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Note on dual superconformal symmetry of the $\mathcal{N} = 4$ super Yang-Mills S matrix

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We present a supersymmetric recursion relation for tree-level scattering amplitudes in $\mathcal{N} = 4$ super Yang-Mills. Using this recursion relation, we prove that the tree-level S matrix of the maximally supersymmetric theory is covariant under dual superconformal transformations. We further analyze the consequences that the transformation properties of the trees under this symmetry have on those of the loops. In particular, we show that the coefficients of the expansion of generic one-loop amplitudes in a basis of pseudoconformally invariant scalar box functions transform covariantly under dual superconformal symmetry, and in exactly the same way as the corresponding tree-level amplitudes.

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I. INTRODUCTION AND BACKGROUND

In an interesting paper [1], Drummond, Henn, Korchemsky, and Sokatchev (DHKS) have proposed that scattering amplitudes in planar $\mathcal{N} = 4$ super Yang-Mills (SYM) theory have a novel superconformal symmetry, termed dual in order to distinguish it from the ordinary superconformal symmetry.

This symmetry has also been explained very recently from the string theory standpoint [2,3] using a T-duality of the superstring theory on $AdS_5 \times S^5$ which involves a bosonic T-duality [4], accompanied by a new fermionic T-duality. The combined effect of these T-dualities is to map the original string sigma model into a dual sigma model identical to the original one. The T-duality exchanges the original with the dual superconformal symmetries; furthermore, the strong coupling calculation of the amplitudes in the dual sigma model turns out to be technically identical to that of a Wilson loop with a special closed contour, constructed by gluing together the momenta of scattered particles following the order of the insertions of the string vertex operators [4]. Surprisingly, calculations of the same Wilson loops in $\mathcal{N} = 4$ SYM at weak coupling at one [5,6] and two loops [7–10] are in perfect agreement with the maximally helicity violating (MHV) scattering amplitudes of the $\mathcal{N} = 4$ theory calculated in [11–14]. See [15] for a recent review on the duality between scattering amplitudes and Wilson loops.

According to the proposal put forward in [1], all tree-level superamplitudes are covariant under dual superconformal symmetry, and their transformations should be precisely the same as those of the supersymmetric expression introduced by Nair [16] which generalizes the usual MHV amplitudes.[1] It is one of the goals of this paper to prove this statement, i.e. to show that all tree-level superamplitudes of the $\mathcal{N} = 4$ theory transform covariantly under this symmetry, and in exactly the same way as the MHV superamplitude.

In order to achieve this goal, we look for a method to compute amplitudes which respects superconformal covariance at the diagrammatic level. We claim that one such method is given by an appropriate supersymmetric extension of the Britto, Cachazo, and Feng (BCF) recursion relation [17,18], which we will write down explicitly.[2]

The original motivation for this claim comes from the explicit inspection of the recursive diagrams for the next-to-MHV (NMHV) split-helicity gluonic amplitudes[3] calculated in [17,21]. As was observed in [1], all gluonic split-helicity amplitudes are covariant. Furthermore, one can easily verify that this covariance is realized separately in each recursive diagram, as a direct inspection of the derivations of [17,21] shows. Note, however, that non-split-helicity amplitudes do not transform covariantly [1] and they have to be packaged together with the split-helicity amplitudes into superamplitudes, which according to [1] should transform covariantly in general. So far this claim has been verified for the case of MHV and NMHV amplitudes. To extend this observation to a full proof of dual superconformal covariance of the tree-level S matrix of the $\mathcal{N} = 4$ theory, we will first write down an appropriate

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[1] A superamplitude can be thought of as a generating function that combines all tree amplitudes with a fixed number of external lines and fixed total helicity into one supersymmetric quantity. More details on this formalism are presented later in this section and in Sec. II.

[2] An $\mathcal{N} = 4$ supersymmetric recursion relation using the triple shifts of [19] has recently been written down in [20].

[3] Split-helicity amplitudes have all positive helicity gluons and all negative helicity gluons adjacent.

supersymmetric recursion relation satisfied by the superamplitudes in the maximally supersymmetric theory.

In the supersymmetric formalism of [16], to each particle in the $\mathcal{N} = 4$ theory one associates the usual commuting spinors λ_α, $\tilde{\lambda}_{\dot{\alpha}}$ (in terms of which the momentum of the ith particle is $p^i_{\alpha\dot{\alpha}} = \lambda^i_\alpha \tilde{\lambda}^i_{\dot{\alpha}}$), as well as anticommuting variables η^A_i, where $A = 1, \ldots, 4$ is an $SU(4)$ index. The supersymmetric amplitude can then be expanded in powers of the $\mathcal{N} = 4$ superspace coordinates η^i_A for the different particles, and each term of this expansion corresponds to a particular scattering amplitude in $\mathcal{N} = 4$ SYM. A term containing p powers of η^i_A corresponds to a scattering process where the ith particle has helicity $h_i = 1 - p/2$ [22]. Explicitly, the n-point MHV superamplitude is [16]

$$\mathcal{A}_{\mathrm{MHV}}(1, \ldots, n) = i(2\pi)^4 \times \frac{\delta^{(4)}(\sum_{i=1}^n \lambda_i \tilde{\lambda}_i)\delta^{(8)}(\sum_{i=1}^n \eta_i \lambda_i)}{\langle 12\rangle \cdots \langle n1\rangle}, \tag{1.1}$$

where, as usual, $\langle ij\rangle := \epsilon_{\alpha\beta}\lambda^\alpha_i \lambda^\beta_j$.

The dual superconformal symmetry becomes more transparent after introducing appropriate dual coordinates [1]. These turn out to be 't Hooft's region (or T-dual) momenta

$$p_{i,\alpha\dot{\alpha}} = (x_i - x_{i+1})_{\alpha\dot{\alpha}}, \tag{1.2}$$

along with their supersymmetric partners $\theta^{A\alpha}_i$ introduced in [1] as

$$\eta^A_i \lambda^\alpha_i = \theta^{A\alpha}_i - \theta^{A\alpha}_{i+1}. \tag{1.3}$$

It is important to note that these coordinates are appropriate for characterizing planar diagrams only, where one can express the momentum carried by one line as the difference of the momenta of the two regions of the plane separated by the line. The dual momenta also play an important role in the discussion of pseudoconformal properties of integral functions in [23].

The dual momenta x_i, $i = 1, \ldots, n$, are such that the momenta of each particle are null, i.e. $(x_i - x_{i+1})^2 = 0$, and momentum conservation becomes automatic in this formalism. It therefore makes perfect sense to act with inversions on the dual momenta, which transform as[4]

$$x_{i,\alpha\dot{\beta}} \to \frac{x_{i,\beta\dot{\alpha}}}{x_i^2}. \tag{1.4}$$

From this transformation and the on-shell condition of the momenta, which can be written as $(x_i - x_{i+1})_{\dot{\alpha}\beta}\lambda^\beta_i = 0$, one can derive the transformation of λ^α_i under a dual inversion, with the result [1]

$$\lambda^\alpha_i \to \kappa_i (x_i)^{\dot{\alpha}\beta}\lambda_{i\beta}, \tag{1.5}$$

where κ_i is an arbitrary. For the particular choice $\kappa_i = 1/x_i^2$ the transformation becomes [1]

$$\lambda^\alpha_i \to (x_i^{-1})^{\dot{\alpha}\beta}\lambda_{i\beta}. \tag{1.6}$$

Similarly, the differences of fermionic variables θ_i of adjacent particles are constrained to be *on shell*, namely

$$(\theta_i - \theta_{i+1})\lambda_i = 0, \tag{1.7}$$

and the θ_i transform under inversions as [1]

$$\theta^{A\alpha}_i \to (x_i^{-1})^{\dot{\alpha}\beta}\theta^A_{i,\beta}. \tag{1.8}$$

For completeness, we also present the transformation of the variables η^A which can be deduced from the transformation above [1],

$$I[\eta^A_i] = \frac{x_i^2}{x_{i+1}^2}(\eta^A_i - \theta^A_i x_i^{-1}\tilde{\lambda}_i). \tag{1.9}$$

Using these transformations, it is easy to see that the MHV superamplitude (1.1) transforms covariantly under inversions,

$$\mathcal{A}_{\mathrm{MHV}}(1, 2, \ldots, n) \to \mathcal{A}_{\mathrm{MHV}}(1, 2, \ldots, n)\prod_{k=1}^n x_k^2. \tag{1.10}$$

After introducing a supersymmetric version of BCF on-shell recursion relations, we will show that this transformation property (1.10) is maintained for any tree-level superamplitude in $\mathcal{N} = 4$ SYM.

After this short discussion of dual superconformal properties of tree-level amplitudes, we now move on to consider loop amplitudes. There the situation is more subtle due to the appearance of infrared divergences in the scattering amplitudes, which manifest themselves as ultraviolet divergences in the dual Wilson loops, due to the presence of cusps in the contour. Interestingly, it was shown in [7,8] that by performing dual conformal transformations on the lightlike Wilson loops in the $\mathcal{N} = 4$ theory one can derive anomalous Ward identities, which turn out to be consistent with the Bern, Dixon, and Smirnov (BDS) ansatz [24] for the exponentiated form of the n-point MHV scattering amplitude of the $\mathcal{N} = 4$ theory. In the four- and five-point case, the solution to the Ward identity is actually unique up to a finite constant, whereas for $n \geq 6$ particles, there is room for a conformally invariant discrepancy function, compared to the BDS ansatz, which was indeed found to be nonzero in [10,14]. In this paper we focus our attention on the coefficients of the expansion of one-loop amplitudes in $\mathcal{N} = 4$ SYM in terms of integral box functions, and to their transformation properties under dual superconformal

[4]Special conformal transformations are obtained as an inversion followed by a translation, and a further inversion. Combining this with supersymmetry transformations, one generates all the superconformal transformations. Since the dual supersymmetries are either manifest or are related to ordinary special superconformal symmetries [1], which obviously are symmetries of tree-level $\mathcal{N} = 4$ SYM, invariance of the S matrix under the full dual superconformal symmetry requires only showing invariance under dual inversions.

transformations. We will show that these coefficients are covariant under superconformal symmetry and exhibit the same transformation properties as those of the tree-level superamplitudes. The main tool in this analysis is the use of quadruple cuts of [25] which, crucially, can be performed in four dimensions, since all one-loop amplitudes of the maximally supersymmetric theory are four-dimensional cut constructible [26]. This simplifies the analysis considerably, bypassing dimensional regularization. As an added bonus of this analysis, we will obtain an independent proof of the covariance of the tree-level superamplitudes.

