PHYSICAL REVIEW D **79,** 016005 (2009)

On the π and K as $q\bar{q}$ bound states and approximate Nambu-Goldstone bosons

Shmuel Nussinov[1,2] and Robert Shrock[3]

[1]*School of Physics and Astronomy, Tel Aviv University, Tel Aviv, Israel*
[2]*Schmid College of Science, Chapman University, Orange, California 92866, USA*
[3]*C.N. Yang Institute for Theoretical Physics, Stony Brook University, Stony Brook, New York 11794, USA*
(Received 20 November 2008; published 23 January 2009)

We reconsider the two different facets of π and K mesons as $q\bar{q}$ bound states and approximate Nambu-Goldstone bosons. We address several topics, including masses, mass splittings between π and ρ and between K and K^*, meson wave functions, charge radii, and the $K - \pi$ wave function overlap.

DOI: 10.1103/PhysRevD.79.016005 PACS numbers: 11.30.Rd, 12.38.−t, 12.39.Jh, 12.40.Yx

I. INTRODUCTION

Quantum chromodynamics (QCD) is most predictive in the perturbative, short-distance regime. Yet our understanding of long distance, nonperturbative properties of QCD keeps improving. Lattice QCD has been used to compute the hadronic spectrum and matrix elements for weak transitions, the $N_c \to \infty$ limit and new variants thereof have been widely applied, and heavy-quark symmetry has helped to elucidate heavy-heavy $Q\bar{Q}$ and heavy-light $Q\bar{q}$ hadrons. The $SU(2)_L \times SU(2)_R$ global chiral symmetry and the spontaneous breaking of this symmetry to the vectorial, isospin $SU(2)_V$, which are mandatory in QCD [1–13], underlie isospin symmetry, the fact that the pions are light almost Nambu-Goldstone bosons (NGB's) [14,15], and the usefulness of chiral perturbation theory. Another specific advance was the resolution of the $U(1)_A$ problem by instantons [16], thus explaining why the η' is not light.

The remarkable success of the nonrelativistic ("naive") quark model (NQM) treating the u, d, and s quarks as nonrelativistic, spin-$1/2$ fermions is of great interest [17–19]. This model dictated the low-lying flavor SU(3) multiplets and many aspects of their electroweak interactions. Spontaneous chiral symmetry breaking (SχSB) generates dynamical, constituent masses of order the QCD scale $\Lambda_{\rm QCD} \simeq 250$ MeV for the light, almost massless u and d quarks and increments, by this amount, the hard Lagrangian mass of the s quark, $m_s \simeq 120$ MeV to produce a constituent s quark mass. Our results are not sensitively dependent upon the values of the constituent quark masses; we use the values [20,21]

$$M_u = M_d \equiv M_{ud} \simeq 340 \text{ MeV} \tag{1.1}$$

and

$$M_s \simeq 470 \text{ MeV}. \tag{1.2}$$

The dynamical mass generation of the constituent quark masses in QCD can, for example, be shown via analysis of the Dyson-Schwinger equation for the quark [22]. One may roughly characterize the range of the QCD interactions responsible for the $\langle\bar{q}q\rangle$ condensate as r_0. The effective size of a constituent quark, consisting of the bare valence quark and its entourage of gluons and $q\bar{q}$ pairs, is expected to be of order r_0. A small r_0, less than 0.2 fm, say, then yields constituent quarks of that size moving inside a hadron of size approximately 1 fm under the influence of a smooth confining potential, making the NQM plausibly justified.

Some of the mechanisms suggested for generating spontaneous chiral symmetry breaking—in particular, Nambu-Jona Lasino-type (NJL) models [14,23], and those involving instantons [16,24,25]—can have relatively short range. However, in the approach of Casher [1] and Banks and Casher [2], SχSB results from confinement. In this approach there is no separation of scales between the constituent quark size and hadron sizes.

The almost massless Nambu-Goldstone pion—the other consequence of spontaneous chiral symmetry breaking—generates arguably the single most serious difficulty for the NQM, namely, what has been called the "ρ-π puzzle" (e.g., [26]) This is the challenge of simultaneously explaining the pion as a $q\bar{q}$ bound state and an approximate NGB, and relating it to the ρ. There is an analogous, although less severe, problem for the K and K^*. Although this difficulty is most clearly manifest within the NQM, it transcends this nonrelativistic model. Thus, also the original Massachusetts Institute of Technology (MIT) bag model with relativistic quarks confined in a spherical cavity requires large hyperfine interactions to try to split the masses of the π and ρ and, like the NQM, fails to explain the almost massless pions [27]. To get a sufficiently light pion in the MIT bag model, it is necessary to argue that substantial contributions due to the fluctuations in the center-of-mass should be subtracted [28] (see also [29,30]).

In this paper we shall revisit the problem of understanding the dual nature of the pion (and kaon) as $q\bar{q}$ bound states and as collective, almost massless Nambu-Goldstone bosons. We suggest that this can be understood because the strong chromomagnetic hyperfine interaction that splits the mass of the ρ and π keeps the valence q and $\bar{q}$ in the pion rather close to each other. The dynamics of this close $q\bar{q}$ pair can be modeled by random walk methods, and we show that these can reproduce the successful relation that

$m_\pi^2 \propto m_q$ and can explain relations that have been challenging to understand, such as the dependence of the pion wave function at the origin on the pion mass. The organization of the paper is as follows. In Sec. II we elaborate on the ρ-π puzzle in the context of the nonrelativistic quark model. In Sec. III we present a heuristic picture that gives some new insight into this puzzle by helping to explain the π as both a $q\bar{q}$ bound state and an approximate Nambu-Goldstone boson. In Sec. IV we consider the $K - \pi$ transition form factor, $f_+(q^2)$. Its deviation from unity at vanishing momentum transfer is governed by the Ademollo-Gatto theorem [31], and analogous deviations from heavy-quark universality are derived in a general context. In Sec. V we comment on the systematics of quark mass differences inferred from $(Q\bar{s})$ and $(Q\bar{q})$ mesons, where $q = u$ or d. This has been elaborated independently by Karliner and Lipkin [32]. Still, we feel that it is of sufficient interest to present it here from our point of view.

II. THE ρ-π PUZZLE

In this section we briefly review the ρ-π puzzle. The reader who is already familiar with this may wish to skim this section or skip it and proceed immediately to Sec. III where we present our new physical picture of the pion.

Several pieces of data suggest that the pion, which is lighter than the ρ by approximately 640 MeV, is otherwise rather similar to the ρ, as expected in the nonrelativistic quark model for the ${}^1S_0$ pseudoscalar partner of the ${}^3S_1$ vector meson. These data can be summarized as follows:

(i) One measure of the size of a hadron is provided by the magnitude of the charge radius. The charge radii of the π^+ and K^+ are given by [20]

$$(\langle r^2\rangle_{\pi^+})^{1/2} = 0.672 \pm 0.008 \text{ fm} \tag{2.1}$$

and

$$(\langle r^2\rangle_{K^+})^{1/2} = 0.560 \pm 0.031 \text{ fm}. \tag{2.2}$$

These are rather similar and, indeed, are not very different from the charge radius of the proton,

$$(\langle r^2\rangle_{p})^{1/2} = 0.875 \pm 0.007 \text{ fm}. \tag{2.3}$$

(ii) Similar π and ρ sizes and a somewhat smaller kaon are suggested by the total cross sections on protons at a typical laboratory energy above the resonance region. Averaged over meson charges, at a lab energy $E_{\rm lab} = 10$ GeV, these are [20]

$$\sigma_{\pi N} \simeq 26.5 \text{ mb} \tag{2.4}$$

and

$$\sigma_{KN} \simeq 21 \text{ mb}. \tag{2.5}$$

Although one obviously does not have beams of ρ mesons available experimentally, owing to the very short lifetime of the ρ, it is possible to estimate what the cross section for $\rho - N$ scattering would be if one did have such beams. Diffractive ρ production data and vector meson dominance yield the estimate [33]

$$\sigma_{\rho N} \simeq 27 \pm 2 \text{ mb}. \tag{2.6}$$

This cross section is essentially the same, to within the experimental and theoretical uncertainties, as $\sigma_{\pi N}$ at the same energy [and these are approximately equal to $(2/3)\sigma_{NN}$ at this energy].

(iii) The nonrelativistic quark model was able to fit the measured values of the proton and neutron magnetic moments $\mu_p = 2.793\mu_N$ and $\mu_n = -1.913\mu_N$ [where $\mu_N = e/(2m_p)$] and the ratio $\mu_p/\mu_n \simeq -3/2$, as well as the values of the hyperon magnetic moments, in terms of Dirac magnetic moments of constituent quarks. It also explained decays such as $\omega \to \pi^0 + \gamma$ and $\rho \to \pi\gamma$ as quark spin flip ${}^3S_1 \to {}^1S_0$ electromagnetic transitions. The optimal overlap of the ρ and π wave functions implied by this confirms the similarity of the vector and pseudoscalar meson ground-state wave functions.

(iv) The amplitudes for semileptonic $K_{\ell 3}$ decays involve the vector part of the weak $|\Delta S| = 1$ current and contain the product of V_{us} with the $f_+(q^2)$ transition form factor. In the limit of SU(3) flavor symmetry $m_u = m_d = m_s$, so that $m_K = m_\pi$, the conserved vector current (CVC) property implies that $f_+(0) = 1$. Experimentally, $f_+(q^2 = 0)$ is very close to unity. The success of these fits implies almost optimal overlap between the wave functions of the pion and kaon.

In the nonrelativistic quark model, one can rewrite the two-body quark-antiquark Hamiltonian as an effective one-body problem with the usual reduced mass

$$\mu_{i\bar{j}} = \frac{M_i M_j}{M_i + M_j} \tag{2.7}$$

for the $q_i\bar{q}_j$ pseudoscalar meson. (The context will make clear where the notation μ refers to a magnetic moment and where it refers to a reduced mass.) The corresponding bound-state wave function is denoted $\psi_\pi(\mathbf{r})$, $\psi_K(\mathbf{r})$, etc., where $\mathbf{r} = \mathbf{r}_{q_i} - \mathbf{r}_{\bar{q}_j}$ is the relative coordinate in the bound state. With the above-mentioned typical values $M_{ud} = 340$ MeV and $M_s = 470$ MeV, one has

$$\mu_\pi = \frac{M_{ud}}{2} = 170 \text{ MeV} \tag{2.8}$$

and

$$\mu_K = \frac{M_{ud} M_s}{M_{ud} + M_s} = 200 \text{ MeV}, \tag{2.9}$$

where it is understood that the choices for the input values of the constituent quark masses in these formulas depend somewhat on the method that one uses to infer their values [21]. In the nonrelativistic quark model, since a bound state involving a larger effective reduced mass is expected to be smaller, one has some understanding of the fact that $\sqrt{\langle r^2\rangle_{K^+}} \simeq 0.83\sqrt{\langle r^2\rangle_{\pi^+}}$. The deviation of $f_+(0)$ from unity is also in accord with this difference of reduced masses.

For heavy $Q\bar{Q}$ quarkonium systems one can use the nonrelativistic Schrödinger equation to describe a number of properties of the bound states [18,19]. This use is justified by the fact that in the $c\bar{c}$ and $b\bar{b}$ systems the respective heavy-quark masses $m_c \simeq 1.3$ GeV and $m_b \simeq 4.3$ GeV are large compared with $\Lambda_{\rm QCD}$, and the asymptotic freedom of QCD means that α_s gets small for such mass scales. The hyperfine splitting in these $\bar{Q}Q$ systems, being proportional to α_s/m_Q, is small.

