section we briefly review the main results of [16] concerning the behavior of near-critical D-dimensional branes without stiffness interacting with a bulk static spherically symmetric N-dimensional black hole. We do this mainly to explain the setup of the problem and to fix the notations we will use later. The Schwarzschild-Tangherlini metric of the bulk N-dimensional spacetime is

$$dS^2 = g_{\mu\nu}dx^\mu dx^\nu = -FdT^2 + F^{-1}dr^2 + r^2 d\Omega^2_{N-2}, \tag{3}$$

where $F = 1 - (r_g/r)^{N-3}$ and $d\Omega^2_{N-2}$ is the metric of a $(N-2)$-dimensional unit sphere S^{N-2}. We define the coordinates θ_i ($i = 1, \dots, N-2$) on this sphere by the relations

$$d\Omega^2_{i+1} = d\theta^2_{i+1} + \sin^2\theta_{i+1} d\Omega^2_i. \tag{4}$$

We denote by x^μ ($\mu = 0, \dots, N-1$) the bulk spacetime coordinates and by $\zeta^a (a = 0, \dots, D-1)$ the coordinates on the brane world sheet. The functions $x^\mu = X^\mu(\zeta^a)$ determine the brane world sheet describing the embedding of the $(D-1)$-dimensional object (brane) in a bulk N-dimensional spacetime. We assume that $D \leq N-1$.

In the absence of stiffness, the brane configuration in an external gravitational field $g_{\mu\nu}$ can be obtained by solving the equations which follow from the DNG action [32–34]

$$S = \int d^D\zeta \sqrt{-\det \gamma_{ab}}, \tag{5}$$

where $\gamma_{ab} = g_{\mu\nu}X^\mu_{,a}X^\nu_{,b}$ is an induced metric on the brane world sheet. We set the brane tension factor, which does not enter the brane equations, equal to 1. It is well known that an extremum of this action is a minimal surface. Let $n^\mu_{(i)}$ be unit normals to the brane and

$$K^{(i)}_{\alpha\beta} = -\frac{\partial X^\mu}{\partial \zeta^\alpha}\frac{\partial X^\nu}{\partial \zeta^\beta}\nabla_\nu n^{(i)}_\mu \tag{6}$$

be an extrinsic curvature tensor. (∇_ν is a covariant derivative with respect to the bulk metric $g_{\mu\nu}$.) Then the nonstiff brane equations are of the form

$$K^{(i)} = g^{\alpha\beta}K^{(i)}_{\alpha\beta} = 0. \tag{7}$$

For the axially symmetric D-dimensional static brane (with the isometry group $O(D-1)$) the induced metric is ($n = D-2$)

$$\begin{aligned} ds^2 &= \gamma_{ab}d\zeta^a d\zeta^b \\ &= -FdT^2 + [F^{-1} + r^2(d\theta/dr)^2]dr^2 + r^2\sin^2\theta d\Omega^2_n, \end{aligned} \tag{8}$$

and the action (5) reduces to

$$\begin{aligned} S &= \Delta T \mathcal{A}_n \int dr \mathcal{L}, \\ \mathcal{L} &= r^n \sin^n\theta \sqrt{1 + Fr^2(d\theta/dr)^2}. \end{aligned} \tag{9}$$

Here ΔT is the interval of time and $\mathcal{A}_n = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of a unit n-dimensional sphere.

By analyzing the brane equation it is easy to show [16] that for a brane which asymptotically approaches the equatorial plane $\theta = \pi/2$ one has

$$\theta = \frac{\pi}{2} + q(r), \qquad q = \frac{p}{r} + p' \begin{cases} r^{-1}\ln r, & \text{for } n = 1, \\ r^{-n}, & \text{for } n > 1. \end{cases} \tag{10}$$

We call the set of parameters $\{p, p'\}$, which characterizes the solution, the *asymptotic data*.

The same solution can be determined by its behavior near the horizon. A subcritical brane is uniquely specified by the distance of its tip from the horizon. The condition of the brane surface regularity at this point requires that its tangent plane at the tip is orthogonal to the symmetry axis. This fixes the second constant in the solution. Similarly, a regular brane crossing the horizon is orthogonal to the horizon surface, so that a unique constant fixing the solution is the "gravitational" radius of the induced BBH. A solution separating the sub- and supercritical solution is a *critical solution*. We denote by $\{p_*, p'_*\}$ its asymptotic data.