The rest of the paper is organised as follows: in Sec. II we introduce a supersymmetric generalization of the BCF recursion relations, present the $\overline{\text{MHV}}$ three-point superamplitude, and discuss the behavior of superamplitudes under large complex deformations (shifts). In Sec. III we give some simple applications of the supersymmetric recursion relations. Readers who are familiar with the formalism may wish to skip this part. In Sec. IV we use the supersymmetric recursion relations developed in Sec. II and III to prove that all tree-level superamplitudes in $\mathcal{N} = 4$ SYM transform uniformly under dual conformal transformations. Finally, in Sec. V we prove that the coefficients that appear in the expansion of generic one-loop superamplitudes in $\mathcal{N} = 4$ SYM in a basis of scalar box functions transform covariantly in exactly the same way as the corresponding tree-level amplitudes.

II. $\mathcal{N} = 4$ SUPERSYMMETRIC RECURSION RELATIONS

In this section we write down a supersymmetric recursion relation using two-particle shifts.[5] These shifts can be nicely formulated using the dual superspace variables introduced in [1]. The recursion relation using conventional two-particle shifts requires the three-point anti-MHV amplitude as well as the MHV amplitude as input. We will thus require a three-point anti-MHV superamplitude and we propose precisely such a superamplitude in the next subsection. We then address the important issue of the large-z behavior of the $\mathcal{N} = 4$ superamplitudes in Sec. II B, where we prove that the superamplitude calculated with the supersymmetric recursion relation agrees with that obtained by standard methods.

In order to set up the formalism, we briefly review the derivation of the BCF recursion relations. The key property entering these recursion relations is factorization on multiparticle poles (or collinear factorization, for MHV amplitudes). To exploit this efficiently, one considers a particular deformation of an amplitude which shifts the spinors of two of the n massless external particles, labeled here as i and j, as [18]

$$\tilde{\lambda}_i \to \hat{\tilde{\lambda}}_i := \tilde{\lambda}_i + z\tilde{\lambda}_j, \qquad \lambda_j \to \hat{\lambda}_j := \lambda_j - z\lambda_i, \quad (2.1)$$

where z is the complex parameter characterizing the deformation. The spinors λ_i and $\tilde{\lambda}_j$ are left unshifted. The deformations (2.1) are chosen in such a way that the corresponding shifted momenta

$$\begin{aligned} \hat{p}_i(z) &:= \lambda_i \hat{\tilde{\lambda}}_i = p_i + z\lambda_i\tilde{\lambda}_j, \\ \hat{p}_j(z) &:= \hat{\lambda}_j \tilde{\lambda}_j = p_j - z\lambda_i\tilde{\lambda}_j, \end{aligned} \quad (2.2)$$

are on shell for all complex z. Furthermore, $p_i(z) + p_j(z) = p_i + p_j$. Hence the quantity $\mathcal{A}(p_1, \ldots, p_i(z), \ldots, p_j(z), \ldots, p_n)$ is a well-defined one complex parameter family of scattering amplitudes, parametrized by z.

One then considers the following contour integral, where the contour $\mathcal{C}$ is the circle at infinity in the complex z plane,

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} dz \frac{\mathcal{A}(z)}{z}. \quad (2.3)$$

The integral in (2.3) vanishes if $\mathcal{A}(z) \to 0$ as $z \to \infty$.[6] It then follows from Cauchy's theorem that we can write the amplitude we wish to calculate, $\mathcal{A}(0)$, as a sum of residues of $\mathcal{A}(z)/z$,

$$\mathcal{A}(0) = - \sum_{\text{poles of } \mathcal{A}(z)/z \text{ excluding } z=0} \text{Res}\left[\frac{\mathcal{A}(z)}{z}\right]. \quad (2.4)$$

At tree level, $\mathcal{A}(z)$ has only simple poles in z. A pole at $z = z_P$ is associated with a shifted momentum $\hat{P} := P(z_P)$ flowing through an internal propagator becoming null. The residue at this pole is then obtained by factorizing the shifted amplitude on this pole. The result is that

$$\mathcal{A} = \sum_P \sum_h \mathcal{A}_L^h(z_P) \frac{i}{P^2} \mathcal{A}_R^{-h}(z_P), \quad (2.5)$$

where the sum is over the possible assignments of the helicity h of the intermediate state, and over all possible P such that precisely one of the shifted momenta, say $\hat{p}_i$, is contained in P.

The left- and right-hand amplitudes $\mathcal{A}_L$ and $\mathcal{A}_R$ are well-defined amplitudes only for $z = z_P$, when $P(z)$ becomes null. We call $\lambda_{\hat{P}}$ and $\tilde{\lambda}_{\hat{P}}$ the spinors associated to the internal, on-shell momentum $\hat{P}$, so that $\hat{P} := \lambda_{\hat{P}}\tilde{\lambda}_{\hat{P}}$. Notice that the intermediate propagator is evaluated with unshifted kinematics.

[5]As mentioned earlier, an $\mathcal{N} = 4$ supersymmetric recursion relation was written down in [20] for NMHV amplitudes using a set of three antiholomorphic shifts suggested by Risager [19]. In that case it can be seen immediately that the two amplitudes appearing in the corresponding recursion relation must have the MHV helicity configuration. Indeed, the corresponding diagrams are the super MHV diagrams considered in Sec. 5 of [27].

[6]We prove this property for a large portion of the superamplitude in Sect. II B and use supersymmetry to argue that this is enough to determine the entire superamplitude.

Since a momentum invariant involving both (or neither) of the shifted legs i and j does not give rise to a pole in z, the shifted legs i and j must always appear on opposite sides of the factorization channel. In order to limit the number of recursive diagrams, it is very convenient to shift adjacent legs. In this case, the sum over P in (2.5) is just a single sum. In the following we will do this, so that the shifted legs will always be i and $j = i + 1$. We will denote the shift in (2.1) with the standard notation $[ii+1\rangle$.

Now for the supersymmetric version of the BCF recursion relation. First, we notice that it is very easy to describe the shifts (2.1) and (2.2) using dual (or region) momenta. One simply defines

$$\hat{p}_i := x_i - \hat{x}_{i+1}, \qquad \hat{p}_{i+1} := \hat{x}_{i+1} - x_{i+2}, \tag{2.6}$$

where we have introduced a shifted region momentum

$$\hat{x}_{i+1} := x_{i+1} - z\lambda_i\tilde{\lambda}_{i+1}. \tag{2.7}$$

Notice that this is the only region momentum that is affected by the shifts.[7] Therefore in the supersymmetric case we expect that θ_{i+1} is shifted but all other θ's remain unshifted. This implies that

$$\theta_i - \theta_{i+2} = \eta_i\lambda_i + \eta_{i+1}\lambda_{i+1}, \tag{2.8}$$

should remain unshifted. This is in complete similarity to the fact that the sum of the shifted momenta is unshifted, $\hat{p}_i + \hat{p}_{i+1} = p_i + p_{i+1}$. Now, in the case of the $[ii+1\rangle$ shift employed here, we have shifted λ_{i+1} according to (2.1) and so we can achieve this by shifting η_i to

$$\hat{\eta}_i = \eta_i + z\eta_{i+1}, \tag{2.9}$$

and leaving η_{i+1} unshifted. This then gives the shifted θ_{i+1}

$$\hat{\theta}_{i+1} := \theta_{i+1} - z\eta_{i+1}\lambda_i. \tag{2.10}$$

The recursion relation builds up tree-level amplitudes recursively from lower point amplitudes. The starting point of this process is the MHV superamplitude (1.1) (in fact just the three-point MHV superamplitude is needed) together with the three-point anti-MHV superamplitude which we present and discuss in the next section.

The supersymmetric recursion relation follows from arguments similar to those which led to (2.5). We have

$$\mathcal{A} = \sum_P \int d^4\eta_{\hat{P}}\mathcal{A}_L(z_P)\frac{i}{P^2}\mathcal{A}_R(z_P), \tag{2.11}$$

where $\eta_{\hat{P}}$ is the anticommuting variable associated to the internal, on-shell leg with momentum $\hat{P}$.

Note that in the case of superamplitudes it does not make sense to assign individual helicities to the external particles, and every superamplitude is characterized by the number of external particles and its total helicity, which is the sum of the helicities of all external particles. In the recursion relation (2.11) we have an important constraint on $\mathcal{A}_L$ and $\mathcal{A}_R$, namely, the total helicity of $\mathcal{A}_L$ plus the total helicity of $\mathcal{A}_R$ must equal the total helicity of the full amplitude $\mathcal{A}$. This condition replaces the sum over internal helicities in the standard BCF recursion (2.5).

A. Supersymmetric anti-MHV three-point amplitudes

In writing down recursion relations, one needs as a starting point the three-point MHV and $\overline{\text{MHV}}$ amplitudes. Whereas the former are given by the usual Nair formula, we also require a supersymmetric expression for the latter. We claim that this is

$$\mathcal{A}_{\overline{\text{MHV}}}(1,2,3) = i(2\pi)^4\frac{\delta^{(4)}(p_1+p_2+p_3)\delta^{(4)}(\eta_1[23]+\eta_2[31]+\eta_3[12])}{[12][23][31]}. \tag{2.12}$$

For example, the gluonic amplitude $\mathcal{A}(1_g^+, 2_g^+, 3_g^-) = [12]^3/([23][31])$ is immediately obtained by extracting the component $\prod_{A=1}^4 \eta_3^A$ of (2.12).

