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Hyperbolic supersymmetric quantum Hall effect

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Developing a noncompact version of the supersymmetric Hopf map, we formulate the quantum Hall effect on a superhyperboloid. Based on $OSp(1|2)$ group theoretical methods, we first analyze the one-particle Landau problem, and successively explore the many-body problem where the Laughlin wave function, hard-core pseudopotential Hamiltonian, and topological excitations are derived. It is also shown that the fuzzy superhyperboloid emerges at the lowest Landau level.

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I. INTRODUCTION

In the past several years, the understanding of higher dimensional formulations of the quantum Hall effect (QHE) has greatly progressed. The initial study of this direction may date back to the pioneering work of Haldane who formulated the QHE on two-spheres more than two decades ago [1]. Beyond the importance to the study of the QHE itself, in a modern perspective, Haldane's QHE could be appreciated as a physical realization of fuzzy geometry on a curved manifold. However, reasonable higher dimensional generalizations of Haldane's model had not been found until the breakthrough of Zhang and Hu's four-dimensional QHE [2]. Since their discovery, many analyses have been devoted to further generalizations of the QHE on other higher dimensional curved manifolds. Among them, QHEs on complex projective manifolds [3] and higher dimensional spheres [4,5] have been well explored, accompanied with the developments of fuzzy geometry and matrix models [6].

Since the previous investigations are mainly concerned with compact bosonic manifolds, there might be two successive directions to be pursued. One direction would be the exploration on noncompact manifolds. With respect to hyperboloids, several works have already been reported, for the Landau problem [7–10] and for the QHE [8,11–14] as well. The other direction is the exploration on supermanifolds. Ivanov *et al.* launched the construction of the Landau model on compact supermanifolds, such as supersymmetric complex projective spaces [15] and superflag manifolds [16]. Independently, Hasebe and Kimura investigated the Landau problem on superspheres [17,18]. Recently, particular properties of the supersymmetric (SUSY) Landau models are starting to be unveiled, such as nonanticommutative geometry in the lowest Landau level (LLL) [15–17,19,20], enhanced SUSY in higher Landau levels [19–23], and the existence of negative norm states [19,20]. The remedy for the negative norm problem was implicitly suggested in Ref. [19], and well developed in Refs. [21–23] by introducing the appropriate metric in Hilbert space. Many-body problems on supermanifolds, which we call the SUSY QHE, have also been explored in Refs. [19,24–27]. The SUSY QHE was first formulated on a supersphere [24], and next on a superplane [19,25]. Their corresponding bosonic "body" manifolds are, respectively, two-spheres and Euclidean planes, and both of them are maximally symmetric spaces with Euclidean signatures; the former has positive constant curvature, while the latter has zero constant curvature. Recently, it was also found that the setup of the SUSY QHE was applicable to hole-doped antiferromagnetic quantum spin models [28].

In this paper, we explore a formulation of the QHE on a superhyperboloid whose body is the hyperboloid, which has negative constant curvature and is the last two-dimensional maximally symmetric space with a Euclidean signature. For the construction, we introduce a noncompact version of the SUSY Hopf map. The author believes this to be the first case where the noncompact SUSY Hopf map and its related materials are developed. The hyperbolic formulation of the SUSY QHE would be interesting, also from fuzzy geometry and AdS/CFT points of view. The hyperbolic SUSY QHE provides a nice physical realization of the fuzzy superhyperboloid, and, interestingly, the fuzzy hyperboloid or fuzzy (Euclidean) AdS^2 naturally appears in the context of AdS/CFT correspondence [29,30]. The hyperboloid SUSY QHE itself is closely related to the concept of holography. While on spheres a natural definition of the boundary does not exist, there is one on hyperboloids or AdS spaces. Further, edge states in the QHE are described by the chiral CFT formalism [31,32], which reflects bulk properties governed by the Chern-Simons field theory. The bulk-edge correspondence in hyperbolic (SUSY) QHE is expected to demonstrate the concept of "AdS/CFT" in condensed matter physics.

In the first half of this paper, we formulate the QHE on a (bosonic) hyperboloid based on the noncompact Hopf map, and rederive several results reported in Refs. [7–9,11–13]. We provide new ingredients also, such as the pseudopotential Hamiltonian and topological excitations. In the latter half, we extend the discussions to the superhyperboloid case, where we explore the noncompact SUSY Hopf

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map, and construct a formulation of the hyperbolic SUSY QHE. The detailed organization of this paper is as follows. In Sec. II, we briefly review basic properties of the $SU(1,1)$ group. In Sec. III, the noncompact Hopf map is introduced. The one-particle problem on the hyperboloid is discussed in Sec. IV. The noncommutative geometry in the LLL is derived, and the Hall relation is confirmed in Sec. V. In Sec. VI, we discuss the many-body problem on the hyperboloid. In Secs. VII, VIII, IX, X, and XI, with the use of the $OSp(1|2)$ super Lie group, we supersymmetrize the previous discussions. Section XII is devoted to a summary and discussions. Several definitions related to the supermatrix are given in Appendix A. In Appendix B, the Lagrange formalism on the superhyperboloid is provided. The irreducible representations of the $SU(1,1)$ group are summarized in Appendix C.

II. PRELIMINARIES I

A. The $SU(1,1)$ group and algebra

$SU(1,1)$ is topologically equivalent to a not-simply connected noncompact manifold $D\times S^1$ (D represents a disk), and is isomorphic to several groups,

$$SU(1,1)\simeq SL(2,R)\simeq Sp(2,R) \tag{2.1}$$

and

$$SU(1,1)/Z_2\simeq SO(2,1). \tag{2.2}$$

The $SU(1,1)$ group element g is defined so as to satisfy the relation

$$g^\dagger\sigma^3 g=\sigma^3, \tag{2.3}$$

with the constraint

$$\det(g)=1. \tag{2.4}$$

When g is expressed as

$$g=\begin{pmatrix} u & v^* \\ v & u^*\end{pmatrix}, \tag{2.5}$$

the constraint (2.4) becomes

$$uu^*-vv^*=1. \tag{2.6}$$

The inverse of g is given by

$$g^{-1}=\sigma^3 g^\dagger\sigma^3=\begin{pmatrix} u^* & -v^* \\ -v & u\end{pmatrix}\neq g^\dagger=\begin{pmatrix} u^* & v^* \\ v & u\end{pmatrix}. \tag{2.7}$$

Since $SU(1,1)$ is a noncompact group, its unitary representation is infinite dimensional. (The irreducible representations of $SU(1,1)$ are summarized in Appendix C, and detailed discussions can be found in Ref. [33].) In this paper, we deal with a nonunitary representation of the principal discrete series, and hence the generators are generally represented by non-Hermitian and finite dimensional matrices. The $SU(1,1)$ generators are given by

$$s^a=\frac{1}{2}\kappa^a, \tag{2.8}$$

where κ^a are

$$\kappa^1=i\sigma^1,\qquad \kappa^2=i\sigma^2,\qquad \kappa^3=\sigma^3. \tag{2.9}$$

Here, σ^a denote Pauli matrices; non-Hermitian matrices κ^1 and κ^2 are boost generators to the x and y directions, respectively, while the Hermitian matrix κ^3 is the rotation generator on the x-y plane. s^a satisfy the algebra

$$[s^a,s^b]=i\epsilon^{ab}{}_c s^c, \tag{2.10}$$

where ϵ^{abc} represents the three-rank antisymmetric tensor with $\epsilon^{123}=1$, and the indices are raised or lowered by the metric $\eta_{ab}=\eta^{ab}=(+,+,-)$. $-s_a$ also satisfy the $SU(1,1)$ algebra, and are related to s^a as

$$\sigma^3 s^a\sigma^3=-s_a. \tag{2.11}$$

The Casimir operator is given by

$$C=\eta_{ab}s^as^b=s^1s^1+s^2s^2-s^3s^3, \tag{2.12}$$

and its eigenvalues are

$$C=-j(j-1) \tag{2.13}$$

with $j=1,3/2,2,5/2,\cdots$. It should be noticed that the Casimir index j begins from 1 not 0. We summarize the properties of κ^a for later convenience. Their anticommutation relations are given by

$$\{\kappa^a,\kappa^b\}=-2\eta^{ab}, \tag{2.14}$$

and then, with (2.10),

$$\kappa^a\kappa^b=-\eta^{ab}+i\epsilon^{ab}{}_c\kappa^c. \tag{2.15}$$

Their normalizations are

$$\mathrm{tr}\,(\kappa^a\kappa^b)=-2\eta^{ab}. \tag{2.16}$$

The completeness relation is

$$4\eta_{ab}(\kappa^a)_\alpha{}^\beta(\kappa^b)_\gamma{}^\delta=-2\delta_\alpha{}^\delta\delta_\beta{}^\gamma+\delta_\alpha{}^\beta\delta_\gamma{}^\delta. \tag{2.17}$$