Figure 2 illustrates the near-critical behavior of a supercritical brane. (A similar graph for subcritical branes can be easily obtained from this one by evident changes.) A near-critical brane configuration is characterized by a parameter R_0, which is its radius at the intersection with the horizon. For a subcritical brane a similar parameter is Z_0, the proper distance of the tip of the brane from the horizon. We consider a case when R_0 (Z_0) is much smaller than the gravitational radius r_g of the bulk black hole. In the vicinity of the horizon, located at $Z = 0$, that is for $Z \ll r_g$, one has

$$r - r_g \approx \kappa Z^2/2, \qquad F \approx \kappa^2 Z^2, \tag{11}$$

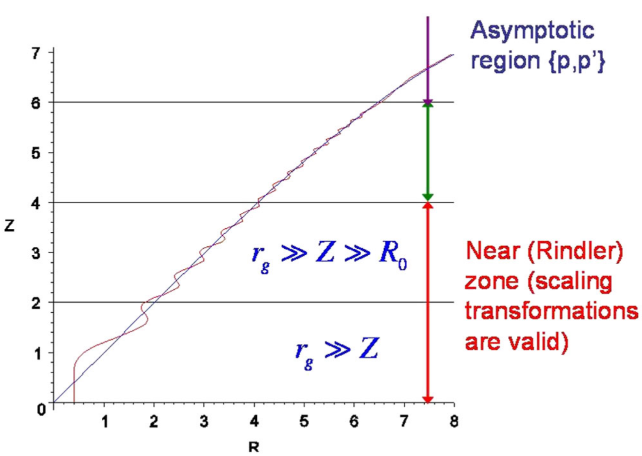

FIG. 2 (color online). This figure schematically shows a configuration of a supercritical brane in the regime when it is close to the critical one. Z is a proper distance from the horizon as a function of the radius R. R_0 is the radius of the surface of the intersection of the brane with the horizon. In the region where $Z \ll r_g$ the curvature surface of the horizon can be neglected (the *Rindler domain*).

where $\kappa = \frac{1}{2}(dF/dr)|_{r_g}$ is the surface gravity. We call this region a *near (or Rindler) zone*. The corresponding induced metric for a near-critical brane in the Rindler zone is

$$ds^2 = -\kappa^2 Z^2 dT^2 + \left[\left(\frac{dZ}{d\lambda}\right)^2 + \left(\frac{dR}{d\lambda}\right)^2\right] d\lambda^2 + R^2 d\Omega_n^2. \tag{12}$$

Here $(Z(\lambda), R(\lambda))$ is a brane equation written in a parametric form. The action (5) for this induced metric is

$$S = \kappa \Delta T \mathcal{A}_n \mathcal{S}, \tag{13}$$

$$\mathcal{S} = \int d\lambda Z R^n \sqrt{(dZ/d\lambda)^2 + (dR/d\lambda)^2}. \tag{14}$$

This action is evidently invariant under the transformations $\lambda \to \tilde{\lambda}(\lambda)$. In the regions where either Z or R is a monotonic function of λ, these functions themselves can be used as parameters. As a result, one obtains two other forms of the action which are equivalent to $\mathcal{S}$

$$\mathcal{S} = \int dZ \mathcal{L}_R = \int dR \mathcal{L}_Z, \tag{15}$$

where

$$\mathcal{L}_R = Z R^n \sqrt{1 + R'^2}, \qquad \mathcal{L}_Z = Z R^n \sqrt{1 + \dot{Z}^2}. \tag{16}$$

Here the prime stands for the derivative with respect to Z, while the dot stands for the derivative with respect to R. The corresponding Euler-Lagrange equations are

$$ZRR'' + (RR' - nZ)(1 + R'^2) = 0, \tag{17}$$

$$RZ\ddot{Z} + (nZ\dot{Z} - R)(1 + \dot{Z}^2) = 0. \tag{18}$$

It is easy to check that the form of Eqs. (17) and (18) is invariant under the following transformations:

$$R(Z) = k\tilde{R}(\tilde{Z}), \qquad Z = k\tilde{Z}, \tag{19}$$

$$Z(R) = k\tilde{Z}(\tilde{R}), \qquad R = k\tilde{R}. \tag{20}$$

Equations (17) and (18) have a simple solution

$$R = \sqrt{n} Z \tag{21}$$

which plays a special role. We call it a *critical solution*. It describes a critical brane which touches the horizon of the bulk black hole at one point, $Z = R = 0$. It separates the two different families of solutions, supercritical and subcritical.

The relation between R_0 and $\{p, p'\}$ is of the form

$$\ln R_0 = \gamma \ln \Delta p + f(\ln \Delta p) + \ldots, \tag{22}$$

where

$$\gamma = \frac{2}{n+2}, \qquad \Delta p = \sqrt{(p - p_*)^2 + (p' - p'_*)^2}, \tag{23}$$

and the function $f(z)$ is periodic, $f(z + \omega) = f(z)$, with the period

$$\omega = \frac{\pi(n+2)}{\sqrt{4+4n-n^2}}. \quad (24)$$

Our aim is to study how the near-critical solutions are modified when a brane is stiff. We assume that the effective width of the brane is much smaller that the gravitational radius of the black hole. In this approximation, as in the case of a nonstiff brane, the main features of the phase transition in the BBH system are determined by the brane behavior in the near zone, that is close to the event horizon of the bulk black hole, where the Rindler approximation is valid.