In order to verify that (2.12) is supersymmetric, we multiply it by the sum of the supercharges $\sum_{i=1}^3 Q_{i;\alpha}^A := \sum_{i=1}^3 \eta_i^A\lambda_{i,\alpha}$. Upon acting on the combination of delta functions in (2.12), one has

$$\begin{aligned}\sum_{i=1}^3 Q_{i;\alpha}^A &\to \frac{-\eta_2^A[31]-\eta_3^A[12]}{[23]}\lambda_{1,\alpha} + \eta_2^A\lambda_{2,\alpha} + \eta_3^A\lambda_{3,\alpha} \\ &= \eta_2^A\frac{\lambda_{2,\alpha}[23]+\lambda_{1,\alpha}[13]}{[23]} \\ &\quad + \eta_3^A\frac{\lambda_{3,\alpha}[23]+\lambda_{1,\alpha}[21]}{[23]} = 0,\end{aligned} \tag{2.13}$$

where the last equality follows from momentum conservation $\lambda_1\tilde{\lambda}_1 + \lambda_2\tilde{\lambda}_2 + \lambda_3\tilde{\lambda}_3 = 0$. As discussed in [1], the condition for the amplitude to be invariant under the second set of supersymmetry generators is

$$\bar{Q}_{A\dot{\alpha}}\mathcal{A}_{\overline{\text{MHV}}}(1,2,3) = \sum_{i=1}^3 \tilde{\lambda}_{i\dot{\alpha}}\frac{\partial}{\partial\eta_i^A}\mathcal{A}_{\overline{\text{MHV}}}(1,2,3) = 0. \tag{2.14}$$

If we act with the operator $\bar{Q}_{A\dot{\alpha}}$ on the argument of the fermionic delta function in (2.12), we obtain

$$\begin{aligned}&\bar{Q}(\eta_1[23]+\eta_2[31]+\eta_3[12]) \\ &\quad = \tilde{\lambda}_1[23]+\tilde{\lambda}_2[31]+\tilde{\lambda}_3[12] = 0,\end{aligned} \tag{2.15}$$

thus proving that $\mathcal{A}_{\overline{\text{MHV}}}$ is invariant also under the $\bar{Q}$ supersymmetries.

Next we would like to show explicitly that (2.12) transforms as a three-point amplitude, *i.e.* that

$$\mathcal{A}_{\overline{\text{MHV}}}(1,2,3) \to x_1^2x_2^2x_3^2\mathcal{A}_{\overline{\text{MHV}}}(1,2,3), \tag{2.16}$$

[7]This is true only if adjacent legs are shifted. If i and j are not adjacent, then region momenta $x_{i+1}\ldots x_j$ are all shifted by $-z\lambda_i\tilde{\lambda}_j$.

under a conformal inversion. This is slightly nontrivial due to the absence of the usual eight-dimensional delta function of supermomentum conservation in (2.12).

The proof is very simple. Firstly, we notice that since

$$\frac{1}{[12][23][31]} \to \frac{x_1^2 x_2^2 x_3^2}{[12][23][31]}, \tag{2.17}$$

we have to show that the combination $\delta^{(4)}(p_1 + p_2 + p_3)\delta^{(4)}(\eta_1[23] + \eta_2[31] + \eta_3[12])$ is invariant under inversions.

In order to see this, we recall that

$$\delta^{(4)}(\eta_1[23] + \eta_2[31] + \eta_3[12]) := \prod_{A=1}^{4}(\eta_1^A[23] + \eta_2^A[31] + \eta_3^A[12]). \tag{2.18}$$

Multiplying and dividing by $\lambda_{1,\alpha}$ for a fixed α, one gets

$$(\eta_1^A[23] + \eta_2^A[31] + \eta_3^A[12])\lambda_{1,\alpha} = [23](\theta_1 - \theta_4)_\alpha^A \tag{2.19}$$

(notice that we have broken the cyclicity of the θ variables). Hence we can write

$$\delta^{(4)}(\eta_1[23] + \eta_2[31] + \eta_3[12]) = \left(\frac{[23]}{\lambda_{1,\alpha}}\right)^4 \prod_{A=1}^{4}(\theta_1 - \theta_4)_\alpha^A, \tag{2.20}$$

at fixed (and arbitrary) α. The transformation properties of (2.20) are manifest, using $\lambda_1 \to x_1^{-1}\lambda_1$, $[23] \to [23]/x_1^2$, and $\theta_1 \to x_1^{-1}\theta_1$, $\theta_4 \to x_4^{-1}\theta_4 = x_1^{-1}\theta_4$, where the last step follows since the expression (2.12) contains a $\delta^{(4)}(p_1 + p_2 + p_3) = \delta^{(4)}(x_1 - x_4)$. Therefore

$$\begin{aligned}\left(\frac{[23]}{\lambda_{1,\alpha}}\right)^4 \prod_{A=1}^{4}(\theta_1 - \theta_4)_\alpha^A \to{}& \frac{1}{(x_1^2)^4}\left(\frac{[23]}{(x_1^{-1})^{\dot\alpha\beta}\lambda_{1,\beta}}\right)^4 \\ &\times \prod_{A=1}^{4}(x_1^{-1})^{\dot\alpha\beta}(\theta_1 - \theta_4)_\beta^A \\ ={}& \frac{1}{(x_1^2)^4}\prod_{A=1}^{4}(\eta_1^A[23] + \eta_2^A[31] \\ &+ \eta_3^A[12]),\end{aligned} \tag{2.21}$$

where the last equality follows in a way completely similar to that used to derive (2.20), except that one multiplies and divides by $x_1^{-1}\lambda_1$. Finally, comparing (2.20) and (2.21), we see that

$$\delta^{(4)}(\eta_1[23] + \eta_2[31] + \eta_3[12]) \to \left(\frac{1}{x_1^2}\right)^4 \delta^{(4)}(\eta_1[23] + \eta_2[31] + \eta_3[12]), \tag{2.22}$$

under conformal inversions. Since $\delta^{(4)}(x_1 - x_4) \to (x_1^2)^4\delta^{(4)}(x_1 - x_4)$, it follows that the combination $\delta^{(4)}(p_1 + p_2 + p_3)\delta^{(4)}(\eta_1[23] + \eta_2[31] + \eta_3[12])$ is invariant, and hence the three-point $\overline{\text{MHV}}$ amplitude (2.12) transforms correctly as (2.17) under inversions.

To conclude this section, we notice that an expression for the three-point $\overline{\text{MHV}}$ has been presented in [28] which reads[8]

$$\begin{aligned}\mathcal{A}_{\overline{\text{MHV}}}(1,2,3) = i(2\pi)^4 &\frac{\delta^{(4)}(p_1 + p_2 + p_3)}{[12][23][31]} \\ &\times \int \prod_{i=1}^{3} d^4\bar\eta_i e^{\sum_{i=1}^{3}\bar\eta_{i;A}\eta_i^A} \\ &\times \delta^{(8)}(\bar\eta_1\tilde\lambda_1 + \bar\eta_2\tilde\lambda_2 + \bar\eta_3\tilde\lambda_3).\end{aligned} \tag{2.23}$$

It is very easy to perform the $\bar\eta$ integrations, and check that (2.23) coincides with our form (2.12) of the three-point $\overline{\text{MHV}}$ superamplitude.

In Secs. III and IV we will use (2.12) in specific examples in order to show how the supersymmetric recursions and the dual momentum superspace formalism work in practice.

B. Large-z behavior of the supersymmetric amplitudes $\mathcal{A}(z)$

In the remainder of this section we want to discuss a crucial ingredient in the derivation of the supersymmetric recursion formula (2.11). The argument leading to (2.5) and its supersymmetric version (2.11) requires that the z-shifted amplitude vanishes as[9] $z \to \infty$. In the case of component gluon amplitudes, this issue was addressed in [18] using MHV diagrams, as well as Feynman diagrams. There, it was shown that when the two gluons associated with the shifted momenta (recall we are using $[i\, i+1\rangle$ shifts) have positive helicity, the amplitude vanishes as $z \to \infty$.

When translated to the supersymmetric case, this argument implies that the z-shifted superamplitude $\mathcal{A}(\eta_i = \eta_{i+1} = 0; z) \to 0$ as $z \to \infty$. The Britto, Cachazo, Feng, and Witten argument then states that the recursion relation is valid for $\eta_i = \eta_{i+1} = 0$. In other words, defining the function

$$f := \mathcal{A}_{\text{recursion}} - \mathcal{A}, \tag{2.24}$$

where by $\mathcal{A}_{\text{recursion}}$ we denote the result of performing the calculation using the supersymmetric recursion formula (2.11), and $\mathcal{A}$ is the correct superamplitude, we have that the function f vanishes whenever $\eta_i = \eta_{i+1} = 0$.

Here, instead of showing directly that the complete superamplitude vanishes at large z, we argue directly, using supersymmetry, that the recursion relation does give the correct full superamplitude, given that we know they agree

[8]We thank Johannes Henn for bringing this to our attention.

[9]The large-z behavior of amplitudes in $\mathcal{N} = 4$ was also addressed in [29] and in [20], and in the very recent paper [30].

for $\eta_i = \eta_{i+1} = 0$. In order to do this, we make use of $\bar{Q}$ supersymmetry (where $\bar{Q}_{A\dot{\alpha}} := \sum_{l=1}^{n} \tilde{\lambda}_{l\dot{\alpha}}\partial/\partial\eta_l^A$), which constrains the form of both the amplitude $\mathcal{A}$ and the result of the recursion relation $\mathcal{A}_{\text{recursion}}$.[10] Hence the difference function f is $\bar{Q}$ supersymmetric,

$$\bar{Q}_{A\dot{\alpha}} f = 0. \tag{2.25}$$

We also notice that $\bar{Q}$ supersymmetry has been efficiently used in [30] to show that superamplitudes in $\mathcal{N} = 4$ SYM ($\mathcal{N} = 8$ supergravity) fall of as $1/z$ $(1/z^2)$ as $z \to \infty$.

In order to exploit the consequences of $\bar{Q}$ supersymmetry, we evaluate (2.25) at $\eta_i = \eta_{i+1} = 0$, and use the fact that f vanishes when $\eta_i = \eta_{i+1} = 0$, to get

$$\left(\tilde{\lambda}_{i\dot{\alpha}} \frac{\partial}{\partial \eta_i^A} + \tilde{\lambda}_{i+1\dot{\alpha}} \frac{\partial}{\partial \eta_{i+1}^A}\right) f \bigg|_{\eta_i=\eta_{i+1}=0} = 0. \tag{2.26}$$

For each A, (2.26) gives two equations which imply that $\partial f/\partial\eta_i = \partial f/\partial\eta_{i+1} = 0$ when $\eta_i = \eta_{i+1} = 0$. Since $\bar{Q}$ commutes with all $\partial/\partial\eta_l$ derivatives, we can repeat the above argument for $\partial f/\partial\eta_i$ and $\partial f/\partial\eta_{i+1}$ to show that all second derivatives of f with respect to η_i, η_{i+1} also vanish when $\eta_i = \eta_{i+1} = 0$. Continued repetition of this argument shows that f, and all its partial derivatives with respect to η_i and η_{i+1}, vanish when $\eta_i = \eta_{i+1} = 0$, and hence f must vanish everywhere.