B. Complex representation

The complex representation is given by

$$\tilde{\kappa}^a\equiv-\kappa^{a*}=\kappa^t_a, \tag{2.18}$$

and related to the original representation by the unitary transformation

$$\tilde{\kappa}^a=R^\dagger\kappa^aR, \tag{2.19}$$

where $R=\sigma^1$. Then, with an $SU(1,1)$ spinor ϕ, its charge conjugation is constructed as

$$\phi_c=R^\dagger\phi^*, \tag{2.20}$$

and the Majorana condition $\phi_c=\phi$ is given by

$$\phi=\sigma^1\phi^*, \tag{2.21}$$

or

$$\phi^{1*} = \phi^2, \qquad \phi^{2*} = \phi^1. \tag{2.22}$$

Without introducing the complex conjugation, the $SU(1,1)$ singlet is constructed as

$$(R^\dagger \varphi^*)^\dagger \sigma^3 \phi = \varphi^t \sigma^1 \sigma^3 \psi = -i\varphi^t \sigma^2 \phi. \tag{2.23}$$

III. NONCOMPACT HOPF MAP

The original (1st) Hopf map is given by

$$S^3 \to S^2 \simeq S^3/S^1, \tag{3.1}$$

and its noncompact version may be introduced as

$$\mathrm{AdS}^3 \to H^2 \simeq \mathrm{AdS}^3/S^1, \tag{3.2}$$

where $\mathrm{AdS}^n \simeq SO(n-1,2)/SO(n-1,1)$, and H^n represents an n-dimensional two-leaf hyperboloid that is equivalent to Euclidean $\mathrm{AdS}^n \simeq SO(n,1)/SO(n)$. H^2 with radius r is simply defined as

$$\eta_{ab}x^a x^b (= x^2 + y^2 - z^2) = -r^2. \tag{3.3}$$

Apparently, H^2 is invariant under the $SO(2,1)$ rotations generated by

$$J^a = -i\epsilon^{abc}x_b \frac{\partial}{\partial x^c}. \tag{3.4}$$

With a special choice of the vector on the hyperboloid $(x,y,z) = (0,0,\pm r)$, the stabilizer group is found to be the $SO(2)$ rotational group around the z axis, and hence $H^2 \simeq SO(2,1)/SO(2)$. With polar coordinates, the coordinates on the two-leaf hyperboloid are parametrized as

$$x = r\sinh\tau\sin\theta, \qquad y = r\sinh\tau\cos\theta, \qquad z = \pm r\cosh\tau, \tag{3.5}$$

where $-\infty < \tau < \infty$ and $0 \le \theta < 2\pi$. $z > 0$ corresponds to the upper leaf, while $z < 0$ corresponds to the lower leaf. In this paper, we focus on the upper leaf, while the treatment of the lower leaf is completely analogous.

The noncompact Hopf map (3.2) is explicitly represented by the mapping from g to x^a:

$$gg^\dagger = \eta_{ab}x^a \sigma^3 \kappa^b. \tag{3.6}$$

Taking the square of both sides and the trace, one may reproduce the hyperboloid constraint

$$\eta_{ab}x^a x^b = -1, \tag{3.7}$$

where (2.3) and (2.16) were used. (For simplicity, we deal with a hyperboloid with unit radius in the following, unless otherwise stated.) With the parametrization of g (2.5), x^a are expressed as

$$x^1 = i(u^*v - v^*u), \quad x^2 = u^*v + v^*u, \quad x^3 = u^*u + v^*v, \tag{3.8}$$

or, more concisely,

$$\phi \to x^a = 2\phi^\dagger \sigma^3 s^a \phi, \tag{3.9}$$

where ϕ represents the "noncompact" Hopf spinor

$$\phi = \begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.10}$$

which satisfies the normalization

$$\phi^\dagger \sigma^3 \phi = u^*u - v^*v = 1. \tag{3.11}$$

From (3.9), the hyperboloid condition is readily derived as

$$\eta_{ab}x^a x^b = -(\phi^\dagger\sigma^3\phi)^2 = -1. \tag{3.12}$$

With the complex representation $\tilde{s}^a = \frac{1}{2}\tilde{\kappa}^a$, (3.9) is rewritten as

$$\phi \to x^a = 2\phi^t \tilde{s}_a \sigma^3 \phi^*. \tag{3.13}$$

Inverting (3.9), the noncompact Hopf spinor is expressed as

$$\phi = \begin{pmatrix} \sqrt{\frac{1+x^3}{2}} \\ \frac{x^2 - ix^1}{\sqrt{2(1+x^3)}} \end{pmatrix} e^{i\chi} = \begin{pmatrix} \cosh\frac{\tau}{2} \\ \sinh\frac{\tau}{2} e^{i\theta} \end{pmatrix} e^{i\chi}, \tag{3.14}$$

where the $U(1)$ phase factor is canceled in the mapping (3.13). The noncompact Hopf spinor is equal to the $SU(1,1)$ coherent state formulated in [34], which satisfies the coherent state equation

$$\eta_{ab}x^a s^b \phi = -\tfrac{1}{2}\phi \tag{3.15}$$

or

$$\eta_{ab}x^a \phi^t \tilde{s}^b = -\tfrac{1}{2}\phi^t. \tag{3.16}$$

A. $U(1)$ connection

The noncompact Hopf map induces the $U(1)$ connection as

$$A = \frac{i}{2}\,\mathrm{tr}(g^\dagger \sigma^3 dg) = i\phi^\dagger \sigma^3 d\phi, \tag{3.17}$$

which is explicitly evaluated as

$$A = dx^a A_a = -\frac{I}{2} dx^a \epsilon_{ab}{}^3 \frac{x^b}{1+x^3}, \tag{3.18}$$

with $I = 1$. In general, I takes an integer, and $I/2$ represents the "monopole" charge. The corresponding field strengths are given by

$$F_{ab} = \partial_a A_b - \partial_b A_a = -\frac{I}{2}\epsilon_{abc}x^c. \tag{3.19}$$

IV. HYPERBOLIC LANDAU PROBLEM

Here, we explore one-particle quantum mechanics on the surface of a hyperboloid in a monopole background.

A. $SU(1,1)$ covariant angular momenta

The $SU(1,1)$ covariant angular momenta are given by

$$\Lambda^a = -i\epsilon^{abc}x_b D_c, \tag{4.1}$$

where D_a denote covariant derivatives

$$D_a = \partial_a + iA_a. \tag{4.2}$$

The algebra of the covariant angular momenta is

$$[\Lambda^a, \Lambda^b] = i\epsilon^{ab}{}_c(\Lambda^c - F^c), \tag{4.3}$$

with $SO(2,1)$ vector field strengths F^a,

$$F^a = -\frac{1}{2}\epsilon^{abc}F_{bc} = -\frac{I}{2}x^a. \tag{4.4}$$

The covariant angular momenta are tangent to the surface of the hyperboloid, and orthogonal to the field strengths

$$\eta_{ab}\Lambda^a F^b = \eta_{ab}F^a\Lambda^b = 0. \tag{4.5}$$

The total angular momenta J^a are constructed as

$$J^a = \Lambda^a + F^a, \tag{4.6}$$

and satisfy the relations

$$[J^a, M^b] = i\epsilon^{ab}{}_c M^c, \tag{4.7}$$

where $M^a = J^a$, Λ^a, and F^a. In particular, when $M^a = J^a$, (4.7) represents the closed $SU(1,1)$ algebra, and the corresponding $SU(1,1)$ Casimir operator is given by

$$C = \eta_{ab}J^aJ^b = \eta_{ab}\Lambda^a\Lambda^b - \frac{I^2}{4}, \tag{4.8}$$

where (4.5) was used. The eigenvalues of the Casimir operator are

$$C = -j(j-1), \tag{4.9}$$

where, due to the existence of field strengths, j takes

$$j = -\frac{I}{2} + n + 1. \tag{4.10}$$

Here n denotes the Landau level (LL) index.

