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Shear viscosity from effective couplings of gravitons

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We calculate the shear viscosity of field theories with gravity duals using Kubo formula by calculating the Green function of dual transverse gravitons and confirm that the value of the shear viscosity is fully determined by the effective coupling of transverse gravitons on the horizon. We calculate the effective coupling of transverse gravitons for Einstein and Gauss-Bonnet gravities coupled with matter fields, respectively. Then we apply the resulting formula to the case of AdS Gauss-Bonnet gravity with F^4 term corrections of Maxwell field and discuss the effect of F^4 terms on the ratio of the shear viscosity to entropy density.

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I. INTRODUCTION

The anti-de Sitter/conformal field theory (AdS/CFT) correspondence [1–4] has been a useful tool in the study of properties of strongly coupled gauge theories. By using AdS/CFT, the shear viscosity of strongly coupled gauge theories can be calculated in the hydrodynamic limit [5–8] on the AdS side. It is found that there is some universality on the value of the ratio of shear viscosity over entropy density, which is always $1/4\pi$ in the regimes described by gravity. This ratio is also conjectured to be a universal lower bound (the KSS bound [6]) for all materials. All known materials in nature by now satisfy this bound. More discussions on the universality and the bound can be found in [9–19].

The universal value of $1/4\pi$ is also obtained in the case with nonzero chemical potentials turned on [20–23]. In [24], the value η/s is also calculated to be $1/4\pi$ for gauge theories with the gravity dual of the Einstein-Born-Infeld theory. With stringy corrections the value of η/s has a positive derivation from $1/4\pi$ but still satisfies the KSS bound [25–31]. However, in [32–34] the authors considered R^2 higher derivative gravity corrections and found that the modification of the ratio of shear viscosity over entropy density to the conjectured bound is negative, which means that the lower bound is violated in this condition. The higher derivative gravity corrections they considered can be seen as generated from stringy corrections given the vastness of the string landscape. A new lower bound $4/25\pi$, which is smaller than $1/4\pi$, is proposed, based on the causality of dual field theory.

In [35,36], the authors calculated the ratio of shear viscosity to entropy density for general gravity theory duals. They identified the value of the ratio with a quotient of effective couplings [37] of two different polarizations of gravitons, κ_{xy} and κ_{rt}, valued on the horizon [35]. This ratio can be different from $1/4\pi$ in general gravity theories. In [38], the authors confirmed the dependence of shear viscosity on the effective coupling of transverse gravitons imposed in [35] using the approach of scalar membrane paradigm. This effective coupling of transverse gravitons valued on the horizon in general gravity theory may be different from the corresponding one in Einstein gravity, which leads to the value of the ratio different from $1/4\pi$.

In this paper we will first confirm the formula of the dependence of the shear viscosity on the effective coupling of transverse gravitons using Kubo formula through a direct calculation of the Green function of transverse gravitons. We reach the same result as using the membrane paradigm fluid in [38]. The calculation of the effective coupling of gravitons should be careful because there are many total derivatives in the effective action of gravitons which do not affect the equation of motion of gravitons. Then we will calculate the effective coupling of transverse gravitons for Einstein gravity and Gauss-Bonnet gravity coupled to matter fields separately. In the case of Einstein gravity it would be easy to show that the value of the ratio is not affected if matter fields are minimally coupled. However, in the case of Einstein-Gauss-Bonnet gravity, it has already been observed that the ratio has corrections if chemical potentials are turned on [39]. We will also calculate, in the Gauss-Bonnet gravity, the F^4 corrections of Maxwell field to the ratio and find that the ratio ranges from $(1-4\lambda)/4\pi$ to $1/4\pi$ for $\varepsilon \geq -\pi G l^2/72$. For $\varepsilon < -\pi G l^2/72$, $(1-4\lambda)/4\pi \leq \eta/s \leq (1-4\lambda-\lambda\pi G l^2/18\varepsilon)/4\pi$, and the ratio cannot reach $1/4\pi$ because the temperature has a lower bound above zero. Here ε is a parameter given shortly.

The organization of this paper is as follows. We will first derive the dependence of shear viscosity on the effective coupling of transverse gravitons using Kubo formula in Sec. II. In Sec. III we calculate the effective coupling of transverse gravitons for Einstein and Gauss-Bonnet gravities coupled with matter fields, respectively. Then in

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Sec. IV we apply the resulting formula of the dependence of shear viscosity on the effective coupling of transverse gravitons to AdS Gauss-Bonnet gravity with F^4 corrections of Maxwell field. Section V is devoted to conclusions and discussions.

II. THE DEPENDENCE OF SHEAR VISCOSITY ON THE EFFECTIVE COUPLING

It was first noted in [35] that the shear viscosity is determined by the effective coupling of transverse gravitons. In [38] the authors confirmed this by using the scalar membrane paradigm fluid. In this section we obtain this result by calculating the shear viscosity through the energy momentum/graviton correspondence using the Kubo formula [9,13]

$$\eta = \lim_{\omega\to 0}\frac{1}{2\omega i}(G^A_{xy,xy}(\omega,0) - G^R_{xy,xy}(\omega,0)), \tag{1}$$

where η is the shear viscosity, and the retarded Green's function is defined by

$$G^R_{\mu\nu,\lambda\rho}(k) = -i\int d^4x e^{-ik\cdot x}\theta(t)\langle[T_{\mu\nu}(x), T_{\lambda\rho}(0)]\rangle. \tag{2}$$