For light $\bar{q}q$ systems, however, the situation is different. Let us denote

$$\langle 0|J_\lambda^j|\pi^k(p)\rangle = if_\pi \delta^{jk} p_\lambda, \tag{2.10}$$

where j, k are isospin indices and J_λ is the weak charged current, so that $\langle 0|J_\lambda^{1-i2}|\pi^+(p)\rangle = if_\pi p_\lambda$. Similarly, $\langle 0|J_\lambda^{4-i5}|K^+(p)\rangle = if_K p_\lambda$. Experimentally [20],

$$f_\pi = 92.4 \text{ MeV}, \qquad f_K = 113 \text{ MeV}. \tag{2.11}$$

Analogous constants enter in the leptonic decays of the vector mesons. The rate for the decay $M_{i\bar{j}}^+ \to \ell^+\nu_\ell$, where $\ell = \mu$ or e, is

$$\Gamma(M_{i\bar{j}}^+ \to \ell^+\nu_\ell) = \frac{|V_{ij}|^2 G_F^2 f_{M_{i\bar{j}}}^2 m_{M_{i\bar{j}}} m_\ell^2}{4\pi}\left[1 - \frac{m_\ell^2}{m_{M_{i\bar{j}}}^2}\right]^2, \tag{2.12}$$

where here $V_{ij} = V_{ud}$ for $M_{u\bar{d}}^+ = \pi^+$ and V_{us} for $M_{u\bar{s}}^+ = K^+$. Since $M_{i\bar{j}}$ is a $q_i\bar{q}_j$ bound state, this rate is proportional to $|\psi(0)|^2$. With the normalization of $\psi(r)$ in the nonrelativistic quark model determined by the condition $\int d^3r|\psi(r)|^2 = 1$, it follows that for a given $^1S_0$ or $^3S_1$ $q_i\bar{q}_j$ meson M,

$$f_M \propto \frac{|\psi_M(0)|}{m_M^{1/2}}. \tag{2.13}$$

Hence,

$$\frac{|\psi_K(0)|}{|\psi_\pi(0)|} = \frac{f_K}{f_\pi}\left(\frac{m_K}{m_\pi}\right)^{1/2} = 2.3. \tag{2.14}$$

The difficulty of deriving this ratio from the NQM was noted early on as the van Royen–Weisskopf "paradox" [34].

Conventionally, in the context of the quark model, the ρ-π mass difference was explained by means of a very strong chromomagnetic, i.e., color hyperfine (chf) splitting between these particles. The similar, although smaller, mass difference between the K^* and K was also explained by this color hyperfine interaction. Taking account of color, the Hamiltonian for the color hyperfine (chromomagnetic) interaction has the form

$$H_{\rm chf} = \frac{v_{\rm chf}(r)}{M_i M_j}(\vec{\lambda}_i \cdot \vec{\lambda}_j)(\vec{\sigma}_i \cdot \vec{\sigma}_j), \tag{2.15}$$

where the function $v_{\rm chf}(r)$ involves the overlap of the interacting constituent (anti)quarks. Recall that the product $\vec{\sigma}_i \cdot \vec{\sigma}_j$ is equal to 1 and -3 times the identity matrix $\mathbb{I}_{2\times 2}$ when the q_i and $\bar{q}_j$ spins are coupled to $S = 1$ and $S = 0$, respectively. Similarly, the product of $SU(3)_c$ color matrices $\vec{\lambda}_i \cdot \vec{\lambda}_j$ is equal to $-16/3$ and $-8/3$ times $\mathbb{I}_{3\times 3}$ if the colors are coupled as $3 \times \bar{3} \to 1$ (meson) and $3 \times 3 \to \bar{3}$ (baryon), respectively. The resultant 1:(-3) ratio of mass shifts in $S = 1$ and $S = 0$ $q\bar{q}$ mesons or quark pairs in baryons and the 1:(1/2) ratio of the color factor for $q\bar{q}$ mesons versus qq interactions in baryons yield good fits to meson and baryon masses. The dependence of $H_{\rm chf}$ on $1/(M_iM_j)$ is also important for this successful fit. In the NQM as applied here, $v_{\rm chf}(r) \propto \delta^3(\mathbf{r})$, so that the color hyperfine shifts evaluated to first order in $H_{\rm chf}$ are proportional to $|\psi(0)|^2$ (where the subscript M on ψ is suppressed in the notation) [17]. This is analogous to the hyperfine splitting in hydrogen, which is also proportional to the square $|\psi(0)|^2$ of the hydrogenic wave function at the origin. We focus here on the $^1S_0$ and $^3S_1$ isovector mesons, i.e., the π and ρ, absorb the color factor into the prefactor and thus write, for the energy due to $H_{\rm chf}$,

$$E_{\rm hcf} = \frac{A\langle\vec{\sigma}_i \cdot \vec{\sigma}_j\rangle|\psi(0)|^2}{M_i M_j}. \tag{2.16}$$

The meson mass to zeroth order in $H_{\rm chf}$ is denoted m_0. Then the physical masses are

$$m_\rho = m_0 + \frac{A|\psi(0)|^2}{M_{ud}^2} \tag{2.17}$$

and

$$m_\pi = m_0 - \frac{3A|\psi(0)|^2}{M_{ud}^2}. \tag{2.18}$$

Equivalently,

$$m_0 = \frac{3m_\rho + m_\pi}{4} \tag{2.19}$$

and

$$A = \frac{(m_\rho - m_\pi)M_{ud}^2}{4|\psi(0)|^2}. \tag{2.20}$$

Numerically, $m_0 = 620$ MeV and the color hyperfine splitting is

$$(\Delta E)_{\rm chf} = m_\rho - m_\pi = \frac{4A|\psi(0)|^2}{M_{ud}^2} = 640 \text{ MeV}. \tag{2.21}$$

The color hyperfine interaction also plays an important role in splitting the K and π, since their mass difference, $m_K - m_\pi \simeq 360$ MeV, exceeds, by about a factor of 3, the difference in current-quark masses, $m_s - m_d \simeq 120$ MeV. In the NQM, this is attributed to the fact that the color hyperfine interaction energy $-a/M_{ud}^2$ for the π (with $a > 0$) is negative and larger in magnitude than the corresponding energy $-a/(M_{ud}M_s)$ for the K, since $M_s > M_{ud}$.

The large size of the splitting (2.21) shows that the color hyperfine interaction is not a small perturbation on the zeroth-order Hamiltonian value, m_0. Furthermore, the strongly attractive, short-range color hyperfine interaction has the effect of contracting the pion to a size substantially smaller than the size indicated by experimental data on the charge radius and $\sigma_{\pi N}$ scattering cross section. Moreover, since $v_{\rm chf}(r) \propto \delta^3(\mathbf{r})$, which is clearly sensitive to short-distance interactions between quarks, there is a problem of internal consistency when one uses a color hyperfine interaction in the context of the nonrelativistic quark model, since at short distances, because of asymptotic freedom, the light quarks behave in a relativistic quasifree manner with their small, current-quark masses, not as nonrelativistic, massive, constituent quarks.

The fact that $M_{ud}/M_s \simeq 0.7$ has two countervailing effects on the relative charge radii of the π^+ and K^+. First, since the reduced mass μ_K is slightly greater than μ_π [cf. Eqs. (2.8) and (2.9)], it is plausible that the corresponding meson could be somewhat smaller. Yet the very important attractive color hyperfine interaction should make $\sqrt{\langle r^2\rangle_{K^+}}$ larger than $\sqrt{\langle r^2\rangle_{\pi^+}}$.

We proceed to discuss some properties of the color hyperfine interaction further. Since an attractive interaction involving a $\delta^3(\mathbf{r})$ function potential is inconsistent in the nonrelativistic constituent quark model, we model $v_{\rm chf}(r)$ as a spherical square well of depth V_0 and radius r_0. The dimensionless quantity fixing the number of bound states is the ratio of the strength V_0 of a potential to the kinetic energy of a particle bound by this potential in a region of size r_0, namely $E_{\rm kin} \simeq p^2/(2\mu) = \pi^2/(2\mu r_0^2)$ ($E_{\rm kin} = p = \pi/r_0$, relativistically). The ratio $V_0/E_{\rm kin} = 2\mu V_0 r_0^2/\pi^2$. Since μ is determined by Eq. (2.7), we thus fix the product $V_0 r_0^2$. One knows from general QCD theory that at distances $r \ll 1/\Lambda_{\rm QCD}$, the static quark potential has the Coulombic form

$$V_{q\bar{q}} = \frac{C_{2f}\alpha_s}{r} \quad \text{for } r \ll \frac{1}{\Lambda_{\rm QCD}}, \tag{2.22}$$

where $C_{2f} = 4/3$ is the quadratic Casimir invariant for the fundamental representation of $SU(3)_c$ and the logarithmic dependence of the running α_s on r is left implicit. At distances of order $1/\Lambda_{\rm QCD}$, $V_{q\bar{q}}$ has a linear form resulting from the chromoelectric flux tube joining the q and $\bar{q}$,

$$V_{q\bar{q}} = \sigma r \quad \text{for } r \sim \frac{1}{\Lambda_{\rm QCD}}, \tag{2.23}$$

where $\sigma = 1/(2\pi\alpha') \simeq (400\ \text{MeV})^2$ is the string tension with α' the Regge slope. To illustrate the nature of the ρ-π puzzle, let us consider an infinite square-well potential, which provides a simple model of confinement. The (unit-normalized) ground-state wave function is

$$\psi(r) = \left(\frac{\pi}{2r_0^3}\right)^{1/2}\left(\frac{\sin pr}{pr}\right), \tag{2.24}$$

where $p = \pi/r_0$. With this potential, one has

$$\langle r^2\rangle = \int r^2|\psi(r)|^2 d^3r = \left(\frac{1}{3} - \frac{1}{2\pi^2}\right) r_0^2, \tag{2.25}$$

so that $\sqrt{\langle r^2\rangle} = 0.532 r_0$. The measured value of $\langle r^2\rangle_\pi$ then determines $r_0 = 1.26$ fm. Denoting $\langle r^2\rangle \equiv d^2$, one can write $|\psi(0)|^2 = c/d^3$ with c a constant. In this model,

$$c = \frac{\pi}{2}\left(\frac{1}{3} - \frac{1}{2\pi^2}\right)^{3/2} = 0.236. \tag{2.26}$$

Substituting the value of $|\psi(0)|^2$ into Eq. (2.21), we find

$$(\Delta E)_{\rm chf} = \frac{2\pi A}{M_{ud}^2 r_0^3} \tag{2.27}$$

so that

$$A = \frac{(m_\rho - m_\pi)M_{ud}^2 r_0^3}{2\pi} = 3.1. \tag{2.28}$$

Thus, both the large shift in Eq. (2.21) and the rather large value of the coefficient A show that the NQM treatment of the very strong, short-range hyperfine interaction as a perturbation is not really justified. As could be expected on general grounds, such a strong short-range interaction has the effect of producing a pion wave function that is smaller in spatial extent than is experimentally observed. In effect, the pion—which, by definition, is the ground state in the attractive ${}^1S_0$ pseudoscalar channel—is "swallowed." i.e., squeezed into a contracted state of radius much smaller than that of the ρ. The wave function for the ρ itself slightly expands relative the original common unperturbed π and ρ wave functions, due to the repulsive color hyperfine interaction in the ${}^3S_1$ vector channel (which is 1/3 as strong as the attraction in the ${}^1S_0$ channel). The NQM puzzle of a very light pion thus extends also to its expected much smaller size. In general, any extra, attractive, potential that binds the pion more strongly than the ρ yields a π that is smaller than the ρ. The only way to maintain a common shape for the ρ and π wave functions in a nonrelativistic potential model framework is to have $v_{\rm chf}(\mathbf{r})$ constant as a function of r, which is very different from the NQM's form $v_{\rm chf} \propto \delta^3(\mathbf{r})$.

