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Supergravity solutions without triholomorphic $U(1)$ isometries

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We investigate the construction of five-dimensional supergravity solutions that do not have any triholomorphic $U(1)$ isometries. We construct a class of solutions that in various limits of parameters reduces to many of previously constructed five-dimensional supergravity solutions based on both hyper-Kähler base spaces that can be put into a Gibbons-Hawking form and hyper-Kähler base spaces that cannot be put into a Gibbons-Hawking form. We find a new solution which is over triaxial Bianchi type IX Einstein–hyper-Kähler base space with no triholomorphic $U(1)$ symmetry. One special case of this solution corresponds to a five-dimensional solution based on Eguchi-Hanson type II geometry.

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I. INTRODUCTION

It is believed that in the strong coupling limit, many horizonless three-charge brane configurations undergo a geometric transition and become smooth horizonless geometries with black hole or black ring charges [1]. These charges come completely from fluxes wrapping on nontrivial cycles. The three-charge black hole (ring) systems are dual to the states of corresponding conformal field theories: in favor of the idea that nonfundamental-black-hole (-ring) systems effectively arise as a result of many horizonless configurations [2,3]. The simplest eleven-dimensional supergravity metrics (the low-energy limit of M-theory [4]) with three-charge geometries have the form [5]

$$ds_{11}^2 = -(Z_1Z_2Z_3)^{-2/3}(dt+\omega)^2 + (Z_1Z_2Z_3)^{1/3}ds_4^2 + \left(\frac{Z_2Z_3}{Z_1^2}\right)^{1/3}(dx_5^2+dx_6^2) + \left(\frac{Z_3Z_1}{Z_2^2}\right)^{1/3}(dx_7^2+dx_8^2) + \left(\frac{Z_1Z_2}{Z_3^2}\right)^{1/3}(dx_9^2+dx_{10}^2), \quad (1.1)$$

where $x_5, \ldots, x_{10}$ are coordinates of compactification six-torus and ds_4^2 is any hyper-Kähler metric (which is equivalent to a metric with self-dual curvature in four dimensions). To preserve supersymmetry, the base metric ds_4^2 should be hyper-Kähler [6]. The five-dimensional spacetime submetric of (1.1) together with Maxwell field make the bosonic sector of five-dimensional minimal supergravity. In five dimensions, unlike the four dimensions in which the only horizon topology is 2-sphere, we can have different more interesting horizon topologies such as black holes with horizon topology of 3-sphere [7]; black rings with horizon topology of 2-sphere × circle [8,9]; black saturn, a spherical black hole surrounded by a black ring [10]; and black lens, in which the horizon geometry is a lens space $L(p, q)$ [11]. All allowed horizon topologies have been classified in [12–14]. Recently, it was shown how a uniqueness theorem might be proved for black holes in five dimensions [15,16]. It was shown that stationary, asymptotically flat vacuum black holes with two commuting axial symmetries are uniquely determined by their mass, angular momentum, and rod structure. Specifically, the rod structure [17] determines the topology of horizon in five dimensions.

In [18], the authors have used hyper-Kähler Atiyah-Hitchin base space and its ambipolar generalizations to construct five-dimensional, three-charge supergravity solutions that only have a rotational $U(1)$ isometry. The complete solutions are regular around the critical surface of ambipolar base space. These solutions are very remarkable because (ambipolar) hyper-Kähler Atiyah-Hitchin geometries (unlike ambipolar Gibbons-Hawking geometries) do not have any triholomorphic $U(1)$ isometry [triholomorphic $U(1)$ isometry means the $U(1)$ preserves all three complex structures of the hyper-Kähler geometry]. Hence they could be used to study the interesting physical processes, such as merger of two Breckenridge-Myers-Peet-Vafa black holes [19] or the geometric transition of a three-charge supertube of arbitrary shape; that do not respect any triholomophic $U(1)$ symmetry. In [20], the authors also have used the Atiyah-Hitchin metric and constructed a solution to five-dimensional minimal supergravity.

The Atiyah-Hitchin metric is a special case of Bianchi type IX Einstein-Kähler metrics with generic nontriholomorphic $U(1)$ isometries. It is interesting to note that in some special limits, Bianchi type IX space reduces to Taub-NUT and Eguchi-Hanson spaces (the latter geometry will be referred as Eguchi-Hanson type II in this paper).

The Bianchi type IX space has been used in construction of M2 and M5 brane solutions which are realizations of supergravity solutions for localized IIA D2/D6(2), NS5/D6 (5), and IIB NS5/D5(4) intersecting brane systems [21]. By lifting a D6 (D5 or D4)-brane to four-dimensional hyper-Kähler Bianchi type IX geometry embedded in M-theory, these solutions were constructed by placing M2- and M5-branes in the Bianchi type IX background geometry. The special feature of this construction is that the solutions are

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not restricted to be in the near core region of the D6 (D5 or D4)-brane. Moreover, all of the different solutions preserve $1/4$ of the supersymmetry as a result of the self-duality of the Bianchi type IX metric. All previously known M2 and M5 solutions [22–25] are special cases of the solutions presented in [21].

