Quantization of the Myers-Pospelov model: The photon sector interacting with standard fermions as a perturbation of QED

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(Received 29 October 2008; published 10 December 2008)

We study the quantization of the electromagnetic sector of the Myers-Pospelov model coupled to standard fermions. Our main objective, based upon experimental and observational evidence, is to construct an effective theory which is a genuine perturbation of QED, such that setting the Lorentz invariance violation parameters to zero will reproduce it. To this end we provide a physically motivated prescription, based on the effective character of the model, regarding the way in which the model should be constructed and how the QED limit should be approached. This amounts to the introduction of an additional coarse-graining physical energy scale M, under which we can trust the effective field theory formulation. The prescription is successfully tested in the calculation of the Lorentz invariance violating contributions arising from the electron self-energy. Such radiative corrections turn out to be properly scaled by very small factors for any reasonable values of the parameters and no fine-tuning problems are found. Microcausality violations are highly suppressed and occur only in a spacelike region extremely close to the light cone. The stability of the model is guaranteed by restricting to concordant frames satisfying $1 - |\mathbf{v}_{\rm max}| > 6.5 \times 10^{-11}$.

DOI: 10.1103/PhysRevD.78.125011 PACS numbers: 12.20.−m, 04.60.Cf, 11.30.Cp, 11.30.Qc

I. INTRODUCTION

The Myers-Pospelov (MP) model [1] is an effective field theory that incorporates scalars, fermions, and photons in a particle (active) Lorentz invariance violating (LIV) theory. It includes dimension five operators, together with the presence of a fixed timelike direction n^μ selecting a preferred frame. Such direction is assumed to arise from a spontaneous Lorentz symmetry breaking in an underlying theory and endows the model with covariance under observer (passive) Lorentz transformations. The modified free Lagrangian density is

$$\begin{aligned}\mathcal{L}_{MP} = &-\phi^*(\partial^2 + m^2)\phi + i\frac{\rho}{\tilde{M}}\phi^*(n^\mu\partial_\mu)^3\phi \\ &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\xi}{2\tilde{M}}(n^\mu F_{\mu\nu})(n^\alpha\partial_\alpha)(n_\rho\epsilon^{\rho\nu\kappa\lambda}F_{\kappa\lambda}) \\ &+\bar{\Psi}(i\gamma^\mu\partial_\mu - m)\Psi + \frac{1}{\tilde{M}}\bar{\Psi}(n_\alpha\gamma^\alpha)(\eta_1 + \eta_2\gamma_5) \\ &\times(n^\mu\partial_\mu)^2\Psi, \end{aligned} \tag{1}$$

to which we add the electromagnetic interaction via the standard minimal coupling. Such an effective theory is interpreted here as a model to describe the imprints at standard model energies of active LIV, codified by the dimensionless parameters ρ, ξ, η_1, and η_2, which is produced by drastic modifications of the space-time structure at a fundamental scale $\bar{M}$, as suggested by some phenomenological models inspired upon developing quantum gravity theories [2–4] and string theories [5]. Nevertheless, up to now there is no systematic derivation of a semiclassical approximation starting from a fundamental quantum gravity theory, for example, that could determine the exact nature of the possible corrections arising from such space granularity. This situation has prompted the construction and analysis of effective field theory models which capture the basic ingredients that we expect to survive at standard model energies.

The additional Lorentz violating terms in (1) are unique according to the following criteria: (i) quadratic in the same field, (ii) one more derivative than the corresponding kinetic term, (iii) being gauge invariant, (iv) being Lorentz invariant, except for the appearance of n_μ, (v) not reducible to the lower dimension by the equations of motion, and (vi) not reducible to a total derivative [1]. The model has recently been generalized to the non-Abelian case including interactions arising from the fields associated to the standard model [6]. As such, it could be considered as a dimension-five-operator generalization of the standard model extension [7]. In this work, we will concentrate upon the simpler version of Ref. [1], particularly upon the proposed modified electrodynamics in its quantized version. The corresponding classical model has been thoroughly studied in relation to synchrotron radiation in Ref. [8]. Also, the self-energy corrections of the model have been recently analyzed in [9]. Radiative corrections to LIV theories have been studied in Ref. [10] and fine-tuning problems have been discussed in Refs. [9,11].

The point of view adopted in this work is to consider the quantum effective MP model (1) plus the electromagnetic interaction as a perturbation of the Lorentz invariant theory, in the precise sense that after making zero the LIV parameters encoding the corrections we must recover standard QED. Moreover, since all experimental and observational evidence point to negligible LIV [12], the radiative corrections arising from LIV should be accordingly very small. As we will see in the sequel, this basic idea provides a guideline in the way one gives a meaning to the model, particularly in regard to its quantization and to the limiting procedure necessary to recover QED.

Generally speaking, the dimension five operators make the theory of the higher order time-derivative (HOTD) type. This fact shows up in the Lagrangian (1) by the presence of third order time derivatives for the scalars, second order time derivatives for the fermions, and third order time derivatives for the photons. It is well-known that HOTD theories pose many difficulties for their implementation [13,14], the most representative ones being the increase in the number of degrees of freedom with respect to the standard ones, together with the appearance of Hamiltonians which are not positive definite being unbounded from below. In this way, if one requires to treat the additional HOTD terms as a perturbation, a careful strategy is required. Fortunately, a systematic approach to carry out this task already exists in the literature [15,16].

In view of the above considerations, a general strategy to define the quantum field theory extension of the MP model would be the following: (i) As usual, the starting point is the classical version of it given in Ref. [1]. (ii) Next, the application of the procedure in Ref. [16] to the classical HOTD MP model would reduce it to a modified effective theory of the same time-derivative character as classical electrodynamics. The procedure leads to field redefinitions plus additional contributions to the interactions. (iii) Finally, this resulting classical theory would be considered as the correct starting point for quantization, which would be carried along the standard lines. The resulting quantum theory would then provide the basis for the calculation of interacting processes using the perturbative scheme of quantum field theory. Some of these steps have been already carried out in Refs. [17], for the case of the scalar and fermion fields.

Perhaps we should emphasize at this stage that we are dealing with two different classes of perturbations: the first one concerns only the LIV parameters, occurs at the classical level and serves to define the correct starting point for quantization. Once the resulting theory is quantized, the usual quantum field theory interacting processes can be calculated, corresponding to the second class of perturbations. Both approximations should be made consistent when predicting a result to a given order in any of the LIV parameters. In this sense it is clear that we are not producing a quantum version of the full MP model, but only one which is adapted to our basic requirement of describing the LIV corrections as perturbations to QED.

Since the model respects observer (passive) Lorentz transformations we consider the parameters ρ, ξ, η_1, η_2, and $\bar{M}$, to be invariant under them. Nevertheless, the general form of the four-vector describing the preferred frame is $n^\mu = \gamma(1, \mathbf{v})$, with $1/\gamma = \sqrt{1 - \mathbf{v}^2}$, so that highly boosted systems will greatly amplify the values of the LIV parameters which are strongly constrained in earth-based reference frames. Thus we also restrict the observer Lorentz transformations to concordant reference frames which move nonrelativistically with respect to Earth [18]. In the sequel, we will give a quantitative characterization of such allowed observers. A further simplification is introduced by taking into account that the parameters ρ, ξ, η_1, and η_2 are independent. In this way we set the field ϕ together with the parameters ρ, η_1, and η_2 equal to zero. Then we deal with a minimal LIV extension of standard QED.

The paper is organized as follows. In Sec. II, we discuss the classical MP modifications to electrodynamics. There we construct the corresponding Hamiltonian formulation in terms of canonically transformed fields that guarantee the appropriate normalization of the momentum squared terms in the Hamiltonian density, that includes the identification of the interacting sector. Section III deals with the quantization of the model in terms of standard creation-annihilation operators. The modified dispersion relations are identified and the Hamiltonian is shown to be positive definite for momenta $\mathbf{k}$ such that $|\mathbf{k}| < \bar{M}/(2|\xi|)$. In Sec. IV, we construct the modified photon propagator in the Coulomb gauge which is subsequently written in four dimensional notation by incorporating the static Coulomb contribution appearing in the Hamiltonian. Section V contains the physical motivation and specific proposal for our prescription that allows to understand the quantum MP model as a tiny perturbation of QED, according to the experimental and observational evidence of highly suppressed LIV. A coarse-graining mass scale $M \ll \bar{M}$ is further introduced in the problem, dictated by the effective character of the model, and signaling the onset of the modifications in the space-time structure. In Sec. VI, we set up the general structure the electron self-energy calculation including only the modified photon propagator ($\xi \neq 0$) interacting with standard fermions ($\eta_1 = \eta_2 = 0$). The scale M is taken into account via a factor of the Pauli-Villars type, designed to act as the appropriate regulator in the QED limit. Also we perform a power expansion of the self-energy in terms of the external momentum and identify those terms to be subjected to scrutiny regarding their suppressed character and good QED limit in the next section. The general strategy for their calculation is presented in Sec. VII and all the LIV contributions to order ξ^2 are accordingly obtained. One of such calculations is presented in full detail, while we only write the results for the remaining ones. In Sec. VIII, we present a preliminary study of the microcausality violation in the model by identifying the spacelike region where it occurs, together with an estimation of the magnitude of such violation. The final Sec. IX contains a summary of the work. The notation and conventions are stated in Appendix A which, together with Appendix B, contain information relevant for the specific calculations in the paper. In Appendix C, the relationship between the modified photon propagator in different gauges is established. The last Appendix, (D) includes the definitions of the LIV contributions which calculation is not fully developed in the text.

II. THE MODEL

With the simplifications stated above, we consider the modified photon sector

$$\mathcal{L}_\gamma = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\xi}{2\tilde{M}}(n^\mu F_{\mu\nu})(n^\alpha\partial_\alpha)(n_\rho\epsilon^{\rho\nu\kappa\lambda}F_{\kappa\lambda}) - J^\mu A_\nu, \tag{2}$$

where the electromagnetic current J^μ will be subsequently realized in terms of unmodified spin $1/2$ fermions, according to the choice $\eta_1 = \eta_2 = 0$. Our general strategy will be first to quantize the photons and subsequently to consider the interaction, via the standard minimal coupling, with the unmodified quantum fermions.

The equations of motion in the Lorentz gauge are

$$(\eta^{\nu\lambda}\partial^2 - 2g(n\cdot\partial)^2 n_\rho\epsilon^{\rho\nu\sigma\lambda}\partial_\sigma)A_\lambda = J^\nu. \tag{3}$$

In order to get a better control of the LIV modifications, we find it convenient to work in the Hamiltonian scheme, so that we switch to a $3+1$ canonical formulation of the problem. Taking advantage of the remaining observer Lorentz invariance of the model, we choose to work in the rest frame $n^\mu = (1, \mathbf{0})$, where the free modified photon contribution is

$$\mathcal{L}_\gamma = \frac{1}{2}(\dot{A}^i + \partial_i A^0)^2 - \frac{1}{4}F_{ij}F^{ij} + g\epsilon^{ijk}\dot{A}^i\partial_j\dot{A}^k - J^\mu A_\mu, \qquad g = \frac{\xi}{\tilde{M}}. \tag{4}$$

This choice has the advantage that, up to a total derivative, the resulting system is not of the HOTD type. Nevertheless, it exhibits in a simpler setting most of the questions associated to the quantization of the full MP model. In addition, let us emphasize that we will carry the quantization without any approximation in the parameter g.

The canonical approach gives the following momenta

$$\Pi_0 = \frac{\partial\mathcal{L}_\gamma}{\partial\dot{A}_0} = 0, \qquad \Pi_i = \frac{\partial\mathcal{L}_\gamma}{\partial\dot{A}^i} = \dot{A}^i + \partial_i A^0 + 2g\epsilon^{ijk}\partial_j\dot{A}^k, \tag{5}$$

together with their Poisson brackets,

$$\{A^i(t,\mathbf{x}), \Pi_j(t,\mathbf{y})\} = \delta^i_j\delta^3(\mathbf{x}-\mathbf{y}). \tag{6}$$

The next step is to construct the Hamiltonian density $\mathcal{H}_C = \Pi_i\dot{A}^i - \mathcal{L}$, which is

$$\mathcal{H}_C = \Pi_i\dot{A}^i - \tfrac{1}{2}(\dot{A}^i + \partial_i A^0)^2 + \tfrac{1}{4}F_{ij}F^{ij} - g\epsilon^{ijk}\dot{A}^i\partial_j\dot{A}^k + J^\mu A_\mu. \tag{7}$$

In order to write the velocities in terms of the momenta, it is convenient to consider the combination $\Pi_i - \partial_i A^0$ together with the operator

$$M^{ik} = (\delta^{ik} + 2g\epsilon^{ijk}\partial_j), \tag{8}$$

in the second part of Eq. (5). To solve for the velocities, we need the inverse of the operator M^{ik} for which we obtain the exact nonlocal expression

$$(M^{-1})^{ij} = \frac{1}{(1+4g^2\nabla^2)}(\delta^{ij} - 2g\epsilon^{irj}\partial_r + 4g^2\partial_i\partial_j). \tag{9}$$

In this way we solve

$$\dot{A}^i = (M^{-1})^{ij}(\Pi_j - \partial_j A^0), \tag{10}$$

which we substitute in Eq. (7). The result is

$$\mathcal{H}_C = \tfrac{1}{2}(\Pi_p - \partial_p A^0)(M^{-1})^{pr}(\Pi_r - \partial_r A^0) - \tfrac{1}{2}(\partial_i A^0)^2 + \tfrac{1}{4}F_{ij}F^{ij} + J^\mu A_\mu. \tag{11}$$

Integrating by parts and using some of the properties for $(M^{-1})^{ij}$ written in Appendix A, we arrive at

$$\mathcal{H}_C = \tfrac{1}{2}\Pi_p(M^{-1})^{pr}\Pi_r + (\partial_p\Pi_p + J^0)A^0 + \tfrac{1}{4}F_{ij}F^{ij} - J^i A^i. \tag{12}$$

It can be verified that the corresponding Hamilton equation of motion reproduces the correct expression (10) for $\dot{A}^i$.