III. A PHYSICAL PICTURE OF APPROXIMATE NAMBU-GOLDSTONE BOSONS

NJL-type models do succeed in producing a massless or nearly massless pion in a bound-state picture, as was shown first via a solution of the Bethe-Salpeter equation in the ${}^1S_0$ channel in the original work by Nambu and Jona-Lasinio [14] (with an appropriate reinterpretation of the four-fermion operator as involving quarks rather than nucleons in a modern context [23]). However, the coupling of the four-fermion operator is not calculated directly from the underlying QCD theory. Furthermore, this four-fermion operator posits that the spontaneous chiral symmetry breaking is a contact interaction, and thus does not directly include the physically appealing mechanism for SχSB as being a consequence of helicity reversal due to confinement [1].

An important insight for understanding the pion as a $q\bar{q}$ bound state and also an approximate Nambu-Goldstone boson has been the argument by Brodsky and Lepage that the physical pion state contains not just the valence $|q\bar{q}\rangle$ state, but large contributions from higher Fock states such as $|q\bar{q} + ng\rangle$, $|q\bar{q}q\bar{q}\rangle$, $|q\bar{q}q\bar{q} + ng\rangle$ (g = gluon), etc. [35]. These higher Fock space states can account for much of the size of the physical pion. This view is in accord with the Goldstone phenomenon in condensed matter physics, where a Goldstone excitation is a collective state (e.g., a quantized spin wave or magnon in the case of a ferromagnet). This insight has been deepened with further work using the light front formalism [36]. Recently, Brodsky and de Téramond have used AdS/QCD methods to calculate hadron masses including m_π, and also f_π, $\langle r^2\rangle_{\pi^+}$, and the pion electromagnetic form factor $F_{\pi^+}(q^2)$ in the spacelike and timelike regions [37].

Another approach is provided by approximate solutions to Dyson-Schwinger and Bethe-Salpeter equations. In addition to phenomenological four-fermion NJL-type kernels for the Bethe-Salpeter equation [14,23], these have involved quark-gluon interactions in an effort to model QCD [22,38]. These equations capture some of the relevant physics, although they do not directly include effects of confinement or nonperturbative effects due to instantons. Confinement means that both quarks and gluons have maximum wavelengths, i.e., minimum bound-state momenta, which affect chiral symmetry breaking [39]. (Solutions of Dyson-Schwinger and Bethe-Salpeter equations have also been used to investigate the dependence of the hadron mass spectrum on the number of light flavors in a general asymptotically free, vectorial non-Abelian gauge theory [40].)

Thus, there has been continual progress in understanding the pion (and kaon) as both a $q\bar{q}$ bound state and an approximate Nambu-Goldstone boson. Here we would like to present a rather simple heuristic picture of this physics which, we believe, contributes further to this progress. For technical simplicity, we restrict ourselves to the large-N_c limit, in which quark loops have a negligibly small effect. In this limit a simple proof that spontaneous chiral symmetry breaking occurs was constructed by showing that the 't Hooft anomaly matching conditions [3] for massless u and d quarks must be realized in the physical spectrum via a massless Nambu-Goldstone pion rather than massless nucleons [4]. To motivate our picture, we note several elements that were missing in the nonrelativistic quark model approach to the ρ-π puzzle:

(i) We need a natural mechanism for producing a sufficiently strong $q\bar{q}$ interaction in the ${}^1S_0$ channel to reduce the mass of the bound state so that, up to electroweak corrections, it vanishes in the limit of zero current-quark masses.

(ii) We would like the same physical picture to explain how the pion is both a $q\bar{q}$ bound state and an approximate Nambu-Goldstone boson whose masslessness follows from the spontaneous breaking of the $SU(2)_L \times SU(2)_R$ global chiral symmetry down to the diagonal, vectorial $SU(2)_V$ and whose interactions involve derivative couplings, which vanish as $q_\mu \to 0$. As part of this, it is desirable that the picture should yield the Gell-Mann-Oakes-Renner (GMOR) formula for the pion (and kaon) mass [41,42].

(iii) We would like to resolve the van Royen–Weisskopf paradox [34] and explain how the quark wave function of the pion at the origin, $\psi_\pi(0)$, can be $\propto m_\pi^{1/2}$ and thus be consistent with a finite value of f_π in the chiral limit where $m_\pi \to 0$.

(iv) Finally, we would like to understand how an approximate Nambu-Goldstone boson such as the pion, which appears quite different from other hadrons, can have, as indicated by experiment, roughly the same size as these other hadrons.

We next present our new picture and show how it addresses these questions. As is well known, if a quark has zero current-quark mass, the covariant derivative $\bar{q}\not{D}q$ in the QCD Lagrangian, preserves chirality. A dynamical, constituent quark mass can be generated via an approximate solution of the Dyson-Schwinger equation for the quark propagator. In the one-gluon exchange approximation, one finds a nonzero solution for the effective quark mass M if $C_{2f}\alpha_s = (4/3)\alpha_s \gtrsim \mathcal{O}(1)$. In this framework, the dynamical quark mass M is thus the consequence of a sufficiently strong quark-gluon coupling at the relevant scale, $\alpha_s(\mu)$ at $\mu \sim \Lambda_{\rm QCD}$. This dynamical quark mass can also be seen to result from the helicity reversal due to confinement [1]. These two approaches can also be seen to connect with the NJL-type analysis, with $M \simeq G\langle\bar{q}q\rangle \simeq \langle\bar{q}q\rangle/(2\pi f_\pi^2)$, where G denotes the NJL four-fermion coupling. If one restricts oneself to a quenched approximation in which there are no quark loops, then the presence of higher Fock space states with $q\bar{q}$ pairs inside a meson with its valence constituent quarks can be regarded as being due to a kind of *zitterbewegung* motion of these valence quarks. Light-quark mesons which are not approximate NGB's, such as the ρ, can be modeled satisfactorily as being composed simply of a constituent quark and constituent antiquark. The effective size of this constituent $q\bar{q}$ bound state is of order $1/M_{ud}$. The application of the nonrelativistic constituent quark model to such mesons is reasonable, with the constituent quarks moving in an approximately nonrelativistic fashion under the assumed confining potential, with a weakly repulsive hyperfine interaction for the ρ meson.

In the pion, however, the interaction at all scales is (strongly) attractive. This is manifested in the Euclidean pseudoscalar correlator. We recall that several general properties of hadrons have been understood on the basis of Euclidean correlation function inequalities [8–12]. Let us consider the Euclidean correlation function for a pseudoscalar $q\bar{q}$ bound state,

$$P(x, y) = \langle[\bar{u}(x)\gamma_5 d(x)][\bar{d}(y)\gamma_5 u(y)]\rangle. \tag{3.1}$$

Performing the Gaussian fermionic (Grassmann) integration in the path integral yields

$$P(x, y) = \int d\mu(A_\nu(x))S(A)^\dagger(x, y)S(A)(x, y), \tag{3.2}$$

where $d\mu(A_\nu(x))$ is the positive measure of the Euclidean path integral, including the e^{-S_G} factor from the gauge part of the action, where

$$S_G = \frac{1}{4}G_{\mu\nu}G^{\mu\nu}, \tag{3.3}$$

and the fermionic determinant is absent in the quenched approximation used here. In Eq. (3.2), $S(A)(x, y)$ is the propagator of the quark (a light u or d quark) moving from the initial position y to the final position x, in the presence of the background gauge field $A_\mu(x) \equiv \sum_a T_a A^a_\nu(x)$. $S(A)^\dagger(x, y)$ denotes the Hermitian adjoint of $S(A)(x, y)$ in color and Dirac space. We use the relation

$$\gamma_5 S(A)(y, x)\gamma_5 = S(A)^\dagger(x, y). \tag{3.4}$$

This property is unique to γ_5 and is not shared by any of the other 16 Dirac matrices. It ensures that the path integrand is positive for all field configurations, making $P(x, y)$ larger than all other Euclidean (scalar, vector, axial-vector, and tensor) correlators. Asymptotically, when $|x - y| \to \infty$, any correlator $C(x, y)$ behaves, up to a power-law prefactor, as

$$C(x, y) \simeq \exp(-m_0|x - y|), \tag{3.5}$$

where m_0 is the mass of the lightest physical state with the quantum numbers of the correlator considered. This, together with the inequality

$$P(x, y) \sim \exp(-m_\pi|x - y|) \geq \text{any } C(x, y), \tag{3.6}$$

guarantees that the pion is, indeed, the lightest meson. Furthermore, the positivity of $P(x, y)$ for any $|x - y|$ and the fact that $S(x, y)$ is a monotonically decreasing function of $|x - y|$ implies that the effective quark-antiquark potential in the pion (to the extent that this nonrelativistic language is appropriate) is attractive at all relative distances.

As we noted in the previous section, in the nonrelativistic quark model a very strong hyperfine interaction between the quark and antiquark in the pion is needed in order to reduce its mass nearly to zero, and such an interaction tends to produce a wave function for the valence $q\bar{q}$ in the pion that is restricted to a very small spatial extent (almost collapsed). Following this lead, we suggest that , while the spacetime (or Euclidean) picture of a $\bar{q}q$ vector meson is two wool-ball-like single strands of valence quark and antiquark lines, the pion is a double strand, namely, closer valence $\bar{q}$ and q world lines whose motion forms a single wool-ball-like configuration. According to this picture, in the pion, but not in the ρ etc., the valence $\bar{q}$ and q lines with collinear momenta track each other at a distance that is shorter than 1 fm. This is similar to NJL-type models, in which, by construction, the important interaction is of short range. Thus, in our picture the pion qualitatively differs from the ρ as a nearly Nambu-Goldstone particle should and, at the same time, can be consistently considered as a $\bar{q}_i q_j$ state. Here and below, analogous comments, with obvious changes for the heavier m_s, apply for the K and its comparison with the K^*.

It is well known from discussions of the chiral anomaly [4–9] that a massless collinear quark and antiquark of opposite helicity, corresponding to the bilinear operator product $\bar{\psi}\gamma_5\psi$, can mimic the pole of a massless pseudoscalar particle and replace the latter in the calculation of the anomaly. Here we suggest that such a configuration, including the effect of spontaneous chiral symmetry breaking, can represent the massless pion, explain the puzzling strong color hyperfine interaction between the q_i and $\bar{q}_j$, and can account for its behavior as a light, approximate Nambu-Goldstone boson. In general, one would expect from the basic quantum mechanical relation $(\Delta p_i)(\Delta r_i) \gtrsim \hbar$ that restricting the q and $\bar{q}$ to a small interval along some axis $\hat{x}_i$ would entail large momenta along this axis. However, in the presence of an appropriate gluonic field configuration, the gauge-covariant momentum $p_\mu \mathbb{I} - gT^a A^a_\mu$ can vanish.

