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Moduli spaces of Chern-Simons quiver gauge theories and $\mathrm{AdS}_4/\mathrm{CFT}_3$

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We analyze the classical moduli spaces of supersymmetric vacua of 3D $\mathcal{N} = 2$ Chern-Simons quiver gauge theories. We show quite generally that the moduli space of the 3D theory always contains a baryonic branch of a parent 4D $\mathcal{N} = 1$ quiver gauge theory, where the 4D baryonic branch is determined by the vector of 3D Chern-Simons levels. In particular, starting with a 4D quiver theory dual to a 3-fold singularity, for certain general choices of Chern-Simons levels this branch of the moduli space of the corresponding 3D theory is a 4-fold singularity. Our results lead to a simple general method, using existing 4D techniques, for constructing candidate 3D $\mathcal{N} = 2$ superconformal Chern-Simons quivers with AdS_4 gravity duals. As simple, but nontrivial, examples, we identify a family of Chern-Simons quiver gauge theories which are candidate $\mathrm{AdS}_4/\mathrm{CFT}_3$ duals to an infinite class of toric Sasaki-Einstein seven-manifolds with explicit metrics.

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I. INTRODUCTION

Three-dimensional Chern-Simons (CS) gauge theories coupled to matter, with $\mathcal{N} = 2$ supersymmetry or higher, have recently attracted considerable attention, as prominent candidates for field theory duals of AdS_4 vacua of string and M theory [1]. The simplest examples of these vacua are Freund-Rubin $\mathrm{AdS}_4 \times Y_7$ solutions of 11-dimensional supergravity, where Y_7 is a Sasaki-Einstein seven-manifold (or orbifold). Such backgrounds are expected to be anti–de Sitter/conformal field theory (AdS/CFT) dual to the field theory on a large number of M2-branes at a Calabi-Yau (CY) 4-fold singularity. One would then like to answer the question: what are the field theory duals of such solutions? Of course this hinges on the open problem of what are the degrees of freedom on the M2-branes. Progress in this direction has been made in the recent work of Aharony *et al.* (ABJM) [2]. The authors of this reference have identified the gauge theory duals of a class of $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ backgrounds, showing that these are $\mathcal{N} = 6$ (or $\mathcal{N} = 8$) Chern-Simons quivers with two nodes and Chern-Simons levels $(k, -k)$. In fact, the quiver itself is precisely the same as the 4D $\mathcal{N} = 1$ model of [3].

The corresponding situation in type IIB string theory is understood rather better. Here one can construct large classes of $\mathcal{N} = 1$ $\mathrm{AdS}_5/\mathrm{CFT}_4$ duals by considering N D3-branes placed at a conical Calabi-Yau 3-fold singularity X. In many cases the gauge theory may be constructed from the open string degrees of freedom living on the (fractional) branes. In these examples the dual theory is described by a 4D $\mathcal{N} = 1$ quiver gauge theory. The moduli space of vacua of these theories contains a branch (the mesonic branch) which is a symmetric product of the Calabi-Yau singularity X one started with. The gravity dual is then expected to be $\mathrm{AdS}_5 \times Y_5$, where Y_5 is the Sasaki-Einstein base of the Calabi-Yau cone $X = C(Y_5)$, thus closing the circle. The key difference with the M theory setup described in the paragraph above is that D-branes in string theory are currently understood in much greater detail than M-branes in M theory.

In this paper we analyze the classical vacuum moduli spaces (VMS) of $\mathcal{N} = 2$ Chern-Simons quiver gauge theories with arbitrary CS levels. These spaces in general may be rather complicated, containing several branches (i.e. Coulomb, Higgs, or mixed branches). However, motivated by the situation in 4D and the CS quiver theory of [2], we will focus our attention on a particular branch of these theories. If the CS quiver theories we discuss indeed have an interpretation in terms of M2-branes at a CY 4-fold singularity, we believe it is this branch that should reproduce the CY 4-fold as the moduli space of the transverse M2-branes. For simplicity we will take all ranks of the gauge groups equal to N and denote this by $U(N)_{k_1} \times \cdots \times U(N)_{k_n}$, although the results we describe may be easily generalized to the case of arbitrary ranks. We begin with the Abelian theory $N = 1$. We show that the VMS contains a branch that is closely related to the moduli space of a parent 4D $\mathcal{N} = 1$ quiver theory, in a sense that we shall explain more precisely during the course of the paper. In particular, when this parent quiver theory arises from a 3-fold singularity, for certain general choices of Chern-Simons levels the corresponding 3D theory has a branch of the moduli space which is a 4-fold singularity. The discussion is extended to the non-Abelian theories with little modification.

Note that, *a priori*, it is not clear what are the conditions that a Chern-Simons quiver should satisfy in order to flow to a superconformal fixed point in the infrared (IR). The situation ought to be more subtle than is the case in four

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dimensions, where anomalies, NSVZ beta functions, and a-maximization [4] provide important constraints on the IR dynamics.

The results of this paper are a first key step towards identifying candidate $\mathcal{N}=2$ conformal Chern-Simons quiver gauge theories with $AdS_4 \times Y_7$ gravity duals. In particular, they suggest a general method for constructing 3D Chern-Simons quiver gauge theories arising from M2-branes at a given Calabi-Yau 4-fold singularity. As an application, we discuss a family of Chern-Simons quiver gauge theories that are candidate duals to an infinite family of explicit Sasaki-Einstein seven-manifolds, constructed in [5] and further analyzed in [6].

The plan of the rest of the paper is as follows. In Sec. II we recall the field content and Lagrangian of $\mathcal{N}=2$ Chern-Simons theories, with product gauge group and bifundamental matter, i.e. Chern-Simons quiver gauge theories. In Sec. III we analyze the VMS of Abelian quivers. Section IV describes the extension to non-Abelian gauge groups. In Sec. V we discuss the relevance of our results for the construction of superconformal Chern-Simons quivers with AdS_4 duals. Section VI presents an infinite family of Chern-Simons quiver gauge theories which are candidate AdS_4/CFT_3 duals to a corresponding family of explicit Sasaki-Einstein seven-manifolds.

II. FIELD CONTENT AND LAGRANGIANS

We largely follow the notation and discussion in [1,2,7]. A 3D $\mathcal{N}=2$ vector multiplet V consists of a gauge field A_μ, a scalar field σ, a two-component Dirac spinor χ, and another scalar field D, all transforming in the adjoint representation of the gauge group G. This is simply the dimensional reduction of the usual 4D $\mathcal{N}=1$ vector multiplet. In particular, σ arises from the zero mode of the component of the vector field in the direction along which we reduce. The matter fields Φ_a are chiral multiplets, consisting of a complex scalar ϕ_a, a fermion ψ_a, and an auxiliary scalar F_a, which may be in arbitrary representations $\mathcal{R}_a$ of G. An $\mathcal{N}=2$ Lagrangian then consists of the three terms

$$S = S_{\rm CS} + S_{\rm matter} + S_{\rm potential}. \tag{2.1}$$

We describe each of these in turn.

We will be interested in product gauge groups of the form

$$G = \prod_{i=1}^{n} U(N_i). \tag{2.2}$$

It will turn out to be important to allow different Chern-Simons levels k_i for each factor $U(N_i)$. If V_i denotes the projection of V onto the ith gauge group factor, then in component notation the Chern-Simons Lagrangian, in Wess-Zumino gauge, takes the form

$$S_{\rm CS} = \sum_{i=1}^{n} \frac{k_i}{4\pi} \int {\rm Tr}\Big(A_i \wedge dA_i + \frac{2}{3} A_i \wedge A_i \wedge A_i - \bar{\chi}_i \chi_i + 2D_i\sigma_i\Big). \tag{2.3}$$

The Chern-Simons levels k_i are quantized. In particular, for $U(N_i)$ or $SU(N_i)$ gauge group $k_i \in \mathbb{Z}$ are integers if the trace in (2.3) is normalized in the fundamental representation.