III. STIFF BRANES

In this work we consider the Dirac-Nambu-Goto action with minimal stiffness correction terms which play the role of higher curvature corrections [35]:

$$S = -\int d^{n+2}\zeta\sqrt{-\gamma}(1 + BK^2 + C\mathcal{K}^2). \quad (25)$$

Here γ is the determinant of the induced metric given in Eq. (5), $K = \sum_i K^{(i)\mu}_{\mu}$ is the trace of the extrinsic curvature tensor, and $\mathcal{K}^2 = \sum_i K_{(i)\mu\nu}K^{(i)\mu\nu}$ is its square. A minimal model of stiffness corrections involves only quadratic powers of the extrinsic curvature tensor. In this work we will concentrate on the above "truncated" model as a toy model whose solution gives us the static configuration of a stiff brane embedded close to the horizon of the bulk black hole, namely, in the Rindler zone.

In the particular case of a domain wall the stiffness coefficients were calculated in the framework of a microscopic model of a vacuum defect in field theory with spontaneous symmetry breaking [29]. In this work we show that the exact numerical values of the coefficients do not affect the qualitative features of the solution. Nevertheless the sign of the coefficients is important. Positive stiffness coefficients

$$B, C > 0$$

ensure us that the energy density for the static solution

$$\epsilon = -\mathcal{L} = \sqrt{-\gamma}(1 + BK^2 + C\mathcal{K}^2) \quad (26)$$

is positive since K^2 and $\mathcal{K}^2$ are both non-negative.

Before going further let us discuss two interesting special cases of the general theory.

C = 0 case. In this special case the stiff string equations have a simple exact solution. Namely, any solution of the DNG equations (7) is at the same time a solution of the stiff string equations. Indeed, a general variation of the action can be split into the variation along the brane and the transverse one. Variations along the brane surface vanish identically. For transverse variations, multiplying the equations of motion by a normal to the brane gives

$$n^{\mu}_{(i)}\frac{\delta\mathcal{L}}{\delta X^{\mu}} = K^{(i)}(1 + BK^2), \qquad -2B\sqrt{-\gamma}Kn^{\mu}_{(i)}\frac{\delta K}{\delta X^{\mu}} = 0. \quad (27)$$

Now, substituting the DNG equation $K^{(i)} = 0$ into the right-hand side of (27), we see that it vanishes. Hence $K^{(i)} = 0$ is a solution of the stiff string equations.

B + C = 0 case. This case is not interesting for our consideration since it violates the positive energy condition, but it is of mathematical interest. Let us notice that the Gauss-Codazzi relation for a flat bulk spacetime implies

$$\mathcal{R} = K^2 - \mathcal{K}^2. \quad (28)$$

Thus one can rewrite the action (25) in the following form:

$$S = -\int d^{n+2}\zeta\sqrt{-\gamma}(1 + B\mathcal{R} + (B+C)\mathcal{K}^2). \quad (29)$$

For $B + C = 0$ the term with $\mathcal{K}^2$ vanishes.

Let us return to the discussion of the general case. The units of the stiffness coefficients are length-squared. Therefore under scaling transformation of the spatial coordinates

$$R \rightarrow sR, \qquad Z \rightarrow sZ, \quad (30)$$

we have

$$K^2 \rightarrow s^{-2}K^2, \qquad \mathcal{K}^2 \rightarrow s^{-2}\mathcal{K}^2. \quad (31)$$

Using this transformation we can set one of the coefficients to be unit, say $C = 1$. We can think about it as taking the basic unit length of the stiff string to be $\sqrt{C}$. Then we are left only with one free parameter B in the action. We will use this choice later in Secs. VI and VII, when we will discuss the results of the numerical calculations.

For completeness we give here the components of the extrinsic curvature for the brane $(Z(\lambda), R(\lambda))$ with the induced metric (12):

$$K_{\lambda\lambda} = \mathcal{P}^{-1}\mathcal{A}, \qquad K_{TT} = -Z\frac{dR}{d\lambda}\mathcal{P}^{-1},$$
$$K_{\theta\theta} = R\frac{dZ}{d\lambda}\mathcal{P}^{-1}, \quad (32)$$

where

$$\mathcal{P} = \sqrt{(dZ/d\lambda)^2 + (dR/d\lambda)^2}, \quad (33)$$

$$\mathcal{A} = \frac{dZ}{d\lambda}\frac{d^2R}{d\lambda^2} - \frac{dR}{d\lambda}\frac{d^2Z}{d\lambda^2}. \quad (34)$$