These are defined on the field theory side. The advanced Green's function can be related to the retarded Green's function of energy momentum tensor by $G^A_{\mu\nu,\lambda\rho}(k) = G^R_{\mu\nu,\lambda\rho}(k)^*$. In the frame of AdS/CFT correspondence, one is able to compute the retarded Green's function by making a small perturbation of metric. Here we choose spatial coordinates so that the momentum of the perturbation points along the z-axis. Then the perturbations can be written as $h_{\mu\nu} = h_{\mu\nu}(t, z, u)$. In this basis there are three groups of gravity perturbations, each of which is decoupled from others: the scalar, vector, and tensor perturbations [40]. Here we use the simplest one, the tensor perturbation h_{xy}. We use ϕ to denote this perturbation with one index raised $\phi = h^x_y$ and write ϕ in a basis as $\phi(t, u, z) = \phi(u)e^{-i\omega t+ipz}$. To calculate the Green functions of the energy momentum tensor we should first fix a background black hole solution and get the equation of motion for gravitons in this background. In this paper we mainly focus on the case of Ricci-flat black hole backgrounds. The case for black holes with hyperbolic horizon topology has been discussed recently [41,42]. We assume that the background black hole solution is of the form

$$ds^2 = -g(u)(1-u)dt^2 + \frac{1}{h(u)(1-u)}du^2 + \frac{r_+^2}{ul^2}(d\vec{x}^2), \tag{3}$$

where the horizon of the black hole locates at $u = 1$ and the boundary is at $u = 0$, $h(u)$, $g(u)$ are functions of u, regular at $u = 1$ and l is the AdS radius which is related to the cosmological constant by $\Lambda = -6/l^2$. Note here that we impose the conditions $h(u)$ and $g(u)$ are regular at the horizon. This indicates that the Ricci-flat black hole solution we consider here should be a nonextremal solution. The calculations below are not valid for extremal black holes. One can expand the Einstein equation of motion to the first order of ϕ to get the equation of motion of gravitons, and the effective action of gravitons can be obtained by expanding the gravity action to the second order of ϕ. In the frame of Einstein gravity, the equation of motion of ϕ is just the Klein-Gordon equation for massless scalars. The effective action for the transverse gravitons is always equal to

$$S = \frac{1}{16\pi G}\int d^5x\sqrt{-g}\left(-\frac{1}{2}\right)(\nabla_\mu\phi\nabla^\mu\phi) \tag{4}$$

up to some total derivatives in Einstein gravity. However, in gravity theories with higher derivative corrections, it may not still be the one for a minimally coupled massless scalar. Now we consider a specific kind of effective graviton action which is the same as the one considered in [38]. We write the effective action in the momentum space

$$S = \frac{1}{16\pi G}\int\frac{dwdp}{(2\pi)^2}du\sqrt{-g_0}(K(u)\phi'\phi' + w^2K(u)g_{0uu}g_0^{00}\phi^2 - p^2L(u)\phi^2) \tag{5}$$

up to some total derivatives, where a prime stands for the derivative with respect to u, and

$$\phi(t, u, z) = \int\frac{dwdp}{(2\pi)^2}\phi(u;k)e^{-iwt+ipz},$$
$$k = (w, 0, 0, p), \qquad \phi(u;-k) = \phi^*(u;k). \tag{6}$$

This action can be viewed as a minimally coupled massless scalar with an effective coupling $K_{\rm eff}(u) = K(u)/g_0^{uu}$ plus a ϕ^2 term proportional to p^2. Here $g_{0\mu\nu}$ denotes the background metric (3). For Einstein gravity, the effective coupling $K_{\rm eff} = -1/2$ as can be seen in the action (4). However, in general gravity theories, the effective coupling may depend on the radial coordinate u. In general the effective coupling should be regular at the horizon, so $1/K(u)$ should have a simple pole at $u = 1$.

With these assumptions, the equation of motion of the transverse graviton can be derived from the action (5)

$$\phi''(u,k) + A(u)\phi'(u,k) + B(u)\phi(u,k) = 0, \tag{7}$$

where

$$A(u) = \frac{(K(u)\sqrt{-g_0})'}{K(u)\sqrt{-g_0}}, \tag{8}$$

$$B(u) = -g_{0uu}g_0^{00}w^2 + \frac{L(u)}{K(u)}p^2. \tag{9}$$

Substituting the metric function yields

$$B(u) = \frac{w^2}{h(u)g(u)(1-u)^2} + \frac{L(u)}{K(u)}p^2. \quad (10)$$

Because the shear viscosity only involves physics in the zero momentum limit, $L(u)$ would not affect the value of η. The only constraint on $L(u)$ is that it should be regular at the horizon $u = 1$. In fact we can also have an extra term $w^2N(u)\phi^2$ in the action (5), and we assume $N(u)$ is also a function of u, which is regular at the horizon $u = 1$. The addition of such a term will not affect the value of η. Then we follow the standard procedure to solve this Eq. (7). First we impose the incoming boundary condition at the horizon so that

$$\phi(u) = (1-u)^{-i\beta w}F(u), \quad (11)$$

where $F(u)$ is regular at the horizon. β can be calculated by considering (7) in the limit $u \to 1$, which leads to

$$\beta = \frac{1}{\sqrt{h(1)g(1)}}. \quad (12)$$

Because we only need to know the $w \to 0$ behavior of this graviton we can expand the solution in the way

$$F(u) = 1 + i\beta w F_0(u) + O(w^2) + O(p^2). \quad (13)$$

By expanding Eq. (7) to the first order of w, we get the equation of $F_0(u)$

$$F_0''(u) + A(u)F_0'(u) + \frac{1}{(1-u)^2} + \frac{A(u)}{1-u} = 0. \quad (14)$$