We conclude that the recursion formula agrees with the superamplitude for all η. Several, nontrivial checks of this statement can be found in the next section.

III. EXAMPLES

In this section we present some simple applications of the supersymmetric recursion relation.

A. First example, supersymmetric MHV amplitudes

The first example is the case of the MHV amplitude. Here we describe in detail the four-point case, but the generalization to higher numbers of points is straightforward as explained below.

We choose a $[12\rangle$ shift, i.e.

$$\hat{\tilde{\lambda}}_1 = \tilde{\lambda}_1 + z\tilde{\lambda}_2, \qquad \hat{\lambda}_2 = \lambda_2 - z\lambda_1. \tag{3.1}$$

Correspondingly,

$$\hat{p}_1 = x_1 - \hat{x}_2, \qquad \hat{\eta}_1\lambda_1 = \theta_1 - \hat{\theta}_2, \tag{3.2}$$

where

$$\hat{x}_2 = x_2 - z\lambda_1\tilde{\lambda}_2, \qquad \hat{\theta}_2 = \theta_2 - z\eta_2\lambda_1. \tag{3.3}$$

Notice that $\hat{\eta}_1 = \eta_1 + z\eta_2$. Also,

$$\eta_2\hat{\lambda}_2 = \hat{\theta}_2 - \theta_3. \tag{3.4}$$

We begin by considering the very simple four-point case. The two amplitudes on the left and on the right must be MHV and $\overline{\text{MHV}}$. Choosing a $[12\rangle$ shift selects the left-hand amplitude to be MHV, and the right-hand amplitude to be $\overline{\text{MHV}}$,

$$\mathcal{A}_L = \frac{\delta^{(4)}(\hat{1} + 4 + \hat{P})\delta^{(8)}(\hat{\eta}_1\lambda_1 + \eta_4\lambda_4 + \eta_{\hat{P}}\lambda_{\hat{P}})}{\langle 1\hat{P}\rangle\langle\hat{P}4\rangle\langle 41\rangle},$$
$$\mathcal{A}_R = \frac{\delta^{(4)}(\hat{2} + 3 - \hat{P})\delta^{(4)}(\eta_{\hat{P}}[23] + \eta_2[3\hat{P}] + \eta_3[\hat{P}2])}{[\hat{P}2][23][3\hat{P}]}. \tag{3.5}$$

Here we have used the n-point MHV superamplitude (1.1), and the expression for the three-point $\overline{\text{MHV}}$ amplitude in (2.12).

Now we make use of the identity

$$\delta^{(8)}(\hat{\eta}_1\lambda_1 + \eta_4\lambda_4 + \eta_{\hat{P}}\lambda_{\hat{P}})\delta^{(4)}(\eta_{\hat{P}}[23] + \eta_2[3\hat{P}] + \eta_3[\hat{P}2])$$
$$= \delta^{(8)}\Big(\sum_{i\in L,R} \hat{\eta}_i\hat{\lambda}_i\Big)\delta^{(4)}(\eta_{\hat{P}}[23] + \eta_2[3\hat{P}] + \eta_3[\hat{P}2]). \tag{3.6}$$

The second line follows from inserting the solution for $\eta_{\hat{P}}$ of the second δ function into the first δ function and using momentum conservation at the second vertex of the diagram in Fig. 1. Furthermore, with the help of the identities $\sum_i\hat{\eta}_i\hat{\lambda}_i = \sum_i\eta_i\lambda_i$ and $\sum_i\hat{p}_i = \sum_i p_i$, the amplitude can be

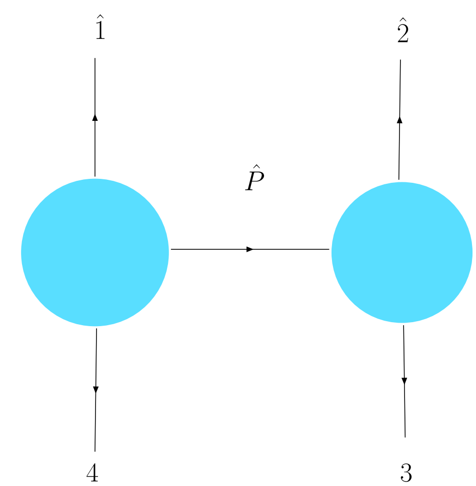

FIG. 1 (color online). Recursive diagram for the MHV four-point amplitude. Given the $[12\rangle$ shift we have chosen, the amplitude on the left must be MHV, and that on the right $\overline{\text{MHV}}$.

[10] That the recursion relation maintains the $\bar{Q}$ supersymmetry can be straightforwardly checked. Applying $\bar{Q}$ on a generic recursive diagram entering (2.11) produces two terms, one where $\bar{Q}$ acts on $\mathcal{A}_L$ and one where $\bar{Q}$ acts on $\mathcal{A}_R$. Noting that the z shift leaves the expression of $\bar{Q}$ unaffected, and because of the invariance of $\mathcal{A}_L$ and $\mathcal{A}_R$ under $\bar{Q}$ supersymmetry, these two terms combine into a contribution proportional to $\tilde{\lambda}_{\hat{P}}\int d^4\eta_{\hat{P}}\partial/\partial\eta_{\hat{P}}(\mathcal{A}_L\mathcal{A}_R)$. This is a total derivative, and hence it vanishes. Therefore each recursive diagram (and hence the recursion relation) maintains the $\bar{Q}$ supersymmetry. The invariance under the Q supersymmetry is manifest because of the presence of an overall delta function of supermomentum conservation.

written as

$$\mathcal{A}(1,2,3,4) = i\delta^{(4)}(\sum_{i\in L,R} p_i)\delta^{(8)}(\sum_{i\in L,R}\eta_i\lambda_i)A(1,2,3,4), \tag{3.7}$$

where

$$A = \frac{1}{P_{23}^2}\frac{1}{\langle 41\rangle[23]\langle 1\hat{P}\rangle\langle\hat{P}4\rangle[\hat{P}2][3\hat{P}]}\int d^4\eta_{\hat{P}}\delta^{(4)}(\eta_{\hat{P}}[23] + \eta_2[3\hat{P}] + \eta_3[\hat{P}2]). \tag{3.8}$$

Completely standard manipulations lead to

$$\langle 1\hat{P}\rangle\langle\hat{P}4\rangle[\hat{P}2][3\hat{P}] = \langle 12\rangle\langle 34\rangle[23]^2, \tag{3.9}$$

hence

$$A(1,2,3,4) = \frac{1}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}. \tag{3.10}$$

Hence we reproduce the expected supersymmetric MHV superamplitude. Finally, we notice that the recursion relation for an n-point MHV superamplitude is a simple generalization of that presented above. The only difference is that the amplitude on the left-hand side of Fig. 1 will be an $(n-2)$-point MHV superamplitude. The algebra is identical to that of the four-point example discussed above and leads to the expected result (1.1).

Before moving on to consider five-point amplitudes, we would like to make a comment on the large-z behavior of the amplitude. On general grounds, it is known that a two-particle shift where the holomorphic spinor associated to a negative helicity gluon, and the antiholomorphic spinor of a positive helicity gluon are shifted, leads in general to a bad large-z behavior of the shifted amplitude [18], i.e. the shifted amplitude $A(z)$ does not vanish as $z \to \infty$. For example, performing such shifts in the gluonic Parke-Taylor formula may lead to a $\mathcal{O}(z^2)$ growth at large z. The interesting fact we wish to point out is that the supersymmetric recursion relation for the MHV superamplitude discussed here is blind to such bad shifts, as the helicities of the particles in the two superamplitudes entering the recursion relations are not specified, and the recursion relation produces the correct result. Note that this is a general property of the $\mathcal{N}=4$ supersymmetric recursion relations. *A priori* this might sound like a contradiction since some of the component amplitudes, which enter the superamplitude as coefficients in the expansion in powers of η's, have bad large-z behavior. However, one has to remember that under a shift not only the λ and $\tilde{\lambda}$ variables are shifted but also the η's [see (2.9)], and that the coefficients of the correct η expansion of the superamplitude are actually certain linear combinations of component amplitudes which do have good large-z behavior.

B. Second example, five-point $\overline{\text{MHV}}$ amplitudes

We continue using the same shifts as in (3.1). The difference with the previous case is that now the two amplitudes on the left- and right-hand side of the propagator will both be MHV superamplitudes.

In this case, the two amplitudes are

$$\mathcal{A}_L = \frac{\delta^{(4)}(\hat{1}+5+\hat{P})\delta^{(8)}(\hat{\eta}_1\lambda_1 + \eta_5\lambda_5 + \eta_{\hat{P}}\lambda_{\hat{P}})}{\langle 1\hat{P}\rangle\langle\hat{P}5\rangle\langle 51\rangle},$$
$$\mathcal{A}_R = \frac{\delta^{(4)}(\hat{2}+3+4-\hat{P})\delta^{(8)}(-\eta_{\hat{P}}\lambda_{\hat{P}} + \eta_2\hat{\lambda}_2 + \eta_3\lambda_3 + \eta_4\lambda_4)}{\langle\hat{P}\,\hat{2}\rangle\langle\hat{2}3\rangle\langle 34\rangle\langle 4\hat{P}\rangle}. \tag{3.11}$$

As usual, the product of two fermionic delta functions in $\mathcal{A}_L$ and $\mathcal{A}_R$ generates a delta function which imposes conservation of the supermomentum $\delta^{(8)}(\sum_i\eta_i\lambda_i)$.