We address the questions posed above, starting with (i). There are two sources of explicit chiral symmetry breaking, namely, finite quark masses and the presence of nonzero electroweak interactions. For our present discussion we shall imagine that, unless otherwise indicated, electroweak interactions are turned off. Then a nonzero pion mass is induced via the explicit chiral symmetry breaking term $m_u\bar{d}d + m_u\bar{u}u$ in the QCD Lagrangian. To see this in our picture, we consider the spacetime evolution of the q_i and $\bar{q}_j$ in a pion, after a Euclidean Wick rotation. In the QCD context, the q_i and $\bar{q}_j$ are connected by a chromoelectric flux tube, and their positions fluctuate within length scales of order $1/\Lambda_{\rm QCD} \simeq 1$ fm. We consider a model in which in the pion, but not in the ρ or other vector mesons, the valence q and $\bar{q}$ track each other at a distance h shorter than $1/\Lambda_{\rm QCD}$, at all times. The constituent quark masses are then less relevant to the dynamics, being gradually replaced, as h gets smaller, by their current-quark masses. The point here is that constituent quark masses are consequences of spontaneous chiral symmetry breaking, which disappears at short distances (large momenta). In QCD, the scale-dependent dynamically generated constituent quark mass decays, as a function of Euclidean momenta p, like

$$M_q \sim \frac{\langle \bar{q}q \rangle}{p^2} \tag{3.7}$$

up to logs [22], where

$$\langle \bar{q}q \rangle \sim 4\pi f_\pi^3 \sim \Lambda_{\rm QCD}^3. \tag{3.8}$$

More generally,

$$M_q \sim \Lambda_{\rm QCD}\left(\frac{\Lambda_{\rm QCD}}{p}\right)^{2-\gamma}, \tag{3.9}$$

where γ denotes the anomalous dimension of the $\bar{q}q$ operator; here we use the property that γ is a power series in the running coupling α_s, and α_s approaches zero at short distances because of the asymptotic freedom of QCD. In our picture it is this "melting away" of the constituent quark masses at short distance which provides, in the NQM language, the very strong hyperfine interactions in the pion.

Next, as an answer to question (ii), we would like to show how the GMOR relation for the pion and other pseudoscalar meson masses (aside from the η'), which embodies the Nambu-Goldstone nature of these pseudoscalar mesons, is naturally expected in our picture. Let the total Euclidean length $R = |x| = \sqrt{\tau^2 + \mathbf{r}^2}$, where $\tau = it$, be the net Euclidean distance traveled by the $q\bar{q}$ double line describing the valence quark-antiquark in the pion. The double line describes a random walk with n straight sections of total length L. When probed at distances that are short compared with $1/\Lambda_{\rm QCD}$, the quark masses are the hard, current-quark masses, $m_u \simeq 4$ MeV and $m_d \simeq 8$ MeV. When $(m_u + m_d)|x| \gtrsim 1$, the quark propagation involves the suppression factor $e^{-(m_u+m_d)|x|}$. Hence, a characteristic length describing this propagation is

$$L \propto \frac{1}{m_u + m_d}. \tag{3.10}$$

Now the end-to-end distance R for a random walk with step sizes d (in any dimension) is given by [43]

$$R^2 \sim d^2 n. \tag{3.11}$$

On average, if the total length of the n-step walk is L and the step length is d, then

$$d \simeq \frac{L}{n}. \tag{3.12}$$

Hence,

$$R^2 \simeq Ld. \tag{3.13}$$

Then the basic correlation function relation, Eq. (3.5), implies that

$$m_\pi \simeq \frac{1}{R} \tag{3.14}$$

and hence that

$$m_\pi^2 \simeq \frac{m_u + m_d}{d}. \tag{3.15}$$

Since the step size d is connected, via the helicity reversal process, to the underlying confinement and dynamical breaking of chiral symmetry, it is natural to equate

$$\frac{1}{d} = -\frac{\langle \bar{q}q \rangle}{f_\pi^2} \tag{3.16}$$

(where we follow the usual phase convention for the quark fields so that, with m_q taken as positive, the condensate $\langle \bar{q}q \rangle < 0$). Combining these with Eq. (3.15), we see that this heuristic analysis yields the GMOR mass relation,

$$m_\pi^2 = -\frac{(m_u + m_d)}{f_\pi^2}\langle \bar{q}q \rangle, \tag{3.17}$$

where $\langle qq \rangle \equiv \langle \sum_{a=1}^{N_c} \bar{q}_a q^a \rangle$ with $q = u$ or $q = d$ (these condensates being essentially equal in QCD). A similar argument, with appropriate replacement of light-quark mass m_u or m_d by m_s, yields the analogous GMOR-type mass relations for the K^+ and K^0,

$$m_{K^+}^2 = -\frac{(m_u + m_s)}{f_K^2}\langle \bar{q}q \rangle \tag{3.18}$$

and

$$m_{K^0}^2 = -\frac{(m_d + m_s)}{f_K^2}\langle \bar{q}q \rangle, \tag{3.19}$$

where $\langle \bar{q}q \rangle \simeq \langle \bar{s}s \rangle$ for $q = u, d$.

The nearby q and $\bar{q}$ paths in our picture generate $q - \bar{q}$ color interactions that depend on the difference of these paths. This suggests the possibility of using this picture to infer the derivatively coupled form of pion interactions appropriate for a Nambu-Goldstone particle. This derivative coupling means that in the static limit, these Nambu-Goldstone bosons become noninteracting.

We next address point (iii) above, concerning the relation $f_\pi \sim |\psi(0)|/\sqrt{m_\pi}$. Since the matrix element (2.10) as it enters in the $\pi^+ \to \ell^+\nu_\ell$ decay amplitude obviously involves the annihilation of the u and $\bar{d}$ quarks in the π^+ to produce the virtual timelike W^+ that, in turn, produces the $\ell^+\nu_\ell$ pair, it clearly depends on the $u\bar{d}$ wave function in the pion evaluated at the origin of the relative coordinate, $|\psi_\pi(0)|$. The question here concerns what happens in the chiral limit, where $m_\pi \to 0$. For this discussion we again imagine that electroweak interactions are turned off, except that we take into account the couplings leading to the $\pi^+_{\ell 2}$ decay. Now the pion wave function at a given time involves the intersection of the worldlines of its constituent q and $\bar{q}$ with the $t = 0$ hyperplane in the full $\mathbb{R}^4$ Wick-rotated spacetime. This wave function has many Fock space components. The matrix element (2.10) involves the annihilation of the valence $q_i \bar{q}_j$ component by the axial-vector current. Higher Fock space components in the pion wave function correspond to additional crossings of the $t = 0$ hyperplane. A measure of the contributions of these additional components can be obtained from our random walk representation. We note that for the present purpose it is essentially a one-dimensional random walk that is relevant, since we are inquiring about passages across a hyperplane, namely, that defined by the condition $t = 0$, of codimension 1 in the full Euclidean $\mathbb{R}^4$. Now in general, the number of times that a one-dimensional random walk with n steps returns to the origin is asymptotically $\propto \sqrt{n}$ for large n. The contribution of the valence $q\bar{q}$ component of the full pion wave function to the annihilation probability $|\psi_\pi(0)|^2$ is thus reduced by the factor $1/\sqrt{n}$. By Eq. (3.11), $n^{-1/2} \propto R^{-1}$ and by Eq. (3.14), $R^{-1} \simeq m_\pi$, so $|\psi_\pi(0)|^2$ is reduced by the factor m_π. This means that $|\psi_\pi(0)| \propto \sqrt{m_\pi}$ in the chiral limit, thereby canceling the $\sqrt{m_\pi}$ in the denominator of Eq. (2.13), and yielding a finite value of f_π. Similar remarks apply for f_K in the hypothetical limit of $m_s \to 0$ as well as $m_{u,d} \to 0$. Thus, our picture provides a plausible resolution of the van Royen and Weisskopf paradox [Eq. (2.13)] [34].

Finally, we address issue (iv) concerning the similar size of the π and ρ. We should emphasize from the very outset that this is challenging. The qualitatively different physical pictures involved give an indication of the complexity in the calculation of charge radii. On the one hand, if the size is controlled by the relatively small separation h in our picture with double $q_i \bar{q}_j$ lines, then the pion should be much smaller than the ρ. On the other hand, since the distance $R \simeq 1/m_\pi$ controls the overall pion size, it follows this size can become, at least formally, unbounded in the chiral limit $m_\pi \to 0$. (In practice, pion wave functions centered within a distance R of each other would overlap and become entangled.) This divergence in R as $m_\pi \to 0$ is not an artifact of our picture; the range of the residual strong force mediated, at long distance, by pion exchange, formally diverges in this chiral limit. The property that the pion charge radius also diverges in the chiral limit is a natural concomitant of this divergence in the pion size. Let us elaborate on this.

The charge radius (squared) of a hadron is

$$\langle r^2 \rangle = \int \rho(r) r^2 d^3 r, \tag{3.20}$$

where $\rho(r)$ denotes the charge density. The quantity $\sqrt{|\langle r^2 \rangle|}$ gives one measure of the size of a composite particle [44]. This is especially clear for a meson such as the π^+ or K^+, where the u and, respectively, $\bar{d}$ or $\bar{s}$ both contribute positively to the integrand in Eq. (3.20) [44,45].

The charge radius squared is proportional to the slope of the electromagnetic form factor $F(q^2)$ at $q^2 = 0$ [46]:

$$\langle r^2 \rangle = 6 \frac{dF(q^2)}{dq^2} \bigg|_{q^2=0}. \tag{3.21}$$

The latter form factor satisfies a t-channel dispersion relation ($t \equiv q^2$)

$$F(t) = \int dt \frac{\mathrm{Im}[F(t')]}{t - t'}. \tag{3.22}$$

In particular, for the case under consideration, $F(t) = F_{\pi^+}(t)$, the integration is from $t' = (2m_\pi)^2$ to $t' = \infty$. In the vector meson dominance approximation for $F(t)$, one commonly replaces the $\mathrm{Im}[F(t')]$ by a delta function corresponding to the approximation of zero width for the relevant vector meson. Here, using ρ dominance for $F_{\pi^+}(t)$, one replaces $\mathrm{Im}[F_{\pi^+}(t')]$ by a delta function $\propto \delta(t' - m_\rho^2)$. This narrow-width approximation, together with the known value $F_{\pi^+}(0) = 1$, yields

$$F_{\pi^+}(q^2) = \frac{m_\rho^2}{m_\rho^2 - q^2}, \tag{3.23}$$

so that

$$\sqrt{\langle r^2 \rangle_{\pi^+}} = \frac{\sqrt{6}}{m_\rho} = 0.62 \text{ fm}. \tag{3.24}$$

This is close to the experimentally measured value, given above in Eq. (2.1) [33,37]; quantitatively, it is smaller than this experimental value by only 7%. (One can also include the effects of the ρ width, but this will not be necessary for our discussion here.) An analogous vector meson dominance prediction for the K^+ charge radius works very well also [33]. *A priori*, one might worry that an additional threshold contribution from $t' \simeq (2m_\pi)^2$ might dominate and lead to $\langle r^2 \rangle_{\pi^+} \simeq 1/m_\pi$. However, this does not happen here because of the derivative coupling of soft pions, as Nambu-Goldstone bosons. In the particular case here, another reason why this does not happen is that there is a $\sqrt{t' - 4m_\pi^2}$ factor in $\mathrm{Im}(F(t'))$ that arises from the P-wave nature of the $\pi\pi$ amplitude. Nevertheless, $\langle r^2 \rangle_{\pi^+}$ does diverge as $\langle r^2 \rangle_{\pi^+} \sim \ln(1/m\pi)$ in the chiral limit where $m_\pi \to 0$ [47].