The canonical variables can be written in the convenient form

$$\Pi^T_i = \dot{A}^i_T + 2g\epsilon^{ijk}\partial_j\dot{A}^k_T, \qquad \Pi^L_i = \dot{A}^i_L + \partial_i A^0, \qquad A^i_L = \frac{1}{\nabla^2}\partial_i(\partial_k A^k), \tag{13}$$

where we are using the standard definition for a transverse (T) and longitudinal (L) decomposition of a vector field $\mathbf{U} = \mathbf{U}_T + \mathbf{U}_L$, where $\nabla\cdot\mathbf{U}_T = 0$, $\nabla\times\mathbf{U}_L = 0$. In the case of the velocities the separation leads to

$$\dot{A}^i_T = \frac{1}{W^2}(\delta^{ij} - 2g\epsilon^{irj}\partial_r)\Pi^T_j, \qquad \dot{A}^i_L = \frac{1}{W^2}\Pi^L_i + \partial_i\left(\frac{4g^2}{W^2}\partial_j\Pi^L_j - A^0\right), \tag{14}$$

with the notation $W = \sqrt{1+4g^2\nabla^2}$. As in the usual case, A_0 is a Lagrange multiplier leading to the Gauss law as a secondary constraint

$$\Theta = \partial_i\Pi_i + J^0 = 0, \tag{15}$$

which can also be understood as arising from the time-derivative $\dot{\Pi}_0$ of the primary constraint $\Pi_0 \approx 0$. The evolution $\dot{\Theta} \approx 0$ leads to current conservation in such a way that we have only two first class constraints as in the standard case. In terms of transverse and longitudinal variables the Gauss law is written as

$$\partial_i \Pi_i^L + J^0 = 0. \tag{16}$$

The equation of motion

$$\partial^i F_{i0} = +J^0,$$

yields

$$A^0 = -\frac{1}{\nabla^2}(J^0 + \partial_0 \partial_i A^i). \tag{17}$$

At this stage we select the Coulomb gauge by choosing

$$\begin{aligned} A^0 &= -\frac{1}{\nabla^2} J^0, \qquad \Pi_0 = 0, \\ \partial_k A^k &= 0 \to A_L^i(x) = 0, \qquad \Pi_i^L = -\frac{1}{\nabla^2}\partial_i J^0. \end{aligned} \tag{18}$$

The dynamical variables are contained only in the transverse modes

$$\Pi_i^T = \dot{A}_T^i + 2g\epsilon^{ijk}\partial_j \dot{A}_T^k, \qquad A_T^i(x), \tag{19}$$

which satisfy the Dirac brackets

$$\{A_T^i(t,\mathbf{x}), \Pi_m^T(t,\mathbf{y})\} = \left(\delta^{im} - \frac{\partial_x^i \partial_x^m}{\nabla_x^2}\right)\delta^3(\mathbf{x}-\mathbf{y}). \tag{20}$$

Using repeated integration by parts in the Hamiltonian, together with the transversality condition, we arrive at

$$H_C = \int d^3x \left(\frac{1}{2}\Pi_p^T (M^{-1})^{pr}\Pi_r^T + \frac{1}{2}J^0\left(-\frac{1}{\nabla^2}\right)J^0 + \frac{1}{4}F_{ij}F^{ij} - J^i A_T^i\right). \tag{21}$$

Our final goal is to express the dynamical fields in terms of creation-annihilation operators, corresponding to modified frequency modes, satisfying standard bosonic commutation relations that will reproduce the field commutation relations arising from the correspondence principle applied to the respective Dirac brackets. To this end it is necessary that the relation $\Pi = \dot{A}$ holds, which is equivalent to require that the kinetic term of the Hamiltonian density be normalized as $\frac{1}{2}\Pi^2$. In order to achieve this we perform the canonical transformation ($\mathbf{A}_T \to \bar{\mathbf{A}}_T, \mathbf{\Pi}^T \to \bar{\mathbf{\Pi}}^T$) given by

$$\begin{aligned} A_T^i &= \frac{\sqrt{1+W}}{\sqrt{2}W}\left[\delta^{iq} - \frac{2g}{(1+W)}\epsilon^{imq}\partial_m\right]\bar{A}_T^q, \\ \Pi_r^T &= \frac{\sqrt{1+W}}{\sqrt{2}}\left[\delta^{rq} + \frac{2g}{(1+W)}\epsilon^{rmq}\partial_m\right]\bar{\Pi}_q^T. \end{aligned} \tag{22}$$

The nonzero transverse Dirac brackets for the variables $\bar{A}_T^i$ and $\bar{\Pi}_j^T$ are the same as in Eq. (20) in virtue of the canonical character of the transformation. Rewriting the Hamiltonian (21) in terms of the new variables leads to

$$H_C = \int d^3x \left(\frac{1}{2}\bar{\Pi}_p^T\bar{\Pi}_p^T + \frac{1}{2}\bar{A}_T^r\left(-\frac{\nabla^2}{W^2}\right)[\delta^{rp} - 2g\epsilon^{rnp}\partial_n]\bar{A}_T^p + \frac{1}{2}J^0\left(-\frac{1}{\nabla^2}\right)J^0 - J^i A_T^i(\bar{A}_T)\right). \tag{23}$$

Let us emphasize that in the last interaction term A_T^i is a functional of the dynamical field $\bar{A}_T^j$. In this sense, the electromagnetic vertex will be modified with respect to the latter field but will retain the usual structure with respect to the former. In this way, some care is required when implementing the perturbation theory starting from the zeroth order Hamiltonian written in terms of $\bar{A}_T^i$ and $\bar{\Pi}_j^T$.

III. THE QUANTUM THEORY

Now we have the basic ingredients to proceed with the quantization of the modified photon field. We start from the usual plane wave expansion of the operator $\bar{A}_T^i(x)$

$$\bar{A}_T^i(x) = \int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3}}\sum_{\lambda=\pm1}\sqrt{\frac{1}{2\omega_\lambda(\mathbf{k})}}[a_\lambda(\mathbf{k})\varepsilon^i(\lambda,\mathbf{k})e^{-ik(\lambda)\cdot x} + a_\lambda^\dagger(\mathbf{k})\varepsilon^{i*}(\lambda,\mathbf{k})e^{+ik(\lambda)\cdot x}], \tag{24}$$

in terms of creation-annihilation operators $a_\lambda^\dagger(\mathbf{k})$, $a_\lambda(\mathbf{k})$, respectively. The notation is

$$[k(\lambda)]_\mu = (\omega_\lambda(\mathbf{k}), -\mathbf{k}), \qquad k(\lambda)\cdot x = \omega_\lambda(\mathbf{k})x^0 - \mathbf{k}\cdot\mathbf{x}, \tag{25}$$

where the modified normal frequencies will be consistently determined. The properties of the polarization vectors $\varepsilon^i(\lambda,\mathbf{k})$, $\lambda = \pm 1$, chosen in the circularly polarized (helicity) basis, are collected in Appendix B. The momenta are given by

$$\bar{\Pi}_i^T(x) = \int \frac{d^3\mathbf{k}}{\sqrt{(2\pi)^3}}\sum_{\pm\lambda}\sqrt{\frac{1}{2\omega_\lambda(\mathbf{k})}}[(-i\omega_\lambda)a_\lambda(\mathbf{k})\varepsilon^i(\lambda,\mathbf{k}) \times e^{-ik(\lambda)\cdot x} + (i\omega_\lambda)a_\lambda^\dagger(\mathbf{k})\varepsilon^{i*}(\lambda,\mathbf{k})e^{+ik(\lambda)\cdot x}]. \tag{26}$$

Assuming the standard creation-annihilation commutation rules

$$[a_\lambda(\mathbf{k}), a_{\lambda'}^\dagger(\mathbf{k}')] = \delta_{\lambda\lambda'}\delta^3(\mathbf{k}-\mathbf{k}') \tag{27}$$

and starting from (24) and (26) we recover the basic field commutator at equal times

$$[\bar{A}_T^i(t,\mathbf{x}), \bar{\Pi}_j^T(t,\mathbf{y})] = i\left(\delta^{ij} - \frac{\partial_{xi}\partial_{xj}}{\nabla^2}\right)\delta^3(\mathbf{x}-\mathbf{y}), \tag{28}$$

which is the expected result after the canonical transformation. The corresponding equations of motion are

$$\bar{\Pi}_q^T = \partial_0\bar{A}_T^q, \qquad \dot{\bar{\Pi}}_r^T = \frac{\nabla^2}{W^2}[\delta^{rp} - 2g\epsilon^{rnp}\partial_n]\bar{A}_T^p. \tag{29}$$

Going to the momentum space we can obtain the modified dispersion relations from

$$\omega^2 \bar{\mathbf{A}}_T(k) = \frac{|\mathbf{k}|^2}{1 - 4g^2|\mathbf{k}|^2}[\bar{\mathbf{A}}_T(k) - 2gi\mathbf{k} \times \bar{\mathbf{A}}_T(k)], \quad (30)$$

which reduces to the diagonalization

$$i\mathbf{k}\times \to \lambda|\mathbf{k}|, \qquad \mathbf{A_T} \to \mathbf{A_T^\lambda}, \quad (31)$$

when the vector potential is expressed in the helicity basis. In this way

$$\omega_\lambda^2 \bar{\mathbf{A}}_T^\lambda = \left(\frac{|\mathbf{k}|^2}{1 - 4g^2|\mathbf{k}|^2}\right)[1 - 2\lambda g|\mathbf{k}|]\bar{\mathbf{A}}_T^\lambda, \quad (32)$$

yielding the modified energy-momentum relation

$$\omega_\lambda^2(\mathbf{k}) = \frac{|\mathbf{k}|^2}{[1 + 2\lambda g|\mathbf{k}|]}, \quad (33)$$

which is exact in g. With no loss of generality we assume from now on that $g > 0$. Let us notice that the four-vector $[k(\lambda = +1)]_\mu$ is spacelike, while $[k(\lambda = -1)]_\mu$ is timelike. At this stage we are confronted with two problems that arise rather frequently in LIV theories: (i) On one hand, the frequency $\omega_-(\mathbf{k})$ will become imaginary when $|\mathbf{k}| > 1/(2g)$ and diverges when $|\mathbf{k}| = |\mathbf{k}|_{\max} = 1/(2g)$. From an intuitive point of view, we consider $1/(2g)$ as the analogous of the value $|\mathbf{k}|_{\max} = \infty$ in the standard case and we will cut all momentum integrals at this value. The introduction of the coarse-graining scale $M \ll \bar{M}$, explained in more detail in Sec. V, effectively produces the more stringent and smooth cutoff

$$g|\mathbf{k}| < gM \ll 1. \quad (34)$$

(ii) On the other hand, since $[k(\lambda = +1)]_\mu$ is spacelike, we can always perform an observer Lorentz transformation such that $\omega_+(\mathbf{k})$ becomes negative thus introducing stability problems in the model. For a given momentum $\mathbf{k}$ this occurs for $1/\sqrt{1 + 2g|\mathbf{k}|} < |\mathbf{v}| < 1$. Then, the condition (34) leads to the requirement that the allowed concordant frames in which the quantization will remain consistent are such that $\gamma < 1/\sqrt{2gM}$, with respect to the rest frame.

Our next step is to verify that the resulting free ($J^\mu = 0$) Hamiltonian is in fact positive definite and has the expected expression in terms of the previously introduced creation-annihilation operators. Let us begin with the kinetic term

$$H_{\rm KE} = \frac{1}{2}\int d^3x \bar{\Pi}_i^T \bar{\Pi}_i^T, \quad (35)$$

which leads to

$$H_{\rm KE} = \frac{1}{2}\int d^3\mathbf{k} \sum_\lambda \left[\left(-\frac{\omega_\lambda(\mathbf{k})}{2}\right) a_\lambda(\mathbf{k}) a_\lambda(-\mathbf{k}) e^{-i2\omega_\lambda(\mathbf{k})t} + \frac{\omega_\lambda(\mathbf{k})}{2} a_\lambda^\dagger(\mathbf{k}) a_\lambda(\mathbf{k}) + \text{H.c.}\right], \quad (36)$$

in terms of the creation-annihilation operators.

The potential term contribution is

$$H_{\rm POT} = \frac{1}{2}\int d^3x \bar{A}_T^r \left(-\frac{\nabla^2}{W^2}\right)[\delta^{rp} - 2g\epsilon^{rnp}\partial_n]\bar{A}_T^p, \quad (37)$$

which analogously reduces to

$$H_{\rm POT} = \frac{1}{2}\int d^3\mathbf{k} \sum_{\pm\lambda} \left[\left(\frac{\omega_\lambda(\mathbf{k})}{2}\right)[a_\lambda(\mathbf{k}) a_\lambda(-\mathbf{k})] e^{-i2\omega_\lambda(\mathbf{k})t} + \frac{\omega_\lambda(\mathbf{k})}{2} a_\lambda^\dagger(\mathbf{k}) a_\lambda(\mathbf{k}) + \text{H.c.}\right]. \quad (38)$$

Here we have made use of the dispersion relations (33), together with Eqs. (B3) and (B4). This leads to the expected final expression ($H = H_{\rm KE} + H_{\rm POT}$).

$$H = \frac{1}{2}\int d^3\mathbf{k} \sum_\lambda [a_\lambda(\mathbf{k}) a_\lambda^\dagger(\mathbf{k}) + a_\lambda^\dagger(\mathbf{k}) a_\lambda(\mathbf{k})]\omega_\lambda(\mathbf{k}), \quad (39)$$

arising from the cancellation of the time dependent terms and including the modified frequencies (33). Thus the Hamiltonian is Hermitian as far as the frequencies remain real, which is the case in the region $|\mathbf{k}| < 1/(2g)$.