B. One-particle Hamiltonian

The one-particle Hamiltonian is

$$H = \frac{1}{2M}\eta_{ab}\Lambda^a\Lambda^b, \tag{4.11}$$

in which the radial kinetic term does not exist, since the particle is confined on the surface of the hyperboloid. With (4.8) and (4.10), the energy eigenvalues are easily derived as

$$E_n = \frac{1}{2M}\left(I\left(n+\frac{1}{2}\right) - n(n+1)\right). \tag{4.12}$$

Equation (4.12) coincides with the result in Refs. [7–9,11–13]. Unlike the case of the sphere [1], the hyperboloid Landau level energy has the maximum

$$E_{\max} = \frac{I^2}{8M} + \frac{1}{8M} \tag{4.13}$$

at $n = I/2 - 1/2$. Meanwhile, the LLL energy is the same in the case of the sphere,

$$E_{\mathrm{LLL}} = E_{n=0} = \frac{I}{4M}. \tag{4.14}$$

However, the hyperboloid LLL energy is *not* the minimum, since (4.12) is unbounded as found at $n \to \infty$. By recovering the radius r and taking the thermodynamic limit, $I, r \to \infty$ with fixed I/r^2, Eq. (4.12) reproduces the LL energies on the Euclidean plane,

$$E_n \to \omega(n + \tfrac{1}{2}), \tag{4.15}$$

where $\omega = I/Mr^2$.

The eigenstates in the LLL are constructed by the symmetric products of the components of the noncompact Hopf spinor,

$$u_{m_1,m_2} = \sqrt{\frac{I!}{m_1!m_2!}}u^{m_1}v^{m_2}, \tag{4.16}$$

where $m_1, m_2 \geq 0$ and $m_1 + m_2 = I$. Since we are concerned with the nonunitary representation, the degeneracy in the LLL becomes finite, and we define the filling fraction as

$$\nu = N/D \to 1/m, \tag{4.17}$$

where $N = I + 1$ denotes the number of all particles, and $D = mI + 1$ denotes the number of all states, respectively. The right arrow corresponds to the thermodynamic limit.

C. Coherent state on a hyperboloid

With J^a of $I = 1$, the noncompact Hopf spinor satisfies

$$J^a\phi = -s^a\phi, \tag{4.18}$$

and, in the LLL, the $SU(1,1)$ operators are effectively represented as

$$J^a = -\phi^t\tilde{s}_a\frac{\partial}{\partial\phi}, \tag{4.19}$$

where $\tilde{s}^a = s_a^t$. Since $-\tilde{s}^a$ obey the $SU(1,1)$ commutation relations, so do J^a. The one-particle state aligned with the direction $\Omega^a(\chi)$ on the hyperboloid satisfies the relation

$$[\Omega^a(\chi)\cdot J_a]\phi_\chi(\phi) = \frac{I}{2}\phi_\chi(\phi), \tag{4.20}$$

and ϕ_χ is constructed as

$$\phi_\chi(\phi) = (\chi^\dagger\sigma^3\phi)^I = (\alpha^* u - \beta^* v)^I, \tag{4.21}$$

where $\chi = (\alpha, \beta)^t$ is related to $\Omega^a(\chi)$ by the relation

$$\Omega^a(\chi) = \chi^\dagger\sigma^3\kappa^a\chi. \tag{4.22}$$

V. HYPERBOLIC NONCOMMUTATIVE GEOMETRY AND HYPERBOLIC HALL LAW

The kinetic term is quenched in the LLL, and the LLL limit is realized by simply neglecting Λ^a. Then, in the limit, from (4.6), one may deduce the relation

$$x^a \longrightarrow X^a = -\alpha L^a, \tag{5.1}$$

with $\alpha = 2/I$. While, originally, x^a are the c-number coordinates on the hyperboloid, they are effectively regarded as the $SU(1,1)$ operators in the LLL, and they satisfy the algebra

$$[X^a, X^b] = -i\alpha\epsilon^{abc}X_c, \tag{5.2}$$

which defines the fuzzy hyperboloid [29,30]. From (5.2), the equations of motion are derived as

$$I^a = \frac{d}{dt}X^a = -i[X^a, V] = -\alpha\epsilon^{abc}x_b E_c, \tag{5.3}$$

with the electric field $E_a = -\partial_a V$, so one may find the hyperbolic Hall law

$$\eta_{ab}I^a E^b = 0. \tag{5.4}$$

VI. HYPERBOLIC QUANTUM HALL EFFECT

A. Hyperbolic Laughlin-Haldane wave function

In Haldane's original setup, the Laughlin wave function is given by the $SU(2)$ singlet made of the (compact) Hopf spinors [1], and indeed, such a spherical Laughlin-Haldane wave function can also be constructed from the stereographic projection from the Laughlin wave function on the Euclidean plane. The Laughlin-Haldane wave function on a hyperboloid could similarly be derived: we may adopt the $SU(1,1)$ singlet made of the noncompact Hopf spinors

$$\Phi = \prod_{i<j}(\phi_i^t\sigma^3 R\phi_j)^m = \prod_{i<j}(u_i v_j - v_i u_j)^m, \tag{6.1}$$

which is consistent with the results in Refs. [8,11]. The last expression of (6.1) is superficially equivalent to the spherical Laughlin-Haldane function, but here, the noncompact Hopf spinors are used. Since any two-body state described by the Laughlin-Haldane wave function does not have an $SU(1,1)$ angular momentum greater than $m(N-2)$, the hard-core pseudopotential Hamiltonian is constructed as

$$H_{\text{H.c.}} = \sum_{i<j}\sum_{m(N-2)+1\leq J}^{m(N-1)} V_J P_J(i,j), \tag{6.2}$$

where $V_J > 0$ denotes the pseudopotential, and P_J represents the projection operator to the two-body subspace with the $SU(1,1)$ Casimir index J,

$$\begin{aligned} P_J(i,j) &= \prod_{J'\neq J}\frac{\eta_{ab}(J^a(i)+J^b(i))(J^b(j)+J^b(j))+J'(J'-1)}{J'(J'-1)-J(J-1)} \\ &= \prod_{J'\neq J}\frac{2\eta_{ab}J^a(i)J^b(j)-I(\frac{I}{2}-1)+J'(J'-1)}{J'(J'-1)-J(J-1)}. \end{aligned} \tag{6.3}$$

In the last equation, we have used $\eta_{ab}J^aJ^b = -j(j-1)_{j=-I/2+1} = -I/2(I/2-1)$.