The matter (kinetic) term takes a simple form in superspace, namely,

$$\begin{aligned} S_{\rm matter} &= \int d^3x d^4\theta \sum_a {\rm Tr}\bar{\Phi}_a e^V \Phi_a \\ &= \int d^3x \sum_a \mathcal{D}_\mu \bar{\phi}_a \mathcal{D}^\mu \phi_a - \bar{\phi}_a \sigma^2 \phi_a + \bar{\phi}_a D \phi_a \\ &\quad + \text{fermions}, \end{aligned} \tag{2.4}$$

where in the second line we have expanded into component fields, and we have not written the terms involving the fermions ψ_a. The auxiliary fields σ and D are here understood to act on ϕ_a in the appropriate representation $\mathcal{R}_a$, just as for the covariant derivatives $\mathcal{D}_\mu$ which contain the gauge field A_μ.

The superpotential term is

$$\begin{aligned} S_{\rm potential} &= \int d^3x d^2\theta W(\Phi) + \text{c.c.} \\ &= -\int d^3x \sum_a \Big|\frac{\partial W}{\partial \phi_a}\Big|^2 + \text{fermions}. \end{aligned} \tag{2.5}$$

At this stage we take the superpotential to be an arbitrary polynomial in the scalar fields ϕ_a, and we have included the couplings in the definition of W. Notice that the coupling constants are in general not related to the Chern-Simons levels, as is necessarily the case for $\mathcal{N}=3$ supersymmetry [7]. In particular, they may be renormalized in the IR.

The resemblance of these theories to 4D $\mathcal{N}=1$ theories should be apparent. Notice, however, that there are no kinetic terms for the gauge fields, which are replaced by the CS terms. The fields in the vector multiplets are therefore auxiliary fields.

III. ABELIAN CHERN-SIMONS QUIVERS

Recall that a quiver is a directed graph on n nodes, with arrow set $\mathcal{A}$ and head and tail maps h, t: $\mathcal{A} \to \{1, 2, \ldots, n\}$. In general we associate a gauge group factor $U(N_i)$ to node $i \in \{1, \ldots, n\}$, with the chiral field Φ_a transforming in the fundamental representation of the gauge group at node $h(a)$ and the antifundamental representation of the gauge group at node $t(a)$. The gauge group is thus given by (2.2). The superpotential W is constructed as the trace of a sum of closed oriented paths in the quiver.

The coefficients in this sum are the (classical) superpotential couplings.

We begin by specializing to the Abelian case with $N_i = 1$ for all i, so that the gauge group is simply

$$G = U(1)^n. \tag{3.1}$$

All of the gauge fields A_i are hence Abelian. The labels $a \in \mathcal{A}$ on the chiral fields Φ_a run over arrows in the quiver, and Φ_a has charge $+1$ under $U(1)_{h(a)}$ and charge -1 under $U(1)_{t(a)}$. Furthermore, the auxiliary fields σ and D are then n-component fields, σ_i and D_i.

The potential $\mathcal{V}$ for the theory is a sum of a D-term potential and an F-term potential [given by (2.5)], so that

$$\mathcal{V} = \mathcal{V}_D + \mathcal{V}_F. \tag{3.2}$$

Here we have defined

$$\mathcal{V}_F = \sum_{a\in\mathcal{A}} \left|\frac{\partial W}{\partial \phi_a}\right|^2, \tag{3.3}$$

whereas the D-term potential takes the form

$$\mathcal{V}_D = -\sum_{i=1}^{n} \frac{k_i}{2\pi} D_i \sigma_i + \sum_{a\in\mathcal{A}} |\phi_a|^2 (\sigma_{h(a)} - \sigma_{t(a)})^2 - \sum_{a\in\mathcal{A}} |\phi_a|^2 (D_{h(a)} - D_{t(a)}). \tag{3.4}$$

Here the first term comes from the CS action (2.3), whereas the second and third terms come from the matter action (2.4). We may rewrite the last term in (3.4) as

$$\begin{aligned} -\sum_{a\in\mathcal{A}} |\phi_a|^2 (D_{h(a)} - D_{t(a)}) &= -\sum_{i=1}^{n} D_i \Big[\sum_{a|h(a)=i} |\phi_a|^2 - \sum_{a|t(a)=i} |\phi_a|^2 \Big] \\ &= \sum_i D_i \mathcal{D}_i, \end{aligned} \tag{3.5}$$

where we have defined the usual 4D $\mathcal{N} = 1$ D term as

$$\mathcal{D}_i = -\sum_{a|h(a)=i} |\phi_a|^2 + \sum_{a|t(a)=i} |\phi_a|^2. \tag{3.6}$$

Integrating out the auxiliary fields D_i then immediately gives

$$\mathcal{D}_i = \frac{k_i \sigma_i}{2\pi}, \tag{3.7}$$

where there is no summation on the right-hand side. Notice that on summing the equalities in (3.7) over all the nodes of the quiver, the left-hand side vanishes. This follows from the fact that nothing is charged under the overall diagonal $U(1)$. We thus find the condition

$$\sum_{i=1}^{n} k_i \sigma_i = 0. \tag{3.8}$$

Substituting (3.7) back into the action the terms involving D_i cancel, because the potential is linear in D_i, leaving only the second term in (3.4). Thus

$$\mathcal{V}_D = \sum_{a\in\mathcal{A}} |\phi_a|^2 (\sigma_{h(a)} - \sigma_{t(a)})^2. \tag{3.9}$$

A. Supersymmetric vacua

In vacuum the fermions are all set to zero, with the scalar fields taking constant vacuum expectation values (VEVs). The potential $\mathcal{V}$, since it is manifestly non-negative, then has an absolute minimum at zero. In fact since both $\mathcal{V}_D$ (3.9) and $\mathcal{V}_F$ (3.3) are both non-negative, each must vanish separately in a supersymmetric vacuum.

The F-term equations are simply

$$\frac{\partial W}{\partial \phi_a} = 0. \tag{3.10}$$

This defines an affine algebraic set

$$\mathcal{Z} = \{dW = 0\} \subset \mathbb{C}^M, \tag{3.11}$$

where in the Abelian case at hand $M = |\mathcal{A}|$. This is exactly as for 4D $\mathcal{N} = 1$ quiver gauge theories.

We next turn to the D-term equations. Again, since (3.9) is a sum of non-negative terms, the potential is minimized at zero. One set of solutions is clearly given by

$$\sigma_1 = \sigma_2 = \cdots = \sigma_n \equiv s. \tag{3.12}$$

Here $s \in \mathbb{R}$ is arbitrary. As will become clear, this is an interesting branch of the moduli space, since the final result when the corresponding 4D quiver theory is dual to a 3-fold singularity will be a 4-fold singularity. In general there could be other branches, obtained by instead setting certain $\phi_a = 0$ and thus allowing for more general σ_i. It is simple to write down examples of quivers which have such branches. However, we believe that for the quivers relevant for the AdS_4/CFT_3 correspondence, it is the above branch that should reproduce the CY 4-fold geometry as the moduli space of transverse M2-branes. In any case, we will not consider the other branches of the VMS, if indeed there are any, in the present paper.