The Bianchi type IX spaces were used recently for construction cohomogeneity two metrics of G_2 holonomy which are foliated by twistor spaces [26]. The twistor spaces are two-sphere bundles over Bianchi type IX Einstein metrics with self-dual Weyl tensor.

In this paper, we use self-dual Bianchi type IX space as the base space to construct some new five-dimensional supergravity solutions with generic nontriholomorphic $U(1)$ isometries. We would like to stress that, in general, constructing solutions with nontriholomorphic $U(1)$ isometries is a rather complicated, tedious, and challenging task. To our knowledge, for classical black holes and black rings, only two solutions exist [27,28]. The outline of this paper is as follows. In Sec. II, we give a brief review of five-dimensional supergravity and equations of motion. In Sec. III, we present Bianchi type IX space and show how the space reduces to different well-known spaces that were used previously for constructing five-dimensional supergravity solutions. We consider in detail the case of triaxial Bianchi type IX space that could be considered for construction of a new class of supergravity solutions. In section IV we consider a class of supergravity solutions over triaxial Bianchi type IX space and present the analytic expressions for the solutions near the center of space-time and also at infinity. We also provide the results of numerical integration of equations of motion and discuss the behaviors of our solutions.

II. FIVE-DIMENSIONAL MINIMAL SUPERGRAVITY

The bosonic part of five-dimensional minimal supergravity is Einstein-Maxwell theory with a Chern-Simon term and is given by the action [29]

$$S = \frac{-1}{4\pi G}\int\left(\frac{1}{4}R * 1 + \frac{1}{2}F \wedge *F + \frac{2}{3\sqrt{3}}F \wedge F \wedge A\right). \tag{2.1}$$

The equations of motion are

$$R_{\mu\nu} + 2F_{\mu\lambda}F_{\nu}{}^{\lambda} - \frac{1}{3}g_{\mu\nu}F^2 = 0, \tag{2.2}$$

$$d * F + \frac{2}{\sqrt{3}}F \wedge F = 0. \tag{2.3}$$

A bosonic solution is supersymmetric if it admits a supercovariantly constant, symplectic Majorana Killing spinor ε^{μ} obeying

$$D_{\mu}\varepsilon^a + \frac{1}{4\sqrt{3}}(\gamma_{\mu}{}^{\nu\lambda} - 4\delta^{\nu}_{\mu}\gamma^{\lambda})F_{\nu\lambda}\varepsilon^a = 0. \tag{2.4}$$

From a single commuting ε^a, a scalar f, a one-form V, and three two-forms $\Phi^{ab} \equiv \Phi^{(ab)}$ could be constructed [29]; given by

$$f\varepsilon^{ab} = \bar{\varepsilon}^a\varepsilon^b, \tag{2.5}$$

$$V_{\mu}\varepsilon^{ab} = \bar{\varepsilon}^a\gamma_{\mu}\varepsilon^b, \tag{2.6}$$

$$\Phi^{ab}_{\mu\nu} = \bar{\varepsilon}^a\gamma_{\mu\nu}\varepsilon^b. \tag{2.7}$$

The solutions could be classified depending on the Killing vector V_{μ} to be timelike or null. If we consider the case in which f is not zero and $V = \frac{\partial}{\partial t}$ is a timelike Killing vector field, then the metric can be written as

$$ds^2 = -f^2(dt + \omega)^2 + \frac{1}{f}ds_B^2, \tag{2.8}$$

where $ds_B^2 = h_{mn}dx^m dx^n$ is the metric of the four-dimensional hyper-Kähler base space B [29]. We note that the metric $\frac{1}{f}ds_B^2$ is obtained by projecting the full five-dimensional metric ds^2 perpendicular to the orbits of Killing vector field V.

If we define

$$\mathbf{e}^0 = f(dt + \omega), \tag{2.9}$$

then $\mathbf{e}^0 \wedge \eta$ defines a positive orientation for the five-dimensional metric (2.8), where η is a positive orientation on the base space B. The two-form $d\omega$ only has components tangent to the base space B and so it can be split into self-dual and anti-self-dual parts with respect to the metric of base space

$$d\omega = \frac{G^+}{f} + \frac{G^-}{f}. \tag{2.10}$$

Taking the differentials of f and V and then the exterior derivatives of obtained equations, and using the fact that V is a Killing vector, leads to the following result for the two-form F

$$F = \frac{\sqrt{3}}{2}\left(-\frac{1}{f^2}V \wedge df + \frac{1}{3}G^+ + G^-\right), \tag{2.11}$$

or

$$F = \frac{\sqrt{3}}{2}de^0 - \frac{1}{\sqrt{3}}G^+, \tag{2.12}$$

where G^+ is given by

$$G^+ = \frac{f}{2}(d\omega + *_B d\omega). \tag{2.13}$$

From the Bianchi identity and equation of motion (2.3), we get the following equations:

$$dG^+ = 0, \tag{2.14}$$

$$\nabla^2 \frac{1}{f} = \frac{4}{9}(G^+)^2 = \frac{2}{9}(G^+)_{mn}(G^+)^{mn}, \tag{2.15}$$

where ∇^2 is the Laplace operator on base space B.