B. Excitations

Operators for excitations (quasiparticle and quasihole) on a hyperboloid are, respectively, given by

$$A(\chi) = \prod_i^N \chi^\dagger R^\dagger \frac{\partial}{\partial\phi_i} = \prod_i^N\left(\alpha^*\frac{\partial}{\partial v_i} + \beta^*\frac{\partial}{\partial u_i}\right), \tag{6.4a}$$

$$A^\dagger(\chi) = \prod_i^N \phi_i^t R\sigma^3\chi = \prod_i(\alpha v_i - \beta u_i), \tag{6.4b}$$

where χ specifies the point $\Omega^a(\chi)$ at which excitations are generated, by the relation (4.22). Their commutation relations are evaluated as

$$\begin{aligned} &[A(\chi), A^\dagger(\chi)] = 1, \\ &[A(\chi), A(\chi')] = [A^\dagger(\chi), A^\dagger(\chi')] = 0, \end{aligned} \tag{6.5}$$

and $A(\chi)$ and $A^\dagger(\chi)$ are interpreted as annihilation and creation operators, respectively. The creation operator satisfies the following commutation relation with angular momentum,

$$[\Omega_a(\chi)J^a, A^\dagger(\chi)] = -\frac{N}{2}A^\dagger(\chi). \tag{6.6}$$

In particular, at the bottom of the upper leaf $\Omega^a = (0,0,1)$, Eq. (6.6) becomes

$$[J^z, A^\dagger(\chi)] = \frac{N}{2}A^\dagger(\chi), \tag{6.7}$$

which implies that the generation of the quasihole pushes each of the particles to the z direction by $1/2$, and the quasihole is identified with a charge deficit. At $\nu = 1/m$, there are m states per particle, and the quasihole carries the fractional charge $1/m$.

VII. PRELIMINARIES II

For the construction of the spherical SUSY QHE [17,24], the $UOSp(1|2)$ group was used. The bosonic subgroup of $UOSp(1|2)$ is $SU(2)$, and the graded Hermitian conjugate was adopted to impose a consistent Majorana condition. Meanwhile, for the case of the hyperbolic SUSY QHE, we use the $OSp(1|2)$ group whose subgroup is $SU(1,1)$, and the conventional Hermitian conjugate is adopted [35].

A. The $OSp(1|2)$ group and algebra

Here, we sketch basic structures of the $OSp(1|2)$ group. The $OSp(1|2)$ group element g is defined so as to satisfy the relation

$$g^\dagger k g = k, \tag{7.1}$$

and the constraint

$$\mathrm{sdet}\,(g) = 1. \tag{7.2}$$

Here,

$$k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{7.3}$$

and the superdeterminant (sdet) is defined in Appendix A. The g is parametrized as

$$g = \begin{pmatrix} u & v^* & \eta^* u + \eta v^* \\ v & u^* & \eta u^* + \eta^* v \\ \eta & -\eta^* & 1 - \eta^*\eta \end{pmatrix}, \tag{7.4}$$

where u and v are Grassmann-even quantities, and η is a Grassmann-odd quantity. The inverse of g is *not* its simple Hermitian conjugate, but

$$\begin{aligned} g^{-1} &= k g^\dagger k \\ &= \begin{pmatrix} u^* & -v^* & -\eta^* \\ -v & u & -\eta \\ -\eta u^* - \eta^* v & \eta^* u + \eta v^* & 1 - \eta^*\eta \end{pmatrix} \\ &\neq g^\dagger \\ &= \begin{pmatrix} u^* & v^* & \eta^* \\ v & u & -\eta \\ \eta u^* + \eta^* v & \eta^* u + \eta v^* & 1 - \eta^*\eta \end{pmatrix}. \end{aligned} \tag{7.5}$$

With (7.4), the constraint (7.2) is restated as

$$u^* u - v^* v - \eta^* \eta = \psi^\dagger k \psi = -\psi^t k' \psi^* = 1, \tag{7.6}$$

where ψ denotes the noncompact SUSY Hopf spinor

$$\psi = \begin{pmatrix} u \\ v \\ \eta \end{pmatrix}, \tag{7.7}$$

and

$$k' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{7.8}$$

The $OSp(1|2)$ algebra is constructed as

$$[l^a, l^b] = i\epsilon^{ab}{}_c l^c, \qquad [l^a, l^\alpha] = \tfrac{1}{2}(\kappa^a)_\beta{}^\alpha l^\beta, \qquad \{l^\alpha, l^\beta\} = \tfrac{1}{2}(\epsilon^t \kappa_a)^{\alpha\beta} l^a, \tag{7.9}$$

where

$$\epsilon = \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \epsilon^t = \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{7.10}$$

The $OSp(1|2)$ Casimir operator is given by

$$C = \eta_{ab} l^a l^b - \epsilon_{\alpha\beta} l^\alpha l^\beta, \tag{7.11}$$

and its eigenvalues are

$$C = -j(j - \tfrac{1}{2}), \tag{7.12}$$

with $j = 1/2, 1, 3/2, 2, \cdots$. It is noted that the Casimir index begins from $1/2$ not 0. The fundamental representation of the $OSp(1|2)$ algebra is

$$l^a = \frac{1}{2}\begin{pmatrix} \kappa^a & 0 \\ 0 & 0 \end{pmatrix}, \qquad l^\alpha = \frac{1}{2}\begin{pmatrix} 0 & \tau^\alpha \\ -(\epsilon\tau^\alpha)^t & 0 \end{pmatrix}, \tag{7.13}$$

and is normalized as

$$\mathrm{str}\,(l^a l^b) = -\tfrac{1}{2}\eta^{ab}, \qquad \mathrm{str}(l^\alpha l^\beta) = \tfrac{1}{2}\epsilon^{\alpha\beta}, \qquad \mathrm{str}(l^a l^\alpha) = 0, \tag{7.14}$$

where the supertrace (str) is defined in Appendix A. When l^a and l^α satisfy the $OSp(1|2)$ algebra,

$$-l_a, \;\; d^\alpha \equiv (\sigma^1)_\beta{}^\alpha (l^\beta)^t \tag{7.15}$$

also satisfy the algebra. $-l_a$ and d^α are related to l^a and l^α as

$$-l_a = k l^a k, \qquad d^\alpha = k l^\alpha k, \tag{7.16}$$

with k (7.3).

B. Complex representation

The complex representation of (7.13) is constructed as

$$\tilde{l}^a = -l^{a*}, \qquad \tilde{l}^\alpha = \epsilon_{\alpha\beta} d^\beta, \tag{7.17}$$

and related to l^a and l^α by the unitary transformation

$$\tilde{l}^a = \mathcal{R}^\dagger l^a \mathcal{R}, \qquad \tilde{l}^\alpha = \mathcal{R}^\dagger l^\alpha \mathcal{R}, \tag{7.18}$$

where

$$\mathcal{R} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.19}$$

The properties of $\mathcal{R}$ are summarized as

$$\mathcal{R} = \mathcal{R}^t = \mathcal{R}^\dagger = \mathcal{R}^{-1}, \tag{7.20}$$

$$\mathcal{R}^2 = (\mathcal{R}^t)^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.21}$$

Then, the charge conjugation of ψ is determined as

$$\psi_c = \mathcal{R}^\dagger \psi^*, \tag{7.22}$$

and, without using complex conjugation, the $OSp(1|2)$

singlet can be constructed as

$$(\psi_c)^\dagger k\psi' = \psi^t \mathcal{R} k\psi' = -(uv' - vu' + \eta\eta'). \quad (7.23)$$

For later convenience, we introduce another complex representation,

$$j^a = l_a^*, \qquad j^\alpha = \epsilon_{\alpha\beta} l^\beta \quad (7.24)$$

whose original representation is $-l_a$ and d^α. Equations (7.15) and (7.24) are related by the unitary transformation

$$j^a = \mathcal{R}^\dagger(-l_a)\mathcal{R}, \qquad j^\alpha = \mathcal{R}^\dagger d^\alpha \mathcal{R}. \quad (7.25)$$

It should be noticed that j^a and j^α are linearly dependent on l^a and l^α, while $\tilde{l}^a$ and $\tilde{l}^\alpha$ are not, because $\tilde{l}^\alpha = (-1)^{\alpha+1}(l^\alpha)^t$ cannot be expressed by linear combinations of l^a and l^α. In the following, j^a and j^α will be used rather than $\tilde{l}^a$ and $\tilde{l}^\alpha$. While (7.23) is not invariant under the $OSp(1|2)$ transformation generated by j^a and j^α,

$$\psi^t k \mathcal{R} \psi' = uv' - vu' - \eta\eta' \quad (7.26)$$

is invariant. The two complex representations (7.17) and (7.24) are simply related as

$$j^a = k\tilde{l}^a k, \qquad j^\alpha = k\tilde{l}^\alpha k. \quad (7.27)$$

Further, they are related to l^a and l^α by the unitary transformation

$$j^a = \mathcal{K}^t l^a \mathcal{K}, \qquad j^\alpha = \mathcal{K}^t l^\alpha \mathcal{K} \quad (7.28)$$

where

$$\mathcal{K} = \mathcal{R}k = k'\mathcal{R} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (7.29)$$