The conditions (3.7) then become

$$\mathcal{D}_i = \frac{k_i s}{2\pi}. \tag{3.13}$$

Note then that (3.8) implies

$$\sum_{i=1}^{n} k_i = 0. \tag{3.14}$$

This is hence a necessary condition on the Chern-Simons levels for a Chern-Simons quiver theory to admit the above vacua with $s \neq 0$. If (3.14) does not hold then s is identically zero and note that we reduce to the usual 4D space of D-term equations with zero Fayet-Iliopoulos (FI) parameters. This would usually be called the Higgs branch. Indeed, the VMS equations (3.13) may be regarded as promoting a 4D FI parameter to a VEV. The FI parameter is $\zeta_i = k_i s/2\pi$, and thus the direction is determined by the vector of CS levels, while the scale is determined by the VEV s of the auxiliary scalars. Thus, provided the vector $k = (k_1, \ldots, k_n) \neq 0$ and (3.14) holds, the 3D space of absolute minima of the potential is always one real dimension higher than the 4D space of minima for the corresponding quiver theory.

We may conveniently rewrite the 3D D-term equations (3.13) in a 4D language as follows. We begin by noting that the n-vector k is, more invariantly, an element of the dual Lie algebra $\mathfrak{t}_n^* \cong \mathbb{R}^n$ of $G = U(1)^n$, so

$$k \in \mathfrak{t}_n^*. \tag{3.15}$$

There is hence a kernel

$$\ker(k) \subset \mathfrak{t}_n \cong \mathbb{R}^n, \tag{3.16}$$

given by vectors that contracted with k give zero. Provided $k \neq 0$, this defines a vector subspace of dimension $n - 1$. Then the 3D D-term equations (3.13) may be written as

$$\sum_{i=1}^{n} v_i \mathcal{D}_i = 0, \qquad v \in \ker(k). \tag{3.17}$$

Note that this gives the correct VMS even when $k = 0$. Also notice that the vector $v = (1, 1, \ldots, 1) \in \ker(k)$ if (3.14) holds. Since the D term for this vector, and only for this direction, is identically zero, we see that (3.17) imposes $(n - 2)$ linearly independent equations for $k \neq 0$ satisfying (3.14). In fact from now on we assume the latter conditions to hold.

B. Gauge symmetries

In vacuum the gauge fields are set to zero.[1] Constant gauge transformations therefore map vacua to vacua, and to compute the space of gauge-equivalent solutions we must also identify by these gauge transformations. We have already noted that the overall diagonal $U(1)$ acts trivially, and thus naively it seems one should quotient the space of F-term and D-term solutions by the action of $U(1)^{n-1} \cong U(1)^n/U(1)$ to obtain the VMS. However, there is an immediate problem with this: the VMS would then be odd-dimensional, which is incompatible with supersymmetry. The resolution of this apparent puzzle becomes clear on examining the CS action more carefully, precisely as in [2] (see also [8]).

We define

$$a = \sum_{i=1}^{n} A_i, \qquad b = \frac{1}{h}\sum_{i=1}^{n} k_i A_i, \tag{3.18}$$

where we have introduced

$$h = \gcd\{k_i\}. \tag{3.19}$$

The Abelian CS action for the gauge fields $A = (A_1, \ldots, A_n)$ is

$$S_{\rm CS}(A) = \frac{1}{4\pi}\sum_{i=1}^{n}\int k_i A_i \wedge dA_i. \tag{3.20}$$

Now consider making the simultaneous variations

$$A_i \longrightarrow A_i + \lambda, \qquad i = 1, \ldots, n \tag{3.21}$$

with λ an arbitrary one-form. This induces the variations

$$\delta_\lambda a = n\lambda, \tag{3.22}$$

$$\delta_\lambda b = 0, \tag{3.23}$$

where the second equation follows from (3.14). The variation of the CS action is hence

$$\delta_\lambda S_{\rm CS}(A) = \frac{2}{4\pi}\sum_{i=1}^{n}\int \lambda \wedge k_i dA_i, \tag{3.24}$$

where note there are two terms to vary in each summand, but they give equal contributions after integrating by parts. We may rewrite this as

$$\delta_\lambda S_{\rm CS}(A) = \frac{2h}{4\pi}\int \lambda \wedge db. \tag{3.25}$$

We thus conclude that

$$S_{\rm CS}(A) = \frac{h}{2\pi n}\int b \wedge f + S', \tag{3.26}$$

where we have defined $f = da$, and by definition

$$\delta_\lambda S' = 0. \tag{3.27}$$

Since the overall $U(1)$ decouples from the matter, we see that the first "bf" term in the action (3.26) describes completely the action for the gauge field a. We may thus introduce a Lagrange multiplier

$$S_\tau = -\frac{1}{2\pi}\int d\tau \wedge f \tag{3.28}$$

and treat f, rather than a, as the basic variable. Integrating out f then imposes[2]

$$b = \frac{n}{h} d\tau. \tag{3.29}$$

The gauge invariance of the theory is now

$$b \longrightarrow b + d\theta, \qquad \tau \longrightarrow \tau + \frac{h}{n}\theta, \tag{3.30}$$

$$A_i \longrightarrow A_i + d\theta_i, \qquad \sum_{i=1}^{n} k_i\theta_i = 0. \tag{3.31}$$

[1]We will modify this statement slightly below.

[2]We note a factor of 2 difference with the corresponding equation in [2].

The gauge transformations (3.31) are precisely those that do not act on b. The transformation (3.30) of b instead arises from

$$A_i \to A_i + d\theta_i, \qquad \sum_{i=1}^{n} k_i\theta_i = h\theta. \tag{3.32}$$

Consider now the character

$$\chi_k\colon U(1)^n \to U(1)$$
$$;(\mathrm{e}^{i\theta_1}, \ldots, \mathrm{e}^{i\theta_n}) \mapsto \exp\left(i\sum_{i=1}^{n} k_i\theta_i\right). \tag{3.33}$$

The gauge transformation of b in (3.30) thus maps to $\exp(ih\theta)$. This lies in the kernel of (3.33) if and only if

$$\theta = \frac{2\pi l}{h}, \tag{3.34}$$

where $l = 1, \ldots, h$. On the other hand, if we assume for the moment that τ has period $2\pi/n$, then gauge fixing $\tau = 0$ leaves a residual gauge symmetry in (3.30) that is precisely the same as (3.34). The transformations (3.31) also lie in the kernel of (3.33) of course. Thus, assuming that τ has period $2\pi/n$, we see that the group of constant gauge transformations acting on the VMS is precisely the kernel of (3.33). This defines an Abelian group $\ker\chi_k \subset U(1)^n$ of rank $n-1$. Note that due to (3.14) this contains the overall diagonal $U(1)$, which acts trivially. Thus the effectively acting group of gauge symmetries is the quotient

$$H_k = \ker\chi_k/U(1) \cong U(1)^{n-2} \times \mathbb{Z}_h. \tag{3.35}$$

It thus remains to justify that the period of τ is indeed $2\pi/n$.[3] As is well-known, the periodicity for τ is related to the flux quantization condition on f via the coupling (3.28). In the above vacua we have set all gauge fields to zero, and thus $f = 0$. However, since nothing is charged under the overall diagonal $U(1)$ gauge group, one may in fact turn on a diagonal gauge field in the above vacua. To see this, note that with nonzero gauge fields but constant ϕ_a there is an additional term in the expression for energy

$$\sum_{a\in\mathcal{A}} |\phi_a|^2 (A_{h(a)} - A_{t(a)})^2. \tag{3.36}$$

This comes directly from the kinetic term for the ϕ_a. Thus, in Euclidean signature, and on the branch we consider, the total energy of the vacuum vanishes if and only if $A_1 = \cdots = A_n$, which is a diagonal flux.[4] Note this is closely related to (3.12). The quantization condition on each F_i is the usual Dirac condition

$$\frac{1}{2\pi}\int_\Sigma F_i \in \mathbb{Z}, \tag{3.37}$$

where Σ is any two-cycle. If Σ is a two-sphere in $\mathbb{R}^3$, such a flux would signify the presence of magnetic monopoles inside this two-sphere. Since all F_i are equal, we thus see that

$$\frac{1}{2\pi}\int_\Sigma f \in n\mathbb{Z}, \tag{3.38}$$

which then leads to a period of $2\pi/n$ for τ. Note that this analysis depends on the branch of the vacuum moduli space we are considering. On different branches, the periodicity of τ may *a priori* be different.