We next sketch an estimate of the pion charge radius in our picture. As in the previous section, the higher Fock space states play a key role in this estimate. Consider the $t = 0$ slice of the Wick-rotated Minkowski space. The quantity $\langle r^2 \rangle_{\pi^+}$ can be computed as a sum of the contributions of the various Fock space components of the pion wave function. In our picture these are generated by crossings of the $t = 0$ hyperplane by the $q_i \bar{q}_j$ random-walking double worldline of the pion. Let the kth such crossing be at $\mathbf{r}_k$. The first crossing corresponds to a π^+, say, moving forward in time. Hence, we have a charge $+1$ at this location. At the second crossing, the pion line is reversed, and we have a -1 charge at $\mathbf{r}_2$, etc. The definition Eq. (3.20) above then yields $\langle r^2\rangle_{\pi^+} \simeq \sum_{k=1}^{\infty}(-1)^k r_k^2$, where here $r_k \equiv |\mathbf{r}_k|$. Recalling that the kth visit to the $t = 0$ plane happens typically after $n \simeq k^2$ steps of the random-walking double line, with individual step size d, we deduce that, on average, $r_k^2 \simeq k^2 d^2$. By itself, this would yield, for $\langle r^2\rangle_{\pi^+}$, the sum $\sum_{k=1}^{\infty}(-1)^k k^2 d^2$. The terms in the above oscillating sum diverge as $k \to \infty$. However, to get the actual sum, we must take into account the fact that the contributions are regularized by the exponential $\exp[-(m_u + m_d)L] \simeq \exp[-(m_u + m_d)k^2 d]$ controlling the total length of the random-walking double line. Using $m_u + m_d = dm_\pi^2$ from Eq. (3.15) above and defining

$$b \equiv m_\pi d, \tag{3.25}$$

we can rewrite the charge radius as

$$\langle r^2\rangle_{\pi^+} \simeq \sum_{k=1}^{\infty}(-1)^k k^2 d^2 e^{-k^2 b^2}. \tag{3.26}$$

Since k gets very large in the chiral limit, there are strong cancellations between successive terms, rendering an accurate estimate difficult. We can at least investigate the nature of the leading divergence in $\langle r^2\rangle_{\pi^+}$. To do this, we replace the above sum, after subtracting and adding an r_0 term and symmetrizing, by an integral over the variable $\xi = kb = km_\pi d$:

$$\begin{aligned} I(b) &= \frac{1}{m_\pi^2}\int_{-\infty}^{\infty} d\xi \xi^2 \exp\left(\frac{i\pi\xi}{b} - \xi^2\right) \\ &= \frac{\sqrt{\pi}}{2m_\pi^2}\left(1 - \frac{\sqrt{\pi}}{2(m_\pi d)^2}\right)\exp\left[-\left(\frac{\pi}{2m_\pi d}\right)^2\right]. \end{aligned} \tag{3.27}$$

The key observation here is that, while we have, as expected, an explicit $1/m_\pi^2$ factor in front, the integral $I(b)$ and any finite derivative thereof, contain the factor $\exp[-\pi^2/(2m_\pi d)^2]$, which vanishes with an essential zero in the chiral limit $m_\pi \to 0$. This can be seen as a consequence of the strong cancellations between different terms contributing to the sum, which we approximated as an integral. Thus, our calculation shows the absence of a divergence of the power-law form $1/m_{\pi^+}^2$ in $\langle r^2\rangle_{\pi^+}$ and is consistent with the chiral perturbation theory result that $\langle r^2\rangle_{\pi^+}$ diverges like $\ln(1/m_\pi)$ in this limit. For the real world with nonzero current-quark masses for u and d, our analysis above naturally yields a value of $\langle r^2\rangle_{\pi^+} \sim d^2$, since this was the r_0 term in the sum. A similar conclusion, with appropriate replacement of the d with the s quark, applies to $\langle r^2\rangle_{K^+}$ in the SU(3) chiral limit $m_u, m_d, m_s \to 0$.

Our picture can also give a plausible explanation of why the pion-nucleon cross section $\sigma_{\pi N}$ at energies above the resonance region can be comparable to the inferred value of $\sigma_{\rho N}$ at the same energies [cf. Eqs. (2.4) and (2.6)]. Relevant to the πN cross section is the fact that the valence quarks in the pion propagate in an extended double-line manner covering an area of order $R^2 \sim 1/m_\pi^2$. However, because of the strong color hyperfine interaction, the separation h of the valence q_i and $\bar{q}_j$ in the pion is rather small in our model. Hence, while in a crossing of two $q_i\bar{q}_j$ pairs at an ordinary hadronic distance $\sim d$, the probability of an interaction is $O(1)$, here, in contrast, it will be $O((h/d)^2)$. In the context of a hadronic string picture, the small pion mass is related to the separation h via $m_\pi \propto \sigma h$, where σ is the hadronic string tension. Hence, one may roughly estimate that the πN cross section $\sigma_{\pi N}$ contains the factors $(\pi R^2)(h/d)^2 \propto \pi/(\sigma d)^2$. Note that the factor of m_π^2 cancels out between numerator and denominator, leaving $\sigma_{\pi N}$ proportional to an expression involving the string tension and a typical hadronic distance scale, which are the same for the π and the ρ. In the preceding we have presented our efforts to show how our picture of a rather tightly bound $q_i\bar{q}_j$ pair undergoing a random walk inside a pion can explain how this particle can exhibit the properties of an approximate Nambu-Goldstone boson while also being understandable as a $\bar{q}q$ bound state. Ultimately, one should be able to find the differences predicted by our picture as compared with other approaches to this physics. One theoretical tool that is relevant here is lattice gauge theory. However, one faces not only the technical difficulty of simulating very light quark masses and light pions. An additional challenge is that in (Euclidean) lattice simulations one first integrates over the fermionic degrees of freedom. Having the two (say u and $\bar{d}$) quark propagators in the same background color field may not allow one to verify that at all intermediate steps the quark and antiquark are really close to each other. One may need to go back to the sum over fermionic paths in order to actually detect the propagators of the nearby $q_i\bar{q}_j$ pair.

One implication of our model with the nearby $q_i\bar{q}_j$ lines separated by a relatively small distance h is that the purely gluonic exchange amplitude for $\pi\pi$ scattering should be rather small. A recent lattice calculation of the $I = 2\pi\pi$ S-wave scattering length obtained the result $a_2 \simeq -0.043/m_\pi$ [48], in agreement with the Weinberg-Tomozawa soft-pion current algebra result $a_2 = -m_\pi/(16\pi f_\pi^2) = -0.044/m_\pi$ [49,50]. In a hypothetical $\pi\pi'$ scattering, where the π' is comprised of $\bar{d}'$ and u' quarks that are degenerate with the ordinary u and d but do not mix with them, the scattering amplitude involves only gluon exchanges, but not quark interchanges. In this case a preliminary lattice calculation has obtained a $\pi\pi'$ scattering length considerably smaller than a_2, in qualitative agreement with our discussion above [51].

IV. SOME COMMENTS ON THE $K \to \pi$ AND HEAVY-QUARK TRANSITION FORM FACTORS

The K mesons undergo semileptonic $K_{\ell 3}$ decays, such as $K^+ \to \pi^0\ell^+\nu_\ell$, $K_L^0 \to \pi^+\ell^-\bar{\nu}_\ell$, and $K_L^0 \to \pi^-\ell^+\nu_\ell$, mediated by the vector part of the weak charged current. The almost conserved vector current (CVC) [conserved apart from SU(3) flavor-breaking effects] helps to fix the corresponding hadronic matrix elements and the Cabibbo-Kobayashi-Maskawa (CKM) quark mixing matrix element $|V_{us}|$ [52] and tests of three-generation CKM unitarity [53]. Denoting the 4-momenta $p = p_K + p_\pi$ and $q = p_K - p_\pi$, one has

$$\langle \pi^0 | (V_-)_\lambda | K^+ \rangle = \langle \pi^- | (V_-)_\lambda | K_L^0 \rangle = \frac{1}{\sqrt{2}} (f_+(q^2) p_\lambda + f_-(q^2) q_\lambda), \quad (4.1)$$

where $(V_-)_\lambda$ is the weak charged current. The contribution from the $f_-(q^2)$ term is proportional to the final lepton mass squared and is negligible for semileptonic decays to final electrons. The $t \equiv q^2$ variation of $f_+(t)$ over the range $m_e^2 \le t \le (m_K - m_\pi)^2$ can be approximated by a linear function of t:

$$f_+(t) = f_+(0)\left(1 + \lambda_+ \frac{t}{m_\pi^2}\right), \quad (4.2)$$

where $\lambda_+ = 0.0288$ [20], in agreement with chiral perturbation theory calculations [47,54] and also with the expectation from simple K^* vector meson dominance. CVC implies that $f_+(q = 0) = 1$ in the hypothetical limit of exact $SU(3)_V$ symmetry, $m_u = m_d = m_s$ and hence $m_K = m_\pi$. A general result which we will elaborate on later is that the corrections to this symmetry-limit value are always second order in $SU(3)_V$ breaking. This is the well-known Ademollo-Gatto theorem [31]. This feature is evident in the explicit estimate [55,56]

$$f_+(0) = 1 - \frac{5(m_K^2 - m_\pi^2)^2}{384\pi^2 f_\pi^2 (2m_K^2 + m_\pi^2)} = 0.985. \quad (4.3)$$

The correction term in Eq. (4.3) arises from multiparticle contributions to the sum-rule corresponding to the commutator $[Q_{4+i5}, Q_{4-i5}] = Q_3 + \sqrt{3} Q_8$, where the subscripts refer to SU(3) flavor generators. Formally, this sum rule underlies the Ademollo-Gatto theorem; the multiparticle contributions are squares of matrix elements of the divergence $\partial_\mu J^\mu_{4+i5}$ of the strangeness-changing weak vector current, making the deviation from universality quadratic in the $SU(3)_V$ symmetry breaking. However, as is also evident in (4.3), in the limit of $SU(3)_L \times SU(3)_R$ chiral symmetry, with $m_\pi \to 0$ and $m_K \to 0$, the correction term is actually of first order [55]. This is a consequence of the fact that in this chiral limit the contributions to the sum rule that involve the exchange and propagation of massless π's and K's lead to a deviation from universality that is linear in m_K^2. The correction in Eq. (4.3) is small partly because of the numerical coefficient arising from the loop diagram involved in the calculation.

It is instructive to see how the Ademollo-Gatto theorem is realized in the NQM, where the form factor $f_+(q^2)$ can be expressed as an overlap of the K^+ and π^+ wave functions, which we shall denote $F_{K\to\pi}(q^2)$. In the NQM, the π^+ and K^+ consist of nonrelativistic constituent quarks $q_i \bar{q}_j$ and for $L = 0$,

$$F_{K\to\pi}(q = 0) = \int d^3 r \, \psi_\pi(\mathbf{r})^* \psi_K(\mathbf{r}). \quad (4.4)$$

In this model the K and π wave functions (which are real) depend on just an overall flavor-independent potential $V(r)$ and on the reduced masses μ_K and μ_π. We recall our notation M_q for the constituent mass of a quark q, and the values $M_u = M_d \equiv M_{ud} \simeq 330$ MeV, $M_s \simeq 470$ MeV.

For simplicity we use the single-term form for the potential:

$$V_{q\bar{q}}(r) = V_0 \left(\frac{r}{r_0}\right)^\nu, \quad (4.5)$$

where ν is an exponent. Special cases include (i) $\nu = -1$, i.e., Coulombic, (ii) $\nu = 0$, with $V(r) \propto \ln r$; (iii) $\nu = 1$, linear; (iv) $\nu = 2$, harmonic oscillator; and (v) $\nu = \infty$, equivalent to an infinite square-well potential. As was noted above, a realistic quark-quark potential has different forms at short distances and at distances of order $1/\Lambda_{\rm QCD} \sim 1$ fm, so it is more complicated than a single-term form. However, the simplification will suffice for our purposes here. The scaling properties of the Schrödinger equation imply that the spatial extent r characterizing the falloff of the wave function scales with the reduced mass μ as [18,57]

$$r \propto \mu^{-(1/(2+\nu))}. \quad (4.6)$$

For the range of μ considered here, the dependence of this characteristic distance on ν is thus maximal for the Coulombic, $\nu = -1$, case and minimal for $\nu = \infty$, where the spatial extent of ψ is determined completely by the width of the infinite square well and is independent of μ.