IV. THE PHOTON PROPAGATOR

In this section, we calculate the free modified photon propagator starting from the definition

$$\begin{aligned} i\bar{\Delta}_{ij}(x, y) &\equiv \langle 0|T(\bar{A}_i^T(x)\bar{A}_j^T(y))|0\rangle \\ &= \theta(x^0 - y^0)\langle 0|\bar{A}_i^T(x)\bar{A}_j^T(y)|0\rangle + \theta(y^0 - x^0) \\ &\quad \times \langle 0|\bar{A}_j^T(y)\bar{A}_i^T(x)|0\rangle, \end{aligned} \quad (40)$$

where $\bar{\Delta}_{ij}(x, y) = \bar{\Delta}_{ij}(x - y)$ as can be seen from the expression

$$\langle 0|\bar{A}_i^T(x)\bar{A}_j^T(y)|0\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_\lambda \frac{1}{2\omega_\lambda(\mathbf{k})} e^{-i(x-y)\cdot k(\lambda)} \times \varepsilon_i(\lambda, \hat{\mathbf{k}})\varepsilon_j^*(\lambda, \hat{\mathbf{k}}). \quad (41)$$

Here we introduce the notation

$$F_{ij}(\lambda, \hat{\mathbf{k}}) = \varepsilon_i(\lambda, \hat{\mathbf{k}})\varepsilon_j^*(\lambda, \hat{\mathbf{k}}), \quad (42)$$

which leads to the second vacuum expectation value in (40)

$$\langle 0|\bar{A}_j^T(y)\bar{A}_i^T(x)|0\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sum_\lambda \frac{1}{2\omega_\lambda(\mathbf{k})} e^{i(x-y)\cdot k(\lambda)} F_{ji}(\lambda, \hat{\mathbf{k}}). \quad (43)$$

We are interested in expressing the propagator

$$\bar{\Delta}_{ij}(z) = \int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot z}\Delta_{ij}(k), \quad (44)$$

with $z^\mu = x^\mu - y^\mu$, in momentum space. To this end, we start from the expression

$$i\bar{\Delta}_{ij}(z) = \int \frac{d^3k}{(2\pi)^3} e^{+i\mathbf{k}\cdot\mathbf{z}}\Big[\theta(z_0)\sum_{\pm\lambda} e^{-i\omega_\lambda z_0}\frac{1}{2\omega_\lambda(\mathbf{k})}F_{ij}(\lambda,\hat{\mathbf{k}}) + \theta(-z_0)\sum_{\pm\lambda} e^{i\omega_\lambda z_0}\frac{1}{2\omega_\lambda(\mathbf{k})}F_{ji}(\lambda,-\hat{\mathbf{k}})\Big], \quad (45)$$

and introduce the standard representation

$$\theta(z_0) = i\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\frac{d\alpha}{2\pi}\frac{e^{-i\alpha z_0}}{\alpha + i\epsilon}, \quad (46)$$

in order to calculate the corresponding Fourier transform. The result is

$$\bar{\Delta}_{ij}(k) = \sum_\lambda\Big(\frac{\varepsilon_i(\lambda,\hat{\mathbf{k}})\varepsilon_j^*(\lambda,\hat{\mathbf{k}})}{2\omega_\lambda(k_0 - \omega_\lambda + i\varepsilon)} - \frac{\varepsilon_j(\lambda,-\hat{\mathbf{k}})\varepsilon_i^*(\lambda,-\hat{\mathbf{k}})}{2\omega_\lambda(k_0 + \omega_\lambda - i\varepsilon)}\Big). \quad (47)$$

Using Eqs. (B5) and (B6), we rewrite the propagator in the form

$$\bar{\Delta}_{ij}(k) = \frac{1}{2}\sum_\lambda \frac{1}{(k_0^2 - (\omega_\lambda - i\varepsilon)^2)}\Big(\Big[\delta_{ij} - \frac{k_ik_j}{|\mathbf{k}|^2}\Big] + i\lambda\Big[\epsilon^{ijm}\frac{k_m}{|\mathbf{k}|}\Big]\Big). \quad (48)$$

After performing the summations according to (B8)–(B10) we arrive at the following expression for the modified photon propagator in the Coulomb gauge

$$\bar{\Delta}_{ij}(k) = \frac{1}{((k^2)^2 - 4g^2|\mathbf{k}|^2k_0^4)}\Big[(k^2 - 4g^2|\mathbf{k}|^2k_0^2)\times\Big[\delta_{ij} - \frac{k_ik_j}{|\mathbf{k}|^2}\Big] - 2g|\mathbf{k}|^2\epsilon^{ijm}ik_m\Big]. \quad (49)$$

Let us verify the correct limits when $g = 0$, where $\omega^+ = \omega^- = \omega = |\mathbf{k}|$. In this case the first sum in the right-hand side of Eq. (48) gives the standard transverse propagator, while the second sum cancels out.

We would like now to extend the above propagator, which is defined in the transverse sector, to the whole four dimensional space in such a way that the current-current interaction is described by

$$\frac{1}{2}\int d^4kJ^\mu(-k)\bar{\Delta}_{\mu\nu}(k)J^\nu(k). \quad (50)$$

This is achieved by incorporating in Eq. (48) the Coulomb term appearing in (23) in a manner analogous to that described in Ref. [19]. The final result is

$$\bar{\Delta}_{\mu\nu} = -\frac{1}{2}\sum_\lambda \frac{1}{(k_0^2 - \omega_\lambda^2 + i\epsilon)}\Big(\eta_{\mu\nu} + \Big(\frac{\omega_\lambda^2}{|\mathbf{k}|^2} - 1\Big)\delta_{\mu 0}\delta_{\nu 0} - i\lambda n_\rho\epsilon_{\mu\nu}{}^{\rho\sigma}\frac{k_\sigma}{|\mathbf{k}|}\Big), \quad (51)$$

where we have reinserted the vector $n^\rho = (1, \mathbf{0})$.

The last step in the construction is to perform the sums over λ in (51) using the corresponding expressions in Appendix B. The result is

$$\bar{\Delta}_{\mu\nu} = -\frac{1}{((k^2)^2 - 4g^2|\mathbf{k}|^2k_0^4)}[\eta_{\mu\nu}(k^2 - 4g^2|\mathbf{k}|^2k_0^2) + 4g^2|\mathbf{k}|^2k_0^2\delta_{\mu 0}\delta_{\nu 0} + 2gn_\rho\epsilon_{\mu\nu}{}^{\rho\sigma}(ik_\sigma)|\mathbf{k}|^2]. \quad (52)$$

The propagator obtained directly from the equations of motion (3) in the Lorentz gauge is

$$\Delta_{\mu\nu}(k) = \frac{1}{((k^2)^2 - 4g^2|\mathbf{k}|^2k_0^4)}\times\Big[-k^2\eta_{\mu\nu} + 2igk_0^2\epsilon^{lmr}k_m\eta_{l\mu}\eta_{r\nu} - \frac{4g^2k_0^4}{k^2}k_lk_r\delta^l_\mu\delta^r_\nu + \frac{4g^2k_0^4|\mathbf{k}|^2}{k^2}\eta_{0\mu}\eta_{0\nu}\Big]. \quad (53)$$

In Appendix C, we have calculated the propagator Δ_{ij} corresponding to the fields A_T^i starting from $\bar{\Delta}_{ij}$ given by (49) and performing the canonical transformation (22). Moreover, the subsequent inclusion of the Coulomb term in Δ_{ij} leads exactly to the four dimensional propagator $\Delta_{\mu\nu}$ in (53). It is important to emphasize that the Hamiltonian (23) has a noninteracting sector described by the fields $\bar{A}_T^i$, $\bar{\Pi}_j^T$ but induces an interaction density given by $J_iA_T^i \to N[J_\mu A^\mu]$, where A^μ propagates according to (53).

V. THE PRESCRIPTION DEFINING THE EFFECTIVE QUANTUM MODEL

The main goal of this work is to study the possibility of defining the MP model as a perturbative extension of standard QED, that is to say as a model which continuously interpolates between a LIV theory and a Lorentz preserving one. This is to a large extent motivated by the very stringent experimental and observational limits set upon the parameters that codify such LIV. A construction exhibiting this interpolating characteristic has been already presented in Ref. [20], but there the LIV was codified by a dimensionless parameter, as opposed to the situation here. As we will explain in the sequel, the effective character of the model requires the introduction of an additional mass scale M that provides the analogous dimensionless parameter (gM).

Another point that requires attention is the upper limit $|\mathbf{k}|_{\max} = 1/(2g)$ set by the modified dispersion relations

(33), which guarantees the absence of imaginary frequencies together with that of a non-Hermitian Hamiltonian. We consider these facts as indications of the effective character of the model. Assuming for a moment that $1/g \approx E_{\rm QG} \approx M_{\rm Planck}$, the above upper limit would mean that one is probing distances of the order of the Planck length, where we expect quantum gravity effects to be so important that the continuum properties of space might be no longer valid, thus invalidating the use of an standard effective field theory. This means that we need to introduce an additional coarse-graining scale M under which we can safely consider space as a continuum and apply effective field theory methods. Thus we require

$$M \ll \frac{1}{g}. \tag{54}$$

In this way, the upper limit $|\mathbf{k}|_{\max} = 1/(2g)$ can be considered as a mathematical limitation in our model, analogous to $|\mathbf{k}|_{\max} = \infty$ in the standard case. The physical limitation of the model is settled by the scale M and requires to be imposed by an adequate smooth regularization procedure that cuts down the corresponding degrees of freedom over this scale, which occurs a long way before energies of the order $\approx 1/g$ are reached. In this manner, the relation (54) imposes a definite prescription to recover standard QED: (i) first set $g \to 0$ for fixed M and (ii) then set $M \to \infty$. Let us emphasize that at the level of the effective model, the theory is finite and certainly will have an explicit dependence upon the physical parameters g and M. Now comes the question on how do we introduce the scale M. Intuitively, we think of M as the parameter that will regularize the divergent integrals that will appear in the limit $g \to 0$ describing standard QED. This suggests that we introduce this parameter via a Lorentz covariant smooth function $\mathcal{I}(k)$, of the Pauli-Villars type, for example, with the same characteristics that one would require in order to regulate standard QED. A natural choice for $\mathcal{I}(k)$ in our calculation of the electron self-energy is

$$\begin{aligned}\frac{1}{k^2 - m^2 + i\epsilon}\mathcal{I}(k) &= \frac{1}{k^2 - m^2 + i\epsilon} - \frac{1}{k^2 - M^2 + i\epsilon} \\ &= \frac{1}{k^2 - m^2 + i\epsilon}\left(\frac{M^2}{M^2 - k^2 - i\epsilon}\right), \\ M &\gg m.\end{aligned} \tag{55}$$

In this way we are also imposing no additional LIV besides that arising from the original modifications to the dynamics encoded in the parameter g.

VI. THE ELECTRON SELF-ENERGY

As a first step in testing the proposed construction, we consider the calculation of the electron self-energy with the dynamical modifications introduced only via the LIV photon propagator. Let us recall that the perturbation theory based upon the Hamiltonian (23) indicates that the photon propagates with $\Delta_{\mu\nu}$ given by (53). Moreover, we will focus upon the LIV contributions that could produce fine-tuning problems associated to the would be divergent contributions arising in the limit $g \to 0$.

The starting point is

$$\begin{aligned}\Sigma^g(p) = -ie^2 \int \frac{d^4k}{(2\pi)^4}\gamma^\mu\left[\frac{(\gamma(p-k)+m)}{((p-k)^2 - m^2 + i\epsilon)}\right] \\ \times \gamma^\nu \Delta_{\mu\nu}(k)\mathcal{I}(k)\theta\left(\frac{1}{2g} - |\mathbf{k}|\right),\end{aligned} \tag{56}$$

where we have introduced the scale M via

$$\mathcal{I}(k) = \frac{M^2}{M^2 - k^2}. \tag{57}$$

The θ function is there to guarantee the reality of the frequencies $\omega_\lambda(|\mathbf{k}|)$ entering the calculation of the photon propagator in Sec. IV. Let us observe that the expression (56) is finite.

Next we find it convenient to expand the self-energy in powers of the external momentum

$$\begin{aligned}\Sigma^g(p) = \Sigma^g_{p=0} + \left(\frac{\partial \Sigma^g}{\partial p^\mu}\right)_{p=0} p^\mu + \frac{1}{2}\left(\frac{\partial^2 \Sigma^g}{\partial p^\mu \partial p^\nu}\right)_{p=0} p^\mu p^\nu \\ + O(p^3),\end{aligned} \tag{58}$$

where each coefficient in the expansion is a matrix written in terms of some elements of the basis in the 4×4 space of the Dirac matrices. We have considered up to second derivatives in the external momentum because the additional corrections to the numerator of the photon propagator (53) of order gk and $(gk)^2$ make those derivatives power counting divergent, as opposed to the QED case. The fact that we are violating Lorentz transformations in the boost sector, while maintaining rotational invariance would naturally split the above expansion into a time plus space structure. The expansion of the above coefficients in the gamma matrix basis will be denoted by

$$\begin{aligned}\Sigma^g_{p=0} = W^g_C \Gamma^C, \qquad \left(\frac{\partial \Sigma^g}{\partial p^\mu}\right)_{p=0} = W^g_{\{\mu\}C}\Gamma^C, \\ \left(\frac{\partial \Sigma^g}{\partial p^\mu \partial p^\nu}\right)_{p=0} = W^g_{\{\mu\nu\}C}\Gamma^C,\end{aligned} \tag{59}$$

where we use the standard basis

$$\begin{aligned}\Gamma^C\!: \Gamma^4 = I, \qquad \Gamma^\mu = \gamma^\mu, \qquad \Gamma^{\mu\nu} = \sigma^{\mu\nu}, \\ \Gamma^5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \qquad \Gamma^{5,\mu} = \gamma^5\gamma^\mu.\end{aligned} \tag{60}$$

This allows us to rewrite the self-energy as

$$\Sigma^g(p) = \left(W^g_C + p^\mu W^g_{\{\mu\}C} + \tfrac{1}{2}p^\mu p^\nu W^g_{\{\mu\nu\}C}\right)\Gamma^C + O(p^3). \tag{61}$$

In order to deal with the calculation of such coefficients it is convenient to separate the modified photon propagator

(53) into its even and odd parts

$$\Delta_{\mu\nu}(k) = \Delta^{\text{even}}_{\mu\nu}(k) + \Delta^{\text{odd}}_{\mu\nu}(k), \tag{62}$$

and rewrite them in the more compact form

$$\begin{aligned}\Delta^{\text{even}}_{\mu\nu}(k) &= \eta_{\mu\nu}F_1 + F_2(|\mathbf{k}|^2\eta_{0\mu}\eta_{0\nu} - k_l k_r \delta^l_\mu \delta^r_\nu) \\ &= \Delta^{\text{even}}_{\nu\mu}(k) = \Delta^{\text{even}}_{\mu\nu}(-k),\end{aligned} \tag{63}$$

$$\Delta^{\text{odd}}_{\mu\nu}(k) = iF_3 k_r \epsilon^{lmr}\eta_{\mu l}\eta_{\nu m} = -\Delta^{\text{odd}}_{\mu\nu}(-k) = -\Delta^{\text{odd}}_{\nu\mu}(k), \tag{64}$$

where

$$\begin{aligned}F_1 &= -\frac{k^2}{((k^2)^2 - 4g^2|\mathbf{k}|^2 k_0^4)}, \\ F_2 &= \frac{4g^2k_0^4/k^2}{((k^2)^2 - 4g^2|\mathbf{k}|^2 k_0^4)}, \\ F_3 &= -\frac{2gk_0^2}{((k^2)^2 - 4g^2|\mathbf{k}|^2 k_0^4)},\end{aligned} \tag{65}$$

are even functions of $\mathbf{k}$ and k_0.