The solution of this linear differential equation can be expressed as a sum of a specific solution and a general solution. The specific solution denoted by $F_{0p}(u)$ is easy to find

$$F_{0p}(u) = \ln(1-u). \quad (15)$$

The equation for the general solution F_{0g} is

$$F_{0g}''(u) + A(u)F_{0g}'(u) = 0. \quad (16)$$

Integrating this equation on both sides, we get

$$F_{0g}'(u) = \frac{C}{K(u)\sqrt{-g_0}}. \quad (17)$$

Further integrating leads to

$$F_{0g}(u) = C\int \frac{1}{K(u)\sqrt{-g_0}}du + D, \quad (18)$$

where C and D are two integration constants. From the assumptions given above we know $F_{0g}'(u)$ should have a simple pole at $u = 1$ because under our assumption of the metric (3), $\sqrt{-g_0}$ has no poles and $K(u)$ has a simple pole at $u = 1$. Then if $K(u)\sqrt{-g_0}$ is a rational function, it should have a factor $(1-u)$, so $F_{0g}'(u)$ can be written as a sum of $b/(1-u)$, where b is a constant, and some function regular at $u = 1$. Thus the integration on $F_{0g}'(u)$ should give

$$F_{0g}(u) = b\ln(1-u) + Z(u), \quad (19)$$

where $Z(u)$ is a function regular at $u = 1$. In many instances, $K(u)\sqrt{-g_0}$ may not be a rational function. For example, in the case of AdS Born-Infeld black holes [24], the metric function is irrational. In those cases, if we can trust the Taylor expansions of $K(u)\sqrt{-g_0}$ in the region $u \in [0, 1]$ to any precision, we still can have (19) as an asymptotic solution to any desired precision. In this paper, we consider the cases where (19) is valid. We define $S(u) = K(u)\sqrt{-g_0}/(1-u)$ and $S(1) = \lim_{u\to 1}K(u)\times\sqrt{-g_0}/(1-u)$, and $S(1)$ should be a finite quantity. Then by comparing (18) and (19), we can decide the value of s and the derivative of the function $Z(u)$. In general, the solution of $Z(u)$ could not be given explicitly, but fortunately, only $Z'(u)$ affects the final result. As a result, we only give $Z'(u)$ here,

$$b = -\frac{C}{S(1)} \quad (20)$$

and

$$Z'(u) = \frac{C}{1-u}\left(\frac{1}{S(u)} - \frac{1}{S(1)}\right). \quad (21)$$

With the specific solution F_{0p} and the general solution F_{0g}, the final solution should be a sum $F_{0p} + jF_{0g}$, where j is a constant needed to be determined. By requiring the solution to be nonsingular at $u = 1$, j should be chosen to be $-1/b$ and $F_0(u)$ can be uniquely determined as

$$F_0(u) = -\frac{1}{b}Z(u). \quad (22)$$

Next we put this solution into (5) to give the on-shell action:

$$S_{\text{on-shell}} = \frac{1}{16\pi G}\int \frac{dwdp}{(2\pi)^2}du((\sqrt{-g}K(u)\phi'\phi)'). \quad (23)$$

Integrating this action gives

$$S_{\text{on-shell}} = \frac{1}{16\pi G}\int \frac{dwdp}{(2\pi)^2}((\sqrt{-g}K(u)\phi'\phi))\bigg|_{u=1}^{u=0}. \quad (24)$$

In the appendix we argue that the other total derivatives in the bulk action and the Gibbons-Hawking surface term contribution exactly cancel on the boundary. Thus the resulting effective action is totally given by the boundary contribution in (24). Following the standard procedure, the retarded Green function can be calculated as

$$G^R_{xy,xy}(k) = \frac{1}{16\pi G}2\sqrt{-g_0}K(u)\phi'^*\phi\bigg|_{u=0}. \quad (25)$$

Substituting the metric (3), the value (12) of β, the solution of ϕ (21) and (22), into (1), we finally get

$$\eta = \frac{1}{16\pi G}\lim_{w\to 0}\frac{2\sqrt{-g_0}K(u)\phi'^*\phi|_{u=0}}{iw} = \frac{1}{16\pi G}\frac{r_+^3}{l^3}(-2K_{\rm eff}(u=1)). \quad (26)$$

Thus we arrive at the conclusion that the shear viscosity is fully determined by the effective coupling of transverse gravitons on the gravity side. In the Einstein case $K_{\rm eff} = -1/2$ and $\eta = r_+^3/16\pi G l^3$, which is the same as the result obtained in the previous calculations in [13]. In gravity theories where the Bekenstein-Hawking entropy area formula holds, we can further get $\eta/s = (-2K_{\rm eff}(u = 1))/4\pi$.

III. EFFECTIVE COUPLING OF TRANSVERSE GRAVITONS

From the previous section we learn that to calculate the shear viscosity of a gravity dual, one only needs to know the effective couplings of the transverse gravitons in this theory. In [35], a formula showing effective couplings of gravitons with different polarizations is given. However, it is not easy to judge which of the couplings are part of a total derivative in the general formula. Thus in this section, we calculate the effective couplings of transverse gravitons separately for Einstein gravity and Gauss-Bonnet gravity with matter fields minimally coupled to gravity.