In order to simplify the expression of the amplitude it proves convenient to use the identity

$$\langle 34\rangle\langle\hat{P}2\rangle\langle\hat{2}3\rangle\langle 4\hat{P}\rangle = \frac{[4\hat{P}][\hat{P}2][23][34]}{[34]^4}\langle 2\hat{P}\rangle^4, \tag{3.12}$$

which is a consequence of momentum conservation. One further notices that $\langle 1|\hat{P}|4] = \langle 15\rangle[54]$, $\langle 5|\hat{P}|2] = \langle 51\rangle[12]$ so that

$$\frac{1}{P_{15}^2}\frac{1}{\langle 1\hat{P}\rangle\langle\hat{P}5\rangle\langle 51\rangle\langle\hat{P}\,\hat{2}\rangle\langle\hat{2}3\rangle\langle 34\rangle\langle 4\hat{P}\rangle} = \frac{1}{\prod_{i=1}^5[ii+1]}\frac{[34]^4}{\langle 15\rangle^4\langle\hat{2}\,\hat{P}\rangle^4}. \tag{3.13}$$

It is then easy to reproduce known component amplitudes from the recursive diagram in Fig. 2. For practical evaluation purposes, it is also convenient to use

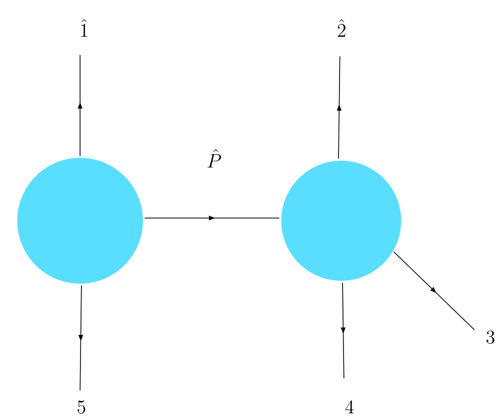

FIG. 2 (color online). Recursive diagram for the five-point $\overline{\text{MHV}}$ amplitude.

$$\delta^{(8)}\Big(\sum_i \eta_i^A \lambda_{i,\alpha}\Big) = \frac{1}{16}\prod_{A=1}^{4}\sum_{i,j}\eta_i^A\eta_j^A\langle ij\rangle, \tag{3.14}$$

in order to extract the relevant contribution from the fermionic delta functions

$$\int d^4\eta_{\hat{P}}\delta^{(8)}(\hat{\eta}_1\lambda_1 + \eta_5\lambda_5 + \eta_{\hat{P}}\lambda_{\hat{P}}) \times \delta^{(8)}(-\eta_{\hat{P}}\lambda_{\hat{P}} + \eta_2\hat{\lambda}_2 + \eta_3\lambda_3 + \eta_4\lambda_4). \tag{3.15}$$

A few examples are in order.

For the split-helicity gluonic amplitude, one picks from (3.15) the contribution proportional to $(\eta_1)^4(\eta_2)^4$, with the result

$$\mathcal{A}(5_g^-, 1_g^-, 2_g^-, 3_g^+, 4_g^+) = i\frac{[34]^3}{[23][45][51][12]}. \tag{3.16}$$

For the gluonic amplitude with helicities $(5^-1^-2^+3^-4^+)$ one picks from (3.15) the coefficient of $(\eta_5)^4(\eta_1)^4$. Further using that $\langle 3\hat{P}\rangle^4/\langle\hat{2}\,\hat{P}\rangle^4 = [24]^4/[34]^4$, one quickly arrives at

$$\mathcal{A}(5_g^-, 1_g^-, 2_g^+, 3_g^-, 4_g^+) = i\frac{[24]^4}{[23][34][45][51][12]}. \tag{3.17}$$

One could further proceed and consider amplitudes involving fermions and scalars. Consider, for example, the amplitude $(5_f^-, 1_g^-, 2_g^-, 3_f^+, 4_g^+)$. Proceeding as before, the fermionic integrations produce a factor of $\langle 51\rangle^3\langle\hat{P}1\rangle \times \langle\hat{P}\,\hat{2}\rangle^3\langle\hat{2}3\rangle$. Standard manipulations lead to $\langle 1\hat{P}\rangle \times \langle\hat{2}\,\hat{P}\rangle = \langle 15\rangle[52]/(\langle 34\rangle[34]$, $\langle\hat{2}3\rangle = -[45]\langle 34\rangle/[25]$, and one quickly finds that

$$\mathcal{A}(5_f^-, 1_g^-, 2_g^-, 3_f^+, 4_g^+) = i\frac{[34]^3[45]}{[12][23][34][45][51]}, \tag{3.18}$$

in agreement with results of [22].

A further check is the derivation of a four-fermion amplitude $(5_{f_1}^-, 1_{f_2}^-, 2_{f_1}^+, 3_g^-, 4_{f_2}^+)$, where f_1 and f_2 denote fermions belonging to two different $\mathcal{N} = 1$ supermultiplets. Similar manipulations lead to the result

$$\mathcal{A}(5_{f_1}^-, 1_{f_2}^-, 2_{f_1}^+, 3_g^-, 4_{f_2}^+) = i\frac{[45][12][24]^2}{[12][23][34][45][51]}, \tag{3.19}$$

in agreement with results of [29].

IV. PROOF OF TREE-LEVEL COVARIANCE

In this section we wish to use a supersymmetric generalization of the BCF recursion relations [17,18] to show that the tree-level S matrix of $\mathcal{N} = 4$ SYM is covariant under dual superconformal transformations. Here we will focus on the dual inversions of the dual superconformal group. As explained earlier, it is most convenient to combine all amplitudes of a fixed total helicity and fixed number of external lines with the help of the dual superspace into one superamplitude, which is a natural generalization of Nair's MHV superamplitude (1.1). It is this superamplitude that we expect to transform uniformly, while the component amplitudes usually do not have simple transformation properties under inversions except for the split-helicity amplitudes [1].

Now assuming that all superamplitudes with up to n external legs transform covariantly, we wish to use superspace generalizations of BCF recursion relations to show that all superamplitudes with $n + 1$ legs also transform covariantly, and hence, by induction, that all superamplitudes with arbitrary numbers of external legs transform covariantly. We will achieve this by showing in the following that actually each diagram in the recursion relation has the correct covariant transformation behavior, inherited from the transformation properties of the two subamplitudes entering the recursion diagram, the propagator, and the bosonic and fermionic delta functions.

While the transformations of the region momenta x_i's are unique, there is a normalization ambiguity in the definition of inversions of the spinor variables λ_i's. In [1] the transformations of the spinors under a conformal inversion were chosen to be $\lambda_i^\alpha \to (x_i^{-1})_{\dot{\alpha}\beta}\lambda_i^\beta$. In the proof of superconformal covariance of tree-level amplitudes constructed using BCF recursion relations, it is however more useful to keep the transformation of λ_i more general and fix the normalizations later. We therefore consider the transformation

$$\lambda_i^\alpha \to \frac{x_i^{\dot{\alpha}\beta}\lambda_{i,\beta}}{\kappa_i}, \tag{4.1}$$

and keep κ_i arbitrary and local (i.e. they can have different values, e.g. x_i^2, x_{i+1}^2, or $\sqrt{x_i^2 x_{i+1}^2}$ for different points i) although as we will see, we will be forced to fix the factors κ_i and κ_{i+1} of the shifted momenta. In order to complete the proof, we consider the transformation properties of amplitudes under this more general transformation.

By considering the explicit expression for the tree-level MHV superamplitude (1.1), one sees that it transforms as

$$\mathcal{A}_{\mathrm{MHV}}(1, 2, \ldots, n) \to \mathcal{A}_{\mathrm{MHV}}(1, 2, \ldots, n)\prod_{k=1}^{n}\frac{\kappa_k^2}{x_k^2}. \tag{4.2}$$

Now we wish to show recursively that in fact all tree-level superamplitudes transform in this way under dual conformal inversions.

Consider building a superamplitude recursively from two superamplitudes with fewer legs, both of which transform like the MHV amplitude above under (4.1) (see Fig. 3),

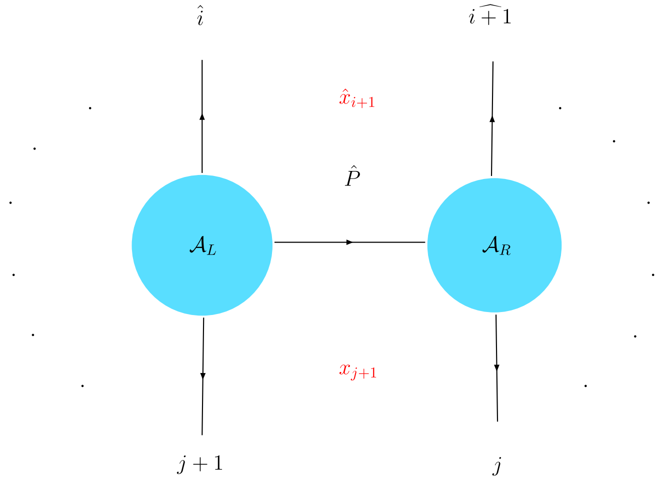

FIG. 3 (color online). Generic recursion diagram used in the proof of covariance.

$$\mathcal{A}_L(j+1, j+2, \ldots, \hat{i}, \hat{P}) \rightarrow \frac{\kappa_{j+1}^2 \ldots \kappa_{i-1}^2 \hat{\kappa}_i^2 \hat{\kappa}_P^2}{x_{j+1}^2 \ldots x_i^2 \hat{x}_{i+1}^2} \times \mathcal{A}_L(j+1, j+2, \ldots, \hat{i}, \hat{P}), \tag{4.3}$$

$$\mathcal{A}_R(\widehat{i+1}, i+2, \ldots, j, -\hat{P}) \rightarrow \frac{\hat{\kappa}_{i+1}^2 \kappa_{i+2}^2 \ldots \kappa_j^2 \hat{\kappa}_P^2}{\hat{x}_{i+1}^2 x_{i+2}^2 \ldots x_{j+1}^2} \times \mathcal{A}_R(\widehat{i+1}, i+2, \ldots, j, -\hat{P}). \tag{4.4}$$

In the recursion we will make use of the shift denoted by $[i i+1\rangle$, i.e.

$$\hat{\tilde{\lambda}}_i = \tilde{\lambda}_i + z\tilde{\lambda}_{i+1}, \qquad \hat{\lambda}_{i+1} = \lambda_{i+1} - z\lambda_i, \tag{4.5}$$

with all other spinors unchanged.