The 3D VMS, or at least the branch satisfying (3.12), is then the Kähler quotient of the space of F-term solutions $\mathcal{Z}$ by H_k at moment map level zero:

$$\mathcal{M}_{3\mathrm{D}}(k) = \mathcal{Z}//H_k. \tag{3.39}$$

Notice this moduli space is acted on by $U(1) \cong U(1)^{n-1}/H_k$, and that a further Kähler quotient by this $U(1)$ would produce the usual mesonic moduli space of the corresponding 4D theory

$$\mathcal{M}_{4\mathrm{D}} = \mathcal{M}_{3\mathrm{D}}(k)//U(1). \tag{3.40}$$

Indeed, if one introduces an FI parameter $\zeta \in \mathbb{R}$ for this $U(1)$ quotient, via (3.40) one obtains a family of mesonic moduli spaces $\mathcal{M}_{4\mathrm{D}}(\zeta k)$, labeled by ζ. As reviewed, for example, in [9], in general the space of FI parameters for a Kähler quotient is a fan, which is a set of convex polyhedral cones glued together along their boundary faces. Inside each cone the quotient spaces are isomorphic as complex manifolds, but have an induced Kähler class that depends linearly on ζ. As one moves from one cone to another along a boundary wall, the moduli space undergoes a form of small birational transformation called a flip. In the case at hand, the CS vector k picks a particular real line through the origin in the space of FI parameters of the corresponding 4D $\mathcal{N} = 1$ theory, where we may interpret $\zeta = s$. Thus the mesonic spaces for $\zeta > 0$ are all isomorphic, with a Kähler class depending linearly on ζ. This will be a (partial) resolution of the mesonic moduli space with $\zeta = 0$. As one passes to $\zeta < 0$ the moduli space undergoes a flip, with again the moduli spaces for $\zeta < 0$ being all isomorphic and the Kähler class depending linearly on ζ. Thus the 3D VMS (3.39) may be obtained by gluing this one-parameter family of 4D mesonic moduli spaces together, with the $U(1) \cong U(1)^{n-1}/H_k$ fibered over each mesonic space in the family.

We also note that (3.39) may be viewed as a (geometric invariant theory) quotient of $\mathcal{Z}$ by the complexified gauge group

$$H_k^{\mathbb{C}} = (\mathbb{C}^*)^{n-2} \times \mathbb{Z}_h. \tag{3.41}$$

In fact we may define $\mathcal{M}_{3\mathrm{D}}(k)$ as an affine variety via

[3] We note that in [8] the authors stated explicitly that they did not have a field theory explanation for this period in their orbifold models.

[4] Equivalently, this is implied by the equations of motion for the ϕ_a.

$$\mathcal{M}_{3D}(k) = \mathcal{Z}//H_k^{\mathbb{C}} \equiv \mathrm{Spec}\mathbb{C}[\mathcal{Z}]^{H_k^{\mathbb{C}}}. \tag{3.42}$$

The equivalence between the two descriptions is standard—see, for example, [10]. Moduli spaces of quivers with relations were first introduced in [11]. Given a quiver with relations, the moduli spaces in [11] are defined by first picking a character of the gauge group, precisely as in (3.33), and then defining the holomorphic (geometric invariant theory) quotient, with respect to this character $k \in \mathbb{Z}^n$, of the set $\mathcal{Z}$ satisfying the relations. This is very closely related to the moduli space (3.42).[5]

C. Example: The ABJM theory

It is straightforward to recover the results of [2] from the above discussion. The quiver has $n = 2$ nodes with four bifundamental fields, which are grouped into two pairs in conjugate representations of the gauge group $G = U(1)^2$. The vector of CS levels is $(k, -k)$, in the notation of [2], so $h = k$. In this Abelian case the superpotential is identically zero, and thus the space of F-term solutions is $\mathcal{Z} \cong \mathbb{C}^4$. Moreover, the group (3.35) is simply $H_k \cong \mathbb{Z}_k$, and one obtains $\mathcal{M}_{3D}(k) = \mathbb{C}^4/\mathbb{Z}_k$ as the 3D VMS. Note in this example that there are certainly no other branches to the VMS. A further quotient of this space by the relative $U(1)$ gives the conifold singularity,[6] which is of course the mesonic moduli space of the 4D theory [3].

IV. NON-ABELIAN CHERN-SIMONS QUIVERS

We now return to the general case where

$$G = \prod_{i=1}^{n} U(N_i). \tag{4.1}$$

In this case ϕ_a is an $N_{h(a)} \times N_{t(a)}$ matrix, and σ_i and D_i are both $N_i \times N_i$ Hermitian matrices. We denote the gauge indices by α, β, so that, for example, the matrix elements of D_i are denoted $D_{i\alpha\beta}$. Here α, $\beta = 1, \ldots, N_i$, so the range of the gauge indices is understood to depend on i in this notation. Thus

$$(D\phi_a)_{\alpha\beta} = \sum_{\gamma=1}^{h(a)} D_{h(a)\alpha\gamma}\phi_{a\gamma\beta} - \sum_{\delta=1}^{t(a)} D_{t(a)\delta\beta}\phi_{a\alpha\delta}, \tag{4.2}$$

where $\alpha = 1, \ldots, h(a)$, $\beta = 1, \ldots, t(a)$. Note carefully the index structure.

Taking the variation of the scalar potential with respect to $D_{i\alpha\beta}$ thus gives the usual 4D D-term equation

$$\frac{k_i\sigma_i}{2\pi} = -\sum_{a|h(a)=i} \phi_a\phi_a^\dagger + \sum_{a|t(a)=i} \phi_a^\dagger\phi_a \equiv \mathcal{D}_i \tag{4.3}$$

with $k_i\sigma_i$ playing the role of a moment map level. Note there is no sum on i here. Also note that σ_i in (4.3) is indeed Hermitian. Substituting back into the potential, the terms involving D_i again cancel because the potential is linear in D_i. Since σ_i is Hermitian, the potential may be written

$$\mathcal{V}_D = \sum_{a\in\mathcal{A}} \sum_{\alpha=1}^{h(a)} \sum_{\beta=1}^{t(a)} (M_a^\dagger)_{\beta\alpha}(M_a)_{\alpha\beta} = \sum_{a\in\mathcal{A}} \sum_{\alpha=1}^{h(a)} \sum_{\beta=1}^{t(a)} |M_{a\alpha\beta}|^2. \tag{4.4}$$

Here we have defined

$$M_a = \sigma\phi_a, \tag{4.5}$$

which in matrix notation is

$$M_a = \sigma_{h(a)}\phi_a - \phi_a\sigma_{t(a)}, \tag{4.6}$$

or in components

$$M_{a\alpha\beta} = \sum_{\gamma=1}^{h(a)} \sigma_{h(a)\alpha\gamma}\phi_{a\gamma\beta} - \sum_{\delta=1}^{t(a)} \sigma_{t(a)\delta\beta}\phi_{a\alpha\delta}. \tag{4.7}$$

The potential is thus minimized at

$$M_a = 0. \tag{4.8}$$

Recall now that the gauge group $U(N_i)$ acts on σ_i by conjugation. So $g_i \in U(N_i)$ acts as

$$\sigma_i \mapsto g_i\sigma_i g_i^{-1}. \tag{4.9}$$

Since σ_i is Hermitian, it is necessarily diagonalizable by an appropriate choice of g_i. The eigenvalues of σ_i are then of course real, and in this gauge we may write

$$\sigma_{i\alpha\beta} = s_{i\alpha}\delta_{\alpha\beta}, \tag{4.10}$$

where there is no sum, and $s_{i\alpha} \in \mathbb{R}$ are the eigenvalues. In such a gauge choice, which always exists, we have

$$M_{a\alpha\beta} = (s_{h(a)\alpha} - s_{t(a)\beta})\phi_{a\alpha\beta}. \tag{4.11}$$

Again, there is no sum in this formula.