A. For Einstein gravity

The action of Einstein-Hilbert gravity with matter fields minimally coupled to the gravity can be written as

$$S = \frac{1}{16\pi G}\int d^5x\sqrt{-g}(R - 2\Lambda + \mathcal{L}_{\rm matter}). \quad (27)$$

Here $\mathcal{L}_{\rm matter}$ is the Lagrangian of the matter fields coupled to gravity which can be the sum of any scalar or gauge fields. We assume the background black hole solution is of the form (3), which implies that the matter fields only depend on the radial coordinate u and we also assume that $\mathcal{L}_{\rm matter}$ depends on the metric only through the coupling of the metric to some ordinary derivatives of matter fields, such as the cases of minimally coupled scalar field and Maxwell fields, where covariant derivatives of matter fields are equivalent to the ordinary derivatives, so that $\delta_{(2)}\mathcal{L}_{\rm matter} = 0$ (see below). The Einstein equation of motion for this action is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}(R - 2\Lambda + \mathcal{L}_{\rm matter}) + \frac{\delta\mathcal{L}_{\rm matter}}{\delta g^{\mu\nu}} = 0. \quad (28)$$

We want to obtain the effective action for the perturbation h_y^x. The effective action for $\phi(u)$ is a sum of two parts: the bulk action $S_{\rm bulk}$ and the Gibbons-Hawking boundary term $S_{\rm GB}$. The Gibbons-Hawking term does not affect the effective coupling, and the bulk effective action should be a sum of a surface contribution and a term proportional to the equation of motion of $\phi(u)$, the latter of which vanishes on shell. We derive the effective action by keeping terms to the second order of $\phi(u)$ in the action:

$$S_{\rm bulk} = \frac{1}{16\pi G}\int d^5x(\delta_{(2)}\sqrt{-g}(R - 2\Lambda + \mathcal{L}_{\rm matter}) + \sqrt{-g}\delta_{(2)}(R - 2\Lambda + \mathcal{L}_{\rm matter})). \quad (29)$$

Here $\delta_{(2)}(\cdots)$ means to only keep terms of the second order of ϕ in $(\cdots)$. We have $\delta_{(2)}\mathcal{L}_{\rm matter} = 0$ because the matter fields only depend on the radial coordinate u and the metric couples to the matter fields only through ordinary derivatives. With the xx component of the on-shell Einstein equation of motion $R - 2\Lambda + \mathcal{L}_{\rm matter} = 2g^{xx}R_{xx}$, we can get the action for $\phi(u)$ to be always the form of (4) up to some total derivatives. Thus $K_{\rm eff} = -1/2$ holds in the whole spacetime and thus, of course, $K_{\rm eff} = -1/2$ on the horizon. Because the Bekenstein-Hawking area entropy formula always holds in Einstein gravity, it is straightforward that the ratio of η/s is always $1/4\pi$ as long as the assumptions in Sec. II are satisfied.

B. For Gauss-Bonnet gravity

In this subsection we calculate the effective coupling of transverse gravitons for Gauss-Bonnet gravity. We consider the action of Einstein gravity with Gauss-Bonnet terms as well as matter fields

$$S = \frac{1}{16\pi G}\int d^5x\sqrt{-g}\bigg(R - 2\Lambda + \frac{\lambda l^2}{2}(R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) + \mathcal{L}_{\rm matter}\bigg). \quad (30)$$

The Einstein equation of motion for this action is

$$R_{\mu\nu} - \frac{g_{\mu\nu}}{2}\bigg(R - 2\Lambda + \mathcal{L}_{\rm matter} + \frac{\lambda l^2}{2}(R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\rho\sigma}R^{\alpha\beta\rho\sigma})\bigg) + \frac{\delta\mathcal{L}_{\rm matter}}{\delta g^{\mu\nu}} + \frac{\lambda l^2}{2}(2RR_{\mu\nu} - 4R_{\rho\mu}R^{\rho}{}_{\nu} - 4R^{\rho\sigma}R_{\rho\mu\sigma\nu} + 2R_{\rho\sigma\lambda\mu}R^{\rho\sigma\lambda}{}_{\nu}) = 0. \quad (31)$$

To simplify calculations, we consider a simpler metric

$$ds^2 = -\frac{g^\star(u)r_+^2}{l^2u}N^2dt^2 + \frac{l^2}{4u^2g^\star(u)}du^2 + \frac{r_+^2}{ul^2}(d\vec{x}^2), \quad (32)$$

which is a specific case of (3) by setting $g(u) = \frac{g^\star(u)r_+^2}{l^2u(1-u)}N^2$ and $f(u) = \frac{4u^2g^\star(u)}{l^2(1-u)}$, so the calculations in Sec. II are still valid for this metric. N^2 is a constant that can be fixed at the boundary, which is defined in order to make the solution conformal to flat Minkowski spacetime on the boundary at $r\to\infty$

$$N^2 = \tfrac{1}{2}(1 + \sqrt{1 - 4\lambda}). \tag{33}$$

By keeping the action to the second order of ϕ we can get the effective action for transverse gravitons

$$\begin{aligned} S_{\text{bulk}} = \frac{1}{16\pi G}\int d^5x\Big[&\delta_{(2)}\sqrt{-g}\Big(R - 2\Lambda + \frac{\lambda l^2}{2} \\ &\times (R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) + \mathcal{L}_{\text{matter}}\Big) \\ &+ \sqrt{-g}\delta_{(2)}\Big(R - 2\Lambda + \frac{\lambda l^2}{2}(R^2 - 4R_{\mu\nu}R^{\mu\nu} \\ &+ R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) + \mathcal{L}_{\text{matter}}\Big)\Big]. \end{aligned} \tag{34}$$