A couple of comments are in order before we proceed. First of all, consider the spinor variables $\lambda_{\hat{P}}$ and $\tilde{\lambda}_{\hat{P}}$ of the internal on-shell leg $\hat{P}$. If we use the DHKS transformation of λ_i and do not introduce κ_i, then from the point of view of $\mathcal{A}_L$ the spinor $\lambda_{\hat{P}}$ would transform under inversions into $\hat{x}_{i+1}\lambda_{\hat{P}}/(\hat{x}_{i+1})^2$, and from the point of view $\mathcal{A}_R$ into $x_{j+1}\lambda_{\hat{P}}/(x_{j+1})^2$, which are not compatible. This is why we have introduced an arbitrary factor into the λ transformations. For the spinor $\lambda_{\hat{P}}$ we have

$$\lambda_{\hat{P}}^{\alpha} \rightarrow \frac{\hat{x}_{i+1}^{\dot{\alpha}\beta}\lambda_{\hat{P},\beta}}{\hat{\kappa}_P} = \frac{x_{j+1}^{\dot{\alpha}\beta}\lambda_{\hat{P},\beta}}{\hat{\kappa}_P}. \tag{4.6}$$

Secondly, the two superamplitudes $\mathcal{A}_L$ and $\mathcal{A}_R$ above depend on unshifted momenta but also on the shifted momenta $\hat{p}_i$, $\hat{p}_{i+1}$, and $\hat{P}$. By assumption these amplitudes are covariant under inversions of the corresponding sets of shifted and unshifted momenta using the assignments of region momenta in Fig. 3. On the other hand, every recursive diagram depends only on unhatted quantities due to the fact that hatted quantities depend via z only on unhatted quantities. To be more specific, z for the recursive diagram given above has to be set to the solution of the equation

$$(P - z\lambda_i\tilde{\lambda}_{i+1})^2 = 0, \tag{4.7}$$

which is $z_P = P^2/[i+1|P|i\rangle$, where $P = P_R := \sum_{l=i+1}^{j} p_l$. It can easily be checked that the two seemingly different definitions of the transformations of hatted quantities as defined above and as *inherited* from the unhatted quantities, combined with the appropriate transformation of $z = z_P$, are actually identical. For the purpose of the proof it is more convenient to work with the inversions of hatted quantities as defined above, hence we will use those in what follows, but the reader should keep in mind that this is completely equivalent to performing all transformations on unhatted quantities.

An important fact to note at this point is that, whereas so far we have kept the κ_i arbitrary, the $[i i+1\rangle$ shift in fact fixes the transformation under inversions of λ_i and λ_{i+1}. To see this, note that $\hat{\lambda}_i = \lambda_i$ and so the transformation $\hat{\lambda}_i^{\alpha} \rightarrow x_i^{\dot{\alpha}\beta}\hat{\lambda}_{i,\beta}/\hat{\kappa}_i$ must be consistent with $\lambda_i^{\alpha} \rightarrow x_i^{\dot{\alpha}\beta}\lambda_{i,\beta}/\kappa_i$ under inversions, requiring $\hat{\kappa}_i = \kappa_i$. A more complicated consistency condition comes from considering the transformation of $\hat{\lambda}_{i+1}^{\alpha}$ and comparing with the transformation of $\lambda_{i+1} - z\lambda_i$. Here the factors κ will in general be functions of the region momenta x and so the shifted factors $\hat{\kappa}$ are simply the same function of the shifted region momenta $\hat{x}$. One solution of these conditions is

$$\kappa_i = x_i^2 \qquad \kappa_{i+1} = x_{i+1}^2, \Rightarrow \hat{\kappa}_i = x_i^2 \qquad \hat{\kappa}_{i+1} = \hat{x}_{i+1}^2, \tag{4.8}$$

which we assume from now on.

Now, in order to use an inductive proof on the number of legs, we consider the contribution to the superamplitude given by the recursive diagram in Fig. 3,

$$\int \frac{d^4P}{P^2} \int d^4\eta_{\hat{P}} \delta^{(4)}(P_L + P)\delta^{(8)}(\hat{\Lambda}_L + \lambda_{\hat{P}}\eta_{\hat{P}}) \times \delta^{(4)}(P_R - P)\delta^{(8)}(\hat{\Lambda}_R - \lambda_{\hat{P}}\eta_{\hat{P}})A_L A_R = \delta^{(4)}(P_L + P_R)\delta^{(8)}(\Lambda_L + \Lambda_R)\frac{1}{P_L^2}\delta^{(4)}(\langle\lambda_{\hat{P}}\hat{\Lambda}_L^A\rangle)A_L A_R, \tag{4.9}$$

where we have defined amplitudes with momentum conservation and supermomentum conservation delta functions removed as $A_{L,R}$,

$$\mathcal{A} = \delta^{(4)}\Big(\sum_k p_k\Big)\delta^{(8)}\Big(\sum_k \eta_k\lambda_k\Big)A. \tag{4.10}$$

We have also introduced the shorthand notation $\Lambda_L := \sum_{l=j+1}^{i} \eta_l\lambda_l$, $\hat{\Lambda}_L := \sum_{l=j+1}^{\hat{i}} \eta_l\lambda_l$, and $P_L := \sum_{l=j+1}^{i} \lambda_l\tilde{\lambda}_l$

as usual. Similarly, we have defined $\Lambda_R := \sum_{l=i+1}^{j} \eta_l \lambda_l = -\Lambda_L$, $\hat{\Lambda}_R := \sum_{l=i+1}^{j} \widehat{\eta_l \lambda_l} = -\hat{\Lambda}_L$, and $P_R := \sum_{l=i+1}^{j} \lambda_l \tilde{\lambda}_l = -P_L$. Notice also that $\hat{\Lambda}_L = \hat{\theta}_{i+1} - \theta_{j+1}$. Finally, we observe that in the last line of (4.9), $\eta_{\hat{P}}$ appearing inside A_L and A_R should be thought of as the solution of the equation $\hat{\Lambda}_L + \lambda_{\hat{P}} \eta_{\hat{P}} = 0$.

Using (4.6) and the standard transformations (1.4) and (1.8) of the x_i and the θ_i under inversions, we find

$$\frac{1}{P^2} = \frac{1}{(x_{i+1} - x_{j+1})^2} \rightarrow x_{i+1}^2 x_{j+1}^2 \frac{1}{P^2}, \tag{4.11}$$

$$\delta^{(4)}(\langle \lambda_{\hat{P}} \hat{\Lambda}_L^A \rangle) \rightarrow \frac{1}{\hat{\kappa}_P^4} \delta^{(4)}(\langle \lambda_{\hat{P}} \hat{\Lambda}_L^A \rangle), \tag{4.12}$$

and, hence, together with (4.3) and (4.4) we infer that the recursive diagram in Fig. 3 transforms with weight

$$\begin{aligned} & x_{i+1}^2 x_{j+1}^2 \frac{1}{\hat{\kappa}_P^4} \frac{\kappa_{j+1}^2 \ldots \kappa_{i-1}^2 \hat{\kappa}_i^2 \hat{\kappa}_P^2}{x_{j+1}^2 \ldots x_i^2 \hat{x}_{i+1}^2} \frac{\hat{\kappa}_{i+1}^2 \kappa_{i+2}^2 \ldots \kappa_j^2 \hat{\kappa}_P^2}{\hat{x}_{i+1}^2 x_{i+2}^2 \ldots x_{j+1}^2} \\ & = \prod_{k=1}^{n} \frac{\kappa_k^2}{x_k^2} \frac{\hat{\kappa}_i^2 \hat{\kappa}_{i+1}^2}{\kappa_i^2 \kappa_{i+1}^2} \frac{(x_{i+1})^4}{(\hat{x}_{i+1})^4} = \prod_{k=1}^{n} \frac{\kappa_k^2}{x_k^2}, \end{aligned} \tag{4.13}$$

as required. The last equality follows directly from the values of κ_i, κ_{i+1}, and $\hat{\kappa}_{i+1}$ given in (4.8).

In the analysis of the covariance properties of a generic tree amplitude using recursion relations, we may encounter diagrams where either $\mathcal{A}_L$ or $\mathcal{A}_R$ is the three-point anti-MHV amplitude given in (2.12). This class of diagrams is somewhat special since (2.12) does not contain the standard supermomentum conservation delta function. However, we have shown in (2.17) that (2.12) transforms in the correct way under dual superconformal symmetry, hence recursive diagrams involving a three-point anti-MHV amplitude are in fact not special from the point of view of the covariance properties. For completeness, we discuss now how a generic diagram in this class transforms under conformal inversions.

Let $\mathcal{A}_R$ then be the three-point anti-MHV amplitude. Then the generic recursive diagram in this class is of the form

$$\begin{aligned} & \int \frac{d^4 P}{P^2} \int d^4 \eta_{\hat{P}} \delta^{(4)}(P_L + P) \delta^{(8)}(\hat{\Lambda}_L + \lambda_{\hat{P}} \eta_{\hat{P}}) \\ & \quad \times \mathcal{A}_L \times \delta^{(4)}(P_R - P) \\ & \quad \times \frac{\delta^{(4)}(\eta_{\hat{P}}[\widehat{i+1} j] + \eta_{i+1}[j - \hat{P}] + \eta_j[-\hat{P}\, \widehat{i+1}])}{[\widehat{i+1} j][j - \hat{P}][-\hat{P}\, \widehat{i+1}]} \\ & = \delta^{(4)}(P_L + P_R) \delta^{(8)}(\Lambda_L + \Lambda_R) \\ & \quad \times \frac{1}{P^2} \mathcal{A}_L \frac{[\widehat{i+1} j]^3}{[j - \hat{P}][-\hat{P}\, \widehat{i+1}]}, \end{aligned} \tag{4.14}$$

where $j = i + 2$ since we are dealing with a three-point amplitude on the right. Now the conjugate spinors transform as

$$\tilde{\lambda}_{k,\dot{\alpha}} \rightarrow -\frac{\kappa_k}{x_k^2 x_{k+1}^2} x_{k,\dot{\beta}\alpha} \tilde{\lambda}_k^{\dot{\beta}} \tag{4.15}$$

under inversions (for consistency with the transformation of $p_k = \lambda_k \tilde{\lambda}_k$), hence the square brackets transform as

$$[k k+1] \rightarrow \frac{\kappa_k \kappa_{k+1}}{x_k^2 x_{k+1}^2 x_{k+2}^2} [k k+1]. \tag{4.16}$$

We then find that the diagram transforms with weight

$$x_{i+1}^2 x_{j+1}^2 \frac{\kappa_{j+1}^2 \ldots \kappa_{i-1}^2 \hat{\kappa}_i^2 \hat{\kappa}_P^2}{x_{j+1}^2 \ldots x_i^2 \hat{x}_{i+1}^2} \frac{\hat{\kappa}_{i+1}^2 \kappa_j^2}{\hat{\kappa}_P^2 \hat{x}_{i+1}^2 x_j^2 x_{j+1}^2} = \prod_{k=1}^{n} \frac{\kappa_k^2}{x_k^2} \tag{4.17}$$

[using (4.8)], precisely as required.