For $\nu = 2$, i.e., the harmonic oscillator potential, which we write as $V = kr^2/2$, the wave function is proportional to a Hermite polynomial, and, for the ground state, it is

$$\psi = \frac{(\mu k)^{3/8}}{\pi^{3/2}} \exp\left(\frac{\sqrt{\mu k} r^2}{2}\right). \quad (4.7)$$

Substituting this into Eq. (4.4), we calculate

$$F_{K\to\pi}(0) = \frac{2^{3/2}(\mu_K \mu_\pi)^{3/8}}{(\sqrt{\mu_K} + \sqrt{\mu_\pi})^{3/2}}. \quad (4.8)$$

Let us define the following measure of flavor SU(3) symmetry breaking:

$$\epsilon = \frac{M_s - M_{ud}}{M_{ud}}. \quad (4.9)$$

The expression for $F_{K\to\pi}(0)$ in Eq. (4.8) has the following Taylor series expansion in ϵ:

$$F_{K\to\pi}(0) = 1 - \frac{3}{256}\epsilon^2 + O(\epsilon^3) \quad \text{for } \nu = 2. \tag{4.10}$$

With the values of μ_π and μ_K given above,

$$F_{K\to\pi}(0) = 0.999 \quad \text{for } \nu = 2. \tag{4.11}$$

For comparison, consider the Coulomb potential with ground state

$$\psi = \frac{e^{-r/a}}{\pi^{1/2}a^{3/2}}, \tag{4.12}$$

where

$$a_B = \frac{1}{C_{2f}\alpha_s\mu} \tag{4.13}$$

is the Bohr radius and $C_{2f} = 4/3$. Substituting this into Eq. (4.4) for the wave function overlap, we find

$$F_{K\to\pi}(0) = \left(\frac{2\sqrt{a_K a_\pi}}{a_K + a_\pi}\right)^3 = \left(\frac{2\sqrt{\mu_K\mu_\pi}}{\mu_K + \mu_\pi}\right)^3 = 0.992. \tag{4.14}$$

This again has a Taylor series expansion of the form $F_{K\to\pi}(0) = 1 - O(\epsilon^2)$, as expected from the Ademollo-Gatto theorem. Since the r dependences of the logarithmic ($\nu = 0$) and linear ($\nu = 1$) potentials are intermediate between the harmonic oscillator ($\nu = 2$) and Coulomb ($\nu = -1$) potentials, one expects $F(q = 0)$ to be very close to unity for these potentials as well.

These results do not imply such small deviations from unity for the form factor $f_+(0)$ in K_{e3} decay. The mass difference $m_K - m_\pi \sim 360$ MeV far exceeds the value of $m_s - m_u$ expected in a model with a flavor-independent confining potential. Such considerations would apply better to semileptonic $s \to u$ decays of mesons with heavy c or b spectator quarks. Indeed $m_{D_s} - m_{D_u} = 104$ MeV and $m_{B_s} - m_{B_u} = 89$ MeV, consistent with the current-quark mass difference $m_s - m_u$. Unfortunately, these small mass differences imply tiny branching for these decays $D_s \to D_u\ell^+\nu_\ell$ and $B_s \to B_u\ell^+\nu_\ell$.

The generic form factor is a Lorentz-invariant function of q^2. However, for elastic scattering, one can go to a frame where $q^0 = 0$ so that $q^2 = -|\mathbf{q}|^2$ and write

$$F(q^2) = \int d^3r e^{i\mathbf{q}\cdot\mathbf{r}}\psi_\pi(\mathbf{r})^*\psi_K(\mathbf{r}), \tag{4.15}$$

where $q = (q^0, \mathbf{q})$ is the momentum imparted to the leptons in the decay process, $\mathbf{r} = \mathbf{r}_q - \mathbf{r}_{\bar{q}}$, and ψ_K and ψ_π are the initial and final meson wave functions. In the flavor SU(3) symmetry limit, $\psi_K = \psi_\pi$. For $q = 0$, the normalizations of the wave functions imply the conserved vector current (CVC) value $F(0) = 1$.

The last result is quite general; if the mesons contain, in addition to the valence quarks q_i and $\bar{q}_j$, any number of gluons at the position $\mathbf{R}_s$ and/or $q\bar{q}$ quark pairs at the positions $\mathbf{r}_\ell$, $\mathbf{r}_{\ell'}$, we would have, instead of (4.15),

$$F_{K\to\pi}(q^2) = \int d^3r e^{i\mathbf{q}\cdot\mathbf{r}}\Big[\prod_{\ell,\ell',s} d^3r_\ell d^3r_{\bar{\ell}'}d^3R_s \times \psi_\pi(\mathbf{r}, \mathbf{r}_\ell, \mathbf{r}_{\bar{\ell}'}, \mathbf{R}_s)^*\psi_K(\mathbf{r}, \mathbf{r}_\ell, \mathbf{r}_{\bar{\ell}'}, \mathbf{R}_s)\Big], \tag{4.16}$$

so that again in the flavor SU(3) symmetry limit, for equal wave functions and $q = 0$, we have $F(0) = 1$. Here, ψ_K and ψ_π are the Fock space wave functions with any number of gluons and quark-antiquark pairs. Both quarks and gluons carry spin and color, so that ψ could be a superposition of many color and spin couplings which yield overall color singlets. For notational simplicity we have omitted these above. The general arguments presented below do not depend on the slightly simpler form of (4.16).

As is evident in Eqs. (4.15) and (4.16), deviations from $F(0) = 1$ can be caused in two ways. First, even for elastic transitions with $\psi_{\text{initial}} = \psi_{\text{final}}$, the momentum transfer factor $e^{i\mathbf{q}\cdot\mathbf{r}}$ modulates the positive integrand and decreases F. Second, flavor SU(3) breaking, namely, the difference between m_s and m_q, $q = u, d$, causes the π^+ and K^+ wave functions to be different and hence reduces $f_+(0)$ from unity. To analyze this, we shall use the Cauchy-Schwarz inequality, that for any vector space $\mathcal{V}$ with vectors ψ and ϕ and an inner product $\langle\psi, \phi\rangle$, the property

$$|\langle\psi, \phi\rangle| \le \|\psi\|\|\phi\| \tag{4.17}$$

holds, where $\|\psi\| \equiv \sqrt{\langle\psi, \psi\rangle}$. We apply this to the L^2 Hilbert space of square-integrable functions $\psi(\mathbf{r}, \mathbf{r}_\ell, \mathbf{r}_{\bar{\ell}'}, \mathbf{R}_s)$ with the inner product

$$\langle\psi, \phi\rangle = \int d^3r\Big[\prod_{\ell,\ell',s} d^3r_\ell d^3r_{\bar{\ell}'}d^3R_s\,\psi(\mathbf{r}, \mathbf{r}_\ell, \mathbf{r}_{\bar{\ell}'}, \mathbf{R}_s)^*\phi(\mathbf{r}, \mathbf{r}_\ell, \mathbf{r}_{\bar{\ell}'}, \mathbf{R}_s)\Big]. \tag{4.18}$$

Thus,

$$F_{K\to\pi}(\mathbf{q} = 0) = \langle\psi_\pi, \psi_K\rangle. \tag{4.19}$$

Using this, we have

$$|F_{K\to\pi}(\mathbf{q}=0)|^2 = \left|\int d^3r\Big[\prod_{\ell,\ell',s} d^3r_\ell d^3r_{\ell'} d^3R_s \psi_\pi(\mathbf{r},\mathbf{r}_\ell,\mathbf{r}_{\bar{\ell}'},\mathbf{R}_s)^* \psi_K(\mathbf{r},\mathbf{r}_\ell,\mathbf{r}_{\bar{\ell}'},\mathbf{R}_s)\Big]\right|^2$$
$$\le \Big[\int d^3r\Big[\prod_{\ell,\ell',s} d^3r_\ell d^3r_{\ell'} d^3R_s |\psi_\pi(\mathbf{r},\mathbf{r}_\ell,\mathbf{r}_{\bar{\ell}'},\mathbf{R}_s)|^2\Big]\Big]\Big[\int d^3r\Big[\prod_{\ell,\ell',s} d^3r_\ell d^3r_{\ell'} d^3R_s |\psi_K(\mathbf{r},\mathbf{r}_\ell,\mathbf{r}_{\bar{\ell}'},\mathbf{R}_s)|^2\Big]\Big], \quad (4.20)$$

where we write these for the general case of Eq. (4.16) above.

Universal form factors at the no-recoil point for semileptonic decays of mesons containing heavy quarks, e.g., $B_d \to D^-\ell^+\nu_\ell$, follow from the fact that for $m_b > m_c \gg \Lambda_{\rm QCD}$, the heavy quark is a static source of color [transforming as a color SU(3) triplet] with common wave functions for all of the light degrees of freedom in either the B or D mesons [58]. With flavor-independent primary QCD interactions, the difference between ψ_K and ψ_π is due to the different u and s masses only. From the Cauchy-Schwarz inequality, one sees that $F(\mathbf{q}=\mathbf{0})$, as a function of m_s and $m_s - m_u$, is extremal (maximal) at $m_s - m_u = 0$. Hence, the deviation from unity is of order $O((m_s - m_u)^2)$, which is the Ademollo-Gatto theorem [31]. The analogous theorem for heavy quarks is derived by the same type of reasoning [59,60]. Let us rewrite (4.20) as

$$F_{B\to D}(\mathbf{q}=\mathbf{0})[m_i(\text{“light”}), v = m_i/M_Q], \quad (4.21)$$

where m_i(“light”) refers to the masses of the degrees of freedom that are light relative to m_Q, namely $\Lambda_{\rm QCD}$, m_s, etc. and M_Q denotes the mass of the lighter among the heavy quarks, namely m_c in the present case. Again, F is extremal for $v = 0$, and the deviations from universality at the no-recoil point are of order $O(v^2) = O(1/m_Q^2)$, i.e., $O(1/m_c^2)$ in $b \to c$ transitions. Indeed, if the current-quark masses $m_u = m_d = 0$, then, when $m_s \to 0$, the chiral symmetry group is enlarged from $SU(2)_L \times SU(2)_R$ to $SU(3)_L \times SU(3)_R$ [and the QCD condensates would then break these to the respective diagonal subgroups $SU(2)_V$ and $SU(3)_V$].

The generalized Ademollo-Gatto theorem can be formulated in Hamiltonian lattice QCD. The π and K wave functions are replaced by wave functionals with arbitrary patterns of excited links, corresponding to gluonic excitations, and/or extra $q\bar{q}$ pairs. Now consider the matrix element of the strangeness-changing vectorial weak charge, $Q_{u,\bar{s}} = \int d^3x V^0$, where V^μ denotes the associated current. This converts flavors $s \to u$ for the valence quarks. Since $[Q,H] \neq 0$, this changes the energy of the state operated on by $m_K - m_\pi$. However, since Q is an integral over all space, it does not change the 3-momentum of the state on which it operates. Hence, if it operates on a K at rest, it should produce a π at rest also. The matrix element of interest is the overlap of two wave functionals computed for valence quark mass $m_q = m_s$ and for $m_q = m_u$. By the Cauchy-Schwarz inequality, which holds for these wave functionals, the overlap is smaller than unity, achieving its maximum value when $\Delta = m_s - m_u = 0$. Hence, repeating the same arguments as above, we find that

$$F(\mathbf{q}=\mathbf{0}) = 1 - O(\Delta^2). \quad (4.22)$$

V. MASS COMPARISONS INVOLVING HEAVIER HADRONS

We proceed to discuss the systematics of mass differences $m(Q\bar{s}) - m(Q\bar{u})$ for various J^{PC} mesons. Some related work is in Refs. [32,61]. In this context, we recall that modern lattice estimates have yielded a somewhat smaller value of the current-quark mass $m_s \sim 120$ MeV than some older current algebra estimates, which tended to be centered around 180 MeV [47]. In the nonrelativistic quark model [with a flavor-independent nonrelativistic quark-(anti)quark interaction potential], the mass difference between analogous hadrons differing only by having an s quark replaced by a u or d quark should, up to small binding changes due to the different reduced constituent masses, differ by $m_s - m_{ud}$. The real world is more complicated, for several reasons. First, the concept of quark masses and differences needs to be carefully defined. The masses run with the distance or momentum scale at which they are probed. The constituent quarks can be considered to be extended quasiparticles, confined to hadrons with sizes of order $1/\Lambda_{\rm QCD}$. As the MIT-SLAC deep inelastic scattering experiments showed dramatically, as one increases the momentum scale at which one probes such a quark beyond $\Lambda_{\rm QCD}$, it acts quasifree, without the attendant strong coupling to gluons to which it is subject for momenta less than $\Lambda_{\rm QCD}$. As this momentum scale increases considerably beyond $\Lambda_{\rm QCD}$, the quark mass then goes over to approximately the current-quark mass, since the QCD gauge coupling becomes small. Since different hadrons have somewhat different effective scales, this modifies the extracted mass difference.