Having assumed that the matter fields couple to the metric only through ordinary derivatives of matter fields, and the matter fields solution only depends on the radial coordinate u, we can see that the variation $\delta_{(2)}\mathcal{L}_{\text{matter}}$ vanishes. Note that the xx component of the equations of motion (31)

$$\begin{aligned} R_{xx} - \frac{g_{xx}}{2}\Big(&R - 2\Lambda + \mathcal{L}_{\text{matter}} + \frac{\lambda l^2}{2}(R^2 - 4R_{\rho\sigma}R^{\rho\sigma} \\ &+ R_{\rho\sigma\lambda\theta}R^{\rho\sigma\lambda\theta})\Big) + \frac{\delta\mathcal{L}_{\text{matter}}}{\delta g^{xx}} + \frac{\lambda l^2}{2}(2RR_{xx} - 4R_{\rho x}R^{\rho}{}_{x} \\ &- 4R^{\rho\sigma}R_{\rho x\sigma x} + 2R_{\rho\sigma\lambda x}R^{\rho\sigma\lambda}{}_{x}) = 0, \end{aligned} \tag{35}$$

and $\delta\mathcal{L}_{\text{matter}}/\delta g^{xx} = 0$ for the solution (32) we are considering. Substituting the above equation to (34), we find that the effective action for transverse gravitons can be fully expressed using background metrics and the derivatives of metrics. Thus we can determine the effective coupling of the transverse gravitons without knowing the explicit form of matter fields. The bulk action for the transverse graviton is therefore

$$\begin{aligned} S_{\text{bulk}} = \frac{1}{16\pi G}\int d^5x\Big[&\sqrt{-g}\delta_{(2)}\Big(R - 2\Lambda + \frac{\lambda l^2}{2} \\ &\times (R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})\Big) \\ &+ \delta_{(2)}\sqrt{-g}(2R_x{}^x + \lambda l^2(2RR_x{}^x - 4R_{\rho x}R^{\rho x} \\ &- 4R^{\rho\sigma}R_{\rho x\sigma}{}^x + 2R_{\rho\sigma\lambda x}R^{\rho\sigma\lambda x}))\Big]. \end{aligned} \tag{36}$$

Substituting the metric (32) into the bulk action, we finally get the effective coupling of the transverse graviton h_y^x as

$$K_{\text{eff}}(u) = -\tfrac{1}{2}(1 - 2\lambda g^{\star}(u) + 2\lambda u g^{\star\prime}(u)). \tag{37}$$

We see that this effective coupling depends on the background metric and the first derivative of the metric, and it is independent of an explicit form of matter fields. The effect of matter fields is reflected in the metric function $g^{\star}(u)$. Thus we obtain a universal formula of the shear viscosity for the AdS Gauss-Bonnet gravity with arbitrary minimally coupled matter fields, which only depends on the value of the metric and the first derivative of the metric on the horizon.

IV. EFFECTS OF F^4 TERMS IN GAUSS-BONNET THEORY

In the case of Einstein gravity, the effective coupling of transverse gravitons is a constant and not affected by minimally coupled matter fields. However, for Gauss-Bonnet gravity, the effective coupling of transverse gravitons (37) depends on the value of the metric and its first derivative. Thus when matter fields are coupled, the value of the ratio η/s may be different from the case of pure Gauss-Bonnet gravity. In [39], when the Maxwell field is added, the ratio η/s gets a positive correction, compared to the pure AdS Gauss-Bonnet gravity case. Now we apply the resulting formulas (26) and (37) to the case of the Gauss-Bonnet-Maxwell theory with F^4 terms correction.

The effective action of the theory we are considering is [43]

$$\begin{aligned} S &= S_{\text{grav}} + S_{\text{matter}} \\ &= \frac{1}{16\pi G}\int d^5x\sqrt{-g}\Big(R - 2\Lambda + \frac{\lambda l^2}{2}(R^2 - 4R_{\mu\nu}R_{\mu\nu} \\ &\quad + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma})\Big) + \int d^5x\sqrt{-g}\Big(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &\quad + c_1(F_{\mu\nu}F^{\mu\nu})^2 + c_2F_{\mu\nu}F^{\nu\lambda}F_{\lambda\rho}F^{\rho\mu}\Big), \end{aligned} \tag{38}$$

where c_1 and c_2 are two constants and $\Lambda = -6/l^2$. We consider the Ricci-flat black hole solutions with only the F_{tr} component of the Maxwell fields nonvanishing. In this assumption, the solution only depends on a combination ε of c_1 and c_2 [43], where

$$\varepsilon \equiv 2c_1 + c_2. \tag{39}$$

The Ricci-flat black hole solution is

$$\begin{aligned} &ds^2 = -H(r)N^2dt^2 + H^{-1}(r)dr^2 + \frac{r^2}{l^2}d\vec{x}^2, \\ &F_{\text{tr}} = f(r), \end{aligned} \tag{40}$$

where

$$H(r) = \frac{r^2}{2\lambda l^2}\Big(1 - \sqrt{1 - 4\lambda\Big(1 - \frac{ml^2}{r^4} - 16\pi G\frac{I(r)l^2}{3r^4}\Big)}\Big), \tag{41}$$

$$I(r) = 2\int dr r^3\Big(\frac{f(r)^2}{4} + 3\varepsilon f(r)^4\Big), \tag{42}$$

and m is an integration constant, which is related to the mass of the black hole solution, $f(r)$ is given by the root of

$$8\varepsilon f(r)^3 + f(r) - \frac{Q}{r^3} = 0. \tag{43}$$

Here Q is the electric charge of the black hole. The horizon r_+ corresponds to the biggest root of $H(r) = 0$, that is to say at r_+, one has