A. Branches of relative dimension 1

There are various ways of satisfying (4.8). One solution is to take

$$s_{i\alpha} = s \tag{4.12}$$

independently of i and α. Since the overall diagonal $U(1)$ decouples, the sum of the *traces* of the 4D D terms $\mathcal{D}_i$ is zero. In fact this may be seen directly in the definition (4.3) on noting that

$$\mathrm{Tr}\,(\phi\phi^\dagger) = \mathrm{Tr}(\phi^\dagger\phi) \tag{4.13}$$

for any $M \times N$ matrix ϕ. In summing the traces of the $\mathcal{D}_i$ the above two terms appear precisely once each for each bifundamental, with opposite sign, hence the result. Thus the branch (4.12) exists as a solution to (4.3) for nonzero s

[5] The moduli spaces in [11] are projective versions of (3.42).
[6] Note the result of this further quotient does not depend on k.

only if

$$\sum_{i=1}^{n} k_i N_i = 0. \tag{4.14}$$

In fact precisely this condition arises also in the mathematics literature [11]. Indeed, notice this branch has one dimension higher than the mesonic moduli space for the 4D theory, precisely as in [11]. Thus when $N_i = N\tilde{N}_i$ this branch, when it exists, is not obviously interpreted as the moduli space of N M2-branes. To complete the discussion of these branches we should also analyze the gauge symmetries. Since the solution space to the D terms above is one dimension higher than the mesonic moduli space, the gauge group we divide by should be codimension one in G. Indeed, notice that picking (4.12) in fact leaves the gauge symmetry group completely unbroken. The discussion is then very similar to the Abelian case. We may introduce the Abelian gauge fields

$$a = \sum_{i=1}^{n} \mathrm{Tr}A_i, \qquad b = \frac{1}{h}\sum_{i=1}^{n} k_i \mathrm{Tr}A_i. \tag{4.15}$$

The Chern-Simons action is

$$S_{\mathrm{CS}}(A) = \frac{1}{4\pi}\sum_{i=1}^{n}\int k_i \mathrm{Tr}\Big(A_i \wedge dA_i + \frac{2}{3}A_i^3\Big). \tag{4.16}$$

Varying

$$A_i \to A_i + \lambda 1_{N_i\times N_i} \tag{4.17}$$

leaves b invariant if (4.14) holds. The variation of the CS action is then

$$\delta_\lambda S_{\mathrm{CS}}(A) = \frac{h}{2\pi}\sum_{i=1}^{n}\int \lambda \wedge db, \tag{4.18}$$

precisely as in the Abelian case, and hence

$$S_{\mathrm{CS}}(A) = \frac{h}{2\pi \sum_{i=1}^{n} N_i}\int b\wedge f + S'. \tag{4.19}$$

Introducing τ precisely as before, and defining $M = \sum_{i=1}^{n} N_i$, the gauge invariance of the theory is

$$b \to b + d\theta, \qquad \tau \to \tau + \frac{h}{M}\theta, \tag{4.20}$$

$$A_i \to g_i A_i g_i^{-1} - i(dg_i)g_i^{-1}, \qquad \prod_{i=1}^{n}(\det g_i)^{k_i} = 1. \tag{4.21}$$

The discussion of monopoles proceeds as before, implying that τ has period $2\pi/M$, and thus the group of constant gauge symmetries H_k that we quotient by is the kernel of the character

$$\chi(k)\colon \prod_{i=1}^{n} U(N_i) \to U(1)$$
$$: (g_1, \ldots, g_n) \mapsto \prod_{i=1}^{n}(\det g_i)^{k_i}. \tag{4.22}$$

Finally, we end up with a moduli space branch that is precisely analogous to the quiver moduli spaces in [11]. In particular, this branch has one complex dimension higher than the mesonic moduli space one obtains by taking a Kähler quotient of the space of non-Abelian F-term solutions by the full gauge group G. This is what the terminology "relative dimension 1" means at the beginning of this subsection.

B. Branches of relative dimension N

Suppose now for simplicity that $N_i = N$ for all i.[7] Then an alternative way to satisfy (4.8) is to take

$$\phi_{a\alpha\beta} = 0, \qquad \alpha \neq \beta, \tag{4.23}$$

$$s_{i\alpha} = s_\alpha, \qquad \forall i. \tag{4.24}$$

This imposes that the bifundamentals ϕ_a are all diagonal, and that the N eigenvalues of σ_i are independent of i. This leads to N VEVs s_α, $\alpha = 1, \ldots, N$. Indeed, note that provided the σ_i are invertible (which at a generic point they will be) we may write (4.8) as

$$\phi_a = \sigma_{h(a)}^{-1}\phi_a\sigma_{t(a)}. \tag{4.25}$$

On diagonalizing each σ_i this implies that if $\phi_{a\alpha\beta} \neq 0$ we must have

$$s_{h(a)\alpha} = s_{t(a)\beta}. \tag{4.26}$$

Thus generic $\{\phi_a\}$ reduce us to the branch in the previous subsection, whereas diagonal, but otherwise generic, ϕ_a lead to (4.23) and (4.24). Note, however, that just as for the Abelian case, we might allow for even less constrained σ by instead further restricting certain subsets of the ϕ_a to be zero. This branch structure thus in general appears rather complicated. However, for now we focus on (4.23) and (4.24).

For generic (pairwise nonequal) eigenvalues in (4.24) the subgroup of the gauge symmetry group G preserving this diagonal gauge choice for σ_i is

$$K = \Big(\prod_{i=1}^{n} U(1)^N\Big) \times S_N \cong U(1)^{nN} \times S_N. \tag{4.27}$$

Here the S_N permutes the diagonal elements of all the matrices, so as to preserve (4.24). When some of the eigenvalues become equal, note that this symmetry group becomes enhanced to a non-Abelian group. By restricting to diagonal bifundamentals (4.23), the superpotential

[7] The generalization to arbitrary N_i should be a straightforward extension.

clearly reduces to N copies of the $N = 1$ superpotential, and thus the space of F-term solutions is simply $\mathcal{Z}^N$. Similarly, the CS action for the gauge group (4.27) is N copies of the Abelian $N = 1$ CS action, with the overall $U(1)$ decoupling in each copy separately. Thus one clearly obtains N copies of the $N = 1$ VMS, with the permutation group S_N in (4.27) simply permuting the copies. Thus this branch of the VMS is the symmetric product

$$\mathcal{M}_{3\mathrm{D},N}(k) = \mathrm{Sym}^N \mathcal{M}_{3\mathrm{D},1}(k) \tag{4.28}$$

where $\mathcal{M}_{3\mathrm{D},1}(k)$ is the Abelian moduli space. Notice this branch is the moduli space found in [2] for the ABJM theory. Note also that this moduli space is N complex dimensions higher than the mesonic moduli space, compared to 1 complex dimension higher for the branch discussed in the previous subsection. It seems reasonable, given the discussion above, that the various branches that generally exist in between these two extremes have relative dimensions between 1 and N, and thus the branch (4.28) is in fact the highest dimensional branch of the full VMS. It has a natural physical interpretation as the moduli space of N pointlike objects on $\mathcal{M}_{3\mathrm{D}}(k) = \mathcal{M}_{3\mathrm{D},1}(k)$. The full VMS appears to be quite a complicated object in general. It would be interesting to investigate more carefully the structure we have outlined above. In particular, there may be a more elegant method for analyzing the full moduli space than the simple discussion above.