From the general expressions for the contributions in (58), together with the symmetry properties of the propagator plus the symmetrical integration over the three-momenta it is possible to determine that the nonzero contributions to $\Sigma^g(p)$ are

$$\begin{aligned}\Sigma^g(p) = {}& AI + i\tilde{A}\gamma^i\gamma^j\gamma^k\epsilon^{ijk} + p^0B\gamma^0 \\ &- p^i(C\gamma^i - i\tilde{C}\gamma^j\gamma^k\epsilon^{ijk}) \\ &+ \tfrac{1}{2}(p^0)^2(DI + i\tilde{D}\gamma^i\gamma^j\gamma^k\epsilon^{ijk}) \\ &+ \tfrac{1}{2}\mathbf{p}^2(EI + i\tilde{E}\gamma^i\gamma^j\gamma^k\epsilon^{ijk}) \\ &+ i\tilde{F}p^0p^i(\gamma^0\gamma^j\gamma^k\epsilon^{ijk}) + O(p^3).\end{aligned} \tag{66}$$

We will be interested in analyzing only those terms that could give rise to a finite and possibly unsuppressed LIV contribution when $g \to 0$. In this limit we should recover QED, which is parity conserving so that we know that the electron self-energy must have the form

$$\Sigma^{g=0}(p) = W_0 I + W_1(p^\mu\gamma_\mu) + \frac{W_3}{2}p_\mu p^\mu I + O(p^3). \tag{67}$$

From this perspective, all parity violating terms $\tilde{A}$, $\tilde{C}$, $\tilde{D}$, and $\tilde{E}$ in (66) are subject to scrutiny and they should be finally suppressed. On the other hand, the parity conserving contributions can be rearranged in the following way

$$\begin{aligned}\Sigma^g_+(p) = {}& AI + (B - C)p^0\gamma_0 + C(p^\mu\gamma_\mu) \\ &+ \frac{(D + E)}{2}(p^0)^2 I - \frac{E}{2}p_\mu p^\mu I + O(p^3),\end{aligned} \tag{68}$$

so that according to our prescription we expect

$$\begin{aligned}&\lim_{g\to 0}(B - C) = 0, \qquad \lim_{g\to 0}(D + E) = 0, \\ &\lim_{g\to 0}(A, C, -E) = (W_0, W_1, W_3).\end{aligned} \tag{69}$$

The general strategy to evaluate the required integrals is the following. The structure of the denominators $\mathcal{D}$ entering in them is of the form

$$\mathcal{D} = [(k^2)^2 - 4g^2|\mathbf{k}|^2k_0^4 + i\epsilon][k^2 - m^2 + i\epsilon], \tag{70}$$

which can be rewritten

$$\begin{aligned}\mathcal{D} = {}& (1 - 4g^2|\mathbf{k}|^2)[k_0{}^2 - (\omega_-^2(\mathbf{k}) - i\epsilon)] \\ &\times [k_0{}^2 - (\omega_+^2(\mathbf{k}) - i\epsilon)][k_0{}^2 - (E^2(\mathbf{k}) - i\epsilon)].\end{aligned} \tag{71}$$

Within the region of integration ($|\mathbf{k}| < 1/(2g)$), the poles in the complex k_0 plane have the form

$$k_{01} = \mathcal{E}(|\mathbf{k}|) - i\epsilon, \qquad k_{02} = -\mathcal{E}(|\mathbf{k}|) + i\epsilon, \tag{72}$$

with $\mathcal{E}(|\mathbf{k}|) > 0$. Here $\mathcal{E}(|\mathbf{k}|)$ stands for any of the involved energies $\omega_\pm(\mathbf{k})$ and $E(\mathbf{k}) = \sqrt{\mathbf{k}^2 + m^2}$. In this way, it is always possible to perform a Wick rotation to the Euclidean signature such that $k_0 = ik_4$. Because of the remaining rotational symmetry, together with the symmetrical integration over $\mathbf{k}$, one is finally left with only two integration variables which are k_4 and $|\mathbf{k}|$ that can be conveniently rewritten in polar form.

VII. THE LIV CONTRIBUTIONS

In this section, we present a detailed calculation of the corrections $W^g_{\{\mu\}C}$ to the electron self-energy arising from the even sector of the photon propagator $\Delta_{\mu\nu}$ corresponding to the $(B - C)$ term in Eq. (68). The calculation of the remaining contributions goes along similar lines and we only give the final results.

A. General structure of the contributions

As a first step, it is convenient to split them into the following temporal and spatial pieces

$$\begin{aligned}\left(\frac{\partial\Sigma}{\partial p^0}\right)_{p=0} &= W_{\{0\}M}\Gamma^M \\ &= -ie^2\int\frac{d^4k}{(2\pi)^4}\gamma^\mu\left[\frac{\gamma_0}{[k^2 - m^2 + i\epsilon]} \right. \\ &\qquad \left. + \frac{2k^0(m - \gamma k)}{[k^2 - m^2 + i\epsilon]^2}\right]\gamma^\nu\Delta_{\mu\nu}(k)\mathcal{J}(k),\end{aligned} \tag{73}$$

$$\left(\frac{\partial\Sigma}{\partial p^i}\right)_{p=0} = W_{\{i\}M}\Gamma^M$$
$$= ie^2\int\frac{d^4k}{(2\pi)^4}\gamma^\mu\left[\frac{\gamma^i}{[k^2-m^2+i\epsilon]} - \frac{2k^i(m-\gamma k)}{[k^2-m^2+i\epsilon]^2}\right]\gamma^\nu\Delta_{\mu\nu}(k)\mathcal{J}(k), \quad (74)$$

and further separate each contribution according to the even and odd pieces of the photon propagator $\Delta_{\mu\nu}(k)$. To simplify the notation we have introduced

$$\mathcal{J}(k) = \mathcal{I}(k)\theta\left(\frac{1}{2g} - |\mathbf{k}|\right). \quad (75)$$

Taking the even part of the photon propagator in (73), the temporal component of the derivative is

$$\left(\frac{\partial\Sigma}{\partial p^0}\right)^{\text{even}}_{p=0} = W^{\text{even}}_{\{0\}M}\Gamma^M$$
$$= -i\gamma^0e^2\int\frac{d^4k}{(2\pi)^4}\left[\frac{1}{(k^2-m^2+i\epsilon)} - \frac{2k_0^2}{(k^2-m^2+i\epsilon)^2}\right](-2F_1)\mathcal{J}(k), \quad (76)$$

where we have used

$$\gamma^\mu\gamma^0\gamma^\nu\Delta^{\text{even}}_{\mu\nu} = -2\gamma^0F_1. \quad (77)$$

The function F_1 was introduced in Eq. (65) and from (76) we see that the only contribution is given by the component $W^{\text{even}}_{\{0\}0}$. Let us define the quantity

$$W^{\text{even}}_{\{0\}0} \equiv B$$
$$= -2ie^2\int\frac{d^4k}{(2\pi)^4}\left[\frac{1}{(k^2-m^2+i\epsilon)} - \frac{2k_0^2}{(k^2-m^2+i\epsilon)^2}\right]\frac{k^2}{((k^2)^2-4g^2|\mathbf{k}|^2k_0^4)}\mathcal{J}(k). \quad (78)$$

The spatial contribution is (no sum over i)

$$\left(\frac{\partial\Sigma}{\partial p^i}\right)^{\text{even}}_{p=0} = W^{\text{even}}_{\{i\}M}\Gamma^M$$
$$= -ie^2\int\frac{d^4k}{(2\pi)^4}\left[\frac{1}{(k^2-m^2+i\epsilon)} + \frac{2(k^i)^2}{(k^2-m^2+i\epsilon)^2}\right](\gamma^\mu\gamma^i\gamma^\nu\Delta^{\text{even}}_{\mu\nu})\mathcal{J}(k), \quad (79)$$

where we use

$$\gamma^\mu\gamma^i\gamma^\nu\Delta^{\text{even}}_{\mu\nu} = 2\gamma^i(-F_1 + (k_i^2 - |\mathbf{k}|^2)F_2). \quad (80)$$

The rotational invariance of the three-momentum integration leads to

$$W^{\text{even}}_{\{i\}i} \equiv -C$$
$$= 2ie^2\int\frac{d^4k}{(2\pi)^4}\left[\frac{1}{(k^2-m^2+i\epsilon)} + \frac{2|\mathbf{k}|^2/3}{(k^2-m^2+i\epsilon)^2}\right]\frac{\left(k^2 - \frac{8g^2|\mathbf{k}|^2k_0^4}{3k^2}\right)}{((k^2)^2-4g^2|\mathbf{k}|^2k_0^4)}\mathcal{J}(k). \quad (81)$$

B. Calculation of the $(B - C)$ contribution

From Eqs. (78) and (81), we have

$$B - C = 4ie^2\int\frac{d^4k}{(2\pi)^4} \times \frac{(k_0^2+\frac{1}{3}\mathbf{k}^2)k^2 - \frac{4g^2}{3k^2}|\mathbf{k}|^2k_0^4[k^2-m^2+\frac{2}{3}\mathbf{k}^2]}{(k^2-m^2+i\epsilon)^2((k^2)^2-4g^2|\mathbf{k}|^2k_0^4)} \times \mathcal{J}(k). \quad (82)$$

In order to calculate the noncovariant integrals of the above type, together with those in Appendix D, we give some details of the procedure sketched at the end of the previous section. Basically we implement the following steps.

(i) First, we perform a Wick rotation to a Euclidean signature, such that

$$k_0 = ik_4, \qquad k^2 = -(k_4^2+\mathbf{k}^2) = -k_E^2,$$
$$d^4k_E = i4\pi|\mathbf{k}|^2dk_4d|\mathbf{k}|. \quad (83)$$

(ii) Second, since we are maintaining rotational invariance we are left with only two variables

$$-\infty < k_4 < +\infty, \qquad 0 < |\mathbf{k}| < \frac{1}{2g}. \quad (84)$$

In this two-dimensional space we introduce the following polar coordinates

$$k_4 = r\cos\alpha, \qquad |\mathbf{k}| = r\sin\alpha, \quad (85)$$

where $k_E^2 = r^2$. Next we have to integrate over the rectangular strip defined by (84) and we choose first to integrate over r and subsequently over α. In this way we have

$$\int d^4k = i\int d^4k_E = i4\pi\int_0^\pi d\alpha\sin^2\alpha\int_0^{1/(2g\sin\alpha)} r^3dr. \quad (86)$$

Applying the above procedure to Eq. (82) we have

$$B - C = -4e^2 \int \frac{d^4k_E}{(2\pi)^4} \frac{[k_E^2(k_4^2 - \frac{1}{3}\mathbf{k}^2) + \frac{4g^2}{3k_E^2}|\mathbf{k}|^2 k_4^4(-k_E^2 - m^2 + \frac{2}{3}|\mathbf{k}|^2)]}{(k_E^2 + m^2)^2((k_E{}^2)^2 - 4g^2|\mathbf{k}|^2k_4^4)} \frac{M^2}{(M^2 + k_E^2)} \theta\left(\frac{1}{2g} - |\mathbf{k}|\right). \tag{87}$$

Introducing the polar coordinates (85) yields

$$B - C = -\frac{e^2}{\pi^3}\int_0^\pi d\alpha \sin^2\alpha \int_0^{1/(2g\sin\alpha)} r^3 dr \frac{[(\cos^2\alpha - \frac{1}{3}\sin^2\alpha) + \frac{4}{3}g^2\sin^2\alpha\cos^4\alpha(-r^2 - m^2 + \frac{2}{3}r^2\sin^2\alpha)]}{(r^2 + m^2)^2(1 - 4g^2r^2\sin^2\alpha\cos^4\alpha)} \frac{M^2}{(M^2 + r^2)}. \tag{88}$$

The required radial integrals are

$$K_1(\alpha) = \int_0^{1/(2g\sin\alpha)} \frac{r^3}{(r^2 + m^2)^2(1 - r^2c^2(\alpha))} \frac{M^2}{(M^2 + r^2)} dr, \tag{89}$$

$$K_2(\alpha) = \int_0^{1/(2g\sin\alpha)} \frac{r^5}{(r^2 + m^2)^2(1 - r^2c^2(\alpha))} \frac{M^2}{(M^2 + r^2)} dr, \tag{90}$$

which can be exactly calculated, yielding

$$K_1 = \frac{M^2}{2}\left[\frac{(c^2m^4 + M^2)}{\rho^2(\Delta^2)^2}\ln\left(\frac{\Lambda_m^2}{m^2}\right) - \frac{M^2}{(\Delta^2)^2\eta}\ln\left(\frac{\Lambda_M^2}{M^2}\right) + \frac{c^2\ln(1 - c^2b^2)}{\rho^2\eta} + \frac{b^2}{\rho\Lambda_m^2\Delta^2}\right], \tag{91}$$

$$K_2 = \frac{M^2}{2}\left[-\frac{(2m^2M^2 + m^4(c^2M^2 - 1))}{\rho^2(\Delta^2)^2}\ln\left(\frac{\Lambda_m^2}{m^2}\right) + \frac{M^4}{(\Delta^2)^2\eta}\ln\left(\frac{\Lambda_M^2}{M^2}\right) - \frac{\ln(1 - c^2b^2)}{\rho^2\eta} - \frac{b^2m^2}{\rho\Lambda_m^2\Delta^2}\right]. \tag{92}$$