The action (25) for the induced metric (12) takes the following form:

$$S = -\kappa\Delta T\mathcal{A}_n\int d\lambda\mathcal{L},$$
$$\mathcal{L} = \mathcal{L}_0 + B\mathcal{L}_1 + C\mathcal{L}_2, \qquad \mathcal{L}_0 = ZR^n\mathcal{P},$$
$$\mathcal{L}_1 = ZR^n\mathcal{P}\Big(\frac{\mathcal{A}}{\mathcal{P}^3} + \frac{dR/d\lambda}{Z\mathcal{P}} + \frac{n\,dZ/d\lambda}{R\mathcal{P}}\Big)^2, \tag{35}$$
$$\mathcal{L}_2 = \frac{R^n(dR/d\lambda)^2}{Z\mathcal{P}} + \frac{nZ(dZ/d\lambda)^2R^{n-2}}{\mathcal{P}} + \frac{ZR^n\mathcal{A}^2}{\mathcal{P}^5}.$$

The action (25) is evidently invariant under transformations $\lambda \to \tilde{\lambda}(\lambda)$ as in the nonstiff case. In a general case, a variation of this action gives equations containing fourth derivatives, while the corresponding constraint equations are of the third order [36].

IV. EULER-LAGRANGE EQUATIONS FOR STIFF BRANES

In the regions where either Z or R is a monotonic function of λ, one of the coordinates can be used as a parameter. As a result, one obtains two additional forms of the action:

$$\mathcal{S} = -\int dZ\mathcal{L}_R = -\int dR\mathcal{L}_Z, \tag{36}$$

where

$$\mathcal{L}_R = ZR^n\mathcal{P}\Big(1 + B\Big[\frac{R''}{\mathcal{P}^3} + \frac{R'}{Z\mathcal{P}} + \frac{n}{R\mathcal{P}}\Big]^2 + C\Big[\frac{R''^2}{\mathcal{P}^6} + \frac{R'^2}{Z^2\mathcal{P}^2} + \frac{n}{R^2\mathcal{P}^2}\Big]\Big), \tag{37}$$

$$\mathcal{L}_Z = ZR^n\mathcal{P}\Big(1 + B\Big[\frac{1}{Z\mathcal{P}} - \frac{\ddot{Z}}{\mathcal{P}^3} + \frac{n\dot{Z}}{R\mathcal{P}}\Big]^2 + C\Big[\frac{\ddot{Z}^2}{\mathcal{P}^6} + \frac{1}{Z^2\mathcal{P}^2} + \frac{n\dot{Z}^2}{R^2\mathcal{P}^2}\Big]\Big). \tag{38}$$

$\mathcal{P}$ in the first relation means $\mathcal{P} = \sqrt{1 + R'^2}$, while in the second equation one has $\mathcal{P} = \sqrt{1 + \dot{Z}^2}$.

As in the nonstiff case, the form with $R(Z)$ is useful for the description of the supercritical brane while $Z(R)$ is more suitable for the description of subcritical branes.

The equation for $R(Z)$ takes the following form:

$$-2bZ^3R^3(1 + R'^2)^2R^{(4)} + 4bZ^2R^2(R'^2 + 1)[5ZRR'R'' - (R'^2 + 1)(nZR' + R)]R^{(3)} + F(R, R', R'', Z) = 0, \tag{39}$$

where

$$\begin{aligned}F(R, R', R'', Z) = {}& 5bZ^3R^3(1 - 6R'^2)R''^3 + 3bZ^2R^2(1 + R'^2)[5RR' + nZ(4R'^2 - 1)]R''^2 + Z^3R^3(1 + R'^2)^3R'' \\ & - ZR(1 + R'^2)^2(2[2b + 3B]nZRR' + bR^2[R'^2 - 2] + nZ^2[b + 3B - 3Bn + 2b(n - 2)R'^2])R'' \\ & + (1 + R'^2)^3(R^3R'[Z^2 + Z^2R'^2] - nZR^2[Z^2 + Z^2R'^2]) - (1 + R'^2)^3([b - 3B(n - 1)]nZ^2RR' \\ & - bnZR^2R'^2 + bR^3R'[R'^2 + 2] + n[n - 2]Z^3[b + B(n - 1) + 2bR'^2])\end{aligned} \tag{40}$$

and $b = B + C$.

A similar equation for $Z(R)$ is

$$-2bR^3Z^3(1 + \dot{Z}^2)^2Z^{(4)} + 4bR^2Z^2(1 + \dot{Z}^2)[5RZ\dot{Z}\ddot{Z} - (1 + \dot{Z}^2)(nZ + R\dot{Z})]Z^{(3)} + G(Z, \dot{Z}, \ddot{Z}, R) = 0, \tag{41}$$

where

$$\begin{aligned}G(Z, \dot{Z}, \ddot{Z}, R) = {}& 5bR^3Z^3(1 - 6\dot{Z}^2)\ddot{Z}^3 + 3bR^2Z^2(1 + \dot{Z}^2)[5nZ\dot{Z} + R(4\dot{Z}^2 - 1)]\ddot{Z}^2 + R^3Z^3(1 + \dot{Z}^2)^3\ddot{Z} \\ & - RZ(1 + \dot{Z}^2)^2(2[3B + 2b]nRZ\dot{Z} - bR^2[2\dot{Z}^2 - 1] + nZ^2[2b(n - 2) + (3B + b - 3Bn)])\ddot{Z} \\ & - R^2Z^2(R - nZ\dot{Z})(1 + \dot{Z}^2)^4 + (1 + \dot{Z}^2)^3(-bnR^2Z\dot{Z} + nRZ^2[b - 3B(n - 1)]\dot{Z}^2 + bR^3[1 + 2\dot{Z}^2] \\ & + 2bn(n - 2)Z^3\dot{Z} + n(n - 2)Z^3\dot{Z}^3[b + B(n - 1)]).\end{aligned} \tag{42}$$