V. IN SEARCH OF CONFORMAL CHERN-SIMONS QUIVERS

The results we have discussed so far in this paper are rather general: we have discussed the classical vacuum moduli spaces of $\mathcal{N} = 2$ CS quivers, where the bifundamental matter and superpotential are arbitrary. When the Chern-Simons quiver arises from a parent 4D quiver gauge theory dual to a 3-fold singularity, namely, the matter content and interactions of the 3D theory are formally the same as those of the 4D theory, our results imply that the VMS contains (the symmetric product of) a complex four-dimensional branch of the corresponding baryonic moduli space. More precisely, we have found that a necessary condition for such supersymmetric vacua to exist is that the sum of the CS levels vanishes:

$$\sum_{i=1}^{n} k_i = 0. \tag{5.1}$$

The space $\mathcal{Z}$ of F-term solutions is in general a fairly complicated object, with several branches of different dimension. For the class of 4D quiver gauge theories arising from D3-branes at toric Calabi-Yau singularities, this space has recently been studied in [12].[8] In this reference it is shown that in these examples, with $N = 1$, $\mathcal{Z}$ is a complex $(n + 2)$-dimensional affine toric variety. Moreover, there exists a particular branch (the irreducible component) that is argued to be itself an affine Calabi-Yau toric variety. This may be described as a Kähler quotient at level zero ${}^{\mathrm{irr}}\mathcal{Z} = \mathbb{C}^c//U(1)^{c-n-2}$, where c is a number determined from the data of the quiver. The mesonic moduli space of the theory is obtained by performing a further Kähler quotient, and results in the Calabi-Yau 3-fold

$$\mathcal{M}_{4\mathrm{D}} = {}^{\mathrm{irr}}\mathcal{Z}//U(1)^{n-1}. \tag{5.2}$$

Taking the same quiver and replacing the kinetic terms for the gauge fields with Chern-Simons terms with CS level vector $k = (k_1, \ldots, k_n)$ obeying (5.1), we obtain instead a branch of the 3D VMS, namely,

$$\mathcal{M}_{3\mathrm{D}}(k) = {}^{\mathrm{irr}}\mathcal{Z}//H_k. \tag{5.3}$$

This a Calabi-Yau *4-fold*. To see this, notice that the group $U(1)^{n-1}$ necessarily preserves the holomorphic volume form of ${}^{\mathrm{irr}}\mathcal{Z}$, since $\mathcal{M}_{4\mathrm{D}}$ is Calabi-Yau. Thus, in particular, the subgroup H_k preserves this volume form also.

Ultimately, we are interested in conformal field theories. These are candidate gauge theory duals of AdS_4 vacua of string or M theory. Such CFTs, however, will generically be strongly coupled,[9] and at present there are no techniques available to perform independent field theory calculations. Note this is different from four dimensions, where a-maximization [4] is an important tool for testing the existence of conjectured IR fixed points. Using the AdS/CFT correspondence, the issue of conformal invariance in the IR may be translated into the question of whether the theory has an $\mathrm{AdS}_4 \times Y_7$ gravity dual, where Y_7 is a Sasaki-Einstein seven-manifold. These backgrounds arise as the near-horizon limit of a large number of M2-branes, placed at the singularity of the Calabi-Yau cone $C(Y_7)$. Thus, a necessary condition for this situation to hold is that the 3D gauge theory contains this Calabi-Yau 4-fold as a (generic) component of its VMS. This suggests that (5.1) is in fact a necessary condition for conformal invariance. However, there may be additional conditions, yet to be discovered, that a Chern-Simons quiver gauge theory should satisfy in order to flow to a dual conformal fixed point in the IR. Understanding these conditions is clearly an interesting direction for future research.

Notice that different theories may lead to the same moduli space $\mathcal{M}_{3\mathrm{D}}(k)$. This may sound surprising at first, but one should bear in mind that this phenomenon already exists in 4D. There, Seiberg duality implies that different gauge theories all flow to the same conformal field theory in the IR. In fact, instead of thinking of gauge theory duals of some AdS_5 solution, we should more precisely think of

[8] In [12] this is referred to as the master space, and is denoted $\mathcal{F}^\flat$.

[9] The ABJM theory is a notable exception, since it has a weakly coupled limit for large k.

classes of Seiberg-dual gauge theories. Similar dualities exist for 3D theories—see [13] for a recent discussion. However, we are led to consider the possibility that for appropriate values of the Chern-Simons levels, (infinite) families of 4D quivers (e.g. the $Y^{p,q}$ quivers [14]), may all have the same AdS_4 duals, when viewed as 3D Chern-Simons quivers. It is unclear to us whether this will actually be the case, or rather further analysis will reveal that these quivers do not flow to conformal field theories in three dimensions. It will be interesting to analyze this further.

The results discussed here lead to a simple general method for constructing candidate 3D $\mathcal{N} = 2$ superconformal Chern-Simons quivers with AdS_4 gravity duals, using well-developed 4D techniques.[10] We illustrate this in the following section.

VI. EXAMPLE: CHERN-SIMONS QUIVER GAUGE THEORIES FOR THE $Y^{p,k}(\mathbb{C}P^2)$ METRICS

In this final section we discuss a simple class of examples of the construction described in this paper.[11] These are candidate gauge theory duals of the explicit Sasaki-Einstein metrics presented in [5]. Other examples may be treated in a similar manner—we briefly comment on various simple extensions at the end of the paper.

Recall from [5,6] that the $Y^{p,k}(\mathbb{C}P^2)$ metrics enjoy an $SU(3) \times U(1)^2$ isometry, and that the corresponding Calabi-Yau cones are described by a gauged linear sigma model on $\mathbb{C}^5$, with a set of $U(1)$ charges characterized by two integers. These properties motivate considering a quiver gauge theory with 3 nodes and $SU(3)$ symmetry. As we shall see, this seemingly naive hypothesis leads to a consistent picture. We thus begin with the 4D quiver gauge theory that is AdS_5/CFT_4 dual to the orbifold $S^5/\mathbb{Z}_3$, where the $\mathbb{Z}_3 \subset U(1)$ is embedded along the Hopf $U(1)$. Equivalently, this is the theory on N D3-branes placed at the singularity of the canonical complex cone over $\mathbb{C}P^2$, which is the orbifold $\mathbb{C}^3/\mathbb{Z}_3$. The quiver has 3 nodes, with a $U(N)$ gauge group at each node, and 9 bifundamental fields, X_i, Y_i, Z_i, $i = 1, 2, 3$ going from nodes 1 to 2, 2 to 3, and 3 to 1, respectively. This is shown in Fig. 1.

The superpotential takes the $SU(3)$-invariant form

$$W = \epsilon_{ijk}\mathrm{Tr}(X_i Y_j Z_k). \qquad (6.1)$$

The F-term equations $dW = 0$ are hence

[10]It is only a candidate because it is possible that the 3D theory will have to obey additional properties in order to flow to a dual conformal field theory in the IR, as discussed above. Our analysis here is purely classical.

[11]This section has been added in a revised version (v2) of the paper. In Ref. [15], which appeared before the present version but after the first version, the authors also discuss the quiver below. However, they did not make the connection with the explicit metrics in [5,6].