The properties of $\mathcal{K}$ are similar to those of $\mathcal{R}$:

$$\mathcal{K}^t = \mathcal{K}^\dagger = \mathcal{K}^{-1}, \quad (7.30)$$

but $\mathcal{K} \neq \mathcal{K}^t$, and

$$\mathcal{K}^2 = (\mathcal{K}^t)^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = kk' = k'k. \quad (7.31)$$

k and k' are constructed from the products of $\mathcal{K}$ and $\mathcal{R}$ as

$$k = \mathcal{R}\mathcal{K}, \qquad k' = \mathcal{K}\mathcal{R}, \quad (7.32)$$

and related as

$$k = \mathcal{R}^t k' \mathcal{R} = \mathcal{K}^t k' \mathcal{K}, \qquad k' = \mathcal{R}^t k \mathcal{R} = \mathcal{K}^t k \mathcal{K}. \quad (7.33)$$

VIII. THE NONCOMPACT SUSY HOPF MAP

The (original) SUSY Hopf map

$$S^{3|2} \to S^{2|2} \simeq S^{3|2}/S^1 \quad (8.1)$$

was introduced in Ref. [38], and the accompanying bundle structure has been well examined in Refs. [17,39,40]. Here, we explore the noncompact version of it,

$$\mathrm{AdS}^{3|2} \to H^{2|2} \simeq \mathrm{AdS}^{3|2}/S^1, \quad (8.2)$$

where the superhyperboloid $H^{2|2}$ or Euclidean $\mathrm{AdS}^{2|2}$ is defined so as to satisfy the condition

$$\eta_{ab}x^a x^b - \epsilon_{\alpha\beta}\theta^\alpha\theta^\beta = -1. \quad (8.3)$$

Apparently, the condition is invariant under the $OSp(1|2)$ transformations generated by

$$\begin{aligned} L^a &= -i\epsilon^{abc}x_b\partial_c + \tfrac{1}{2}(\kappa^a)_\beta{}^\alpha\theta^\beta\partial_\alpha, \\ L^\alpha &= \tfrac{1}{2}(\epsilon^t\kappa^a)^{\alpha\beta}x_a\partial_\beta - \tfrac{1}{2}(\kappa^a)_\beta{}^\alpha\theta^\beta\partial_a, \end{aligned} \quad (8.4)$$

and $H^{2|2}$ manifestly possesses the $OSp(1|2)$ symmetry. The noncompact SUSY Hopf map is explicitly constructed as

$$g \to gk^3g^\dagger = \delta_{ab}x^a k^b + (\sigma^1)_{\alpha\beta}\theta^\alpha k^\beta, \quad (8.5)$$

where $k^a = kl^a$ and $k^\alpha = kl^\alpha$ are

$$k^1 = \frac{1}{2}\begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad k^2 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$k^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad k^{\theta_1} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \quad (8.6)$$
$$k^{\theta_2} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Though k^a and k^α are "Hermitian" in the sense that

$$k^{a\dagger} = k^a, \qquad k^{\alpha\dagger} = (\sigma^1)_\beta{}^\alpha k^\beta, \quad (8.7)$$

they do not form a closed algebra. With the normalization (7.14), it is not difficult to see that x^a and θ^α [introduced by (8.5)] indeed satisfy the superhyperboloid condition (8.3). With (7.4), x^a and θ^α are expressed as

$$\begin{aligned} x^1 &= i(u^*v - v^*u), \qquad x^2 = u^*v + v^*u, \\ x^3 &= u^*u + v^*v, \qquad \theta^1 = u^*\eta - \eta^*v, \\ \theta^2 &= \eta^*u - \eta v^*, \end{aligned} \quad (8.8)$$

or, more compactly,

$$x^a = 2\psi^\dagger k^a\psi, \qquad \theta^\alpha = 2\psi^\dagger k^\alpha\psi, \quad (8.9)$$

where ψ is the noncompact SUSY Hopf spinor (7.7). From the "Hermiticity" of k^a and k^α, x^a and θ^α are "real" in the sense that

$$x^{a*} = x^a, \qquad \theta^{\alpha*} = (\sigma^1)_\beta{}^\alpha\theta^\beta. \quad (8.10)$$

Namely, $\theta = (\theta^1, \theta^2)^t$ is an $SO(2,1)$ Majorana spinor. From the noncompact SUSY Hopf map (8.9) and the constraint (7.6), it is readily confirmed that x^a and θ^α

satisfy the condition (8.3), since

$$\eta_{ab}x^a x^b - \epsilon_{\alpha\beta}\theta^\alpha\theta^\beta = -(\psi^\dagger k\psi)^2 = -1. \quad (8.11)$$

With the complex representation, the noncompact SUSY Hopf map (8.9) is restated as

$$x^a = 2\psi^t k'^a \psi^*, \qquad \theta^\alpha = 2\psi^t k'^\alpha \psi^*, \quad (8.12)$$

where

$$k'^a \equiv j^a k' = -\eta_{ab}k^b, \qquad k'^\alpha \equiv j^\alpha k' = (\sigma^1)_\beta{}^\alpha k^\beta. \quad (8.13)$$

Inverting (8.9), the noncompact SUSY Hopf spinor is expressed by x^a and θ^α, up to the $U(1)$ phase factor, as

$$\psi = \frac{1}{\sqrt{2(1+x^3)}}\begin{pmatrix} (1+x^3)(1-\frac{1}{4(1+x^3)}\theta\epsilon\theta) \\ (x^2 - ix^1)(1+\frac{1}{4(1+x^3)}\theta\epsilon\theta) \\ (1+x^3)\theta^1 + (x^2 - ix^1)\theta^2 \end{pmatrix}\cdot e^{i\chi}, \quad (8.14)$$

which satisfies the supercoherent equation

$$\eta_{ab}l^a\psi x^b - \epsilon_{\alpha\beta}l^\alpha\psi\theta^\beta = -\tfrac{1}{2}\psi, \quad (8.15)$$

or, in the complex representation,

$$\eta_{ab}x^a\psi^t j^b - \epsilon_{\alpha\beta}\theta^\alpha\psi^t j^\beta = \tfrac{1}{2}\psi^t. \quad (8.16)$$

Thus, the noncompact SUSY Hopf spinor is equivalent to the $OSp(1|2)$ supercoherent state in Ref. [41].