III. THE (TRIAXIAL) BIANCHI TYPE IX SPACE

The Bianchi type IX metric $ds^2_{\mathrm{B.IX}}$ is locally given by the following metric with an $SU(2)$ or $SO(3)$ isometry group [30]

$$ds^2_{\mathrm{B.IX}} = e^{2\{A(\zeta)+B(\zeta)+C(\zeta)\}}d\zeta^2 + e^{2A(\zeta)}\sigma_1^2 + e^{2B(\zeta)}\sigma_2^2 + e^{2C(\zeta)}\sigma_3^2, \tag{3.1}$$

where σ_i's are Maurer-Cartan one-forms (see Appendix). The metric (3.1) satisfies Einstein equations provided

$$\frac{d^2A}{d\zeta^2} = \frac{1}{2}\{e^{4A} - (e^{2B} - e^{2C})^2\}, \tag{3.2}$$

$$\frac{d^2B}{d\zeta^2} = \frac{1}{2}\{e^{4B} - (e^{2C} - e^{2A})^2\}, \tag{3.3}$$

$$\frac{d^2C}{d\zeta^2} = \frac{1}{2}\{e^{4C} - (e^{2A} - e^{2B})^2\}, \tag{3.4}$$

and

$$\frac{dA}{d\zeta}\frac{dB}{d\zeta} + \frac{dB}{d\zeta}\frac{dC}{d\zeta} + \frac{dC}{d\zeta}\frac{dA}{d\zeta} = \frac{1}{2}\{e^{2(A+B)} + e^{2(B+C)} + e^{2(C+A)}\} - \frac{1}{4}\{e^{4A} + e^{4B} + e^{4C}\}. \tag{3.5}$$

Moreover self-duality of the curvature implies

$$\frac{dA}{d\zeta} = \frac{1}{2}\{e^{2B} + e^{2C} - e^{2A}\} - \alpha_1 e^{B+C}, \tag{3.6}$$

$$\frac{dB}{d\zeta} = \frac{1}{2}\{e^{2C} + e^{2A} - e^{2B}\} - \alpha_2 e^{A+C}, \tag{3.7}$$

$$\frac{dC}{d\zeta} = \frac{1}{2}\{e^{2A} + e^{2B} - e^{2C}\} - \alpha_3 e^{A+B}, \tag{3.8}$$

where three constant numbers α_i, $i = 1, 2, 3$ satisfy

$$\alpha_i\alpha_j = \varepsilon_{ijk}\alpha_k. \tag{3.9}$$

We note that Eqs. (3.6), (3.7), and (3.8) are integrals of Eqs. (3.2), (3.3), (3.4), and (3.5). The possible solutions of Eq. (3.9) are

(I) $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 1)$
(II) $(\alpha_1, \alpha_2, \alpha_3) = (1, -1, -1)$
(III) $(\alpha_1, \alpha_2, \alpha_3) = (-1, 1, -1)$
(IV) $(\alpha_1, \alpha_2, \alpha_3) = (-1, -1, 1)$
(V) $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$.

Here we consider all five cases:

Case I:

Choosing $(\alpha_1, \alpha_2, \alpha_3) = (1, 1, 1)$ in Eqs. (3.6), (3.7), and (3.8) yields the Atiyah-Hitchin metric [31] in the form of (3.1) with

$$e^{2A(\zeta)} = \frac{2}{\pi}\frac{\vartheta_2\vartheta_3'\vartheta_4'}{\vartheta_2'\vartheta_3\vartheta_4}, \tag{3.10}$$

$$e^{2B(\zeta)} = \frac{2}{\pi}\frac{\vartheta_2'\vartheta_3\vartheta_4'}{\vartheta_2\vartheta_3'\vartheta_4}, \tag{3.11}$$

$$e^{2C(\zeta)} = \frac{2}{\pi}\frac{\vartheta_2'\vartheta_3'\vartheta_4}{\vartheta_2\vartheta_3\vartheta_4'}, \tag{3.12}$$

where the ϑ's are Jacobi theta functions with complex modulus $i\zeta$. The Jacobi theta functions are given explicitly in the Appendix. Since the Atiyah-Hitchin base space and its ambipolar generalizations have been considered in explicit construction of the most general five-dimensional supersymmetric solutions [18], we do not study them in this article.

Cases II, III, and IV:

All these three cases are not distinct from case I, since they could be obtained by substitutions, $e^A \to -e^A$, $e^B \to -e^B$, and $e^C \to -e^C$ respectively.