Let us denote

$$l = \max(\sqrt{B}, \sqrt{C}). \tag{43}$$

l has dimensionality of the length. The extrinsic curvature corrections are dominant for $R, Z \lesssim l$ where the stiff brane differs significantly from the DNG brane. The significant effect of the stiffness is localized in the region where the original DNG brane is extremely bent. This happens in the neighborhood of the point $R = Z = 0$ which is defined by the length scale l. For $R, Z \gg l$ the solution for the stiff brane-BH system approaches the DNG brane-BH system. In particular, $R = \sqrt{n}Z$ is the attractor solution for the DNG brane-BH system and therefore it should be also an attractor for the stiff brane-BH system.

V. NEAR-CRITICAL MODES

Let us study linear perturbations to the attractor solution for the case of a stiff brane. Our objective is to obtain the modes in the neighborhood of the attractor far away from the singular region $Z \gg l$ but still located in the Rindler zone. The modes will guide us later in the setting of the boundary conditions for stiff branes.

Let us substitute in the equation for R (39) the following expression:

$$R(Z) = \sqrt{n}Z + \rho(Z) \tag{44}$$

and keep only linear terms in $\rho(Z)$. Then we obtain the following linearized equation:

$$a_0^s(Z)\rho + a_\Omega 1^s(Z)\rho' + a_2^s(Z)\rho'' + a_3^s(Z)\rho^{(3)} + a_4^s(Z)\rho^{(4)} = 2C(n-1)(n+1)^2 n^{-(1/2)}Z, \tag{45}$$

$$\begin{aligned} a_0^s(Z) &= (n+1)[2B(n-5) - C(n+7)] + a_0(Z),\\ a_1^s(Z) &= (n+1)Z[2B(16-n) - 17C(n-1)] + a_1(Z),\\ a_2^s(Z) &= -(n+1)Z^2[2B(n+1) + C(2n-1)] + a_2(Z),\\ a_3^s(Z) &= -4(B+C)(n+1)Z^3,\\ a_4^s(Z) &= -2(B+C)Z^4. \end{aligned} \tag{46}$$

Here $a_i(Z)$ are the coefficients in the linearized DNG brane equations:

$$a_0(Z) = (n+1)^2 Z^2, \qquad a_1(Z) = (n+1)^2 Z^3, \qquad a_2(Z) = (n+1)Z^4. \tag{47}$$

Now let us take the limit $Z \gg l$ and as a result we obtain the following equation:

$$(n+1)^2[Z\rho + Z^2\rho'] + (n+1)Z^3\rho'' - 2(B+C) \times [2(n+1)Z^2\rho^{(3)} + Z^3\rho^{(4)}] = 2C(n-1)(n+1)^2 n^{-(1/2)}. \tag{48}$$

The leading term of the particular solution at large Z is

$$\rho_P = \frac{C(n^2-1)}{\sqrt{n}Z} + \mathcal{O}\left(\frac{1}{Z^2}\right), \tag{49}$$

and therefore it does not have an effect on the attractor $R = \sqrt{n}Z$ at large Z, as expected. A general solution of (48) is a sum of this particular solution and a general solution of the homogeneous equation obtained from (48) by omitting the right-hand side. It is interesting that this homogeneous solution depends only on the sum $B + C$ of the stiffness coefficients.

Since the homogeneous equation is of the fourth order, it has four linearly independent solutions (asymptotic modes). Two of the asymptotic modes reproduce the asymptotic solutions for the DNG brane:

$$\rho \sim Z^{-(1/2)(n \pm \sqrt{n^2-4n-4})}. \tag{50}$$

The other two modes appear only for the stiff brane:

$$\rho \sim \exp\left(\pm\sqrt{\frac{n+1}{2(B+C)}}Z\right). \tag{51}$$

The additional two modes above are added due to the stiffness corrections. One of the additional modes is an unstable mode which takes the solution away from the attractor. This mode should be eliminated by appropriate boundary conditions in order to reproduce the DNG solutions at large distances.

Thus we arrive to a boundary value problem. Let us take, for example, the subcritical configuration where the solution can be written as $Z(R)$ (a similar discussion is applicable to the supercritical configuration with some evident changes). Since Eq. (41) is of the fourth order, we need four initial values (in case of an initial value problem). The configuration is axially symmetric with the symmetry axis $R = 0$ and it is plausible that the stiff brane solution preserves the same axial symmetry. Consider a brane passing through the point $Z(0) = Z_0$ (the proper distance of the tip of the brane from the horizon). The axial symmetry and the regularity of the brane at $R = 0$ enforces $\dot{Z}(0) = 0$ and $Z^{(3)}(0) = 0$, while $\ddot{Z}(0)$ remains a free parameter. This free parameter will allow us to eliminate the unstable mode by finding the specific value for $\ddot{Z}(0)$.