The notation is

$$\rho = 1 + m^2c^2(\alpha), \qquad \eta = 1 + M^2c^2(\alpha),$$
$$\Delta^2 = m^2 - M^2, \qquad c(\alpha) = 2g\sin\alpha\cos^2\alpha,$$
$$\Lambda_m^2 = m^2 + b^2(\alpha), \qquad \Lambda_M^2 = M^2 + b^2(\alpha), \tag{93}$$
$$b(\alpha) = \frac{1}{2g\sin\alpha}.$$

In order to simplify the results by including only the dominant terms, we will expand the above expressions in powers of g^2. This is justified since the expressions (91) and (92) are free of poles. Up to order g^2, the remaining integrals over α will be of the form

$$\sin^p\alpha\cos^q\alpha, \qquad \sin^p\alpha\cos^q\alpha\ln(1 - \cos^4\alpha),$$
$$\sin^p\alpha\cos^q\alpha\ln(\sin\alpha), \tag{94}$$

with q, q integers. These integrals contribute only with finite numerical factors, which are not very relevant in order to establish the correct QED limit of the LIV terms and only the final numerical results will be presented. Nevertheless, we will isolate the exact g^2 independent contribution and we will show that the angular integration produces a zero contribution, thus eliminating any indication of fine-tuning. In all the remaining contributions proportional to g^2, we will further expand in powers of m/M and retain only the dominant terms. In this way we will need the approximate expressions

$$K_1 = \left(\frac{M^4}{(m^2 - M^2)^2}\ln\left(\frac{M}{m}\right) + \frac{M^2}{2(m^2 - M^2)}\right) + 2(gm)^2\sin^2\alpha\cos^4\alpha(1 + 4\ln(2gm\sin\alpha)) - 2(gM)^2\sin^2\alpha\Big(1 + \cos^4\alpha(2\ln(2gM\sin\alpha) - \cos^4\alpha\ln(1 - \cos^4\alpha))\Big), \tag{95}$$

$$g^2K_2 = -(gM)^2(\ln(2Mg\sin\alpha) + \tfrac{1}{2}\ln(1 - \cos^4\alpha)) + (gm)^2(2\ln(2gm\sin\alpha) + \tfrac{1}{2}). \tag{96}$$

It is important to observe that the exact g^2 independent term, contained in the first bracket of Eq. (95) gives a zero contribution in virtue of the angular integral factor

$$\int_0^\pi \sin^2\alpha\left(\cos^2\alpha - \frac{1}{3}\sin^2\alpha\right)d\alpha = 0. \tag{97}$$

Performing numerically, the remaining angular integrations in the proposed approximation we obtain

$$B - C = \frac{e^2}{\pi^2}\left\{(gM)^2(-0.070 + 0.010\ln(gM)) - (gm)^2 \times\left(0.016 + 0.021\ln(gm) + 0.031\ln\left(\frac{m}{M}\right)\right)\right\}. \tag{98}$$

The remaining contributions from the even sector are

$$A = \frac{e^2 m}{\pi^2}\left(\frac{M^2}{2(m^2 - M^2)}\ln\left(\frac{M}{m}\right) + (gM)^2(0.75 + 0.047\ln(gM)) - (gm)^2(0.014 + 0.047\ln(gm))\right), \tag{99}$$

$$D + E = -\frac{e^2}{8\pi^2}(g^2 m)\left(\ln\left(\frac{m}{M}\right) - \frac{3}{4}\right). \tag{100}$$

Finally, the odd contributions are

$$\tilde{A} = \frac{e^2}{6\pi^2} g(M^2(0.018 + 0.063\ln(gM)) - m^2(0.026 + 0.063\ln(gm))), \qquad \tilde{C} = \frac{e^2}{48\pi^2}(gm)\left(\ln\left(\frac{m}{M}\right) + \frac{1}{2}\right), \tag{101}$$

$$\tilde{D} = \tilde{F} = \frac{5ge^2}{48\pi^2}\left(\ln\left(\frac{M}{m}\right) + \frac{4}{5}\right), \qquad \tilde{E} = \frac{ge^2}{12\pi^2}\left(\frac{13}{24}\ln\left(\frac{M}{m}\right) + \frac{17}{32}\right). \tag{102}$$

The results obtained above, in the framework of our prescription to recover QED, have precisely the expected property that reduce to zero when we turn off the LIV correction parametrized by g, keeping M fixed. Also, the results are consistent with the fact that the unsuppressed contribution which we still expect to diverge even after we set $g = 0$ and subsequently $M \longrightarrow \infty$, comes in the term A written in Eq. (99). This term corresponds precisely to the mass renormalization contribution in standard QED.

VIII. MICROCAUSALITY VIOLATION

In this section we provide an estimation of the microcausality violation associated to our model. A comprehensive study of such violations is out of the scope of the present work. Microcausality violation has been previously studied in the fermionic sector of the extended standard model[18].

We work directly in the Coulomb gauge associated to our reference system where $n^\mu = (1, \mathbf{0})$. We only consider points x and x' which produce a spacelike interval $(x - x')^2 < 0$. Unfortunately, we cannot perform a passive Lorentz transformation to reach the system where $x_0 - x'_0 = 0$, which might simplify the calculation. This is because such transformation will change $n^\mu = (1, \mathbf{0})$ into $n'^\mu = \gamma(1, \mathbf{v})$ and then our system will turn out to be manifestly of the HOTD type, thus requiring the application of the perturbative process of Ref. [16], which we have avoided in our particular reference frame.

Even in the standard QED case, there is a drawback when working in the Coulomb gauge, which is basically due to the apparent causality violation of the theory arising from the instantaneous character of the scalar potential. When dealing with the commutator $[A_T^i(x), A_T^j(x')]$, which is the naive starting point to test microcausality, this problem shows up because this commutator is proportional to $(\delta^{ij} - \frac{\partial^i\partial^j}{\nabla^2})D(x - x')$ where $D(x - x')$ is a function that has support only in the light cone. Nevertheless, the operator $1/\nabla^2$, which is just a shorthand for the Green function $1/|\mathbf{r} - \mathbf{r}'|$, acting upon $D(x - x')$ produces nonzero results outside the light cone, thus yielding an apparent violation of microcausality. The canonical way of dealing with this problem is to calculate the commutators of the gauge invariant fields $\mathbf{E}$ and $\mathbf{B}$ for spacelike separation. We will follow the same route here and we will discuss only the commutator

$$[\bar{\Pi}_i^T(x), \bar{\Pi}_j^T(x')] = -\partial_0^2[D^{ij}(x - x')] \equiv \Omega^{ij}(x - x'), \tag{103}$$

which is the analogue of the electric fields commutator in standard QED, with $\bar{\Pi}_i^T(x)$ being gauge invariant. Here

$$D^{ij}(x - x') = [\bar{A}_T^i(x), \bar{A}_T^j(x')]. \tag{104}$$

A direct calculation starting from Eq. (32) leads to

$$D^{ij}(x - x') = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\sum_\lambda \frac{1}{2\omega_\lambda(\mathbf{k})} \times (\varepsilon^i(\lambda, \mathbf{k})\varepsilon^{j*}(\lambda, \mathbf{k})e^{-ik(\lambda)\cdot(x-x')} - \varepsilon^{i*}(\lambda, \mathbf{k})\varepsilon^j(\lambda, \mathbf{k})e^{ik(\lambda)\cdot(x-x')}), \tag{105}$$

and we denote $z^\mu = (x^\mu - x^{\mu\prime})$ in the sequel. Using the relations (B5) and (B6) from Appendix B we arrive at

$$D^{ij}(z) = \sum_\lambda \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2\omega_\lambda(\mathbf{k})}\left[\frac{1}{2}\left[\delta^{ij} - \frac{k^i k^j}{|\mathbf{k}|^2}\right] - \frac{i\lambda}{2}\left[\epsilon^{ijm}\frac{k^m}{|\mathbf{k}|}\right]\right]e^{-i(\omega_\lambda z_0 - \mathbf{k}\cdot\mathbf{z})} - \sum_\lambda \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2\omega_\lambda(\mathbf{k})}\left[\frac{1}{2}\left[\delta^{ij} - \frac{k^i k^j}{|\mathbf{k}|^2}\right] + \frac{i\lambda}{2}\left[\epsilon^{ijm}\frac{k^m}{|\mathbf{k}|}\right]\right]e^{i(\omega_\lambda z_0 - \mathbf{k}\cdot\mathbf{z})}, \tag{106}$$

which can be rewritten as

$$D^{ij}(z) = \left[\delta^{ij} - \frac{\partial^i\partial^j}{|\nabla|^2}\right]\frac{1}{2}\sum_\lambda \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2|\omega_\lambda(\mathbf{k})|} \times [e^{-i(\omega_\lambda z_0 - \mathbf{k}\cdot\mathbf{z})} - e^{i(\omega_\lambda z_0 - \mathbf{k}\cdot\mathbf{z})}] - \epsilon^{ijm}\partial^m\frac{1}{2}\sum_\lambda \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{\lambda}{2|\mathbf{k}||\omega_\lambda(\mathbf{k})|} \times [e^{-i(\omega_\lambda z_0 - \mathbf{k}\cdot\mathbf{z})} - e^{i(\omega_\lambda z_0 - \mathbf{k}\cdot\mathbf{z})}]. \tag{107}$$

Let us remark that this expression contains the correct limit when $g = 0$. In this case, $\omega_+ = \omega_+ = |\mathbf{k}|$, so that the contributions of each term in the $\sum_\lambda$ are the same. After the summation, the first line of (107) reproduces the definition of the standard function $D(x - x')$, while the second line is proportional to $\sum_\lambda \lambda = 0$.

Starting from (103) yields

$$\Omega^{ij}(z) = -[\delta^{ij}\partial_0^2 - \partial^i\partial^j]\frac{1}{2}\sum_\lambda \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{\sqrt{1+2\lambda g|\mathbf{k}|}}{2|\mathbf{k}|} \times [e^{-i(\omega_\lambda z_0 - \mathbf{k}\cdot\mathbf{z})} - e^{i(\omega_\lambda z_0 - \mathbf{k}\cdot\mathbf{z})}] - \left(g\partial^i\partial^j + \frac{1}{2}\epsilon^{ijm}\partial^m\right)\frac{1}{2}\sum_\lambda \int \frac{d^3\mathbf{k}}{(2\pi)^3} \times \left[\frac{\lambda}{\sqrt{1+2\lambda g|\mathbf{k}|}}\right][e^{-i(\omega_\lambda z_0 - \mathbf{k}\cdot\mathbf{z})} - e^{i(\omega_\lambda z_0 - \mathbf{k}\cdot\mathbf{z})}], \tag{108}$$

where we have rearranged the above expression in such a way that the first and second lines of (108) recover the standard QED result in the limit $g \to 0$, while the third and fourth lines are equal to zero. In this way, the microcausality violation is encoded in the functions

$$V_1(z) = \frac{1}{2}\sum_\lambda \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{\sqrt{1+2\lambda g|\mathbf{k}|}}{2|\mathbf{k}|} \times [e^{-i(\omega_\lambda z_0 - \mathbf{k}\cdot\mathbf{z})} - e^{i(\omega_\lambda z_0 - \mathbf{k}\cdot\mathbf{z})}], \tag{109}$$

$$V_2(z) = \frac{1}{2}\sum_\lambda \int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\frac{\lambda}{\sqrt{1+2\lambda g|\mathbf{k}|}}\right] \times [e^{-i(\omega_\lambda z_0 - \mathbf{k}\cdot\mathbf{z})} - e^{i(\omega_\lambda z_0 - \mathbf{k}\cdot\mathbf{z})}], \tag{110}$$

which are now acted by local operators only.

Since we expect microcausality violations, we will estimate their impact arising only from the function V_1. Notice that $V_1(z) = -V_1(-z)$ as can be seen from the expression (109). After performing the angular integrations we obtain

$$V_1 = \frac{1}{2(2\pi)^2}\frac{1}{ir}\sum_\lambda \int_0^{1/2g} dk \frac{\sqrt{1+2\lambda gk}}{2} \times [e^{-i(k/\sqrt{1+2\lambda gk})z_0} - e^{i(k/\sqrt{1+2\lambda gk})z_0}][e^{ikr} - e^{-ikr}], \tag{111}$$

where $k = |\mathbf{k}|$ and we have enforced the upper limit $1/2g$ in order to have real frequencies $\omega_\lambda(k)$ according to Eq. (33). The spacelike character of the interval is written as $-r < z_0 < r$. To proceed, we introduce the phases

$$\Phi_{1\lambda}(k) = k\left(r - \frac{1}{\sqrt{1+2\lambda gk}}z_0\right), \quad \Phi_{2\lambda}(k) = k\left(r + \frac{1}{\sqrt{1+2\lambda gk}}z_0\right), \tag{112}$$

in terms of which we rewrite V_1 as

$$V_1 = \frac{1}{2(2\pi)^2}\frac{1}{ir}\frac{1}{2}\sum_\lambda \int_0^{1/2g} dk\sqrt{1+2\lambda gk} \times [e^{i\Phi_{1\lambda}} + e^{-i\Phi_{1\lambda}} - e^{i\Phi_{2\lambda}} - e^{-i\Phi_{2\lambda}}]. \tag{113}$$

In order to make an estimate of the region where microcausality violations occur, we concentrate in the calculation of the momentum integrals appearing in Eq. (113). We apply the stationary phase method to the generic integral

$$I_\lambda = \int_0^{1/2g} dk f_\lambda(k) e^{-i\Phi_\lambda(k)}, \qquad f_\lambda(k) = \sqrt{1+2\lambda gk}, \tag{114}$$

where the relevant phases are given in Eq. (112). The general result for such an integral is

$$I_\lambda = f_\lambda(\bar{k}) e^{-i\Phi_\lambda(\bar{k})} \int_0^{1/2g} dk e^{(-i(1/2)[d^2\Phi/dk^2]_{k=\bar{k}})(k-\bar{k})^2}, \tag{115}$$

where $\bar{k}$ is the momenta that makes the phase stationary within the interval $[0, 1/2g]$.