Case V:

By choosing $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$ the differential equations (3.2) through (3.8) can be solved exactly. We find the solutions

$$A(\zeta) = \frac{1}{2}\ln\left(c^2\frac{\mathfrak{cn}(c^2\zeta, k^2)\mathfrak{dn}(c^2\zeta, k^2)}{\mathfrak{sn}(-c^2\zeta, k^2)}\right), \tag{3.13}$$

$$B(\zeta) = \frac{1}{2}\ln\left(c^2\frac{\mathfrak{cn}(c^2\zeta, k^2)}{\mathfrak{dn}(c^2\zeta, k^2)\mathfrak{sn}(-c^2\zeta, k^2)}\right), \tag{3.14}$$

$$C(\zeta) = \frac{1}{2}\ln\left(c^2\frac{\mathfrak{dn}(c^2\zeta, k^2)}{\mathfrak{cn}(c^2\zeta, k^2)\mathfrak{sn}(-c^2\zeta, k^2)}\right), \tag{3.15}$$

where $\mathfrak{sn}(z, k)$, $\mathfrak{cn}(z, k)$, and $\mathfrak{dn}(z, k)$ are the standard Jacobi elliptic SN, CN, and DN functions. We review Jacobi theta functions as well as Jacobi elliptic functions lore in the Appendix. We change the coordinate ζ in the metric (3.1) to the coordinate r by

$$r = \frac{2c}{\sqrt{\mathfrak{sn}(c^2\zeta, k^2)}}, \tag{3.16}$$

hence, we find the metric in the form (that we call the triaxial Bianchi type IX space) [21]

$$ds^2_{\mathrm{tri.B.IX}} = \frac{dr^2}{\sqrt{F(r)}} + \frac{r^2}{4}\sqrt{F(r)}\left(\frac{\sigma_1^2}{1-\frac{a_1^4}{r^4}} + \frac{\sigma_2^2}{1-\frac{a_2^4}{r^4}} + \frac{\sigma_3^2}{1-\frac{a_3^4}{r^4}}\right), \tag{3.17}$$

where

$$F(r) = \prod_{i=1}^{3}\left(1 - \frac{a_i^4}{r^4}\right), \tag{3.18}$$

and a_1, a_2, and a_3 are three parameters with no loss of generality; we choose them such that $0 = a_1 \leq a_2 = 2kc \leq a_3 = 2c$. We note that the coordinate r must be greater or equal to a_3. Here $0 \leq k \leq 1$ is the square root of modulus of different types of Jacobi elliptic functions and $c > 0$. If $k > 1$, all we need is to interchange the 2 and 3 directions. The metric function (3.18) is positive definite for $r \geq a_3 = 2c$, and the change of coordinate (3.16) guarantees this requirement. For simplicity, we choose coordinate ζ in the range $[0, \alpha_{(c)(k)(1)}]$ where $\alpha_{(c)(k)(m)}$ is the m-th positive root of $\mathfrak{sn}(c^2\zeta, k^2)$. Any other range of the form $[\alpha_{(c)(k)(2n)}, \alpha_{(c)(k)(2n+1)}]$ with $n = 1, 2, 3, \ldots$ or $[-\alpha_{(c)(k)(2n)}, -\alpha_{(c)(k)(2n-1)}]$ can be chosen equivalently.

For the special value of $k = 0$, where the smaller two a's coincide, we find the metric (3.17) reduces to the following metric:

$$ds^2_{\mathrm{EHI}} = \frac{dr^2}{h(r)} + \frac{r^2}{4}h(r)\{d\theta^2 + \sin^2\theta d\phi^2\} + \frac{r^2}{4h(r)}(d\psi + \cos\theta d\phi)^2, \tag{3.19}$$

which is the Eguchi-Hanson type I metric with $h(r) = (1 - \frac{(2c)^4}{r^4})^{1/2}$. In the other extreme case where $k = 1$, the larger two a's coincide and we obtain the Eguchi-Hanson type II metric

$$ds^2_{\mathrm{EHII}} = \frac{dr^2}{h^2(r)} + \frac{r^2}{4}h^2(r)\sigma_1^2 + \frac{r^2}{4}(\sigma_2^2 + \sigma_3^2), \tag{3.20}$$

which is of the same form as the well-known Eguchi-Hanson metric

$$ds^2_{\mathrm{EH}} = \frac{r^2}{4g(r)}[d\psi + \cos(\theta)d\phi]^2 + g(r)dr^2 + \frac{r^2}{4}(d\theta^2 + \sin^2(\theta)d\phi^2), \tag{3.21}$$

by making the substitution $2c = a$ and $h(r) = \frac{1}{\sqrt{g(r)}}$ in (3.20). We note that only for special values of $k = 0$ and $k = 1$, the metric (3.17) admits a triholomorphic $U(1)$ isometry; hence it could be put into Gibbons-Hawking form. In both special cases of $k = 0$ and $k = 1$, the five-dimensional supergravity solutions can be constructed simply by four harmonic functions on the base space. The case with $k = 1$ was considered explicitly in [32], where the authors constructed five-dimensional supersymmetric black ring solutions on the hyper-Kähler Eguchi-Hanson type II base space. Their solutions have the same two angular momentum components and the asymptotic structure on time slices is locally Euclidean. The circle-direction of the black ring is along the equator on a two-sphere bolt on the base space. The case with $k = 0$ gives a separable five-dimensional metric for Eguchi-Hanson type I manifold with a time direction.

By increasing the parameter k as $0 < k < 1$, we obtain triaxial Bianchi type IX metrics with a generic nontriholomorphic $U(1)$ isometry. In the next section, we solve the five-dimensional supergravity equations and find the solutions.