A. $U(1)$ connection

The noncompact SUSY Hopf map (8.5) or (8.9) is invariant under the $U(1)$ gauge transformation:

$$g \to g e^{2i\alpha l^3} \quad (8.17)$$

or

$$\psi \to e^{i\alpha}\psi. \quad (8.18)$$

Such gauge freedom induces a $U(1)$ connection on a superhyperboloid as

$$A = i\mathrm{str}(k^3 g^\dagger k dg) = i\psi^\dagger k d\psi. \quad (8.19)$$

Accompanied with the $U(1)$ gauge transformation (8.17), A is transformed as

$$A \to A + d\alpha, \quad (8.20)$$

as expected. With the explicit form of the noncompact SUSY Hopf spinor (8.14), the components of the $U(1)$ gauge field

$$A = dx^a A_a + d\theta^\alpha A_\alpha \quad (8.21)$$

are evaluated as

$$A_a = -\frac{I}{2}\epsilon_{ab}{}^3\frac{x^b}{1+x^3}\left(1+\frac{2+x^3}{2(1+x^3)}\theta\epsilon\theta\right),$$
$$A_\alpha = -i\frac{I}{2}x^a(\theta\kappa_a\epsilon)_\alpha, \quad (8.22)$$

with $I = 1$. $I/2$ represents the "supermonopole" charge with integer I. Their complex conjugations are given by

$$A_a^* = A_a, \qquad A_\alpha^* = -(\sigma^1)_\alpha{}^\beta A_\beta. \quad (8.23)$$

The super field strengths,

$$F_{ab} = \partial_a A_b - \partial_b A_a, \qquad F_{a\alpha} = \partial_a A_\alpha - \partial_\alpha A_a,$$
$$F_{\alpha\beta} = \partial_\alpha A_\beta + \partial_\beta A_\alpha, \quad (8.24)$$

are also evaluated as

$$F_{ab} = -\frac{I}{2}\epsilon_{abc}x^c\left(1+\frac{3}{2}\theta\epsilon\theta\right),$$
$$F_{a\alpha} = -i\frac{I}{2}(\theta\kappa_b\epsilon)_\alpha(\delta^b{}_a - 3x_a x^b), \quad (8.25)$$
$$F_{\alpha\beta} = -iI(\kappa_a\epsilon)_{\alpha\beta}x^a\left(1+\frac{3}{2}\theta\epsilon\theta\right).$$

IX. HYPERBOLIC SUSY LANDAU PROBLEM

The Landau problem is inspected on the surface of a superhyperboloid in the supermonopole background.

A. $OSp(1|2)$ covariant angular momenta

There are two kinds of covariant angular momenta: one is bosonic and the other is fermionic,

$$\Lambda^a = -i\epsilon^{abc}x_b D_c + \tfrac{1}{2}(\kappa^a)_\beta{}^\alpha\theta^\beta D_\alpha,$$
$$\Lambda^\alpha = \tfrac{1}{2}(\epsilon^t\kappa^a)^{\alpha\beta}x_a D_\beta - \tfrac{1}{2}(\kappa^a)_\beta{}^\alpha\theta^\beta D_a, \quad (9.1)$$

where the covariant derivatives are defined by

$$D_a = \partial_a + iA_a, \qquad D_\alpha = \partial_\alpha + iA_\alpha. \quad (9.2)$$

The covariant angular momenta satisfy the relations

$$[\Lambda^a, \Lambda^b] = i\epsilon^{ab}{}_c(\Lambda^c - F^c),$$
$$[\Lambda^a, \Lambda^\alpha] = \tfrac{1}{2}(\kappa^a)_\beta{}^\alpha(\Lambda^\beta - F^\beta), \quad (9.3)$$
$$\{\Lambda^\alpha, \Lambda^\beta\} = \tfrac{1}{2}(\epsilon^t\kappa_a)^{\alpha\beta}(\Lambda^a - F^a),$$

where

$$F^a = -\frac{I}{2}x^a, \qquad F^\alpha = -\frac{I}{2}\theta^\alpha, \quad (9.4)$$

which are the angular momenta of the supermonopole gauge fields, and are orthogonal to the covariant angular momenta

$$\eta_{ab}\Lambda^a F^b - \epsilon_{\alpha\beta}\Lambda^\alpha F^\beta = \eta_{ab}F^a\Lambda^b - \epsilon_{\alpha\beta}F^\alpha\Lambda^\beta = 0. \quad (9.5)$$

The conserved SUSY angular momenta are constructed as

$$J^a = \Lambda^a + F^a, \qquad J^\alpha = \Lambda^\alpha + F^\alpha, \tag{9.6}$$

and they generate the $OSp(1|2)$ transformations

$$[J^a, M^b] = i\epsilon^{ab}{}_c M^c, \qquad [J^a, M^\alpha] = \tfrac{1}{2}(\kappa^a)_\beta{}^\alpha M^\beta, \qquad \{J^\alpha, M^\beta\} = \tfrac{1}{2}(\epsilon^t \kappa_a)^{\alpha\beta} M^a, \tag{9.7}$$

where $M^a = J^a,\ \Lambda^a,\ F^a$ and $M^\alpha = J^\alpha,\ \Lambda^\alpha,\ F^\alpha$. The corresponding $OSp(1|2)$ Casimir operator is given by

$$\eta^{ab} J^a J^b - \epsilon_{\alpha\beta} J^\alpha J^\beta = \eta^{ab}\Lambda^a\Lambda^b - \epsilon_{\alpha\beta}\Lambda^\alpha\Lambda^\beta - \frac{I^2}{4}, \tag{9.8}$$

where (9.5) and

$$\eta_{ab} F^a F^b - \epsilon_{\alpha\beta} F^\alpha F^\beta = -\frac{I^2}{4} \tag{9.9}$$

were used. The Casimir operator takes the eigenvalues

$$\eta_{ab} J^a J^b - \epsilon_{\alpha\beta} J^\alpha J^\beta = -j(j - \tfrac{1}{2}) \tag{9.10}$$

with

$$j = -\frac{I}{2} + n + \frac{1}{2}. \tag{9.11}$$

Here, n denotes the super LL index.

B. One-particle Hamiltonian

The one-particle Hamiltonian is given by

$$H = \frac{1}{2M}(\eta_{ab}\Lambda^a\Lambda^b - \epsilon_{\alpha\beta}\Lambda^\alpha\Lambda^\beta), \tag{9.12}$$

and is a supersymmetric Hamiltonian in the sense that it is invariant under the $OSp(1|2)$ transformation. From (9.8) and (9.10), its energy eigenvalues are derived as

$$E_n = \frac{1}{2M}\left(I\left(n + \frac{1}{4}\right) - n\left(n + \frac{1}{2}\right)\right). \tag{9.13}$$

The energy takes the maximum

$$E_{\max} = \frac{I^2}{8M} + \frac{1}{32M} \tag{9.14}$$

at $n = I/2 - 1/4$, and the LLL energy is

$$E_{\mathrm{LLL}} = E_{n=0} = \frac{I}{8M}, \tag{9.15}$$

which is equal to the LLL energy on a supersphere [17], and is also equal to the half of the LLL energy in the original hyperbolic case (4.14). Just as in the original hyperboloid case, the energy eigenvalues on a superhyperboloid have the maximum, but are unbounded from below. Since we are concerned with the nonunitary representation of the $OSp(1|2)$ group, the degeneracy in the LLL becomes finite and the LLL bases are constructed from the symmetric products of the components of the noncompact SUSY Hopf spinor as

$$u_{m_1m_2} = \sqrt{\frac{I!}{m_1!m_2!}} u^{m_1} v^{m_2}, \qquad \eta_{n_1n_2} = \sqrt{\frac{I!}{n_1!n_2!}} u^{n_1} v^{n_2} \eta, \tag{9.16}$$

where $m_1, m_2, n_1, n_2 \geq 0$, and $m_1 + m_2 = n_1 + n_2 + 1 = I$. The degeneracy in the LLL is explicitly given by

$$D = (I + 1) + I = 2I + 1, \tag{9.17}$$

and thus, the super LLL is almost doubly degenerate compared to the original (bosonic) case. The filling fraction is usually defined by N/D, where D denotes the total number of states $D = D_B + D_F$ (D_B and D_F are the total numbers of bosonic and fermionic states, respectively), but for later convenience, we define the filling fraction as in the original (bosonic) case,

$$\nu = N/D_B = I/(mI + 1) \to 1/m, \tag{9.18}$$

where the right arrow represents the thermodynamic limit.