In conclusion, we have found that each recursive diagram with shifts $[i i+1\rangle$ contributing to a generic superamplitude transforms covariantly under dual conformal inversions once we assume that $\mathcal{A}_L$ and $\mathcal{A}_R$ transform as superamplitudes. From this, and from the arbitrariness of the choice of the legs i and $i+1$, we conclude by induction that all tree-level superamplitudes in $\mathcal{N} = 4$ SYM transform covariantly as the MHV amplitudes, i.e. as in (4.2).

Note that in the conventions of [1] we would have to set $\kappa_k = x_k^2$ for all k, and the last line of (4.13) would become just

$$\prod_{k=1}^{n} x_k^2, \tag{4.18}$$

which shows that this recursive diagram and hence the whole amplitude transforms uniformly with weight 1 under inversions.

V. COVARIANCE OF THE COEFFICIENTS OF ONE-LOOP AMPLITUDES

In this section we discuss how generic one-loop amplitudes in $\mathcal{N} = 4$ SYM inherit the transformation properties under dual superconformal symmetry from the tree-level amplitudes. It is a well-known fact that all one-loop amplitudes in $\mathcal{N} = 4$ SYM can be expanded in a basis of integral functions which consists only of so-called one-loop scalar boxes, with coefficients that are rational functions of the kinematic variables [11]. We will show in the following that the coefficients of the expansion[11] of an arbitrary $\mathcal{N} = 4$ SYM superamplitude in terms of box functions are given by conformally covariant functions which transform in the same way as the corresponding tree-level superamplitude.

[11]The precise definition of the basis will be given shortly.

This claim is motivated by the special form of the coefficients in the expansion of the split-helicity gluonic amplitudes at one loop calculated in [31,32]. Inspection of the results of these papers shows that these coefficients are covariant under conformal inversions, as they are made of spinor brackets consisting of strings of spinors always belonging to adjacent legs. Another simple example is provided by the infinite sequence of one-loop MHV superamplitudes in $\mathcal{N}=4$ SYM. This superamplitude was calculated in [11], and rederived in [33] using $\mathcal{N}=4$ supersymmetric MHV diagrams, and is written as a sum of two-mass easy box functions, all with the same coefficient.[12] This coefficient is equal to the tree-level MHV superamplitude, which is of course covariant.

Before we proceed, it is important to make a comment on the basis of integral functions that we expand in. The natural basis to consider in the context of dual conformal symmetry is that given by the so-called scalar box functions F_i, which are (pseudo)conformally invariant [23], and are related to the more standard scalar box integrals I_i by a kinematic prefactor [11]. The external momenta at the four corners of a given box function, K_1, K_2, K_3, and K_4 (see Fig. 4), are in general sums of momenta p_i of external particles of the n-point amplitude under consideration. Alternatively, the momenta $K_{1\ldots4}$ can be expressed in terms of the region momenta $x_{1\ldots4}$ as in Fig. 4, e.g. $K_1=x_{12}$, where $x_{ij}:=x_i-x_j$. Then, up to a numerical constant, the relation between the F's and the I's is[13]

$$I_i=\frac{F_i}{\sqrt{R_i}},$$
$$R_i=(x_{13}^2x_{24}^2)^2-2x_{13}^2x_{24}^2x_{12}^2x_{34}^2-2x_{13}^2x_{24}^2x_{23}^2x_{41}^2+(x_{12}^2x_{34}^2-x_{23}^2x_{41}^2)^2. \tag{5.1}$$

It will be useful for later to quote here the transformation of the kinematic factor $\sqrt{R_i}$ under dual conformal inversions:

$$\sqrt{R_i}\rightarrow\frac{\sqrt{R_i}}{x_1^2x_2^2x_3^2x_4^2}. \tag{5.2}$$

Obviously we can expand the amplitude in either basis. We write (schematically)

$$A_{1\text{-loop}}=\sum\mathcal{B}_iI_i=\sum\tilde{\mathcal{B}}_iF_i. \tag{5.3}$$

We will show that it is the supersymmetric generalization of the coefficients $\tilde{\mathcal{B}}_i=\mathcal{B}_i/\sqrt{R_i}$ that have uniform covariant transformation properties under dual superconformal transformations just as the corresponding tree-level amplitudes, while the $\mathcal{B}_i$ have mixed transformation properties.

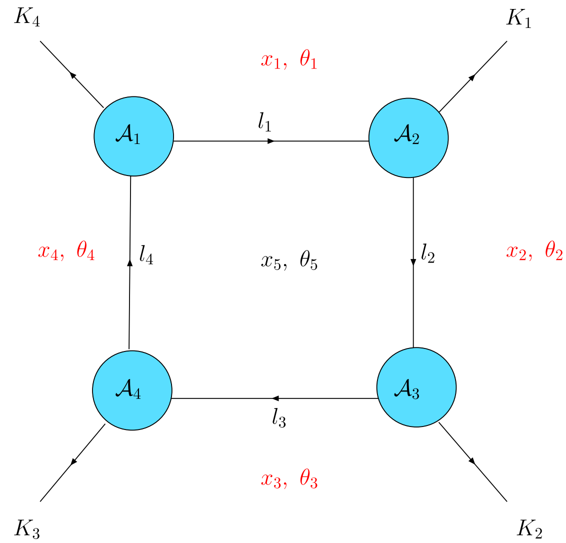

FIG. 4 (color online). Quadruple cut of a one-loop superamplitude in $\mathcal{N}=4$ SYM. The four blobs represent tree-level $\mathcal{N}=4$ superamplitudes. The $K_{1\ldots4}$ correspond to sums of momenta p_i of the external particles.

In order to prove this statement, we now discuss in more detail quadruple cuts of one-loop amplitudes. As mentioned above, all one-loop amplitudes in $\mathcal{N}=4$ SYM are expressed in terms of box functions only [11] and their coefficients can be calculated most efficiently with quadruple cuts [25]. This technique allows one to calculate the coefficients of the box functions one by one, and the problem of finding general one-loop amplitudes in $\mathcal{N}=4$ SYM is reduced to a purely algebraic one, as the coefficients turn out to be given by products of four tree-level amplitudes. Importantly, quadruple cuts freeze the one-loop integration completely and, hence, one can stay in four dimensions, without introducing any regularization.

A generic quadruple cut box is of the form

$$\int d^4l\,\delta^{(+)}(l^2)\delta^{(+)}((l-K_1)^2)\delta^{(+)}((l-K_1-K_2)^2)\times\delta^{(+)}((l+K_4)^2), \tag{5.4}$$

or, reexpressing it in terms of the region momenta in Fig. 4,

$$\int d^4x_5\,\delta^{(+)}(x_{51}^2)\delta^{(+)}(x_{52}^2)\delta^{(+)}(x_{53}^2)\delta^{(+)}(x_{54}^2). \tag{5.5}$$

Under conformal inversions, the delta functions transform in the same way as ordinary propagators, except for the sign of the energy component, which is flipped, so that $\delta^{(+)}((x-y)^2)\rightarrow x^2y^2\delta^{(-)}((x-y)^2)$. Therefore,

[12] An explanation of why these box functions appear all with the same coefficient—equal to 1, if one factors out the tree amplitude—was given in terms of the Wilson loop/MHV amplitudes duality in [6].

[13] In (5.1) we use a collective index i to denote the box function with external momenta $K_{1\ldots4}$, as in Fig. 4.

$$\int d^4x_5\,\delta^{(+)}(x_{51}^2)\delta^{(+)}(x_{52}^2)\delta^{(+)}(x_{53}^2)\delta^{(+)}(x_{54}^2) \to (x_1^2x_2^2x_3^2x_4^2)\int d^4x_5\,\delta^{(-)}(x_{51}^2)\delta^{(-)}(x_{52}^2)\delta^{(-)}(x_{53}^2) \times \delta^{(-)}(x_{54}^2). \tag{5.6}$$

Furthermore, the quadruple cut of the corresponding scalar box function F is invariant under dual conformal inversions.

The coefficient of the scalar box integral I appearing in the expansion of the amplitude is then evaluated as [25]

$$\mathcal{B} = \frac{1}{n_S}\sum_{S,J} n_J(\mathcal{A}_1\mathcal{A}_2\mathcal{A}_3\mathcal{A}_4), \tag{5.7}$$

where n_S is the number of solutions S to the cut condition, and the sum is extended to particles of all spin J in the $\mathcal{N}=4$ theory which can run in the loop. n_J is the number of particles of spin J. $\mathcal{A}_i$, $i=1,\ldots,4$, are the four tree-level amplitudes at the four corners of the quadruple cut, as in Fig. 4.