IV. SUPERGRAVITY SOLUTIONS OVER BIANCHI TYPE IX BASE SPACE

From Eq. (2.14), we can write $G^+ = \lambda d\Gamma$ where Γ is a one-form and λ is a constant. We take the following ansatz for one-forms Γ and ω [14]:

$$\Gamma = p(r)\sigma_1, \tag{4.1}$$

$$\omega = \psi(r)\sigma_1, \tag{4.2}$$

where p and ψ are two functions of r [or ζ through Eq. (3.16)]. We find then

$$G^+ = 2\lambda\left(\frac{2}{r^2}p(r)\mathbf{e}^{(2)} \wedge \mathbf{e}^{(3)} - \frac{p'(r)}{r}\mathbf{e}^{(r)} \wedge \mathbf{e}^{(1)}\right), \tag{4.3}$$

where

$$\mathbf{e}^{(1)} = \frac{r}{2}\frac{\sqrt[4]{F(r)}}{\sqrt{1 - \frac{a_1^4}{r^4}}}\sigma_1,$$

$$\mathbf{e}^{(2)} = \frac{r}{2}\frac{\sqrt[4]{F(r)}}{\sqrt{1 - \frac{a_2^4}{r^4}}}\sigma_1,$$

$$\mathbf{e}^{(3)} = \frac{r}{2}\frac{\sqrt[4]{F(r)}}{\sqrt{1 - \frac{a_3^4}{r^4}}}\sigma_1,$$

and

$$\mathbf{e}^{(r)} = -\frac{dr}{\sqrt[4]{F(r)}}$$

are vierbeins for the metric (3.1). Since G^+ is self-dual, we find

$$p(r) = \frac{p_0}{r^2}, \tag{4.4}$$

hence we find simple analytic forms for one-form Γ and self-dual two-form G^+

$$\Gamma = \frac{p_0}{r^2}\sigma_1, \tag{4.5}$$

$$G^+ = \frac{4\lambda p_0}{r^2}(\mathbf{e}^{(r)} \wedge \mathbf{e}^{(1)} + \mathbf{e}^{(2)} \wedge \mathbf{e}^{(3)}). \tag{4.6}$$

The Laplace operator on base space (3.17) simply is given by

$$\nabla^2 = \frac{1}{r^3}\partial_r\{r^3\sqrt{F(r)}\partial_r\}, \tag{4.7}$$

so from Eq. (2.15), we find

$$\frac{1}{f(r)} = -n\int\frac{dr}{r^7\sqrt{F(r)}} + f_1\int\frac{dr}{r^3\sqrt{F(r)}} + f_2, \tag{4.8}$$

where $n = \frac{32}{9}\lambda^2 p_0^2$ and f_1, f_2 are two constants. Although we cannot express the metric function $f(r)$ in a closed analytic form (since the integrals cannot be expressed in terms of known functions), we can find all necessary information about five-dimensional supergravity solutions by numerically integrating. To determine the one-form ω, we use Eq. (2.13) together with (4.3) and we find the first order differential equation for $\psi(r)$ as

$$\psi'(r) - \frac{2\psi(r)}{r} = -\frac{4\lambda p_0}{r^3 f(r)}. \tag{4.9}$$

To solve this differential equation, we multiply it by $1/r$ and then we find the solution as

$$\psi(r) = -4\lambda p_0 r^2\int\frac{dr}{r^5 f(r)} + \psi_0 r^2, \tag{4.10}$$

where ψ_0 is a constant. We consider now three different cases corresponding to different values for the parameter k, i.e. $k = 1$, $k = 0$, and $0 < k < 1$. In the first two cases, the reduced triaxial Bianchi type IX space gets an enhanced triholomorphic symmetry, while in the last case there is no triholomorphic symmetry.

First, we consider the special case $k = 1$ in which triaxial Bianchi type IX space (3.17) reduces to the Eguchi-Hanson type II metric (3.20). As a result of reduction, the nontriholomorphic isometry breaks and the reduced space admits an enhanced triholomorphic Killing vector field and it is asymptotically locally Euclidean with a self-dual curvature. The metric has a single removable bolt singularity if ψ is restricted to the interval $(0, 2\pi)$ and the topology of the manifold is S^3/Z_2 asymptotically; hence the manifold is asymptotically locally Euclidean. We note that near bolt singularity ($r = 2c(1 + \epsilon^2)$, where $\epsilon \ll 1$), where the Killing vector field $\frac{\partial}{\partial\psi}$ vanishes, the metric reduces to

$$ds^2_{r=2c(1+\epsilon^2)} = 4c^2\{\epsilon^2(d\psi + \cos(\theta)d\phi)^2 + d\epsilon^2\} + c^2(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{4.11}$$

$$\approx z^2 d\psi^2 + dz^2 + c^2(d\theta^2 + \sin^2(\theta)d\phi^2). \tag{4.12}$$