We illustrate the calculation for the case of $\Phi_{1\lambda}$. The remaining cases are completely similar and only the final results are written. The exact expression for the momentum $\bar{k}$ that extremizes $\Phi_{1\lambda}$ is given by the equation

$$\frac{r}{z_0} = \frac{1+\lambda g\bar{k}}{(1+2\lambda g\bar{k})^{3/2}}. \tag{116}$$

Observe that $\bar{k}$ appears always in the combination $g\bar{k}$ so that the solution will be of the form

$$\bar{k} = \frac{1}{g}x\left(\frac{r}{z_0}\right), \tag{117}$$

where $x(\frac{r}{z_0})$ solves the corresponding equation obtained from (116). This is a complicated function of $\frac{r}{z_0}$ and to make some analytical progress the following approximation is made. We found that in the range of $\frac{r}{z_0} = 1 + \epsilon$ with $\epsilon \ll 1$, the exact curve $x(\frac{r}{z_0})$ is well approximated by the straight line

$$\bar{k}_{1\lambda} = -\frac{\lambda}{2g}\left(\frac{r}{z_0} - 1\right), \qquad \frac{r}{z_0} > 1, \tag{118}$$

resulting from the expansion of the phase to order k^2 in Eq. (112), which is

$$\Phi_{1\lambda}(k) = kr - (k - k^2\lambda g)z_0. \tag{119}$$

This means that we are considering a spacelike region close to the light cone such that

$$(1-\epsilon)r < |z_0| < r. \tag{120}$$

A posteriori we will verify that our results in fact fall within the range of the approximation. For this purpose, it is convenient to rewrite the condition (120) by stating that the maximum allowed fractional deviation $|\frac{\Delta z_0}{r}|$ has to satisfy

$$\left|\frac{\Delta z_0}{r}\right| < \epsilon. \tag{121}$$

From now on it is convenient to separate the cases according to the sign of z_0. For $z_0 > 0$ the extremum (118) has to satisfy the condition

$$0 < \bar{k}_{1\lambda} = -\frac{\lambda}{2g}\left(\frac{r}{z_0} - 1\right) < \frac{1}{2g}. \tag{122}$$

We observe that we have no solution for $\lambda = +1$. The choice $\lambda = -1$ requires

$$\frac{r}{2} < z_0. \tag{123}$$

In this way we have

$$\bar{k}_{1-} = \frac{1}{2g}\left(\frac{r}{z_0} - 1\right), \qquad \Phi_{1-}(\bar{k}_{1-}) = \frac{(r - z_0)^2}{4g z_0}, \qquad \frac{r}{2} < z_0 < r, \tag{124}$$

$$\sqrt{1 + 2\lambda g \bar{k}_{1-}} = \sqrt{2 - \frac{r}{z_0}}, \qquad \left[\frac{d^2\Phi_{1-}(k)}{dk^2}\right]_{k=\bar{k}_{1-}} = -2g z_0. \tag{125}$$

The case $z_0 < 0$ produces

$$\Phi_{1\lambda}(k) = kr + (k - k^2\lambda g)|z_0|, \tag{126}$$

with

$$\bar{k}'_{1\lambda} = \frac{\lambda}{2g}\left(\frac{r}{|z_0|} + 1\right), \qquad \frac{r}{|z_0|} > 1. \tag{127}$$

The condition

$$0 < \bar{k}'_{1\lambda} = \frac{\lambda}{2g}\left(\frac{r}{|z_0|} + 1\right) < \frac{1}{2g} \tag{128}$$

cannot be satisfied neither for $\lambda = -1$, nor for $\lambda = +1$. The former leads to negative $\bar{k}'_{1\lambda}$, while the latter requires $\frac{r}{|z_0|} < 0$. In other words there is no solution for $z_0 < 0$.

The case of $\Phi_{2\lambda}(k)$ has a solution only for $z_0 < 0$ and $\lambda = -1$. The results are

$$\bar{k}_{2-} = \frac{1}{2g}\left(\frac{r}{|z_0|} - 1\right), \qquad \Phi_{2-}(\bar{k}_{2-}) = \frac{(r - |z_0|)^2}{4g|z_0|}, \qquad \frac{r}{2} < |z_0| < r, \qquad z_0 < 0, \tag{129}$$

$$\sqrt{1 + 2\lambda g \bar{k}_{2-}} = \sqrt{2 - \frac{r}{|z_0|}}, \qquad \left[\frac{d^2\Phi_{2-}(k)}{dk^2}\right]_{k=\bar{k}_{2-}} = -2g|z_0|. \tag{130}$$

Substituting in (113) yields

$$\begin{aligned}
V_1(z) = {} & \frac{1}{4(2\pi)^2}\frac{1}{ir}\theta(z_0)\sqrt{\frac{2z_0 - r}{z_0}}\left[e^{i((r-z_0)^2/4gz_0)}\int_0^{1/2g} dk e^{-igz_0(k-\bar{k}_{1-})^2}\right] \\
& + \frac{1}{4(2\pi)^2}\frac{1}{ir}\theta(z_0)\sqrt{\frac{2z_0 - r}{z_0}}\left[e^{-i((r-z_0)^2/4gz_0)}\int_0^{1/2g} dk e^{+igz_0(k-\bar{k}_{1-})^2}\right] \\
& - \frac{1}{4(2\pi)^2}\frac{1}{ir}\theta(-z_0)\sqrt{\frac{2z_0 + r}{z_0}}\left[e^{-i((r+z_0)^2/4gz_0)}\int_0^{1/2g} dk e^{+igz_0(k-\bar{k}_{2-})^2}\right] \\
& - \frac{1}{4(2\pi)^2}\frac{1}{ir}\theta(-z_0)\sqrt{\frac{2z_0 + r}{z_0}}\left[e^{+i((r+z_0)^2/4gz_0)}\int_0^{1/2g} dk e^{-igz_0(k-\bar{k}_{2-})^2}\right],
\end{aligned} \tag{131}$$

where we can verify that $V_1(z) = -V_1(-z)$. Though this will not be relevant for our estimation of the microcausality violations, we can go one step further and estimate the remaining integrals in the following way. Introducing the change of variables $u = \sqrt{g}(k - \bar{k}_{1-})$ we obtain

$$\begin{aligned}
I_{1\pm} &= \int_0^{1/2g} dk e^{\pm igz_0(k-\bar{k}_{1-})^2} \\
&= \frac{1}{\sqrt{g}}\int_{-\sqrt{g}\bar{k}_{1-}}^{\sqrt{g}((1/2g)-\bar{k}_{1-})} du e^{\pm iz_0u^2}.
\end{aligned} \tag{132}$$

Substituting the value of $\bar{k}_{1-}$ results in

$$\begin{aligned}
I_{1\pm} &= \frac{1}{\sqrt{g}}\int_{-(1/2\sqrt{g})((r/z_0)-1)}^{(1/2\sqrt{g})(2-r/z_0)} du e^{\pm iz_0u^2} \\
&\simeq \frac{1}{\sqrt{g}}\int_{-\infty}^{\infty} du e^{\pm iz_0u^2} = \sqrt{\frac{\pi}{2gz_0}}(1 \pm i).
\end{aligned} \tag{133}$$

The expression for

$$I_{2\pm} = \int_0^{1/2g} dk e^{\pm igz_0(k-\bar{k}_{2-})^2} = \int_0^{1/2g} dk e^{\mp ig|z_0|(k-\bar{k}_{2-})^2} \tag{134}$$

can be obtained from (133) changing z_0 by $|z_0|$, so that we obtain

$$I_{2\pm} = \sqrt{\frac{\pi}{2g|z_0|}}(1 \mp i). \tag{135}$$

Then we have

$$\begin{aligned} V_1(z) = {} & \frac{1}{4(2\pi)^2}\frac{1}{ir}\theta(z_0)\sqrt{\frac{2z_0 - r}{z_0}}\sqrt{\frac{\pi}{2gz_0}} \\ & \times [e^{i((r-z_0)^2/4gz_0)}(1-i) + e^{-i((r-z_0)^2/4gz_0)}(1+i)] \\ & - \frac{1}{4(2\pi)^2}\frac{1}{ir}\theta(-z_0)\sqrt{\frac{2z_0 + r}{z_0}}\sqrt{\frac{\pi}{2g|z_0|}} \\ & \times [e^{-i((r+z_0)^2/4gz_0)}(1-i) + e^{+i((r+z_0)^2/4gz_0)}(1+i)]. \end{aligned} \tag{136}$$

Next we analyze the regions where microcausality is violated and provide an estimation of the amount of such violation. In our approximation, such violations occur when the functions $e^{\pm i((r-|z_0|)^2/4gz_0)}$ do not oscillate rapidly enough to make $V_1(z)$ equal zero in the spacelike region. Thus we take the condition for having microcausality violations to be the region where the phases change slowly, that is to say where

$$\frac{(r - |z_0|)^2}{4g|z_0|} < 1, \tag{137}$$

in which case the oscillations are very much suppressed. Let us concentrate now in the case $z_0 > 0$ (the case $z_0 < 0$ can be discussed in a similar way). We first examine the curves that limit the region of interest by considering the equality in Eq. (137). For a given r, the solutions of such an equation are

$$\begin{aligned} z_{0+} &= r + 2g + 2\sqrt{g^2 + gr}, \\ z_{0-} &= r + 2g - 2\sqrt{g^2 + gr}. \end{aligned} \tag{138}$$

We observe that z_{0+} is always above the line $z_0 = r$, while z_{0-} is always below. Also notice that both curves tend to the light cone when $g \longrightarrow 0$. The condition (137) is satisfied when

$$r - \left(2\sqrt{g^2 + gr} - 2g\right) < z_0 < r, \tag{139}$$

because this region includes the case $z_0 \longrightarrow r$ which clearly satisfies (137). That is to say, (139) determines the spacelike region where $V_1(z)$ is not zero, thus leading to microcausality violations. For a given r, the range of z_0 within that region is given by $\Delta z_0 = r - z_{0-}$. Then we can quantify the maximum time interval for which such violations occur by

$$\frac{\Delta z_0}{|z_0|} \simeq \frac{\Delta z_0}{r} = \frac{1}{r}\left(2\sqrt{g^2 + gr} - 2g\right). \tag{140}$$

The expression in the right-hand side of (140) is a monotonically decreasing function of r with the following end points

$$\left[\frac{\Delta z_0}{r}\right]_{r\to 0} = 1, \qquad \left[\frac{\Delta z_0}{r}\right]_{r\to\infty} = 0. \tag{141}$$

That is to say, for the whole region $r > r_0$ we can guarantee that

$$\frac{\Delta z_0}{r} < \left[\frac{\Delta z_0}{r}\right]_{r=r_0}. \tag{142}$$

Recall that $g = \xi/\bar{M}$, where ξ is bounded by 10^{-10} when we choose $\bar{M} = M_P = 10^{19}$ GeV ($L_P = 10^{-33}$ cm) [21]. Thus, taking $g = 10^{-43}$ cm and always considering the region $r \gg g$, where we can trust the effective theory, we make some numerical estimations of the relation (142), which are given in Table I. The third and fourth values of Table I correspond to distances given by $r_0 = 10^{11}L_P$ and $r_0 = 10^6 L_P$, which set a lower limit beyond which space becomes granular, according to the models considered in Refs. [4,22], respectively. The calculated microcausality violations in Table I fall comfortably within the range determined by (121) required for the approximation to order k^2 in the phases to be correct.

TABLE I. Upper bound on fractional microcausality violation $|\Delta z_0|/r$ for distances $r > r_0$.

r_0 cm	$|\Delta z_0|/r<$
1	6.3×10^{-22}
10^{-10}	6.3×10^{-17}
10^{-22}	6.3×10^{-11}
10^{-27}	2.0×10^{-8}
10^{-33}	2.0×10^{-5}

IX. FINAL REMARKS

In this work we have proposed a consistent quantization of the electromagnetic sector of the MP model, [1] coupled to standard fermions, such that it can be realized as a perturbative correction of standard QED. By this we mean that in the limit where the LIV parameter $g = \xi/\tilde{M}$ goes to zero one should recover the same quantum corrections arising in QED. Even though this sector of the MP model is not of the higher order time-derivative type, up to a total derivative, some subtleties appear in the quantization of the photon field. The correct perturbative prescription is achieved by recognizing the effective character of the model via the introduction of a coarse-graining scale $M \ll 1/g$, under which we assume that space retains the usual attributes which allow the construction of a standard effective field theory. Such cutoff scale is incorporated, in a smooth way, by means of a Lorentz covariant function of the Pauli-Villars type, which plays the role of a standard regulator in the QED limit and makes sure that all LIV is codified in the parameter g. The mathematical translation of this physical picture amounts to the following prescription in order to properly recover QED: first take $g = 0$, for constant M, and subsequently set $M \to \infty$. The prescription has been tested in the calculation of LIV contributions arising from the electron self-energy, which indeed provides the expected results. In this way, the fine-tuning problems found in Refs. [9,11] disappear and one in fact recovers the correct zero limit for all the LIV corrections, which are indeed shown to be very small perturbations in accordance with the experimental and observational evidence.

Some comments regarding the plausibility of the scale M in relation with the very stringent constraints already found for LIV are now in order. The combinations of parameters $g = \xi/\bar{M}$, $\eta_{1,2}/\bar{M}$, denoted collectively by $\Xi/\bar{M}$, appearing in Eq. (1) are considered as remnants of a more fundamental quantum gravity (QG) theory, which include effects that make space no longer describable in terms of a continuum. Such parameters could arise, for example, in the process of calculating expectation values of well-defined QG operators in semiclassical states that describe Minkowski space-time, which would be necessary to derive the exact nature of the induced corrections to standard particle dynamics at low energies. Let us emphasize that what is bounded by experiments or observations is the ratio $\Xi/\bar{M}$, so that a neat separation of the scale $\bar{M}$ and the correction coefficients Ξ, which could even be zero if no corrections arise, is not possible until a semiclassical calculation is correctly performed starting from a full quantum theory. Initially, the naive expectation was that taking $\bar{M} = M_{\rm Planck}$ will be consistent with Ξ values of order one, which is certainly not the case. Nevertheless, we should not rule out rather unexpected values of Ξ or $\bar{M}$ until the correct calculation is done.