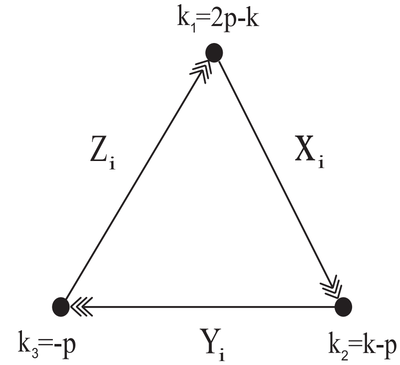

FIG. 1. Quiver diagram for the candidate CS gauge theory duals of $Y^{p,k}(\mathbb{C}P^2)$.

$$X_i Y_j = X_j Y_i, \qquad Y_i Z_j = Y_j Z_i, \qquad Z_i X_j = Z_j X_i. \qquad (6.2)$$

Notice the equations with $i = j$ are redundant. Henceforth we set $N = 1$, so that the bifundamental VEVs are simply coordinates on $\mathbb{C}^9$. Since the equations (6.2) set one monomial equal to another monomial, it is a standard result that the affine variety $\mathcal{Z} = \{dW = 0\} \subset \mathbb{C}^9$ is a toric variety—see, for example, [16].

We may equivalently realize $\mathcal{Z}$ as the affine geometric invariant theory quotient [or equivalently Kähler quotient by $U(1) \subset \mathbb{C}^*$ at level zero]

$$\mathcal{Z} = \mathbb{C}^6//\mathbb{C}^*_{(1,1,1,-1,-1,-1)}. \qquad (6.3)$$

Here the subscript vector denotes the weights of the $\mathbb{C}^*$ action on $\mathbb{C}^6$. Thus, if we introduce coordinates u_i, v_i on $\mathbb{C}^6$, $i = 1, 2, 3$, then the u_i have charges $+1$ and the v_i have charges -1 under the $\mathbb{C}^*$ action. The quotient (6.3) is then defined algebraically as

$$\mathcal{Z} = \mathrm{Spec}\mathbb{C}[u_1, u_2, u_3, v_1, v_2, v_3]^{\mathbb{C}^*}. \qquad (6.4)$$

In words, $\mathcal{Z}$ is the affine variety whose holomorphic functions are precisely the $\mathbb{C}^*$-invariant functions on $\mathbb{C}^6$. This ring of invariant functions is spanned by

$$x_i = u_1 v_i, \qquad y_i = u_2 v_i, \qquad z_i = u_3 v_i. \qquad (6.5)$$

This embeds $\mathcal{Z}$ into $\mathbb{C}^9$, and one easily sees that the relations between the x_i, y_i, and z_i are indeed precisely the F-term relations (6.2). This proves the equivalence of the two descriptions of $\mathcal{Z}$.[12]

For the 3D CS quiver theory, we introduce a CS vector (k_1, k_2, k_3), where $k_3 = -k_1 - k_2$, so that (3.14) holds. In order to obtain the 4D VMS, which is the orbifold $\mathbb{C}^3/\mathbb{Z}_3$, we would quotient $\mathcal{Z}$ by $(\mathbb{C}^*)^3/\mathbb{C}^* \cong (\mathbb{C}^*)^2$. In 4D terms, these are the two anomalous baryonic symmetries of the theory.[13] However, to compute the moduli space of the 3D

[12]See also [12].

[13]There is also a discrete nonanomalous baryonic symmetry. A complete discussion of the discrete symmetries of this theory may be found in [17].

CS theory we instead quotient by the kernel of the map

$$(\mathbb{C}^*)^3 \ni (\lambda_1, \lambda_2, \lambda_3) \mapsto \lambda_1^{k_1}\lambda_2^{k_2}\lambda_3^{k_3} \in \mathbb{C}^*. \qquad (6.6)$$

This kernel, after dividing by the diagonal $\mathbb{C}^*$ which acts trivially, is isomorphic to $\mathbb{C}^* \times \mathbb{Z}_h$, where $h = \gcd(k_1, k_2)$. For simplicity we begin by choosing the CS levels so that $h = 1$. The nontrivial $\mathbb{C}^*$ in the kernel of (6.6) is then generated by the weight vector $(-k_2, k_1, 0)$. The charges of the bifundamentals X_i, Y_i, Z_i under a $\mathbb{C}^*$ action with weights $(q_1, q_2, q_3) \in \mathbb{Z}^3$ are $q_2 - q_1$, $q_3 - q_2$, $q_1 - q_3$, respectively. Thus the charges under the $\mathbb{C}^*$ of interest are $k_1 + k_2$, $-k_1$, $-k_2$, respectively. This determines a $\mathbb{C}^*$ action on $\mathcal{Z}$, which we may lift to an action on $\mathbb{C}^6$ by assigning charges $(k_1 + k_2, -k_1, -k_2, 0, 0, 0)$ to the coordinates $(u_1, u_2, u_3, v_1, v_2, v_3)$ on $\mathbb{C}^6$. Altogether, we thus see that the 3D VMS, for $\gcd(k_1, k_2) = 1$, is the affine quotient of $\mathbb{C}^6$ by $(\mathbb{C}^*)^2$ with charges

$$Q = \begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ k_1 + k_2 & -k_1 & -k_2 & 0 & 0 & 0 \end{pmatrix}. \qquad (6.7)$$

Notice that this quotient preserves the $SU(3)$ symmetry. We now make an $SL(2, \mathbb{Z})$ transformation via

$$\begin{pmatrix} 1 & -k_1 - k_2 \\ 0 & 1 \end{pmatrix}, \qquad (6.8)$$

thus giving an equivalent quotient with charges

$$Q' = \begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & -2k_1 - k_2 & -k_1 - 2k_2 & k_1 + k_2 & k_1 + k_2 & k_1 + k_2 \end{pmatrix}. \qquad (6.9)$$

We then change variables by defining

$$k_1 = 2p - k, \qquad k_2 = k - p \qquad (6.10)$$

to obtain

$$Q' = \begin{pmatrix} 1 & 1 & 1 & -1 & -1 & -1 \\ 0 & -3p + k & -k & p & p & p \end{pmatrix}. \qquad (6.11)$$

Thus

$$\mathcal{M}_{3D}(2p - k, k - p, -p) = \mathbb{C}^6//(\mathbb{C}^*)^2_{Q'}. \qquad (6.12)$$

This realizes the VMS $\mathcal{M}_{3D}$ explicitly as a toric CY 4-fold. We may compute the toric diagram in the usual manner,[14] obtaining the normal vectors

$$\begin{aligned} w_0 &= [0, 0, k - p], & w_1 &= [0, 0, 0], \\ w_2 &= [0, 0, p], & w_3 &= [1, 0, 0], \\ w_4 &= [0, 1, 0], & w_5 &= [-1, -1, k]. \end{aligned} \qquad (6.13)$$

We now note that the 5 vectors $w_1, \ldots, w_5$ precisely define the toric diagram obtained in [6] for the cone over the explicit Sasaki-Einstein manifolds $Y^{p,k}(\mathbb{C}P^2)$ of [5]. The vertices $w_1, \ldots, w_5$ define a compact convex lattice polytope $\mathcal{P}$ in $\mathbb{R}^3$, shown in Fig. 1 of Ref. [6]. Of course, in (6.13) we have 6 vectors, after including the vertex w_0. However, adding this vertex will define the *same* affine toric variety as $\mathcal{P}$, provided the vertex lies inside the polytope $\mathcal{P}$. In this case, we simply obtain a nonminimal presentation of the toric variety, with the additional vertex w_0 corresponding to a blow-up mode of the singularity. One easily sees from [6] that w_0 lies inside $\mathcal{P}$ provided $p \leq k \leq 2p$. Thus, provided k lies within this range, the VMS for the CS quiver gauge theory above with CS levels $(2p - k, k - p, -p)$ is precisely the cone over the explicit Sasaki-Einstein manifold $Y^{p,k}(\mathbb{C}P^2)$.