So the space has the topology of $\mathbb{R}^2 \otimes S^2$ with the radial length equal to $\sqrt{z^2 + c^2}$. If we change the coordinates to

$$R = \frac{1}{2c}\sqrt{r^4 - 16c^4\sin^2\theta}, \tag{4.13}$$

$$\Theta = \tan^{-1}\left(\frac{\sqrt{r^4 - 16c^4}}{r^2}\tan\theta\right), \tag{4.14}$$

$$\Phi = \psi, \tag{4.15}$$

$$\Psi = 2\phi, \tag{4.16}$$

where $2c \le R < \infty$, $0 \le \Theta \le \pi$, $0 \le \Phi \le 2\pi$, $0 \le \Psi \le 4\pi$, then the Eguchi-Hanson type II metric transforms into the Gibbons-Hawking two-center form

$$ds^2_{\mathrm{EHII}} = H(R,\theta)(dR^2 + R^2(d\Theta^2 + \sin^2\Theta d\Phi^2)) + \frac{1}{H(R,\Theta)}\left(\frac{c}{4}d\Psi + Y(R,\theta)d\Phi\right)^2, \tag{4.17}$$

where $H(R,\Theta) = \frac{c}{4}\{1/(R - R_1) + 1/(R - R_2)\}$ and $Y(R,\theta) = \frac{c}{4}\{(R\cos\theta - 2c)/\sqrt{R^2 + 4c^2 - 4Rc\cos\Theta} + (R\cos\theta + 2c)/\sqrt{R^2 + 4c^2 + 4Rc\cos\Theta}\}$. Here $R_1 = (0, 0, 2c)$ and $R_2 = (0, 0, -2c)$ are Euclidean position vectors of two nut singularities.

In this special case, the larger two a's in metric function (3.18) coincide. As a result, we can perform the integrals in (4.8) analytically and we find the five-dimensional supergravity metric function

$$1/f = \frac{\mu^2}{9c^4r^2} - \left(\frac{\mu^2}{72c^6} - \frac{f_1}{16c^2}\right)(\ln(r^2 + 4c^2) - \ln(r^2 - 4c^2)) + f_2, \tag{4.18}$$

and

$$\psi(r) = \frac{2\mu^3}{27c^4r^4} + \frac{\mu f_2}{r^2} + \frac{\mu f_1}{32c^4} - \frac{\mu^3}{144c^8} + \psi_0 r^2 - \frac{\mu(9f_1c^4 - 2\mu^2)}{2304r^2c^{10}}(r^4 - 16c^4)\ln\left(\frac{r^2 + 4c^2}{r^2 - 4c^2}\right), \tag{4.19}$$

where $\mu = \lambda p_0$. These results are exactly in agreement[1] with the results of [6].

To have a regular solution at $r = 2c$, we should set $f_1 = \frac{2\mu^2}{9c^4}$ and then to have a positive definite metric function, we should choose $f_2 \ge -\frac{\mu^2}{36c^6}$. Moreover, to eliminate the timelike curves at $r \to \infty$, we should set $\psi_0 = 0$. However, there are always closed timelike curves near $r = 2c$.

Second, we consider the special case $k = 0$ which triaxial Bianchi type IX space (3.17) reduces to the Eguchi-Hanson type I metric (3.19). As a result of reduction, the nontriholomorphic isometry again breaks and the reduced space admits an enhanced triholomorphic Killing vector field. In this case, to get a real five-dimensional metric function f and one-form ψ, we should choose $\mu = \psi_0 = 0$.

[1] We should note that in [6], the authors used the most negative signature for the five-dimensional metric.

So, the five-dimensional metric function becomes a constant f_2, and the five-dimensional metric is just $-f_2^2 dt^2 + \frac{1}{f_2} ds^2_{\mathrm{EHI}}$.

Third, we consider triaxial Bianchi type IX space with a generic $0 < k < 1$. Although we cannot integrate the integrals in Eq. (4.8) to find an analytic form for the metric function, we can integrate numerically. The behaviors of metric function for different values of parameters are plotted in Figs. 1 and 2. In Fig. 1, we choose f_1 to be zero while in Fig. 2, $f_1 \neq 0$. In both cases, for large r the metric function approaches a constant. We should note that changing the value of k does not change the decaying behavior of function f for $f_1 = 0$, as well as no change for behavior of function f for $f_1 \neq 0$. The change of parameter k only shifts the value of metric function f.

In fact, in the large r limit, the metric function behaves as

$$f(r) \sim \frac{1}{f_2}\left(1 + \frac{f_1}{2f_2 r^2}\right) + O\left(\frac{1}{r^4}\right), \tag{4.20}$$

hence, asymptotically, the five-dimensional metric is

$$ds^2 \rightarrow -\frac{1}{f_2^2}\left(dt + \frac{\lambda p_0 f_2}{r^2}\sigma_1\right)^2 + f_2 ds^2_{\mathrm{tri.B.IX}}. \tag{4.21}$$