Let us assume that we have identified the correct separation in $\Xi_{\rm QG}/M_{\rm QG}$ consistent with the experimental bounds and arising from a correct semiclassical limit of the QG theory. Then we will interpret $M_{\rm QG}$ as the scale in which quantum effects are manifest and where space is characterized by strong fluctuations forbidding its description as a continuum. Nevertheless, another scale M naturally should arise in this approach, which is the one that separates the continuum description of space from a foamy description related to quantum effects. That is to say, for probe energies $E \ll M$ we are definitely within the standard continuum description of space where effective field theory (EFT) methods should apply. For probe energies $E \gg M$ we enter the realm of quantum gravity and there we assume that any EFT has to be replaced by an alternative description. It is natural that a very large number of the basic quantum cells of space characterized by the scale $(1/M_{\rm QG})^3$ will contribute to the much larger cells characterizing the onset of a continuum description, so that we expect $M \ll M_{\rm QG}$.

The maximum allowed momenta $|\mathbf{k}_{\rm max}| \approx M_{\rm QG}/\Xi_{\rm QG}$ in the theory will be mathematically dictated by the positivity of the normal mode's energies, Eq. (34) in our case, and certainly constitutes an extrapolation of the EFT that can be considered as the analogous of taking the maximum momentum equal to infinity in the standard QED case. That is to say, we need to introduce an additional suppression of the excitation modes in our EFT which will be settled by the scale M, thus defining the effective energy range of the model. This is required by the EFT description of excitations in space which demands that the Compton wavelength $1/|\mathbf{k}|$ of the allowed excitations be larger than the scale $1/M$ setting the onset of the continuum. The implementation of this proposal is directly related with our demand that the quantum model constructed from the MP theory be such that it produces a continuous interpolation between those physical results including $\Xi \neq 0$ corrections and those predicted by standard QED ($\Xi = 0$). In order to achieve this, we have proposed the prescription fully described in Sec. V.

Let us now discuss whether or not an estimate of the order of magnitude of the scale M in relation to $M_{\rm QG}$ makes sense. In our specific case the LIV contribution to the electron self-energy produces an additional dimension four contribution to the Lagrangian given by

$$\Delta L = \delta \frac{e^2}{\pi^2} \bar{\Psi} \gamma^0 i \partial_0 \Psi, \tag{143}$$

arising from the $(B - C)$ term in Eq. (68). Our calculation leads to a prediction dominated by

$$|\delta| \sim 10^{-2} \times (gM)^2 |\ln(gM)|, \tag{144}$$

according to Eq. (98). On the other hand, starting from the correction (143) together with bounds from the anisotropy of the inertial mass, the authors of Ref. [9] have established the experimental bound

$$|\delta| < 10^{-21}. \tag{145}$$

In this way, we expect that the scale M is bounded in such a way that the theoretical correction (144) is much less than the experimental bound (145), that is to say when

$$\begin{aligned} gM = \xi \frac{M}{\bar{M}} = 10^{-10} \frac{M}{\bar{M}} \ll 0.65 \times 10^{-10}, \\ \to \qquad M \ll 0.65 \bar{M}. \end{aligned} \tag{146}$$

The above shows that it is safe and consistent with present observations to define a scale M much below the quantum gravity scale $\bar{M}$. Proposals for additional scales M significantly smaller than $\bar{M} = M_P$, that can be understood as signaling the transition between the standard space-time and that associated to the quantum gravity phase, already exists in the literature [4,22].

Next we comment upon the behavior of our result for the electron self-energy under different momentum routings. For arbitrary internal momenta, the basic expression (56) can be rewritten as

$$\Sigma^g(p) = -ie^2 \int \frac{d^4k}{(2\pi)^4} \gamma^\mu S_F(k^\alpha + k_2^\alpha)\gamma^\nu \Delta_{\mu\nu}(-k^\alpha + k_1^\alpha) \times I(-k^\alpha + k_1^\alpha)\theta(1/(2g) - |\mathbf{k_1} - \mathbf{k}|), \tag{147}$$

where $k_1^\alpha - k^\alpha$ $(k^\alpha + k_2^\alpha)$ is the internal photon (electron) momentum, respectively, with k_1^α and k_2^α being arbitrary momenta satisfying the conservation $k_2^\alpha = p^\alpha - k_1^\alpha$. Here $S_F(k^\alpha)$ denotes the standard fermion propagator. Since the integral (147) is finite we are allowed to make the change of variables $k^\alpha - k_1^\alpha \to k^\alpha$, which reduces the integral to the form (56) and shows its invariance under momentum rerouting.

The stability of the model is guaranteed by restricting the observer Lorentz covariance to concordant frames characterized by boosts factors up to $\gamma = 1/\sqrt{2gM}$. Using the bound (146), the maximum allowed boost factor is $\gamma_{\max} = 8.8 \times 10^4$, which corresponds to a maximum relative velocity such that $1 - |\mathbf{v}_{\max}| > 6.5 \times 10^{-11}$. This condition certainly includes concordant frames that move nonrelativistically with respect to Earth.

We have made a preliminary estimation of the microcausality violations in the model by looking at the commutator of two gauge invariant momentum operators (which are the extension of the electric field operators in standard QED) for spacelike separation $r > z_0$. The value of the corresponding function has been calculated using the stationary phase approximation and the condition for having microcausality violations requires that the exponentials oscillate very slowly. This means that the associated phases should be of order one or less, which defines a spacelike region extremely close to the light cone, rapidly approaching to it when the LIV parameter $g \to 0$. For a given value of r, the width $|\Delta z_0|$ of such a region is calculated. The fractional value $(|\Delta z_0|/r)_{\max}$, which sets the upper limit for the allowed microcausality violation is subsequently estimated, leading to a typical value of $|\Delta z_0|/r < 6.3 \times 10^{-17}$ for distances r larger than the Compton wavelength of the electron.

In this paper, we have studied the construction of the quantum MP effective model emphasizing the recovering of the correct QED limit in relation with the absence of fine-tuning problems. A summary of our results has been presented in Ref. [23]. Within the restrictions imposed, we have established the basis of a sound perturbative scheme to proceed with the calculation of additional radiative processes. We defer for further work the analysis of the predictive power of the model in relation to LIV corrections to physical observables.

ACKNOWLEDGMENTS

C. M. R. acknowledges support from DGAPA-UNAM. L. F. U. is partially supported by Projects CONACYT 55310 and DGAPA-UNAM-IN109107. J. D. V. acknowledges support from Projects CONACYT 47211-F and DGAPA-UNAM-IN109107.

APPENDIX A

The conventions used in this work for the Dirac algebra are those of Ref. [24] and we take $\hbar = c = 1$. Also we have

$$\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1), \qquad \delta^{ij} = +1, i = j, \qquad \delta^{ij} = 0, i \neq j,$$
$$\epsilon^{0ijk} = \epsilon^{ijk}, \qquad \epsilon^{0123} = \epsilon^{123} = -\epsilon_{123} = +1,$$
$$\mathbf{A} = (A^i = -A_i), \qquad \mathbf{k} = (k^i = -k_i), \qquad \nabla = \left(\frac{\partial}{\partial x^i} = \partial_i\right),$$
$$\partial_r \Leftrightarrow -ik_r, \qquad \partial_t \Leftrightarrow -ik_0, \qquad \mathbf{A}\cdot\mathbf{B} = A^iB^i, \qquad (\mathbf{A}\times\mathbf{B})^i = \epsilon^{ijk}A^jB^k, \qquad (\nabla\times\mathbf{B})^i = \epsilon^{ijk}\partial_jB^k. \tag{A1}$$

In addition let us summarize some useful properties of the operator $(M^{-1})^{ik}$ introduced in Sec. II,

$$(M^{-1})^{ji}(M^{-1})^{ir} = \frac{1}{(1+4g^2\nabla^2)^2}[(1-4g^2\nabla^2)\delta^{jr} - 4g\epsilon^{jpr}\partial_p + 4g^2(3+4g^2\nabla^2)\partial_j\partial_r], \tag{A2}$$

$$\frac{1}{(1+4g^2\nabla^2)^2}[(M^{-1})^{pi}\epsilon^{ijk}\partial_j(M^{-1})^{kr}] = (1-4g^2\nabla^2)\epsilon^{pjr}\partial_j + 4g(\delta^{pr}\nabla^2 - \partial_r\partial_p), \tag{A3}$$

$$\partial_i\partial_j(M^{-1})^{ij} = \frac{1}{(1+4g^2\nabla^2)}(\nabla^2 + 4g^2(\nabla^2)^2) = \nabla^2, \tag{A4}$$

$$\partial_j(M^{-1})^{ij} = \frac{1}{(1+4g^2\nabla^2)}(1+4g^2\nabla^2)\partial_i = \partial_i, \tag{A5}$$

$$(M^{-1})^{ij}\Pi_j^T = \frac{1}{(1+4g^2\nabla^2)}(\delta^{ij} + 2g\epsilon^{irj}\partial_r)\Pi_j^T. \tag{A6}$$

APPENDIX B

Here we present the properties of the polarization vectors in the helicity basis ($\lambda = \pm 1$), which are used in the expansion of the photon field in Eq. (24). They satisfy the identities

$$\boldsymbol{\varepsilon}^*(\lambda, \hat{\mathbf{k}}) = \boldsymbol{\varepsilon}(-\lambda, \hat{\mathbf{k}}), \qquad \hat{\mathbf{k}} \cdot \boldsymbol{\varepsilon}(\lambda, \hat{\mathbf{k}}) = 0, \qquad \hat{\mathbf{k}} \times \boldsymbol{\varepsilon}(\lambda, \hat{\mathbf{k}}) = -i\lambda \boldsymbol{\varepsilon}(\lambda, \hat{\mathbf{k}}), \tag{B1}$$

$$\boldsymbol{\varepsilon}^*(\lambda, \hat{\mathbf{k}}) \cdot \boldsymbol{\varepsilon}(\lambda', \hat{\mathbf{k}}) = \delta_{\lambda\lambda'}, \qquad \boldsymbol{\varepsilon}(\lambda, -\hat{\mathbf{k}}) = \boldsymbol{\varepsilon}(-\lambda, \hat{\mathbf{k}}), \qquad \boldsymbol{\varepsilon}(\lambda', -\hat{\mathbf{k}}) \cdot (\mathbf{k} \times \boldsymbol{\varepsilon}(\lambda, \hat{\mathbf{k}})) = -i\lambda|\mathbf{k}|\delta_{\lambda\lambda'}, \tag{B2}$$

$$\varepsilon^r(\lambda, \hat{\mathbf{k}})[\delta^{rp} + i2g\epsilon^{rmp}k_m]\varepsilon^p(\lambda', -\hat{\mathbf{k}}) = [1 + 2g\lambda|\mathbf{k}|]\delta_{\lambda\lambda'}, \tag{B3}$$

$$\varepsilon^{r*}(\lambda, \hat{\mathbf{k}})[\delta^{rp} - i2g\epsilon^{rmp}k_m]\varepsilon^p(\lambda', \hat{\mathbf{k}}) = [1 + 2g\lambda|\mathbf{k}|]\delta_{\lambda\lambda'}. \tag{B4}$$

The following combinations are useful in the construction of the corresponding propagator

$$\varepsilon^i(\lambda, \hat{\mathbf{k}})\varepsilon^{j*}(\lambda, \hat{\mathbf{k}}) = \frac{1}{2}\left[\delta^{ij} - \frac{k^i k^j}{|\mathbf{k}|^2}\right] - \lambda\frac{i}{2}\left[\epsilon^{ijm}\frac{k^m}{|\mathbf{k}|}\right], \tag{B5}$$

$$\varepsilon^i(\lambda, -\hat{\mathbf{k}})\varepsilon^{j*}(\lambda, -\hat{\mathbf{k}}) = \frac{1}{2}\left[\delta^{ij} - \frac{k^i k^j}{|\mathbf{k}|^2}\right] + \lambda\frac{i}{2}\left[\epsilon^{ijm}\frac{k^m}{|\mathbf{k}|}\right], \tag{B6}$$

together with the sums

$$\sum_\lambda \varepsilon^i(\lambda, \hat{\mathbf{k}})\varepsilon^{j*}(\lambda, \hat{\mathbf{k}}) = \sum_\lambda \varepsilon^j(\lambda, -\hat{\mathbf{k}})\varepsilon^{i*}(\lambda, -\hat{\mathbf{k}}) = \delta^{ij} - \frac{k^i k^j}{|\mathbf{k}|^2}. \tag{B7}$$

The final calculation of the propagator in Eq. (51) requires the result of the following sums

$$\sum_\lambda \frac{1}{(k_0^2 - \omega_\lambda^2 + i\epsilon)} = \frac{2(k^2 - 4g^2|\mathbf{k}|^2 k_0^2)}{(k^2)^2 - 4g^2|\mathbf{k}|^2 k_0^4 + i\epsilon}, \tag{B8}$$

$$\sum_\lambda \frac{\omega_\lambda^2}{(k_0^2 - \omega_\lambda^2 + i\epsilon)} = \frac{2\mathbf{k}^2 k^2}{(k^2)^2 - 4g^2|\mathbf{k}|^2 k_0^4 + i\epsilon}, \tag{B9}$$

$$\sum_\lambda \frac{\lambda}{(k_0^2 - \omega_\lambda^2 + i\epsilon)} = -\frac{4g|\mathbf{k}|^3}{(k^2)^2 - 4g^2|\mathbf{k}|^2 k_0^4 + i\epsilon}. \tag{B10}$$

APPENDIX C

The purpose of this Appendix is to compare the propagator obtained directly from the equations of motion in the Lorentz gauge and further expressed in the Coulomb gauge, with the propagator (49) obtained directly in the Coulomb gauge after the canonical transformation (22) is made.