In order to show in full generality that the coefficients of the one-loop superamplitudes in $\mathcal{N}=4$ SYM are dual superconformal covariant, we have to generalize (5.7) in a supersymmetric way by lifting the amplitudes to superamplitudes, and introducing the appropriate fermionic delta functions which impose supermomentum conservation at the four corners of the diagram in Fig. 4. This procedure will also lift the coefficient $\mathcal{B}$ in (5.7) to an appropriate supercoefficient. Doing this, we get the following expression for the quadruple cut,[14] which implicitly defines the supercoefficient $\mathcal{B}$:

$$\delta^{(4)}\Big(\sum_{i=1}^4 K_i\Big)\delta^{(8)}\Big(\sum_{i=1}^4 \Lambda_i\Big)\mathcal{B} := \delta^{(4)}\Big(\sum_{i=1}^4 K_i\Big)\frac{1}{n_S}\sum_S\int\prod_{i=1}^4 d^4\eta_{l_i}\,\delta^{(8)}(\lambda_{l_2}\eta_{l_2}-\lambda_{l_1}\eta_{l_1}+\theta_{12})\delta^{(8)}(\lambda_{l_3}\eta_{l_3}-\lambda_{l_2}\eta_{l_2} + \theta_{23})\delta^{(8)}(\lambda_{l_4}\eta_{l_4}-\lambda_{l_3}\eta_{l_3}+\theta_{34})\delta^{(8)}(\lambda_{l_1}\eta_{l_1}-\lambda_{l_4}\eta_{l_4}+\theta_{41})A_1A_2A_3A_4, \tag{5.8}$$

where, as previously, A_i are the relevant superamplitudes with the momentum and supermomentum delta functions removed.[15] The cut loop momenta are defined as $l_i := \lambda_{l_i}\tilde{\lambda}_{l_i}$, $i=1,\ldots,4$, and we set $\theta_{ij} := \theta_i - \theta_j$. We have also defined $\Lambda_i := \sum_{i\in K_i}\eta_i\lambda_i$.

Next, we replace one of the fermionic delta functions with an overall supermomentum conservation delta function, and then perform the η_{l_1}, η_{l_4} integrations to get

$$\begin{aligned}
&\delta^{(4)}\Big(\sum_{i=1}^4 K_i\Big)\delta^{(8)}\Big(\sum_{i=1}^4 \Lambda_i\Big)\frac{1}{n_S}\sum_S\int\prod_{i=1}^4 d^4\eta_{l_i}\,\delta^{(8)}(\lambda_{l_2}\eta_{l_2}-\lambda_{l_1}\eta_{l_1}+\theta_{12})\delta^{(8)}(\lambda_{l_3}\eta_{l_3}-\lambda_{l_2}\eta_{l_2}+\theta_{23})\\
&\quad\times\delta^{(8)}(\lambda_{l_4}\eta_{l_4}-\lambda_{l_3}\eta_{l_3}+\theta_{34})A_1A_2A_3A_4\\
&\quad= \delta^{(4)}\Big(\sum_{i=1}^4 K_i\Big)\delta^{(8)}\Big(\sum_{i=1}^4 \Lambda_i\Big)\frac{1}{n_S}\sum_S\int d^4\eta_{l_2}d^4\eta_{l_3}\,\delta^{(4)}(\langle l_1\theta_{15}^A\rangle)\delta^{(4)}(\langle l_4\theta_{45}^A\rangle)\delta^{(8)}(\lambda_{l_3}\eta_{l_3}-\lambda_{l_2}\eta_{l_2}+\theta_{23})A_1A_2A_3A_4\\
&\quad= \delta^{(4)}\Big(\sum_{i=1}^4 K_i\Big)\delta^{(8)}\Big(\sum_{i=1}^4 \Lambda_i\Big)\frac{1}{n_S}\sum_S\delta^{(4)}(\langle l_1\theta_{15}^A\rangle)\delta^{(4)}(\langle l_4\theta_{45}^A\rangle)\langle l_2l_3\rangle^4A_1A_2A_3A_4.
\end{aligned}\tag{5.9}$$

We now consider the transformation of this expression under inversions. As in the proof of tree-level covariance presented earlier, we make use of the more general form of the transformations involving unspecified parameters κ_i [see (4.1)].[16]

Under dual conformal inversions, the various quantities in (5.9) transform as

[14] An equivalent supersymmetric extension of the quadruple cuts has been introduced in [34]. There it was used to calculate explicitly supercoefficients of NMHV one-loop amplitudes and four-mass box coefficients of next-to-next-to-MHV one-loop amplitudes, and, furthermore, it was checked that these supercoefficients are covariant under dual superconformal

[15] In (5.8) we consider the case where each of the four tree superamplitudes provides an eight-dimensional delta function of supermomentum conservation. The case where some of the tree amplitudes are three-point $\overline{\text{MHV}}$ superamplitudes requires a special treatment, similar to that presented in (4.14) in the proof of covariance of the tree-level recursion relation.

[16] Note however that no hatted quantities appear here, unlike the case of the recursion relation in Sec. IV.

$$\delta^{(4)}(\langle l_1\theta_{15}^A\rangle)\to\left(\frac{1}{\kappa_{l_1}}\right)^4\delta^{(4)}(\langle l_1\theta_{15}^A\rangle),\qquad \delta^{(4)}(\langle l_4\theta_{45}^A\rangle)\to\left(\frac{1}{\kappa_{l_4}}\right)^4\delta^{(4)}(\langle l_4\theta_{45}^A\rangle),\qquad \langle l_2 l_3\rangle\to\left(\frac{x_5^2}{\kappa_{l_2}\kappa_{l_3}}\right)\langle l_2 l_3\rangle,$$
$$A_1\to\frac{\kappa_4^2\kappa_{l_1}^2\kappa_{l_4}^2}{x_1^2x_4^2x_5^2}A_1,\qquad A_2\to\frac{\kappa_1^2\kappa_{l_1}^2\kappa_{l_2}^2}{x_1^2x_2^2x_5^2}A_2,\qquad A_3\to\frac{\kappa_2^2\kappa_{l_2}^2\kappa_{l_3}^2}{x_2^2x_3^2x_5^2}A_3,\qquad A_4\to\frac{\kappa_3^2\kappa_{l_3}^2\kappa_{l_4}^2}{x_3^2x_4^2x_5^2}A_4. \tag{5.10}$$

For the sake of brevity, in writing the transformations of $A_{1\ldots4}$ we have included only the dependence on the transformation of the region momenta $x_{1\ldots5}$ because all other region momenta are just spectators in this diagram—any transformation properties with respect to them are directly inherited from the superamplitudes entering the quadruple cut.

Inserting the transformations (5.10) into (5.9), we see that the corresponding (super)coefficient $\mathcal{B}$ transforms as

$$\mathcal{B}\to\mathcal{B}\frac{\kappa_1^2\kappa_2^2\kappa_3^2\kappa_4^2}{x_1^4x_2^4x_3^4x_4^4}. \tag{5.11}$$

For any of the standard choices of the κ's, the ratio in (5.11) would give 1, and the coefficient $\mathcal{B}$ would then be invariant with respect to the transformation of the region momenta $x_{1\ldots4}$.

The $\mathcal{B}_i$'s are the coefficients relevant for the expansion in the scalar box integrals I_i basis, which the quadruple cut actually calculates. As mentioned earlier, from the point of view of dual conformal symmetry it is more natural to consider the transformation properties of the coefficients $\tilde{\mathcal{B}}=\mathcal{B}/\sqrt{R}$ of the expansion in terms of scalar box functions F_i. The transformation of these coefficients is immediately obtained using (5.2) and (5.11),

$$\tilde{\mathcal{B}}\to\tilde{\mathcal{B}}\frac{\kappa_1^2\kappa_2^2\kappa_3^2\kappa_4^2}{x_1^2x_2^2x_3^2x_4^2}, \tag{5.12}$$

which, upon making the standard choice for the κ_i, becomes $\tilde{\mathcal{B}}\to\tilde{\mathcal{B}}x_1^2x_2^2x_3^2x_4^2$. Reinstating the transformation properties of the spectator region momenta, (5.12) shows that the supercoefficients $\tilde{\mathcal{B}}$ of the expansion of the superamplitude in terms of the scalar box functions F's transform covariantly under dual inversions just as the tree-level superamplitudes, i.e.

$$\tilde{\mathcal{B}}\to\tilde{\mathcal{B}}\prod_{i=1}^{n}\frac{\kappa_i^2}{x_i^2}. \tag{5.13}$$

We would like to conclude with a few comments.

(1) By performing four-dimensional quadruple cuts we have bypassed the problem of dimensionally regularizing the theory (thus breaking conformal invariance). It is only when the cut box is lifted to a full D-dimensional integral box function that infrared divergences appear (and therefore need to be regulated). However, for the sake of determining the transformation properties of the coefficients of the box functions, one can remain in four dimensions. The MHV anomaly of [1] is of course hiding inside the anomalous transformation properties under dual conformal transformations of the D-dimensional box functions.

(2) It is amusing to note that the covariance of the integral coefficients of the one-loop amplitudes provides also an alternative proof that all tree-level superamplitudes with arbitrary total helicity are dual superconformal covariant. This is a simple consequence of the observation that the universal structure of infrared divergences of one-loop amplitudes

$$\mathcal{A}_{\text{1-loop}}|_{\text{IR}}\sim\mathcal{A}_{\text{tree}}\sum_{i=1}^{n}\frac{(-s_{i,i+1})^{-\epsilon}}{\epsilon^2}, \tag{5.14}$$

can be used to extract recursive expressions for tree amplitudes (see [17,32]). A straightforward generalization of this argument to superamplitudes implies that $\mathcal{A}_{\text{tree}}$ is a linear combination of supercoefficients $\tilde{\mathcal{B}}$ which we just have shown to transform covariantly. Notice that the only input needed for this alternative proof of tree-level covariance is the knowledge that the three-point MHV and $\overline{\text{MHV}}$ tree superamplitudes are covariant. Furthermore, it is not necessary to know the large-z behavior of the superamplitudes.

(3) Finally, we compare the remarks of this section to the approach followed by DHKS in [1]. There, it has been conjectured that a generic n-point amplitude in $\mathcal{N}=4$ SYM can be written by factoring out the corresponding n-point MHV amplitude, as [1]

$$\mathcal{A}_n=\mathcal{A}_{n,\text{MHV}}\mathcal{R}, \tag{5.15}$$

where $\mathcal{R}$ is dual superconformal invariant to all loops. In the approach outlined here, we have restricted ourselves to proving the superconformal covariance of coefficients of the expansion of a generic one-loop amplitude in terms of box functions, without separating explicitly the (anomalous) MHV superamplitude. It would be interesting to see how this approach may provide a link between the superconformal invariance of the amplitudes as discussed in [1], and the conformal properties of the integral functions [23] appearing in the expansion of generic amplitudes in $\mathcal{N}=4$ SYM.

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