By looking at the coefficient of $g_{\psi\psi}$, we see $\partial/\partial\psi$ becomes timelike if $4f^3(r)\psi^2(r) > r^2/4\sqrt{F(r)}$, hence we get closed-timelike curves. At large r limit, $\psi(r) \rightarrow \frac{\mu f_2}{r^2}$ and $f(r) \rightarrow \frac{1}{f_2}$, we conclude there are no closed-timelike curves if r is quite large. On the other extreme limit, close to $r = 2c$, the metric function $F(r)$ has Taylor expansion

$$\sqrt{F} \simeq F_1\sqrt{\epsilon} + F_2\epsilon^{3/2} + O(\epsilon^{5/2}), \tag{4.22}$$

where $F_1 = \sqrt{2\{(1-k^4)/c\}}$, $F_2 = (F_1/8c) \times \{(5 - 13k^4)/(k^4 - 1)\}$, and $\epsilon = r - 2c$. So from Eq. (4.8), we find the following behavior for the five-dimensional metric function $f(r)$ near $r = 2c$:

$$f(r) \simeq \frac{1}{f_2} + \frac{n - 16c^4 f_1}{64 f_2^2 c^7 F_1}\sqrt{\epsilon} + O(\epsilon). \tag{4.23}$$

This result for the metric function $f(r)$ with a generic $0 < k < 1$ is remarkable since it provides a smooth transition from the metric function of Eguchi-Hanson type I based solution with $k = 0$ to the metric function of Eguchi-Hanson type II based solution with $k = 1$.

In the limit of $r \simeq 2c$, the field $\psi(r)$ behaves as

$$\psi(r) \simeq \left(\frac{\lambda p_0 f_2}{4c^2} + 4\psi_0 c^2\right) + \left(-\frac{\lambda p_0 f_2}{4c^3} + 4\psi_0 c\right)\epsilon + O(\epsilon^{3/2}). \tag{4.24}$$

To summarize, we have found the five-dimensional supergravity equations of motion. These equations lead to differential equations for the fields $f(r)$ and $p(r)$ and $\psi(r)$. We have solved the equations for $r \sim 2c$ and at infinity and also found the numerical solutions for the other values of r.

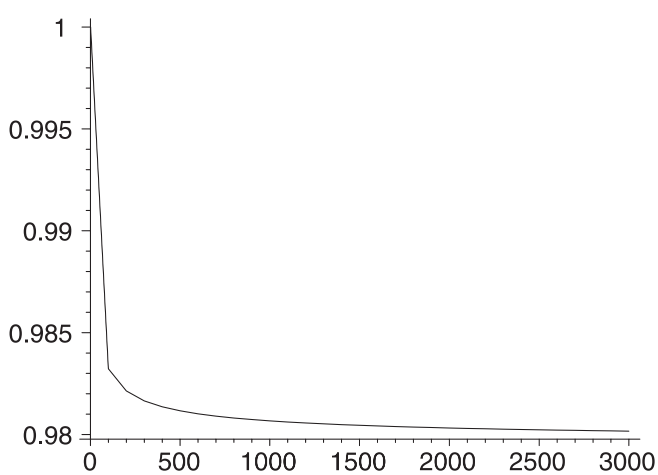

FIG. 1. The metric function f as a function of $\frac{1}{r-2c}$. We set $f_1 = 0$ and $f_2 = 1$.

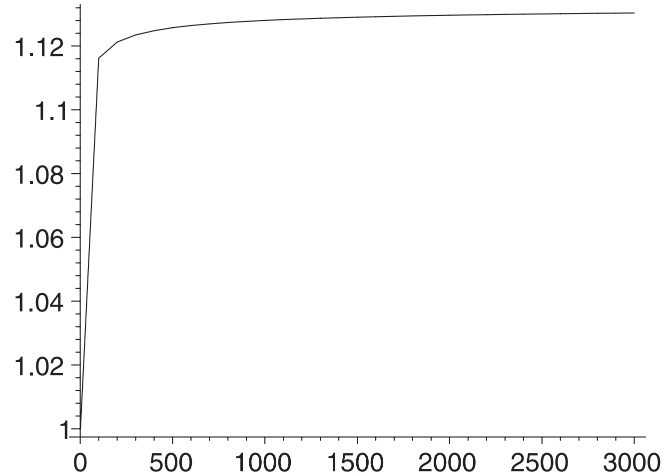

FIG. 2. The metric function f as a function of $\frac{1}{r-2c}$. We set $f_1 = f_2 = 1$.

V. CONCLUSIONS

We have constructed a new class of solutions to five-dimensional supergravity, based on Bianchi type IX Einstein–hyper-Kähler space. The Bianchi type IX Einstein–hyper-Kähler space does not have any triholomorphic $U(1)$ isometries, hence the solutions could be used to study the physical processes that do not respect any triholomorphic Abelian symmetries. We find the solutions to the equations of motion near $r = 2c$ and at infinity. Moreover, by numerical integration, we explicitly find the general behavior of the solutions. Our solutions based on triaxial Bianchi type IX space provide a smooth transition from solutions based on Eguchi-Hanson type I space to corresponding solutions based on Eguchi-Hanson type II space. One feature of the new solution is that in various limits of parameters, it reduces to many of the previously constructed five-dimensional supergravity solutions based on both hyper-Kähler base spaces that can be put into a Gibbons-Hawking form and hyper-Kähler base spaces that cannot be put into a Gibbons-Hawking form.