In conclusion, we have a boundary value problem defining near-critical solutions of the stiff brane equations: In order to obtain the right asymptotic behavior at large R

$$Z \rightarrow \frac{R}{\sqrt{n}}$$

for a brane which passes at $Z(0) = Z_0$ we have to tune the parameter $\ddot{Z}(0) = \ddot{Z}_0$.

VI. NUMERICAL RESULTS FOR STIFF BRANES: $n = 1$ CASE

Using a numerical shooting analysis we obtained the values of $\ddot{Z}_0$ for which there is a solution for the subcritical brane that satisfies the boundary conditions. It starts at Z_0 and asymptotically goes to the attractor. A similar analysis was performed for the supercritical configuration where values of $R''(0)$ were determined as a function of R_0 (the radius of the BBH horizon).

Let us start by examining the case $n = 1$. As we will see, this case is qualitatively different from $n > 1$. For $B = 0$ the action [Eq. (35)] is completely symmetric for the interchange of $R \leftrightarrow Z$. This discrete symmetry of the equations implies the same results for the subcritical and supercritical configurations. For this reason we give here the results for the subcritical configuration $B = 0$ and $n = 1$ (Fig. 3) when for the supercritical configuration the graph is the same (up to the interchange of $R \leftrightarrow Z$). The plot at this figure shows $\ddot{Z}_0$ as a function of Z_0 for near-critical configurations. The finite gap in the neighborhood of the point $R = Z = 0$ demonstrates the first order phase transition in the BBH system. A detailed interpretation of this picture is given below in the discussion of the case $n = 2$ (where the same picture appears only in the subcritical configuration).

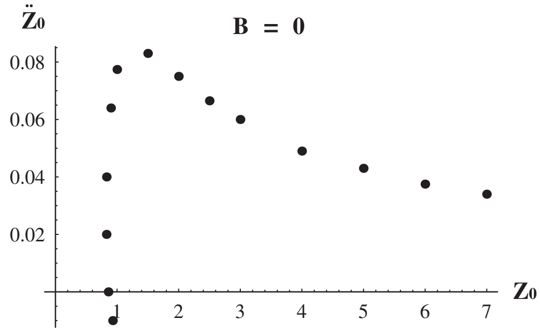

FIG. 3. $\ddot{Z}_0$ as a function of Z_0 for $n = 1$ and $B = 0$. The symmetry of the action in this case implies that the same graph is valid to the supercritical configuration as well—$R''(0)$ as a function of R_0.

For the case of $B \neq 0$ the action (35) is no longer symmetric under the reflection $R \leftrightarrow Z$. Nevertheless we find numerically that the results are symmetric within the used accuracy. Evidence for this symmetry we can find in the linearized equations for a perturbation around the attractor. In Sec. V we studied linearized perturbations to the supercritical configuration

$$R(Z) = \sqrt{n}Z + \rho(Z). \tag{52}$$

Keeping only linear terms in $\rho(Z)$ gave us Eq. (45). For the special case of $n = 1$ this equation reads

$$(B + C)Z^4\rho^{(4)}(Z) + 4(B + C)Z^3\rho^{(3)}(Z) + Z^2(4B + C - Z^2)\rho''(Z) - 2Z^3\rho'(Z) - 2(Z^2 - 4B - 4C)\rho(Z) = 0. \tag{53}$$

In a similar way for the subcritical configuration substitution of

$$Z(R) = \frac{R}{\sqrt{n}} + \zeta(R) \tag{54}$$

into Eq. (41) and keeping only linear terms in $\zeta(R)$ gives the following linear equation ($n = 1$):

$$(B + C)R^4\zeta^{(4)}(R) + 4(B + C)R^3\zeta^{(3)}(R) + R^2(4B + C - R^2)\ddot{\zeta}(R) - 2R^3\dot{\zeta}(R) - 2(R^2 - 4B - 4C)\zeta(R) = 0. \tag{55}$$

Hence the symmetry $R \leftrightarrow Z$ is demonstrated analytically in the linear approximation. This does not imply an exact reflection symmetry of the solutions, but at least makes it possible.