Second, while at the fundamental Lagrangian level the only breaking of flavor symmetry is due to the differences between the current quark masses, this is not the case for the effective potential between the constituent quarks in the naive quark model because of the short-range color hyperfine interactions, though not in the asymptotic, confining part of the potential. This suggests that the mass differences of $Q\bar{s}$ and $Q\bar{q}$ mesons with Q a heavy-quark better estimate the current-quark mass difference $m_s - m_q$ mass difference, with $q = u$ or d, since both the magnitude of the color hyperfine splittings and the effective sizes of the system are smaller there (the latter is a reduced mass effect). Some measured mass differences, averaged over isospin multiplets, are $m(K^*) - m(\rho) \simeq 120$ MeV, $m(\phi) - m(K^*) \simeq 125$ MeV, $m(D_s) - m(D_u) \simeq m(D_s^*) - m(D_u^*) \simeq 100$ MeV, and $m(B_s) - m(B_u) \simeq 90$ MeV. We observe a substantial and fairly systematic tendency of these mass differences to decrease as $m(Q)$ increases. This is in agreement with the lattice gauge theory estimates mentioned above. The pattern in the baryonic spectrum is more complicated, but does not disagree with this general decreasing behavior. As is well known, the large splittings in the $J^P = 1/2+$ baryon octet, viz., $m(\Lambda) - m(N) \simeq 180$ MeV, $m(\Sigma) - m(N) \simeq 255$ MeV, $m(\Xi) - m(\Lambda) \simeq 200$ MeV, and $m(\Xi) - m(\Sigma) \simeq 125$ MeV, can be explained by a color hyperfine interaction, similar to that for the mesons. The equal-spacing mass difference rule in the $J = 3/2$ baryon decuplet with the interval of $\sim$146 MeV can also be explained by the color hyperfine interaction. The relatively large mass difference between $(csu, 1/2+) \equiv \Xi_c$ and $(cud, 1/2^+) \equiv \Lambda_c$ of $m(\Xi_c) - m(\Lambda_c) \simeq 181$ MeV is again in agreement with the expectation based on the large difference in the $s - u$ and $d - u$ color hyperfine interaction, which is evidently not reduced by the presence of the nearby heavy c quark in these baryons. Only the difference of masses of $\Omega_c = (css, 1/2+)$ and $\Xi_c = (csu, 1/2+)$ of 230 MeV appears to be somewhat high. On the basis of this discussion, one expects small mass splittings $m(QQ's) - m(QQ'u)$ between baryons containing two heavy quarks, but this expectation cannot yet be checked.

VI. CONCLUSIONS

In conclusion, we have revisited the ρ-π puzzle, namely, the problem of describing the π meson as a $q_i\bar{q}_j$ bound state and as an approximate Nambu-Goldstone boson and relating its mass and size to those of the ρ meson. We have presented a simple heuristic picture that, we believe, gives insight into this problem. In this picture, the valence q_i and $\bar{q}_j$ quarks in the π are rather tightly bound by the strong color hyperfine interaction that splits the π and ρ masses. We show that this picture can resolve another old puzzle concerning the pion wave function at the origin (van Royen–Weisskopf paradox) and is consistent with the Gell-Mann-Oakes-Renner relation, With appropriate replacement of the u or d quark by the s quark, our picture also applies to the K and its relation to the K^*. Using our model, we present an estimate for the charge radius $\langle r^2\rangle_{\pi^+}$. Our approach gives further insight into the charged-current $K^+ - \pi^+$ transition relevant in $K_{\ell 3}$ decays.

ACKNOWLEDGMENTS

S. N. thanks A. Casher and M. Karliner, and R. S. thanks S. Brodsky, for helpful discussions. The research of R. S. was partially supported by Grant No. NSF-PHY-06-53342.

---

[1] A. Casher, Phys. Lett. **83B**, 395 (1979).
[2] T. Banks and A. Casher, Nucl. Phys. **B169**, 103 (1980).
[3] G. 't Hooft, in *1979 Cargèse Lectures*, NATO Advanced Study Institutes, Ser. B, Vol. 59 (Plenum, New York, 1980), p. 135.
[4] S. Coleman and E. Witten, Phys. Rev. Lett. **45**, 100 (1980).
[5] Y. Frishman, A. Schwimmer, T. Banks, and S. Yankielowicz, Nucl. Phys. **B177**, 157 (1981).
[6] V. A. Novikov, M. A. Shifman, A. I. Vainshtein, and V. I. Zakharov, Nucl. Phys. **B191**, 301 (1981).
[7] C. Vafa and E. Witten, Nucl. Phys. **B234**, 173 (1984).
[8] D. Weingarten, Phys. Rev. Lett. **51**, 1830 (1983).
[9] S. R. Coleman and B. Grossman, Nucl. Phys. **B203**, 205 (1982).
[10] E. Witten, Phys. Rev. Lett. **51**, 2351 (1983).
[11] S. Nussinov, Phys. Rev. Lett. **51**, 2081 (1983); **52**, 966 (1984).
[12] M. Lampert and S. Nussinov, Phys. Rep. **362**, 193 (2002).
[13] Some early studies of spontaneous chiral symmetry breaking in the lattice gauge formulation of QCD and non-Abelian vectorial gauge theories are H. Hamber and G. Parisi, Phys. Rev. Lett. **47**, 1792 (1981); H. Hamber, E. Marinari, G. Parisi, and C. Rebbi, Phys. Lett. **124B**, 99 (1983); J. Kogut, H. W. Wyld, S. H. Shenker, J. Shigemitsu, and D. K. Sinclair, Phys. Rev. Lett. **48**, 1140 (1982); Nucl. Phys. **B225**, 326 (1983). Some analytic studies include B. Svetitsky, S. D. Drell, H. R. Quinn, and M. Weinstein, Phys. Rev. D **22**, 490 (1980); H. Kluberg-Stern, A. Morel, and B. Petersson, Nucl. Phys. **B215**, 527 (1983); H. Kluberg-Stern, A. Morel, O. Napoly, and B. Petersson, Nucl. Phys. **B220**, 447 (1983); I-H. Lee and R. Shrock, Phys. Rev. Lett. **59**, 14 (1987); Phys. Lett. B **201**, 497 (1988). Early reviews include M. Creutz, *Quarks, Gluons, and Lattices* (Cambridge University Press, Cambridge, 1983); J. Kogut, Rev. Mod. Phys. **55**, 775 (1983). Numerically, $\langle\bar{q}q\rangle \simeq -(240\ \mathrm{MeV})^3$.
[14] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961); **124**, 246 (1961).
[15] J. Goldstone, Nuovo Cimento **19**, 154 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962).
[16] G. 't Hooft, Phys. Rev. Lett. **37**, 8 (1976); Phys. Rev. D **14**, 3432 (1976).
[17] After the original works suggesting quarks by Gell-Mann, Zweig, and Neeman, and studies by Dalitz, Morpurgo, and others, an early review of the NQM was J. J. J. Kokkedee, *The Quark Model* (Benjamin, New York, 1969). Some studies of this model after the advent of QCD include A. DeRujula, H. Georgi, and S. L. Glashow, Phys. Rev. D **12**, 147 (1975); M. B. Voloshin and L. B. Okun, JETP Lett. **23**,

333 (1976); A. Le Yaouanc, L. Oliver, O. Pène, and J.-C. Raynal, Phys. Rev. D **15**, 844 (1977); **18**, 1591 (1978); I. Herbst and S. Nussinov, Phys. Rev. D **17**, 1362 (1978); N. Isgur and G. Karl, Phys. Rev. D **18**, 4187 (1978); **19**, 2653 (1979); **20**, 1191 (1979); **21**, 3175 (1980); S. Gasiorowicz and J. L. Rosner, Am. J. Phys. **49**, 954 (1981); I. Cohen and H. Lipkin, Phys. Lett. **106B**, 119 (1981); A. Manohar and H. Georgi, Nucl. Phys. **B234**, 189 (1984); H. J. Lipkin, Phys. Lett. B **233**, 446 (1989); J. M. Richard, Phys. Rep. **212**, 1 (1992); arXiv:nucl-th/0410007, and Refs. [18,19].

[18] C. Quigg and J. Rosner, Phys. Rep. **56**, 167 (1979).

[19] H. Grosse and A. Martin, *Particle Physics and the Schrödinger Equation* (Cambridge University Press, Cambridge, 1997).

[20] See, e.g., http://pdg.lbl.gov, and references therein.

[21] For the light quarks u and d, the constituent quark mass $M_u \simeq M_d$ to within a few MeV, so we shall denote it as M_{ud}. The value of this mass can be estimated roughly as $M_{ud} = m_N/N_c = 310$ MeV or $m_{ud} = m_\rho/2 = 380$ MeV; we shall essentially average these and use the value $M_{ud} = 340$ MeV here, as in Eq. (1.1). M_{ud} and M_s can be obtained by a fit to hadron masses and baryon magnetic moments in the NQM. The current-quark masses, i.e., the masses that quarks would have in the hypothetical absence of strong interactions, are denoted m_i; typical values for these are $m_u \simeq 4$ MeV, $m_d \simeq 8$ MeV, and $m_s \simeq 120$ MeV [20]. These are often called hard quark masses, although, if they are dynamically generated, they themselves are soft at mass scales typically of order 10^2 to 10^3 TeV [62].

[22] Some of the early papers include K. Lane, Phys. Rev. D **10**, 2605 (1974); H. D. Politzer, Nucl. Phys. **B117**, 397 (1976); J. Ball and T.-W. Chiu, Phys. Rev. D **22**, 2550 (1980); J. Cornwall, Phys. Rev. D **26**, 1453 (1982); K. Higashijima, Phys. Rev. D **29**, 1228 (1984).

[23] V. Bernard, R. Brockmann, M. Schaden, W. Weise, and E. Werner, Nucl. Phys. **A412**, 349 (1984); W. Weise, Nucl. Phys. **A434**, 685 (1985); V. Bernard, R. Brockmann, and W. Weise, Nucl. Phys. **A440**, 605 (1985); V. Bernard and U.-G. Meissner, Nucl. Phys. **A489**, 647 (1988); U. Vogl and W. Weise, Prog. Nucl. Part. Phys. **27**, 195 (1991); S. Klevansky, Rev. Mod. Phys. **64**, 649 (1992); V. Miransky, *Dynamical Symmetry Breaking in Quantum Field Theories* (World Scientific, Singapore, 1993); T. Hatsuda and T. Kunihiro, Phys. Rep. **247**, 221 (1994); J. Bijnens, Phys. Rep. **265**, 370 (1996).

[24] C. Callan, R. Dashen, and D. Gross, Phys. Rev. D **16**, 2526 (1977); **17**, 2717 (1978); D. Caldi, Phys. Rev. Lett. **39**, 121 (1977).