It was shown in [5,6] that the metrics $Y^{p,k}(\mathbb{C}P^2)$ exist for integers p, k satisfying the bounds $\frac{3}{2}p \leq k \leq 3p$. In fact, the lower bound here is just a convention. From the explicit analysis in [6], one sees that the range of k may be extended to lie in the interval

$$0 \leq k \leq 3p. \qquad (6.14)$$

However, notice that the gauged linear sigma model quotient is manifestly invariant under the exchange of k with $3p - k$. It is satisfying to find that the explicit metrics [5,6] are also invariant under this exchange. This may be verified by observing that under this transformation the roots x_1, x_2 (recall $h = 3$ in the notation of [6]) of the equations (2.20) in [6] are interchanged. Thus solutions with $k \in [0, \frac{3}{2}p]$ are equivalent to solutions with $k \in [\frac{3}{2}p, 3p]$, which is the range considered in [5]. Hence, without loss of generality, we may take $k \in [\frac{3}{2}p, 3p]$.

To conclude, we have thus constructed an infinite family of CS quiver gauge theories which have explicit candidate Sasaki-Einstein duals, for values of p, k such that[15]

$$\frac{3}{2}p \leq k \leq 2p. \qquad (6.15)$$

Notice then that k_1 and k_2 are non-negative. Given a quiver with CS levels $(k_1 \geq 0, k_2 \geq 0, k_3 \leq 0)$, we may easily determine the values of p, k of the corresponding dual Sasaki-Einstein metric. Using (6.10), we find $p = k_1 + k_2$ and $k = k_1 + 2k_2$. Of course, we may equally pick $p = k_1 + k_2$ and $k = 2k_1 + k_2$. However, from the discussion above, the two choices are in fact completely equivalent, both for the VMS and for the explicit metrics.

It is interesting to examine the two limiting cases of the interval (6.15). At the lower bound, $p = 2r$, $k = 3r$, the CS levels are $(r, r, -2r)$, and the VMS is then a $\mathbb{Z}_r$ orbifold of the quotient of $\mathbb{C}^5$ by the $\mathbb{C}^*$ with charges

$$(2, 2, 2, -3, -3). \qquad (6.16)$$

Notice this case is symmetric under exchanging k and $3p - k$. In fact, this is the cone over the homogeneous Sasaki-Einstein manifold $Y^{2,3}(\mathbb{C}P^2) = M^{3,2}$ [6]. The gauge theory we are proposing here as being dual to this manifold

[14] We refer to [6] for a review of the relevant toric geometry.

[15] Equivalently, $p \leq k \leq \frac{3}{2}p$.

is then different from the proposal made in [18]. For $k = 2p$ we obtain the CS level vector $(0, p, -p)$, and the VMS is then a $\mathbb{Z}_p$ orbifold of the quotient of $\mathbb{C}^5$ by the $\mathbb{C}^*$ with charges

$$(1, 1, 1, -2, -1). \tag{6.17}$$

Notice that $Y^{1,2}(\mathbb{C}P^2)$ is in a sense the first nontrivial member of the $Y^{p,k}(\mathbb{C}P^2)$ family of metrics. Numerical values for the volumes of this particular example were given in [6]. It would be interesting to construct the CS quivers dual to the metrics in [5] with $k \in (2p, 3p]$.

The only check of the conjectured duality we can make at the time of writing is that the VMSs of the CS quiver theories contain the corresponding Calabi-Yau 4-fold geometries as a branch.[16] Combining the geometric discussion above with the results in [6], it is straightforward to give an assignment of R charges of the nine bifundamental fields X_i, Y_i, Z_i.[17] It would be extremely desirable to check the proposed duality further by performing a suitable purely field-theoretic calculation, in the spirit of a-maximization.

Given the above construction, it is natural to conjecture that the CS quiver gauge theories dual to the $Y^{p,k}(B_4)$ manifolds constructed in [5], where B_4 may be any Kähler-Einstein four-manifold, are described precisely by the 4D quivers for the corresponding canonical complex cones over B_4. The remaining possibilities for B_4 are $\mathbb{C}P^1 \times \mathbb{C}P^1$, which was also discussed extensively in [6], and the del Pezzo surfaces dP_n, $n = 3, \ldots, 8$. Notice that the six-dimensional manifolds M_6, obtained in the reduction to type IIA described in [6], are precisely a projective version of these complex cones; these are obtained by compactifying the $\mathbb{C}^*$ fibers to $\mathbb{C}P^1$, as described in [5]. We leave a fuller investigation of these models for future work.

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Note added.—While finalizing this paper for submission to the archive we received [22], which contains comments related to the results presented here. We are grateful to the authors of [22] for informing us about the completion of their work. After submitting the first version of this paper to the archive, the work [15] appeared. This also has overlap with the results we have presented.

[16]Notice that the scalar holomorphic Kaluza-Klein spectrum will automatically be in 1-1 correspondence with the holomorphic functions on the Calabi-Yau cone [19], or its N-fold symmetric product [20,21]; therefore, this matching does not

[17]Note that in doing so one is ignoring the subtleties involved in constructing baryonlike operators in 3D Chern-Simons quivers, where the gauge groups of the UV theory are $U(N)$.

[1] J. H. Schwarz, J. High Energy Phys. 11 (2004) 078.
[2] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, J. High Energy Phys. 10 (2008) 091.
[3] I. R. Klebanov and E. Witten, Nucl. Phys. **B536**, 199 (1998).
[4] K. Intriligator and B. Wecht, Nucl. Phys. **B667**, 183 (2003).
[5] J. P. Gauntlett, D. Martelli, J. F. Sparks, and D. Waldram, Adv. Theor. Math. Phys. **8**, 987 (2006).
[6] D. Martelli and J. Sparks, arXiv:0808.0904.
[7] D. Gaiotto and X. Yin, J. High Energy Phys. 08 (2007) 056.
[8] Y. Imamura and K. Kimura, arXiv:0806.3727.
[9] D. Martelli and J. Sparks, arXiv:0804.3999.
[10] R. P. Thomas, arXiv:math/0512411.
[11] A. D. King, Q. J. Math **45**, 515 (1994).
[12] D. Forcella, A. Hanany, Y. H. He, and A. Zaffaroni, J. High Energy Phys. 08 (2008) 012.
[13] A. Giveon and D. Kutasov, arXiv:0808.0360.
[14] S. Benvenuti, S. Franco, A. Hanany, D. Martelli, and J. Sparks, J. High Energy Phys. 06 (2005) 064.
[15] A. Hanany and A. Zaffaroni, J. High Energy Phys. 10 (2008) 111.
[16] D. Cox, notes available via the Web site, http://www.amherst.edu/~dacox/.
[17] S. Gukov, M. Rangamani, and E. Witten, J. High Energy Phys. 12 (1998) 025.
[18] D. Fabbri, P. Fré, L. Gualtieri, C. Reina, A. Tomasiello, A. Zaffaroni, and A. Zampa, Nucl. Phys. **B577**, 547 (2000).
[19] J. P. Gauntlett, D. Martelli, J. Sparks, and S. T. Yau, Commun. Math. Phys. **273**, 803 (2007).
[20] S. Benvenuti, B. Feng, A. Hanany, and Y. H. He, J. High Energy Phys. 11 (2007) 050.
[21] D. Martelli and J. Sparks, Nucl. Phys. **B759**, 292 (2006).
[22] D. L. Jafferis and A. Tomasiello, J. High Energy Phys. 10 (2008) 101.