VII. NUMERICAL RESULTS FOR STIFF BRANES: $n > 1$ CASE

For $n > 1$ there is no reflection symmetry of the action anymore, and sub- and supercritical solutions behave quite differently. Let us start with the subcritical configuration

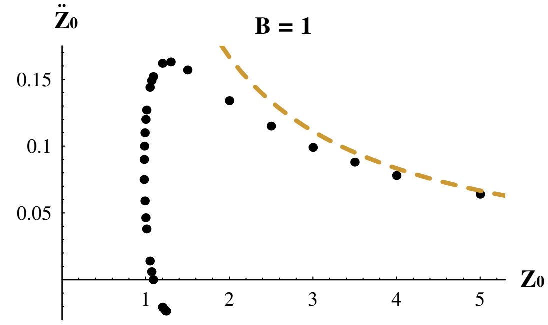

FIG. 4 (color online). $\ddot{Z}_0$ as a function of Z_0 for $n = 2$. The dashed line is the same function for DNG branes (without stiffness terms).

and demonstrate that it exhibits the same qualitative features as $n = 1$. This is a good place to compare the details of the new picture with the nonstiff case.

Consider, for example, the case of $n = 2$ and $B = 1$ in Fig. 4. The value of $\ddot{Z}_0$ is plotted as a function of the position where the brane crosses the axis of symmetry Z_0. In the case of DNG branes, i.e. without stiffness, the dependence of $\ddot{Z}_0$ on Z_0 is determined from the Euler-Lagrange equation (18) to be (see in [16]):

$$\ddot{Z}_0 = \frac{1}{(n + 1)Z_0}. \tag{56}$$

This function is plotted in Fig. 4 for comparison with the case of stiff branes.

Few features can be observed in the graph:

(i) There is a finite gap $0 < Z_0 \lesssim 1$ in which the solution for the embedded stiff brane does not exist at all. Hence the singular point is resolved for the subcritical branch. This is a characteristic feature of first order phase transitions.

(ii) $\ddot{Z}_0$ is bounded, unlike DNG branes [Eq. (56)] for which $\ddot{Z}_0$ is unbounded

(iii) For $1 \lesssim Z_0 \lesssim 1.25$ we see coexistence of two branches of solutions. For any Z_0 in this range there

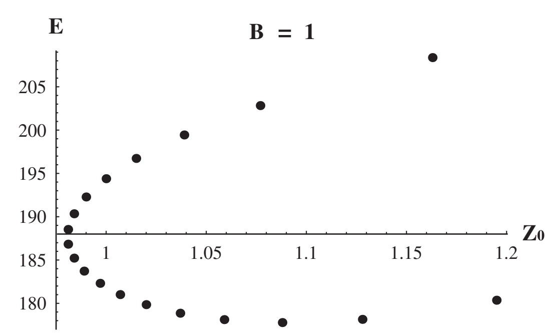

FIG. 5. The energy density integrated for $0 \leq R \leq 5$ as a function of Z_0 comparing two branches in the segment ($1 \lesssim Z_0 \lesssim 1.25$). Note that the minimal energy is obtained at the point which corresponds approximately to $\ddot{Z}_0 = 0$.

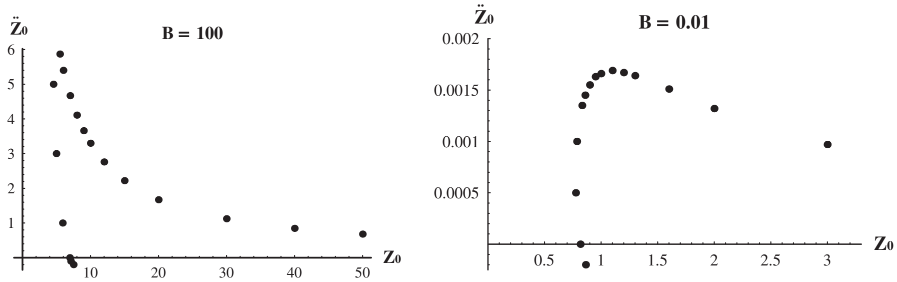

FIG. 6. $\ddot{Z}_0$ as a function of Z_0 for $n = 2$. $B = 100$, 0.01.

are two possible values of $\ddot{Z}_0$ and thus two possible configurations of the stiff brane. One can compare the energy [see (38)] of the two branches. A numerical comparison of the energies (Fig. 5) reveals that the branch with the lower values of $\ddot{Z}_0$ is energetically favored. The branch with higher energy corresponds to a local phase at maximum. This solution should be unstable and separates two stable phases in a first order phase transition.

(iv) There are solutions for stiff branes that satisfy the boundary conditions with negative $\ddot{Z}_0$. Such solutions exist only for $-0.025 \lesssim \ddot{Z}_0 \lesssim 0$.

(v) There exists an "end point" in the plot with minimal value of $\ddot{Z}_0$. For $\ddot{Z}_0$ less than this value a solution does not exist.

(vi) At large values of Z_0 we see that the effects of stiffness are negligible. The points of the stiff branes approach the DNG branes at large values of Z_0.

In order to check that the obtained results are robust we repeated the same calculations for various values of B. In all cases we found that the same qualitative behavior repeats itself: A finite gap in the existence of solutions for $0 < Z_0 \lesssim l$, two branches of solutions in a small neighborhood of $Z_0 \sim l$, etc.