[25] M. A. Nowak, J. J. M. Verbaarschot, and I. Zahed, Phys. Lett. B **228**, 251 (1989); T. A. Appelquist and S. Selipsky, Phys. Lett. B **400**, 364 (1997); M. Velkovsky and E. V. Shuryak, Phys. Lett. B **437**, 398 (1998).

[26] D. G. Caldi and H. Pagels, Phys. Rev. D **14**, 809 (1976); H. Pagels and S. Stokar, Phys. Rev. D **20**, 2947 (1979); P. Langacker and H. Pagels, Phys. Rev. D **19**, 2070 (1979).

[27] A. Chodos, R. Jaffe, K. Johnson, C. Thorn, and V. Weisskopf, Phys. Rev. D **9**, 3471 (1974); T. DeGrand, R. Jaffe, K. Johnson, and J. Kiskis, Phys. Rev. D **12**, 2060 (1975).

[28] J. Donoghue and K. Johnson, Phys. Rev. D **21**, 1975 (1980); K. Johnson, Nucl. Phys. **A374**, 51 (1982).

[29] T. J. Goldman and R. W. Haymaker, Phys. Rev. D **24**, 724 (1981); Phys. Lett. **100B**, 276 (1981).

[30] A. Chodos and C. B. Thorn, Phys. Rev. D **12**, 2733 (1975); G. E. Brown, M. Rho, and V. Vento, Phys. Lett. **84B**, 383 (1979); S. Théberge, A. W. Thomas, and G. A. Miller, Phys. Rev. D **22**, 2838 (1980); G. E. Brown, Prog. Part. Nucl. Phys. **8**, 147 (1982).

[31] M. Ademollo and R. Gatto, Phys. Rev. Lett. **13**, 264 (1964).

[32] M. Karliner and H. Lipkin, Phys. Lett. B **650**, 185 (2007).

[33] T. H. Bauer, R. D. Spital, D. R. Yennie, and F. M. Pipkin, Rev. Mod. Phys. **50**, 261 (1978).

[34] R. van Royen and V. F. Weisskopf, Nuovo Cimento **50**, 617 (1967); **51**, 583 (1967).

[35] G. P. Lepage and S. J. Brodsky, Phys. Lett. **87B**, 359 (1979); Phys. Rev. D **22**, 2157 (1980); S. J. Brodsky and G. P. Lepage, Phys. Scr. **23**, 945 (1981); S. J. Brodsky, Springer Tracts Mod. Phys. **100**, 81 (1982); S. J. Brodsky, C.-R. Ji, A. Pang, and D. G. Robertson, Phys. Rev. D **57**, 245 (1998).

[36] S. J. Brodsky, H.-C. Pauli, and S. Pinsky, Phys. Rep. **301**, 299 (1998).

[37] S. J. Brodsky and G. F. de Téramond, Phys. Lett. B **582**, 211 (2004); G. F. de Téramond and S. J. Brodsky, Phys. Rev. Lett. **94**, 201601 (2005); S. J. Brodsky and G. F. de Téramond, Phys. Rev. Lett. **96**, 201601 (2006); Phys. Rev. D **77**, 056007 (2008); S. J. Brodsky and G. F. de Téramond, arXiv:0802.0514.

[38] K.-I. Aoki, M. Bando, T. Kugo, and M. Mitchard, Prog. Theor. Phys. **85**, 355 (1991); K.-I. Aoki, T. Kugo, and M. Mitchard, Phys. Lett. B **266**, 467 (1991); P. Jain and H. Munczek, Phys. Rev. D **48**, 5403 (1993); C. J. Burden *et al.*, Phys. Rev. C **55**, 2649 (1997); R. Alkofer and L. von Smekal, Phys. Rep. **353**, 281 (2001); P. Maris and C. D. Roberts, Int. J. Mod. Phys. E **12**, 297 (2003).

[39] S. J. Brodsky and R. Shrock, Phys. Lett. B **666**, 95 (2008); arXiv:0803.2541.

[40] T. Banks and A. Zaks, Nucl. Phys. **B196**, 189 (1982); T. Appelquist, D. Karabali, and L. C. R. Wijewardhana, Phys. Rev. Lett. **57**, 957 (1986); T. Appelquist and L. C. R. Wijewardhana, Phys. Rev. D **35**, 774 (1987); **36**, 568 (1987); Y. Iwasaki *et al.*, Phys. Rev. Lett. **69**, 21 (1992); V. Miransky and K. Yamawaki, Phys. Rev. D **55**, 5051 (1997); T. Appelquist, A. Ratnaweera, J. Terning, and L. C. R. Wijewardhana, Phys. Rev. D **58**, 105017 (1998); M. Harada, M. Kurachi, and K. Yamawaki, Phys. Rev. D **68**, 076001 (2003); M. Kurachi and R. Shrock, J. High Energy Phys. 12 (2006) 034; T. Appelquist, G. Fleming, and E. Neil, Phys. Rev. Lett. **100**, 171607 (2008); A. Deuzeman, M. P. Lombardo, and E. Pallante, Phys. Lett. B **670**, 41 (2008).

[41] M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968).

[42] R. Dashen, Phys. Rev. **183**, 1245 (1969).

[43] See, e.g., F. Spitzer, *Principles of Random Walk* (Springer, New York, 1964); J. Rudnick and G. Gaspari, *Elements of the Random Walk* (Cambridge University Press, Cambridge, 2004).

[44] Some illustrative semiclassical examples with total charge $Q = \int \rho(r) d^3 r = 1$ and positive-definite $\rho(r)$ which have constant, exponential, and Gaussian forms, respectively, are the following: (i) if $\rho(r) = 3/(4\pi R^3)$ for $r \le R$ and $\rho(r) = 0$ for $r > R$, then $\langle r^2 \rangle = (3/5)R^2$; (ii) if $\rho(r) = (8\pi R^3)^{-1} e^{-r/R}$, then $\langle r^2 \rangle = 12R^2$; and (iii) if $\rho(r) = (\sqrt{\pi} R)^{-3} e^{-(r/R)^2}$, then $\langle r^2 \rangle = (3/2)R^2$.

[45] The connection between $\sqrt{|\langle r^2 \rangle|}$ and the size of the hadron is less direct for a particle in which the quarks contribute with opposite signs to the integrand of Eq. (3.20). Indeed, because $\langle r^2 \rangle$ flips sign under charge conjugation, $\langle r^2 \rangle = 0$ for a self-conjugate particle, such as π^0. However, isospin invariance implies that the size of π^0 is equal, up to small corrections, to the size of $\pi^{\pm}$. For an electrically neutral, non-self-conjugate meson $M_{i\bar{j}}$, the sign of $\langle r^2 \rangle$ is controlled by the lighter of the q_i and $\bar{q}_j$. For example, $\langle r^2 \rangle_{K^0} = -(0.077 \pm 0.010)$ fm^2 [20], as can be understood from the larger spatial extent of the d compared with the $\bar{s}$, which results from the property that $M_{ud} < M_s$.

[46] More generally, $\langle r^{2\ell} \rangle = \int \rho(r) r^{2\ell} d^3 r$, and

$$\langle r^{2\ell} \rangle = \frac{(2\ell + 1)!}{\ell!} \frac{d^\ell F(q^2)}{(dq^2)^\ell} \bigg|_{q^2=0}$$

with our $(+, -, -, -)$ metric.

[47] J. Gasser and H. Leutwyler, Phys. Rep. **87**, 77 (1982); Ann. Phys. (N.Y.) **158**, 142 (1984); Nucl. Phys. **B250**, 465 (1985); **B250**, 517 (1985); G. Colangelo, J. Gasser, and H. Leutwyler, Nucl. Phys. **B603**, 125 (2001); H. Leutwyler, Nucl. Phys. B, Proc. Suppl. **94**, 108 (2001).

[48] S. R. Beane, P. F. Bedaque, K. Orginos, and M. J. Savage, Phys. Rev. D **73**, 054503 (2006). See also T. Yamazai *et al.* (CP-PACS Collaboration), Phys. Rev. D **70**, 074513 (2004).

[49] S. Weinberg, Phys. Rev. Lett. **17**, 616 (1966); Y. Tomozawa, Nuovo Cimento A **46**, 707 (1966).

[50] S. L. Adler and R. F. Dashen, *Current Algebras and Applications to Particle Physics* (Benjamin, New York, 1968).

[51] P. Bedaque and A. Walker-Loud (private communication).

[52] N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963); M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).

[53] An early three-generation Cabibbo fit was R. E. Shrock and L. L. Wang, Phys. Rev. Lett. **41**, 1692 (1978); this was used for a CKM unitarity fit in R. E. Shrock, S. B. Treiman, and L. L. Wang, *ibid.* **42**, 1589 (1979); [see also V. D. Barger, W. F. Long, and S. Pakvasa, *ibid.* **42**, 1585 (1979)]. For recent reviews, see http://www.lnf.infn.it/wg/vus, Ref. [20], http://www.slac.stanford.edu/xorg/hfag, http://www.utfit.org, and talks at the International Conference on HEP, ICHEP-2008, http://ichep08.com.

[54] U.-G. Meissner, Phys. Rep. **161**, 213 (1988); B. R. Holstein, Int. J. Mod. Phys. A **7**, 7873 (1992); V. Bernard, N. Kaiser, and U.-G. Meissner, Int. J. Mod. Phys. E **4**, 193 (1995); A. Pich, Rep. Prog. Phys. **58**, 563 (1995); J. Bijnens, Phys. Rep. **265**, 370 (1996); P. Post and K. Schilcher, Eur. Phys. J. C **25**, 427 (2002); J. Bijnens and P. Talavera, Nucl. Phys. **B669**, 341 (2003); M. Harada and K. Yamawaki, Phys. Rep. **381**, 1 (2003); A. H. Fariborz, R. Jora, and J. Schechter, Int. J. Mod. Phys. A **20**, 6178 (2005); J. Bijnens, Prog. Part. Nucl. Phys. **58**, 521 (2007); V. Bernard and U.-G. Meissner, Annu. Rev. Nucl. Part. Sci. **57**, 33 (2007).

[55] P. Langacker and H. Pagels, Phys. Rev. Lett. **30**, 630 (1973); H. Pagels, Phys. Rep. **16**, 219 (1975).

[56] W. Wada, Phys. Lett. **49B**, 175 (1974).

[57] Combining Eqs. (4.5) and (4.6), one has $V \propto r^\nu \propto \mu^{-(\nu/(2+\nu))}$. Since the kinetic energy $E_{\rm kin}$ and potential energy $E_{\rm pot}$ satisfy the virial relation $E_{\rm kin} = (\nu/2)E_{\rm pot}$, it follows that the total energy E satisfies $E \propto (1 + \frac{\nu}{2})\mu^{-(\nu/(2+\nu))}$ (negative for a bound state).

[58] S. Nussinov and W. Wetzel, Phys. Rev. D **36**, 130 (1987).

[59] M. Voloshin and M. Shifman, Yad. Fiz. **47**, 801 (1988) [Sov. J. Nucl. Phys. **47**, 511 (1988)]; N. Isgur and M. Wise, Phys. Lett. B **232**, 113 (1989); **237**, 527 (1990); M. E. Luke, Phys. Lett. B **252**, 447 (1990).

[60] For some recent reviews, see, e.g., M. Neubert, Phys. Rep. **245**, 259 (1994); M. Voloshin, Surv. High Energy Phys. **8**, 27 (1995); M. Shifman, in *Proceedings of the Theoretical Advanced Summer Institute, TASI-1995*, edited by D. Soper (World Scientific, Singapore, 1996), p. 409.

[61] M. Karliner, B. Keren-Zur, H. J. Lipkin, and J. L. Rosner, arXiv:0708.4027; arXiv:0804.1575.

[62] N. D. Christensen and R. Shrock, Phys. Rev. Lett. **94**, 241801 (2005).