C. Supercoherent state on a superhyperboloid

The noncompact SUSY Hopf spinor is equivalent to the supermonopole harmonics with the minimum monopole charge $I/2 = 1/2$:

$$J^a_{(I=1)}\psi = (j^a)^t\psi, \qquad J^\alpha_{(I=1)}\psi = (j^\alpha)^t\psi, \tag{9.19}$$

where j^a and j^α are given by (7.24). Therefore, in the LLL, J^a and J^α are effectively represented as

$$J^a = \psi^t j^a \frac{\partial}{\partial\psi}, \qquad J^\alpha = \psi^t j^\alpha \frac{\partial}{\partial\psi}. \tag{9.20}$$

The one-particle state aligned with the direction $(\Omega^a, \Omega^\alpha)$,

$$\Omega^a(\chi) = 2\chi^\dagger k^a \chi, \qquad \Omega^\alpha(\chi) = 2\chi^\dagger k^\alpha \chi, \tag{9.21}$$

is represented as

$$\psi_\chi(\psi) = (\chi^\dagger k\psi)^I = (\alpha^* u - \beta^* v - \xi^* \eta)^I. \tag{9.22}$$

Indeed, ψ_χ satisfies the equation

$$[\eta_{ab}\Omega^a(\chi)J^b - \epsilon_{\alpha\beta}\Omega^\alpha(\chi)J^\beta]\psi_\chi(\psi) = \frac{I}{2}\psi_\chi(\psi). \tag{9.23}$$

X. HYPERBOLIC SUPER FUZZY GEOMETRY AND HYPERBOLIC SUPER HALL LAW

Based on similar discussions developed in Sec. V, one may deduce the noncommutative relation on a superhyperboloid. From the relation (9.6), in the LLL limit ($\Lambda^a, \Lambda^\alpha \to 0$), the coordinates on a superhyperboloid are regarded as the $OSp(1|2)$ operators

$$x^a \to X^a = -\alpha L^a, \qquad \theta^\alpha \to \Theta^\alpha = -\alpha L^\alpha, \tag{10.1}$$

which satisfy the fuzzy superalgebra

$$[X^a, X^b] = -i\alpha\epsilon^{abc}X_c, \qquad [X^a, \Theta^\alpha] = -i\frac{\alpha}{2}(\kappa^a)_\beta{}^\alpha\Theta^\beta,$$
$$\{\Theta^\alpha, \Theta^\beta\} = -\frac{\alpha}{2}(\epsilon^t\kappa^a)^{\alpha\beta}X_a, \tag{10.2}$$

where $\alpha = 2R/I$. The superalgebra (10.2) defines a fuzzy supermanifold that could be called the fuzzy superhyperboloid [42]. From (10.2), the super Hall currents are derived as

$$\begin{aligned} I^a &= \frac{d}{dt}X^a = -i[X^a, V] \\ &= -\alpha\epsilon^{abc}x_bE_c + i\frac{\alpha}{2}(\kappa^a)_\alpha{}^\beta\theta^\alpha E_\beta, \\ I^\alpha &= \frac{d}{dt}\Theta^\alpha = -i[\Theta^\alpha, V] \\ &= i\frac{\alpha}{2}(\epsilon^t\kappa^a)^{\alpha\beta}x_aE_\beta + i\frac{\alpha}{2}(\kappa^a)_\beta{}^\alpha\theta^\beta E_a, \end{aligned} \tag{10.3}$$

where $E_a = -\partial_a V$ and $E_\alpha = \partial_\alpha V$, and the superhyperbolic version of the Hall law is confirmed as

$$\eta_{ab}I^aE^b - \epsilon_{\alpha\beta}I^\alpha E^\beta = 0. \tag{10.4}$$

XI. HYPERBOLIC SUSY QUANTUM HALL EFFECT

A. Hyperbolic SUSY Laughlin-Haldane wave function

It may be natural to adopt the $OSp(1|2)$ singlet function as a hyperbolic SUSY Laughlin-Haldane wave function,

$$\Psi = \prod_{i<j}^N(\psi_i^t k\mathcal{R}\psi_j)^m = \prod_{i<j}(u_iv_j - v_iu_j - \eta_i\eta_j)^m. \tag{11.1}$$

Indeed, (11.1) is invariant under the $OSp(1|2)$ transformations generated by (9.20), and superficially takes the same form of the spherical SUSY Laughlin-Haldane wave function proposed in Ref. [24], but the noncompact SUSY Hopf spinors are used here. The corresponding hard-core pseudopotential Hamiltonian is constructed as

$$H_{\text{H.c.}} = \sum_{i<j}\sum_{m(N-2)+1/2\le J}^{m(N-1)} V_JP_J(i,j). \tag{11.2}$$

Here, P_J is the projection operator of the two-body subspace of the $OSp(1|2)$ Casimir index J:

$$P_J(i,j) = \prod_{J'\neq J}\frac{2\eta_{ab}J^a(i)J^b(j) - \epsilon_{\alpha\beta}J^\alpha(i)J^\beta(j) - \frac{I}{2}(I-1) + J'(J'-\frac{1}{2})}{J'(J'-\frac{1}{2}) - J(J-\frac{1}{2})}, \tag{11.3}$$

where we have used $\eta_{ab}J^aJ^b - \epsilon_{\alpha\beta}J^\alpha J^\beta = -j(j-1/2)_{j=-I/2+1/2} = -I/2(I/2-1/2)$. The hyperbolic SUSY Laughlin-Haldane wave function is rewritten as

$$\Psi = \exp\left(-m\sum_{i<j}^N\frac{\eta_i\eta_j}{u_iv_j - v_iu_j}\right)\cdot\Phi, \tag{11.4}$$

where Φ is the original hyperbolic Laughlin-Haldane wave function (6.1). Expanding the exponential, we obtain

$$\begin{aligned}\Psi = \Phi - m\sum_{i<j}\frac{\eta_i\eta_j}{u_iv_j - v_iu_j}\cdot\Phi + \frac{1}{2}\left(m\sum_{i<j}\frac{\eta_i\eta_j}{u_iv_j - v_iu_j}\right)^2\cdot\Phi \\ + \cdots + \eta_i\eta_2\cdots\eta_N(-m)^{N/2}Pf\left(\frac{1}{u_iv_j - v_iu_j}\right)\cdot\Phi.\end{aligned} \tag{11.5}$$

One may find that both the original Laughlin and the Moore-Read Pfaffian wave functions appear in the expansion: the former appears as the first term, and the latter as the last term. Thus, the two quantum Hall wave functions are "unified" in the SUSY formalism.

B. Excitations

Operators for excitations (quasiparticle and quasihole) on a superhyperboloid are, respectively, constructed as

$$\begin{aligned} A(\chi) &= \prod_i\chi^\dagger\mathcal{R}\frac{\partial}{\partial\psi_i} = \prod_i\chi^\dagger\mathcal{K}k\frac{\partial}{\partial\psi_i} \\ &= \prod_i\left(\alpha^*\frac{\partial}{\partial v_i} + \beta^*\frac{\partial}{\partial u_i} + \xi^*\frac{\partial}{\partial\eta_i}\right), \\ A^\dagger(\chi) &= \prod_i\psi_i^t\mathcal{K}\chi = \prod_i\psi_i^t\mathcal{R}k\chi \\ &= \prod_i(\alpha v_i - \beta u_i + \xi\eta_i), \end{aligned} \tag{11.6}$$

where χ specifies the point on a superhyperboloid by the relation (9.21). Their commutation relations are derived as

$$\begin{aligned} &[A(\chi), A^\dagger(\chi)] = 1, \\ &[A(\chi), A(\chi')] = [A^\dagger(\chi), A^\dagger(\chi')] = 0, \end{aligned} \tag{11.7}$$

which imply that $A(\chi)$ and $A^\dagger(\chi)$ are interpreted as annihilation and creation operators, respectively. The angular momentum of the quasihole follows from

$$[\Omega_a(\chi)J^a - \epsilon_{\alpha\beta}\Omega^\alpha(\chi)J^\beta, A^\dagger(\chi)] = -\frac{N}{2}A^\dagger(\chi), \tag{11.8}$$

which suggests that the excitation carries the fractional charge $1/m$, in the SUSY QHE also.