$$1 - \frac{ml^2}{r_+^4} - 16\pi G \frac{I(r_+)l^2}{3r_+^4} = 0. \tag{44}$$

Before calculating the ratio η/s, we first consider the near boundary behavior of the solution to get the causality constraint for dual field theory. Although the solution of $f(r)$ and $I(r)$ looks complicated, we can see from (43) that while $r \to \infty$, $f(r) \sim Q/r^3$, $I \sim -Q^2/4r^3$. Then the solution near the boundary becomes the same as the one without F^4 terms [39]. As a result, we obtain the same causality constraint as in [39]. Following [33,39], we can calculate the local "speed of graviton"

$$c_g^2(r) = M_2^2 \frac{1 - \sqrt{1 - 4\lambda + M_1}}{2\lambda}\left(3 - 2\frac{1 - 4\lambda + M_2}{1 - 4\lambda + M_1}\right), \tag{45}$$

where

$$M_1 = 16\pi G \frac{4\lambda l^2}{3r^4} I + \frac{4\lambda l^2 m}{r^4} \tag{46}$$

and

$$M_2 = 16\pi G \frac{2\lambda l^2}{3}\left(\frac{f^2}{4} + 3\varepsilon f^4\right). \tag{47}$$

Near the boundary, $f(r) \sim Q/r^3$, $I \sim -Q^2/4r^3$, and $c_g^2(r)$ can be simplified to be

$$c_g^2(r) - 1 = \left(-\frac{5}{2} + \frac{2}{1 - 4\lambda} - \frac{1}{2\sqrt{1 - 4\lambda}}\right)\frac{ml^2}{r^4} + \mathcal{O}\left(\frac{1}{r^5}\right). \tag{48}$$

With this, we obtain the condition to avoid the causality violation

$$-\frac{5}{2} + \frac{2}{1 - 4\lambda} - \frac{1}{2\sqrt{1 - 4\lambda}} < 0. \tag{49}$$

This is the same result as in [33], which implies that there is a condition on the Gauss-Bonnet coefficient $\lambda < 0.09$ in order for the dual theory to obey the causality law.

Now we turn to the η/s ratio. We perform a coordinate transformation $u = r_+^2/r^2$ on (40), which leads to

$$ds^2 = -V(u)N^2 dt^2 + \frac{r_+^2}{4u^3 V(u)} du^2 + \frac{r_+^2}{ul^2} d\vec{x}^2, \tag{50}$$

where $V(u)$ is just the function obtained by changing the variable r in $H(r)$ to u. Putting this metric into the formula (37), we have

$$K_{\rm eff} = -\frac{1}{2}\left(1 - 4\lambda + 2\lambda 16\pi G \frac{I'(1)l^2}{3r_+^4}\right), \tag{51}$$

where

$$I'(1) = -r_+^4\left(\frac{f(u)^2}{4} + 3\varepsilon f(u)^4\right)\bigg|_{u=1}. \tag{52}$$

Note that the area formula of the Bekenstein-Hawking entropy still holds for Ricci-flat black holes in the Gauss-Bonnet gravity [44]. We get the ratio of shear viscosity over entropy density by inserting the root of (43) into (26) and (37)

$$\frac{\eta}{s} = -\frac{K_{\rm eff}}{2\pi} = \frac{1}{4\pi}\left(1 - 4\lambda\left[1 - \frac{8\pi G l^2}{3}\left(\frac{f_+^2}{4} + 3\varepsilon f_+^4\right)\right]\right), \tag{53}$$

where f_+ denotes $f(u)|_{u=1}$, which is the root of the cubic equation (43) at $r = r_+$.

The temperature of the black hole is easy to calculate as

$$T = \frac{1}{2\pi\sqrt{g_{rr}}} \frac{d\sqrt{-g_{tt}}}{dr}\bigg|_{r=r_+}, \tag{54}$$

which gives

$$T = \frac{r_+}{\pi l^2}\left[1 - \frac{8\pi G l^2}{3}\left(\frac{f_+^2}{4} + 3\varepsilon f_+^4\right)\right]. \tag{55}$$

Then the ratio of η/s (53) can be rewritten as

$$\frac{\eta}{s} = -\frac{K_{\rm eff}}{2\pi} = \frac{1}{4\pi}\left(1 - \frac{4\lambda\pi l^2}{r_+} T\right). \tag{56}$$

In fact, this relation can also be deduced from the formulas for $K_{\rm eff}$ (37) and T (54). We can see that the η/s ratio depends on the temperature apparently. As $T \to 0$, the corrections to η/s vanish. Note that although the limit $T \to 0$ is well defined in (56), in fact, some calculations in the above are not valid for extremal black holes since we start from the metric assumption (32) for a nonextremal black hole.

Now we analyze the correction of the F^4 term to the ratio of shear viscosity to entropy density. To do so, we have to study the behavior of the factor $\frac{f_+^2}{4} + 3\varepsilon f_+^4$, in which f_+ depends on r_+ through Eq. (43). We define a new function $P(f)$ as

$$P(f) \equiv 8\varepsilon f^3 + f = \frac{Q}{r^3}. \tag{57}$$

In Fig. 1, we plot $P(f)$ as a function of f. In the plot, the red (vertical) curve denotes $P(f)$ as a function of f in the case of $\varepsilon > 0$, while the blue (horizontal) curve for the case of $\varepsilon < 0$. We can see from the right-hand side of (57) that $P(f) \to 0$ when $r \to \infty$. The boundary condition $f \to 0$ as $r \to \infty$ implies that when $P(f) \to 0$, f must approach to zero, too. Thus the physical part of the curve of $P(f)$ should start from the origin in the figure.