From the equations of motion (3) in the Lorentz gauge we identify the momentum space operator

$$O^{\nu\phi}(k) = -k^2\eta^{\nu\phi} - 2gik_0^2\epsilon^{0\nu\sigma\phi}k_\sigma, \tag{C1}$$

which propagator is

$$\Delta_{\mu\nu}(k) = \frac{1}{(k^2)^2 - 4g^2k_0^4|\mathbf{k}|^2)} \times \left[-k^2\eta_{\mu\nu} + 2igk_0^2\epsilon^{pmq}k_m\eta_{p\mu}\eta_{q\nu} - \frac{4g^2k_0^4}{k^2}k_pk_q\delta_\mu^p\delta_\nu^q + \frac{4g^2k_0^4|\mathbf{k}|^2}{k^2}\eta_{0\mu}\eta_{0\nu}\right], \tag{C2}$$

such that $O^{\nu\mu}\Delta_{\mu\rho} = \delta_\rho^\nu$. Separating the instantaneous Coulomb contribution

$$J^i(-k)\Delta_{ij}(k)J^j(k) = J^\mu(-k)\Delta_{\mu\nu}(k)J^\nu(k) - J^0(-k)\frac{1}{|\mathbf{k}|^2}J^0(k) \tag{C3}$$

and using charge conservation [19], we find the corresponding propagator in the Coulomb gauge

$$\Delta_{ij}(k) = \frac{1}{|\mathbf{k}|^2((k^2)^2 - 4g^2k_0^4|\mathbf{k}|^2)}[k^2(|\mathbf{k}|^2\delta_{ij} - k_ik_j) + 2gk_0^2|\mathbf{k}|^2 i\epsilon^{imj}k_m]. \tag{C4}$$

The Coulomb gauge propagator corresponding to the canonical transformation (22) and which was directly constructed from its vacuum expectation value definition (40) leading to the final result Eq. (49) is

$$\bar{\Delta}_{ij} = \frac{1}{|\mathbf{k}|^2((k^2)^2 - 4g^2k_0^4|\mathbf{k}|^2)}[(k^2 - 4g^2|\mathbf{k}|^2k_0^2) \times (|\mathbf{k}|^2\delta_{ij} - k_ik_j) + 2g|\mathbf{k}|^4 i\epsilon^{imj}k_m]. \tag{C5}$$

Here we show the consistency between (C4) and (C5).

The starting points are the defining relations

$$A = \Delta J_T, \qquad \bar{A} = \bar{\Delta}J_T, \tag{C6}$$

where both A and $\bar{A}$ are in the Coulomb gauge. Whenever it is not confusing, we use the compact notation $A = \Delta J_T \Leftrightarrow A^i = \Delta_{ij}J_T^j$. Moreover, the photon fields are related by the canonical transformation T such that

$$A = T\bar{A}, \qquad A^\dagger = \bar{A}^\dagger T^\dagger. \tag{C7}$$

The invariant object is basically the electromagnetic energy written in terms of the transverse current

$$E = \frac{1}{2}\int d^3xd^3yJ_T^i(x)\Delta_{ij}(x-y)J_T^j(y), \tag{C8}$$

where the sources are real. In momentum space this implies

$$(J_T^i(k))^* = J_T^i(-k), \tag{C9}$$

and Eq. (C8) translates into

$$E = \frac{1}{2}\int d^3k J_T^i(-k)\Delta_{ij}(k)J_T^j(k) = \frac{1}{2}\int d^3k J_T^\dagger \Delta J_T. \tag{C10}$$

Reality of E further demands

$$\Delta^\dagger(k) = \Delta(k). \tag{C11}$$

Expressing E in terms of the fields yields

$$E = \frac{1}{2}\int d^3k J_T^\dagger \Delta J_T = \frac{1}{2}\int d^3k A^\dagger(\Delta^{-1})A, \tag{C12}$$

in such a way that the equivalent description in terms of the barred quantities requires

$$E = \frac{1}{2}\int d^3k \bar{A}^\dagger(\bar{\Delta}^{-1})\bar{A}. \tag{C13}$$

Inserting the transformation (C7), we obtain the relation among the two propagators

$$\Delta = T\bar{\Delta}T^\dagger. \tag{C14}$$

The general structure of a Coulomb gauge propagator in our parity violating theory can be written as

$$\Delta = \Delta_1 K + \Delta_2 S, \qquad \bar{\Delta} = \bar{\Delta}_1 K + \bar{\Delta}_2 S, \tag{C15}$$

where we have introduced the Hermitian matrices

$$K = [K_{ij}] = [|\mathbf{k}|^2\delta_{ij} - k_i k_j], \qquad S = [S_{ij}] = [i\epsilon^{imj}k_m], \tag{C16}$$

with the following properties

$$S^3 = |\mathbf{k}|^2 S, \qquad S^2 = K, \tag{C17}$$

in such a way that each propagator can be written in terms of the matrix S only.

The canonical transformation T has the form

$$T = \alpha I + \beta S = T^\dagger, \tag{C18}$$

in momentum space, with real numerical coefficients

$$\alpha = \frac{\sqrt{1+\sqrt{(1+4a)}}}{\sqrt{2}\sqrt{(1+4a)}},$$
$$\beta = \frac{2g}{\sqrt{2}\sqrt{(1+4a)}}\frac{1}{\sqrt{1+\sqrt{(1+4a)}}}, \qquad a = -g^2|\mathbf{k}|^2. \tag{C19}$$

Substituting in the relation (C14) and after some algebra we obtain the following conditions among the components of the respective propagators

$$\Delta_1 = \frac{1}{(1-4g^2|\mathbf{k}|^2)}[\bar{\Delta}_1 + 2g\bar{\Delta}_2], \tag{C20}$$

$$\Delta_2 = \frac{1}{(1-4g^2|\mathbf{k}|^2)}[\bar{\Delta}_2 + 2g|\mathbf{k}|^2\bar{\Delta}_1]. \tag{C21}$$

From the expressions (C4) and (C5) together with the definition (C15) one can read

$$\Delta_1 = \frac{k^2}{|\mathbf{k}|^2((k^2)^2 - 4g^2k_0^4|\mathbf{k}|^2)}, \quad \Delta_2 = \frac{2gk_0^2}{((k^2)^2 - 4g^2k_0^4|\mathbf{k}|^2)}, \tag{C22}$$

$$\bar{\Delta}_1 = \frac{(k^2 - 4g^2|\mathbf{k}|^2k_0^2)}{|\mathbf{k}|^2((k^2)^2 - 4g^2k_0^4|\mathbf{k}|^2)}, \quad \bar{\Delta}_2 = \frac{2g|\mathbf{k}|^2}{((k^2)^2 - 4g^2k_0^4|\mathbf{k}|^2)}, \tag{C23}$$

and verify that they satisfy the relations (C20) and (C21).

APPENDIX D

In this Appendix, we write the general form of the remaining LIV contributions to the electron self-energy, of which final results are presented in Sec. VII, according to our general scheme of calculation

$$A = 4ie^2m\int\frac{d^4k}{(2\pi)^4}\frac{(k^2 - 2g^2k_0^4|\mathbf{k}|^2/k^2)}{[k^2 - m^2]((k^2)^2 - 4g^2|\mathbf{k}|^2k_0^4)}\mathcal{J}(k), \tag{D1}$$

$$\tilde{A} = \frac{2ie^2}{3}g\int\frac{d^4k}{(2\pi)^4}\frac{|\mathbf{k}|^2k_0^2}{[k^2 - m^2]((k^2)^2 - 4g^2|\mathbf{k}|^2k_0^4)}\mathcal{J}(k), \tag{D2}$$

$$\tilde{C} = \frac{4ie^2}{3}(gm)\int\frac{d^4k}{(2\pi)^4} \times \frac{|\mathbf{k}|^2k_0^2}{(k^2 - m^2)^2((k^2)^2 - 4g^2|\mathbf{k}|^2k_0^4)}\mathcal{J}(k), \tag{D3}$$

$$D + E = 32ie^2m\int\frac{d^4k}{(2\pi)^4}\frac{(k_0^2 + |\mathbf{k}|^2/3)}{(k^2 - m^2)^3} \times \frac{(k^2 - 2g^2k_0^4|\mathbf{k}|^2/k^2)}{((k^2)^2 - 4g^2|\mathbf{k}|^2k_0^4)}\mathcal{J}(k), \tag{D4}$$

$$\tilde{D} = \tilde{F} = -\frac{4ie^2}{3} g \int \frac{d^4k}{(2\pi)^4} \left[\frac{1}{(k^2 - m^2)^2} - \frac{4k_0^2}{(k^2 - m^2)^3} \right] \times \frac{|\mathbf{k}|^2 k_0^2}{((k^2)^2 - 4g^2 |\mathbf{k}|^2 k_0^4)} \mathcal{J}(k), \tag{D5}$$

$$\tilde{E} = \frac{4ige^2}{3} \int \frac{d^4k}{(2\pi)^4} \left[\frac{3}{(k^2 - m^2)^2} + \frac{\frac{4}{3}|\mathbf{k}|^2}{(k^2 - m^2)^3} \right] \times \frac{|\mathbf{k}|^2 k_0^2}{((k^2)^2 - 4g^2 |\mathbf{k}|^2 k_0^4)} \mathcal{J}(k). \tag{D6}$$

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[1] R. C. Myers and M. Pospelov, Phys. Rev. Lett. **90**, 211601 (2003).

[2] R. Gambini and J. Pullin, Phys. Rev. D **59**, 124021 (1999); in *Proceedings of the Second Meeting on CPT and Lorentz Symmetry*, edited by V. A. Kostelecky (World Scientific, Singapore, 2002).

[3] J. Alfaro, H. A. Morales-Técotl, and L. F. Urrutia, Phys. Rev. Lett. **84**, 2318 (2000); Phys. Rev. D **65**, 103509 (2002).

[4] J. Alfaro, H. A. Morales-Tecotl, and L. F. Urrutia, Phys. Rev. D **66**, 124006 (2002).

[5] J. R. Ellis, N. E. Mavromatos, and D. V. Nanopoulos, Phys. Rev. D **61**, 027503 (1999); J. R. Ellis, K. Farakos, N. E. Mavromatos, V. A. Mitsou, and D. V. Nanopoulos, Astrophys. J. **535**, 139 (2000); J. R. Ellis, N. E. Mavromatos, and D. V. Nanopoulos, Phys. Rev. D **62**, 084019 (2000); J. R. Ellis, N. E. Mavromatos, D. V. Nanopoulos, and G. Volkov, Gen. Relativ. Gravit. **32**, 1777 (2000); J. R. Ellis, E. Gravanis, N. E. Mavromatos, and D. V. Nanopoulos, arXiv:gr-qc/0209108; J. Ellis, N. E. Mavromatos, and D. V. Nanopoulos, Phys. Lett. B **665**, 412 (2008).

[6] P. A. Bolokhov and M. Pospelov, Phys. Rev. D **77**, 025022 (2008).

[7] D. Colladay and V. A. Kostelecký, Phys. Rev. D **55**, 6760 (1997); **58**, 116002 (1998).

[8] R. Montemayor and L. F. Urrutia, Phys. Rev. D **72**, 045018 (2005); Phys. Lett. B **606**, 86 (2005).

[9] P. M. Grignino and H. Vucetich, Phys. Lett. B **651**, 313 (2007).

[10] V. A. Kostelecký and R. Potting, Phys. Rev. D **51**, 3923 (1995); A. A. Andrianov and R. Soldati, Phys. Lett. B **435**, 449 (1998); R. Jackiw and V. A. Kostelecký, Phys. Rev. Lett. **82**, 3572 (1999); M. Pérez-Victoria, Phys. Rev. Lett. **83**, 2518 (1999); J. M. Chung, Phys. Lett. B **461**, 138 (1999); W. F. Chen and G. Kunstatter, Phys. Rev. D **62**, 105029 (2000); V. A. Kostelecký, C. D. Lane, and A. G. M. Pickering, Phys. Rev. D 65, 056006 (2002); A. A. Andrianov, P. Giacconi, and R. Soldati J. High Energy Phys. 02 (2002) 030; B. Altschul, Phys. Rev. D **70**, 101701 (R) (2004); J. Alfaro, A. A. Andrianov, M. Cambiaso, P. Giacconi, and R. Soldati, Phys. Lett. B **639**, 586 (2006).

[11] J. Collins, A. Pérez, D. Sudarsky, L. F. Urrutia, and H. Vucetich, Phys. Rev. Lett. **93**, 191301 (2004).

[12] For a review, see for example, D. Mattingly, Living Rev. Rel. **8**, 5 (2005); T. Jacobson, S. Liberati, and D. Mattingly, Ann. Phys. (N.Y.) **321**, 150 (2006); G. Amelino-Camelia, arXiv:gr-qc/0806.0339.

[13] X. Jaen, J. Llosa, and A. Molina, Phys. Rev. D **34**, 2302 (1986).

[14] J. Z. Simon, Phys. Rev. D **41**, 3720 (1990).

[15] D. A. Eliezer and R. P. Woodard, Nucl. Phys. **B325**, 389 (1989).

[16] T.-C. Cheng, P.-M. Ho, and M.-C. Yeh, Nucl. Phys. **B625**, 151 (2002); Phys. Rev. D **66**, 085015 (2002).

[17] C. M. Reyes, L. F. Urrutia, and J. D. Vergara, arXiv:hep-ph/0712.3489; C. M. Reyes, L. F. Urrutia, and J. D. Vergara, AIP Conf. Proc. **977**, 214 (2008).

[18] V. A. Kostelecky and R. Lehnert, Phys. Rev. D **63**, 065008 (2001).

[19] See, for example, S. Weinberg, *The Quantum Theory of Fields I* (Cambridge University Press, Cambridge, 1995), p. 350.

[20] J. Alfaro, Phys. Rev. Lett. **94**, 221302 (2005); Phys. Rev. D **72**, 024027 (2005).

[21] L. Maccione, S. Liberati, A. Celotti, and J. G. Kirk, J. Cosmol. Astropart. Phys. 10 (2007) 013.

[22] F. R. Klinkhamer, JETP Lett. **86**, 73 (2007).

[23] C. M. Reyes, L. F. Urrutia, and J. D. Vergara, arXiv:hep-ph/0810.4346.

[24] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Perseus Books, Cambridge, Massachusetts, 1995).