XII. SUMMARY AND DISCUSSION

Based on the noncompact version of the SUSY Hopf map, we developed a formulation of the QHE on a superhyperboloid, where the conventional definitions of Hermitian and complex conjugations were used, unlike for the spherical SUSY QHE. Using $OSp(1|2)$ group theoretical methods, we derived super Landau level energies and a nonunitary representation of LLL bases. The Landau level on a superhyperboloid has the maximum energy, while the LLL energy is equivalent to that on a supersphere. We constructed the Laughlin wave function, the hard-core pseudopotential Hamiltonian, and fractionally charged excitations on a superhyperboloid. The hyperbolic SUSY Laughlin-Haldane wave function superficially takes the same form as in the spherical QHE, but the noncompact Hopf spinors were used in the present formalism. In the LLL, the hyperbolic fuzzy supergeometry naturally emerges. It was confirmed that the particular properties in the original hyperbolic QHE were also observed in the hyperbolic SUSY QHE.

There might be many directions to be pursued from the present model. One apparent direction is to explore extensions of the QHE on other noncompact manifolds. In particular, the exploration of a noncompact QHE with $SO(3,2)$ symmetry would be interesting, since it is a natural noncompact version of the four-dimensional QHE. As close analogies between the twistor and the QHE have been pointed out in Refs. [44,45], in the LLL of the model, the $SO(3,2)$ symmetry will naturally be enhanced to $SU(2,2)$ conformal symmetry. Then the $SO(3,2)$ version of noncompact QHE appears to realize a more direct relationship with twistor theory. The study of the topological order of the SUSY QHE is another intriguing topic. Since SUSY gives a unified picture of quantum liquids with different topological orders, i.e., Laughlin and Moore-Read states, analyses of the topological order in the SUSY QHE could be important in understanding "transitions" between such topologically different quantum liquids. We hope the hyperbolic SUSY QHE will be a starting point for such stimulating future directions.

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APPENDIX A: SEVERAL DEFINITIONS IN THE SUPERGROUP

When a supermatrix is given by the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{A1}$$

(A and D are Grassmann-even block components, and B and C are Grassmann-odd block components), the superdeterminant is defined as

$$\mathrm{sdet}\, M = \frac{\det(A - BD^{-1}C)}{\det D} = \frac{\det A}{\det(D - CA^{-1}B)}, \tag{A2}$$

and the supertrace is

$$\mathrm{str}\, M = \mathrm{tr} A - \mathrm{tr} D. \tag{A3}$$

(For more details, see Ref. [37] for instance.)

APPENDIX B: LAGRANGE FORMALISM

As a supplement, we argue about the Lagrange formalism, which readily reproduces the results obtained in the Hamiltonian formalism. The one-particle Lagrangian is given by

$$L = \frac{M}{2}(\eta_{ab}\dot{x}^a\dot{x}^b - \epsilon_{\alpha\beta}\dot{\theta}^\alpha\dot{\theta}^\beta) + \dot{x}^a A_a + \dot{\theta}^\alpha A_\alpha, \tag{B1}$$

with the constraint

$$\eta_{ab}x^a x^b - \epsilon_{\alpha\beta}\theta^\alpha\theta^\beta = -1. \tag{B2}$$

In the LLL limit $M \to 0$, the Lagrangian is reduced to

$$L_{\mathrm{eff}} = \dot{x}^a A_a + \dot{\theta}^\alpha A_\alpha = -iI\psi^\dagger k \frac{d}{dt}\psi, \tag{B3}$$

with ψ (7.7) and k (7.3). Regarding ψ as the fundamental quantity, its canonical conjugate momentum is derived as

$$\pi = \partial L_{\mathrm{eff}}/\partial\dot{\psi} = -iIk\psi^*, \tag{B4}$$

where the right derivative was used. Imposing the commutation relations

$$[\psi^A, \pi_B]_\pm = i\delta^A{}_B, \tag{B5}$$

the complex conjugation ψ^* is represented as

$$\psi^* = \frac{1}{I}k' \frac{\partial}{\partial\psi}, \tag{B6}$$

with k' (7.8). Inserting (B6) into the noncompact SUSY Hopf map (8.12), x^a and θ^α are represented as

$$X^a = -\alpha\psi^t j^a \frac{\partial}{\partial\psi}, \qquad \Theta^\alpha = \alpha\psi^t j^\alpha \frac{\partial}{\partial\psi}, \tag{B7}$$

which satisfy the superhyperbolic fuzzy algebra (10.2). Similarly, the normalization condition (7.6) is rewritten as

$$\psi^t \frac{\partial}{\partial\psi} f_{\mathrm{LLL}} = I f_{\mathrm{LLL}}, \tag{B8}$$

and it determines the LLL bases as in Eq. (9.16).

APPENDIX C: IRREDUCIBLE REPRESENTATION OF $SU(1,1)$

Here, we summarize the irreducible representations of the $SU(1,1)$ group. (A more complete discussion is found in Ref. [33].) The irreducible representations are classified as (1) the principal discrete series, (2) the principal continuous series, and (3) the complementary continuous series. The principal discrete and continuous series form the complete bases.

The $SU(1,1)$ Casimir operator is given by the Hermitian operator

$$\eta_{ab}L^aL^b = (L^x)^2 + (L^y)^2 - (L^z)^2, \tag{C1}$$

and its eigenvalues are real numbers that can be negative as well as positive. We express the eigenvalues as

$$-l(l-1). \tag{C2}$$

When l is a real number, the eigenvalue satisfies

$$-l(l-1) \le \tfrac{1}{4}. \tag{C3}$$

Meanwhile, when

$$-l(l-1) > \tfrac{1}{4}, \tag{C4}$$

l can be parametrized as

$$l = \tfrac{1}{2} + i\kappa \tag{C5}$$

with an arbitrary real constant κ, and (C5) provides $-l(l-1) = \frac{1}{4} + \kappa^2 > \frac{1}{4}$. The eigenvalue of L^z is given by a real number m, and simultaneous eigenstates of $\eta_{ab}L^aL^b$ and L^z are introduced as

$$\eta_{ab}L^aL^b|l,m\rangle = -l(l-1)|l,m\rangle, \tag{C6a}$$

$$L^z|l,m\rangle = m|l,m\rangle. \tag{C6b}$$

The raising and lowering of operators are defined by

$$L^{\pm} = L^x \pm iL^y, \tag{C7}$$

and yield the relations

$$L^{+\dagger}L^{+} = \eta_{ab}L^aL^b + (L^z)^2 + L^z, \tag{C8a}$$

$$L^{-\dagger}L^{-} = \eta_{ab}L^aL^b + (L^z)^2 - L^z. \tag{C8b}$$

From the expectation values of (C8) sandwiched by $|l,m\rangle$, the conditions for l and m are derived as

$$0 \le -l(l-1) + m(m+1), \tag{C9a}$$

$$0 \le -l(l-1) + m(m-1). \tag{C9b}$$

1. Principal discrete series

With a real positive l,

$$l > 0, \tag{C10}$$

two independent irreducible representations are introduced:

$$m = l, l+1, l+2, \cdots, \tag{C11}$$

$$m = -l, -l-1, -l-2, \cdots. \tag{C12}$$

Equations (C11) and (C12) are called the positive and negative discrete series, respectively.

2. Principal continuous series

When l takes the form (C5), the irreducible representation is specified as

$$|l,\alpha;m\rangle. \tag{C13}$$

Here, m takes the form

$$m = \alpha, \alpha+1, \alpha+2, \cdots, \tag{C14}$$

or alternatively,

$$m = \alpha, \alpha-1, \alpha-2, \cdots, \tag{C15}$$

with $0 \le \alpha < 1$.

3. Complementary continuous series

When l satisfies the constraint

$$l(l-1) < \alpha(\alpha-1) \tag{C16}$$

or

$$l - \tfrac{1}{2} < |\alpha - \tfrac{1}{2}|, \tag{C17}$$

with the parameters $0 \le \alpha < 1$ and $1/2 < l < 1$, the irreducible representation is specified as

$$|l,\alpha;m\rangle, \tag{C18}$$

where m takes the following values:

$$m = \alpha, \alpha+1, \alpha+2, \cdots, \tag{C19}$$

or alternatively,

$$m = \alpha, \alpha-1, \alpha-2, \cdots. \tag{C20}$$

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