When $Q > 0$, one has $P(f) > 0$. In addition, f should approach to 0 as $r \to \infty$. Therefore, the curves in the region of $P \geq 0$ and $f \geq 0$ correspond to physical solutions in Fig. 1. Namely, in the case of $\varepsilon > 0$, the right-hand part of

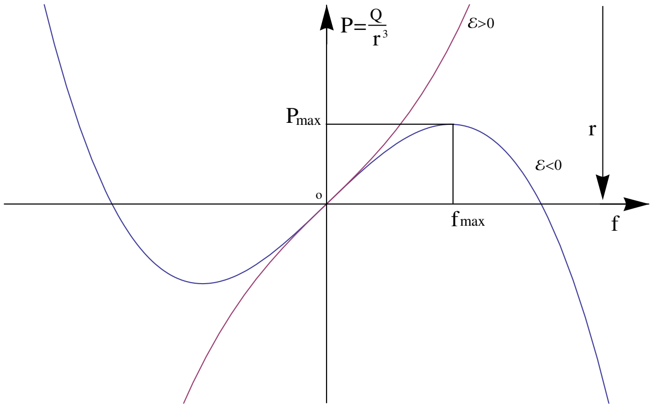

FIG. 1 (color online). $P(f) = 8\varepsilon f^3 + f$ with $\varepsilon > 0$ and $\varepsilon < 0$. The red (vertical) curve denotes $P(f)$ as a function of f in the case of $\varepsilon > 0$ and the blue (horizontal) curve for the case of $\varepsilon < 0$. The fact that $P(f) \to 0$ when $r \to \infty$ and the boundary condition $f \to 0$ as $r \to \infty$ gives that when $P(f) \to 0$, f must approach to zero, too. Thus the physical part of the curve of $P(f)$ must start from the origin in the figure. For $Q > 0$, the curve of $P(f)$ must be in the above of f-axis. Thus we have $0 < f < \infty$ with $\varepsilon > 0$ and $0 < f \le f_{\max}$ for $\varepsilon < 0$. The behavior of $P(f_+)$ as a function of f_+ is the same as $P(f)$.

the red (vertical) curve is of physical meaning, while in the case of $\varepsilon < 0$, the blue (vertical) curve starting from the origin to its peak is physical. $P(f)$ is

$$P_{\max} = \frac{1}{3}\sqrt{\frac{1}{-6\varepsilon}} = \frac{Q}{r_{\min}^3} \tag{58}$$

at the peak, where $f^2 = f_{\max}^2 = -1/24\varepsilon$. This implies that in this case, there is a minimal horizon radius $r_{\min}^3 = Q\sqrt{-6\varepsilon}/3$.

For $Q < 0$, the situation is similar. Without loss of generality, we therefore consider the case of $Q > 0$ only.

The case of $\varepsilon > 0$ is simple. For extremal black holes, one has $T = 0$, while for large black holes, the temperature (54) has the behavior $T = r_+/\pi l^2$ and $f_+ \to 0$. Therefore, in this case, the ratio is in the range from $1/4\pi$ to $(1 - 4\lambda)/4\pi$.

When $\varepsilon < 0$, one has $0 \le f_+^2 \le -1/24\varepsilon$. In this case, the Hawking temperature is in the range from $r_+/\pi l^2$ to $r_+(1 + \pi G l^2/72\varepsilon)/\pi l^2$. In order for the temperature to be positive, one has to impose the constraint $\varepsilon < -\pi G l^2/72$. This constraint excludes the existence of extremal black holes, which requires $\varepsilon > -\pi G l^2/72$. As a result, if $\varepsilon < -\pi G l^2/72$, the ratio is in the range

$$\frac{1}{4\pi}(1 - 4\lambda) \le \frac{\eta}{s} \le \frac{1}{4\pi}\left(1 - 4\lambda - \lambda\frac{\pi G l^2}{18\varepsilon}\right), \tag{59}$$

while if $0 > \varepsilon > -\pi G l^2/72$, due to the existence of an extremal black hole, the situation is the same as the case of $\varepsilon > 0$. Namely, the ratio is in the range

$$\frac{1}{4\pi}(1 - 4\lambda) \le \frac{\eta}{s} \le \frac{1}{4\pi}. \tag{60}$$

As a result, we see that for the arbitrary value of ε, the effect of correction from F^2 and F^4 terms is to alleviate the violation to the universal shear viscosity bound. The ratio ranges from $(1 - 4\lambda)/4\pi$ to $1/4\pi$ for $\varepsilon \ge -\pi G l^2/72$, and from $(1 - 4\lambda)/4\pi$ to $(1 - 4\lambda - \lambda\pi G l^2/18\varepsilon)/4\pi$ for $\varepsilon < -\pi G l^2/72$. The range of the ratio is the same to the case without F^4 terms when $\varepsilon \ge -\pi G l^2/72$. When $\varepsilon < -\pi G l^2/72$, due to the existence of (nonextremal) minimal black holes whose temperature is larger than zero, F^4 terms lead the ratio of η/s to be always smaller than $1/4\pi$.