For illustration we give in Fig. 6 two graphs for two values of B with four orders of magnitude difference ($B = 0.01$, 100).

In addition we checked for various dimensions $n = 4$, 5 and found the same qualitative behavior (see Fig. 7 for $n = 4$ as an example). Despite the fact that $n \leq 4$ is different from $n \geq 5$ since in the former the phase space behavior of the critical solution behaves as of a focal point and in the latter as a node (see [16]). This type of transition in the near-critical solutions has no influence on the neighborhood of the singular point $R = Z = 0$ where the stiffness terms are dominant.

It is surprising that when we repeated similar calculations for *supercritical* stiff branes with $n > 1$ we found that for the supercritical configurations there is no singularity resolution. The stiffness terms break the symmetry between the supercritical and subcritical brane-black hole systems. The supercritical solutions show no gap nor double-branch behavior. As an example let us look at the

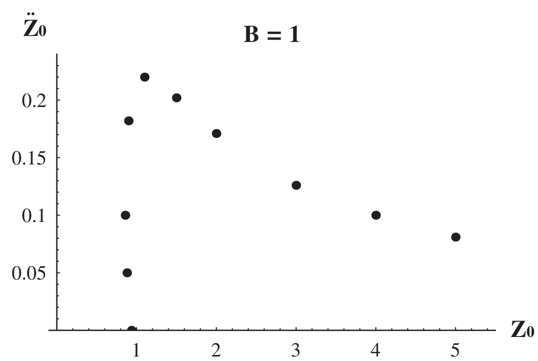

FIG. 7. $\ddot{Z}_0$ as a function of Z_0 for $n = 4$, $B = 1$.

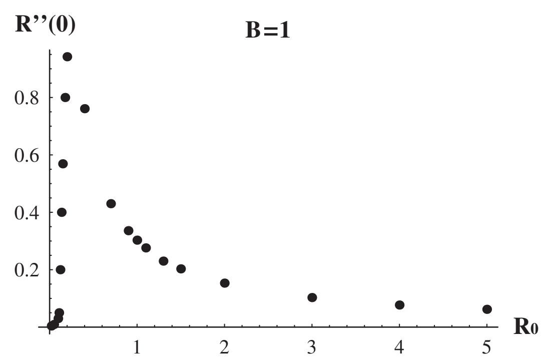

FIG. 8. $R''(0)$ as a function of R_0 (supercritical) for $n = 2$, $B = 1$.

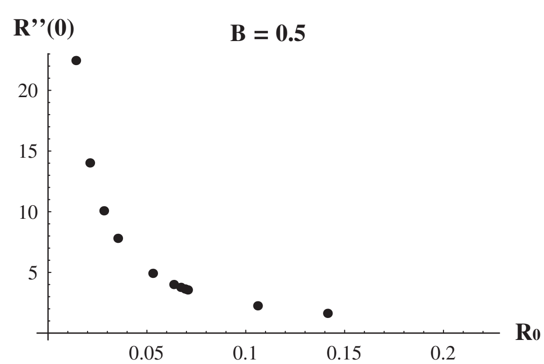

FIG. 9. $R''(0)$ as a function of R_0 (supercritical) for $n = 2$, $B = 0.5$.

supercritical configuration for $n = 2$. For $B < 0.906$ we did not find evidence for the existence of a solution in the vicinity of the point $R(0) = R''(0) = 0$. See Fig. 8 for $B = 1$ and Fig. 9 for $B = 0.5$. We stress that in both cases the curvature singularity still exists.

VIII. DISCUSSION

We observed that due to the stiffness corrections the singularity of the critical solution is resolved for $n = 1$ in a symmetric form (both in the subcritical and supercritical configurations) and for $n > 1$ only for subcritical configurations. We observed this resolution in the creation of a finite gap and a clear signature of a first order phase transition. This signature is observed in a typical hysteresis curve of coexistence of two phases—stable and unstable. For $n = 1$ we see a first order phase transition on both sides of the singularity (supercritical and subcritical configurations) when for $n > 1$ we see only half of this picture—a first order phase transition in the subcritical configuration.

We expect that a similar picture would emerge in merger transitions when higher derivative corrections are included. Inclusion of higher derivative corrections might cause the merger transition to become first order in nature and create a finite gap between the thin black string (“the waist”) and the caged black hole. This way the naked singularity and the violation of cosmic censorship hypothesis that appear in the classical approximation would be resolved. This might also be a natural way to resolve the apparent tension between the suggested scenario for the merger transition and the observation that such a pinch-off can occur only at infinite affine parameter along the horizon [37]. The resolution is the following. When the system approaches the Planckian scale, at a finite time, the first order phase transition takes the system to the second phase. Therefore with quantum corrections the “pathologies” of infinite affine parameter and naked singularity would be resolved.

The asymmetry that we found might be a result of the incompleteness of the truncated model that we used to describe the full effect of quantum corrections. It might be also a hint on an asymmetry which is generic in the topology change in general.

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