We conclude with a few comments about possible directions for future work. We have shown that our new solution (based on triaxial Bianchi type IX space) is asymptotically free of any closed timelike curves. An interesting application of our solutions is to seek their holographic dual theories. In the case of the existence of boundary holographic dual theories to the solutions (or their generalization with cosmological constant), then they would be free of any irregularities associated with the closed timelike curves in the bulk of space-time. Moreover, in deriving our solutions, we have considered the simplest dependence of the five-dimensional metric function on the coordinates (i.e. dependence only on the radial coordinate). The reason for taking the simplest dependence is due to the nonhomogeneity of differential Eq. (2.15) for the metric function; otherwise finding the solutions could be very difficult or even impossible. If we want to find some other solutions (with metric function depending on two or more coordinates), then we should consider the homogenous differential equation ((2.15) with G^+ equal to zero). It is quite possible that in solutions to the homogenous differential equation, the point $r = 2c$ can be converted to a regular hypersurface(s) in five-dimensional space-time and we obtain black hole solutions. The other open issue is study of the thermodynamics of constructed solutions in this paper.

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APPENDIX

In this appendix, we collect some formulas for Maurer-Cartan one-forms, Jacobi theta, and elliptic functions which are crucial for our discussion in Sec. III.

First of all, the Maurer-Cartan one-forms σ_i are given by

$$\sigma_1 = d\psi + \cos\theta d\phi, \tag{A1}$$

$$\sigma_2 = -\sin\psi d\theta + \cos\psi \sin\theta d\phi, \tag{A2}$$

$$\sigma_3 = \cos\psi d\theta + \sin\psi \sin\theta d\phi, \tag{A3}$$

with the property

$$d\sigma_i = \frac{1}{2}\varepsilon_{ijk}\sigma_j \wedge \sigma_k. \tag{A4}$$

We note that the metric on the $\mathbb{R}^4$ [with a radial coordinate R and Euler angles (θ, ϕ, ψ) on an S^3] could be written in terms of Maurer-Cartan one-forms via

$$ds^2 = dR^2 + \frac{R^2}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2), \tag{A5}$$

with $\sigma_1^2 + \sigma_2^2$ as the standard metric of S^2 with unit radius; $4(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$ gives the same for S^3.

The Jacobi theta functions ϑ are given by

$$\vartheta_i(\tau) = \vartheta_i(0|\tau), \tag{A6}$$

where we have used Jacobi-Erderlyi notation

$$\vartheta_1(\nu|\tau) = \vartheta\begin{bmatrix} 1 \\ 1 \end{bmatrix}(\nu|\tau), \tag{A7}$$

$$\vartheta_2(\nu|\tau) = \vartheta\begin{bmatrix} 1 \\ 0 \end{bmatrix}(\nu|\tau), \tag{A8}$$

$$\vartheta_3(\nu|\tau) = \vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}(\nu|\tau), \tag{A9}$$

$$\vartheta_4(\nu|\tau) = \vartheta\begin{bmatrix} 0 \\ 1 \end{bmatrix}(\nu|\tau). \tag{A10}$$

The Jacobi functions with characteristics

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\nu|\tau)$$

are defined by the following series:

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\nu|\tau) = \sum_{n\in Z} e^{i\pi(n-(a/2))\{\tau(n-(a/2))+2(\nu-(b/2))\}}, \tag{A11}$$

where a and b are two real numbers.

The standard Jacobi elliptic SN, CN, and DN functions $\mathfrak{sn}(z, k)$, $\mathfrak{cn}(z, k)$, and $\mathfrak{dn}(z, k)$, are related, respectively, to $\mathfrak{am}(z, k)$, the Jacobi elliptic AM function, by

$$\mathfrak{s}\,\mathfrak{n}(z, k) = \sin(\mathfrak{am}(z, k)), \tag{A12}$$

$$\mathfrak{c}\,\mathfrak{n}(z, k) = \cos(\mathfrak{am}(z, k)), \tag{A13}$$

$$\mathfrak{d}\,\mathfrak{n}(z, k) = \sqrt{1 - k^2\mathfrak{sn}^2(z, k)}. \tag{A14}$$

The Jacobi elliptic AM function is the inverse of the trigonometric form of the elliptic integral of the first kind; which means

$$\mathfrak{a}\,\mathfrak{m}(\mathfrak{f}(\sin\phi, k), k) = \phi, \tag{A15}$$

where $\mathfrak{f}(\varphi, k)$; the elliptic integral of the first kind is given by

$$\mathfrak{f}(\varphi, k) = \int_0^{\sin^{-1}(\varphi)} \frac{d\theta}{\sqrt{1 - k^2\sin^2\theta}}. \tag{A16}$$

The Eguchi-Hanson type I metric also can be written as [33]

$$ds^2_{\mathrm{EHI}} = \tilde{f}^2(r)dr^2 + \frac{r^2}{4}\tilde{g}^2(r)\{d\theta^2 + \sin^2\theta d\phi^2\} + \frac{r^2}{4}(d\psi + \cos\theta d\phi)^2, \tag{A17}$$

where

$$\tilde{f}(r) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{1-\frac{a^4}{r^4}}}\right), \qquad \tilde{g}(r) = \sqrt{\frac{1}{2}\left(1 + \sqrt{1-\frac{a^4}{r^4}}\right)}. \tag{A18}$$

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