V. CONCLUSIONS AND DISCUSSIONS

In a general form, we calculated the shear viscosity through AdS/CFT by calculating the on-shell action of transverse gravitons and confirmed the argument proposed in [35] that the value of η is fully determined by effective couplings of transverse gravitons on the horizon. Then we calculated the effective couplings of Einstein gravity and AdS Gauss-Bonnet gravity with matter fields minimally coupled to the metric separately. We applied the resulting formula for the shear viscosity to the case of AdS Gauss-Bonnet-Maxwell theory with F^4 terms correction of Maxwell field and found that both the F^4 terms and the F^2 terms together give a positive η/s correction, compared to the case without Maxwell field. The ratio ranges from $(1 - 4\lambda)/4\pi$ to $1/4\pi$ for $\varepsilon \ge -\pi G l^2/72$. When $\varepsilon < -\pi G l^2/72$, the correction makes η/s range from $(1 - 4\lambda)/4\pi$ to $(1 - 4\lambda - \lambda\pi G l^2/18\varepsilon)/4\pi$, which is always smaller than $1/4\pi$.

We have learned that the universality of $\eta/s = 1/4\pi$ is valid only for duals of Einstein gravity with arbitrary matter minimally coupled to gravity. Clearly, in a general gravity theory, the effective coupling of transverse gravitons on the horizon can be smaller or bigger than the corresponding value in Einstein gravity. As a result, the universality of $\eta/s = 1/4\pi$ must be violated in a general gravity theory. So far, most studies have been focused on the correction to the universal value $1/4\pi$ due to high derivative terms of gravity, while matter fields are still minimally coupled to gravity. It would be very interesting to see the effect of the nonminimal coupling of matter fields to gravity on the shear viscosity of dual field theory.

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APPENDIX

In this appendix, we argue that the on-shell action on the boundary is just (24) after the Gibbons-Hawking boundary term is included. We assume that the effective action of gravity is

$$S_{\rm bulk} = \int d^5x\sqrt{-g}\mathcal{L}. \tag{A1}$$

The variation of this action of gravity with a boundary ∂M is

$$\delta S_{\rm bulk} = \int d^5x\sqrt{-g}G_{\mu\nu}\delta g^{\mu\nu} + \int_{\partial M} d^4xB.$$

Here B is a boundary contribution whose existence originates from the fact that the derivatives of $\delta g^{\mu\nu}$ are not fixed to 0 on the boundary. Then we have to choose a Gibbons-Hawking term to cancel the contribution of B. We assume the Gibbons-Hawking term to be

$$S_{\rm GB} = \int_{\partial M} d^4xC. \tag{A2}$$

Then if we choose

$$\delta S_{\rm GB} = \int_{\partial M} d^4x\delta C = -\int_{\partial M} d^4xB, \tag{A3}$$

the contribution of B can be eliminated. Here we consider the case where only $\delta g^{xy} \neq 0$ and we choose $g^{xy} = -g^{xx}\phi$ as in the previous sections. Thus the variation of the action related to δg^{xy} should be

$$\delta S_{\rm bulk} = \int d^5x\sqrt{-g}G_{xy}\delta g^{xy} + \int_{\partial M} d^4xB(\delta g^{xy}). \tag{A4}$$

The Gibbons-Hawking term should be designed to eliminate B here, and this should be valid to any order of ϕ. To be consistent with the previous sections, we choose all the functions of g^{xy} here expanded to the second order of ϕ. Then $G_{xy} = 0$ is just the equation of motion (7) for ϕ. We consider our effective action (5) for ϕ, and to keep the equation of motion unaffected, the full bulk part can be the action (5) plus a total derivative

$$S_{\rm bulk} = \frac{1}{16\pi G}\int\frac{dwdp}{(2\pi)^2}du[\sqrt{-g}(K(u)\phi'\phi' + w^2K(u)g_{0uu}g_0^{00}\phi^2 - p^2L(u)\phi^2) + G(\phi, \phi')'], \tag{A5}$$

where $G(\phi, \phi')'$ denotes this total derivative. The variation of this action is then

$$\delta S_{\rm bulk} = \frac{1}{16\pi G}\int\frac{dwdp}{(2\pi)^2}du[(EOM)\delta\phi + 2(K(u) \times\sqrt{-g}\phi'\delta\phi)' + (\delta G(\phi, \phi'))']. \tag{A6}$$

Because we have $\delta\phi = 0$ on the boundary, the term $2(K(u)\sqrt{-g}\phi'\delta\phi)'$, which involves $\delta\phi$, vanishes on the boundary after using the Stokes theorem. Thus the Gibbons-Hawking term C should obey

$$\delta C + \delta G(\phi, \phi') = 0 \tag{A7}$$

on the boundary. To simplify the expression we have chosen $16\pi G = 1$. Thus we can choose

$$C + G(\phi, \phi') = 0. \tag{A8}$$

Thus after integration by parts, the total on-shell effective action becomes a full surface term

$$S_{\text{on-shell}} = S_{\rm bulk} + S_{\rm GB} = \frac{1}{16\pi G}\int\frac{dwdp}{(2\pi)^2}((\sqrt{-g}K(u)\phi'\phi))\Big|_{u=1}^{u=0} \tag{A9}$$

once the Gibbons-Hawking term is included.

In addition, we would stress that because we consider at most two-order derivatives of ϕ in the action, $G(\phi, \phi')$ can be written as a sum of two kinds of terms:

$$G(\phi, \phi') = G_1(u)\phi^2 + G_2(u)\phi\phi'. \tag{A10}$$

$\delta(G_1(u)\phi^2)$ vanishes on the boundary, so the Gibbons-Hawking term C would not involve this part. Thus in the on-shell surface contributions an additional $G_1(u)\phi^2$ term may also exist in (A9). However, this term only contributes a real part to the Green function and thus would not affect the value of η, so we can ignore this term in the calculations.

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