Phase transitions, θ behavior, and instantons in QCD and its holographic model

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In the holographic model of QCD, θ dependence sharply changes at the point of confinement-deconfinement phase transition. In large N QCD such a change in θ behavior can be related to the breakdown of the instanton expansion at some critical temperature T_c. Associating this temperature with confinement-deconfinement phase transition leads to the description of the latter in terms of dissociation of instantons into the fractionally charged instanton quarks. To elucidate this picture, we introduce the nonvanishing chiral condensate in the deconfining phase and assume a specific Lagrangian for the η' field in the confining phase. In the resulting picture the high-temperature phase of the theory consists of the dilute gas of instantons, while the low-temperature phase is described in terms of freely moving fractional instanton quarks.

I. INTRODUCTION

Color confinement, spontaneous breaking of chiral symmetry, the $U(1)$ problem, and the θ dependence are some of the most interesting questions in QCD. At the end of the 1970's, A. M. Polyakov [1] demonstrated charge confinement in QED$_3$. This was the first example where nontrivial dynamics was shown to be a key ingredient for confinement: Instantons (monopoles in 3d) play a crucial role in the dynamics of confinement in QED$_3$. Instantons in four-dimensional QCD were discovered more than 30 years ago [2]. However, their role in QCD$_4$ is still not well understood due to the divergence of the instanton density for large size instantons in the confining phase. Soon after, 't Hooft and Mandelstam [3] suggested a qualitative picture of how confinement could occur in QCD$_4$. The key point in the 't Hooft-Mandelstam approach is the assumption that dynamical monopoles exist and Bose condense. Many papers have been written on this subject since the original formulation [3]; however, the main questions, such as "What are these monopoles?"; "How do they appear in the gauge theories without Higgs fields?"; "How do they interact?"; and " What is the relation (if any) between the 't Hooft-Mandelstam monopoles and instantons?" are still not understood (for a review see [4]).

We reconsider these issues from a slightly different angle by analyzing phase transitions as a function of temperature T at nonzero θ parameter. We study the evolution of the most important field configurations as the phase transition line is crossed. Indeed, understanding θ dependence gives a very good idea about the dynamics of the most important color fluctuations with nontrivial topology. On the other hand, we will see that θ dependence can be studied using an effective Lagrangian approach with color singlet degrees of freedom.

The main result of our work can be formulated as follows. In the holographic model of QCD [5,6] confinement-deconfinement phase transition happens precisely at the value of temperature $T = T_c$ where θ dependence experiences a sudden change in behavior from θ/N in the low-temperature (confining) phase to $e^{-N}\cos\theta$ in the high-temperature (deconfining) phase [7]. Consider now QCD with a large number of colors N. For very high temperatures $T \gg T_c \sim \Lambda_{\text{QCD}}$ the typical size of instantons is very small and the instanton gas is dilute with density of order e^{-N}. Calculations in this region are under complete field theoretic control and the vacuum energy behaves like $e^{-N}\cos\theta$. As the temperature is lowered to be of order T_c the average instanton grows in size and the perturbative expansion around the instanton field configuration becomes unreliable. However, we argue, based on some reasonable assumptions, that in the large N limit the average distance between the instantons remains much larger then the instanton size all the way down to the critical value of temperature $T = T_c$. Below T_c the instanton expansion breaks down and a consistent field theoretic calculation with nonoverlapping instantons is no longer possible. It is then natural to assume that at $T = T_c$ there is a sharp transition in θ behavior, which can be associated with confinement-deconfinement transition, just as in the holographic model. The value of T_c can be estimated by a one-instanton calculation.

To elucidate the physics of the transition we consider a model where the chiral condensate does not vanish in the deconfining phase. The holographic model of QCD is a good example where this phenomenon occurs. On the field theoretic side this can be achieved by coupling fundamental matter to the hidden gauge group whose dynamically generated energy scale is higher than that of QCD. In the presence of nonvanishing chiral condensate, its phase φ

(which can be canonically normalized to yield η') is a perfect probe of the topological charges of the constituents on both sides of the phase transition line. This is a consequence of uniqueness of the η' meson: it always enters the system in combination $(\theta - \varphi)$ irrespectively, whether it is in the confining or the deconfining phase. One should remark here that our results do not really depend on the value of N_f as long as $N_f \ll N$.

The plan of the paper is as follows. We start in Sec. II by reviewing recent work on the holographic model of QCD where we note that θ behavior sharply changes at the point of confinement-deconfinement phase transition. We return from the holographic model to quantum field theory in Sec. III where we argue that instanton expansion breaks down sharply at some critical temperature T_c and estimate its value in terms of Λ_{QCD}. Sections IV and V are devoted to the physical interpretation of the phase transition. Here we attempt to answer the following question: what happens to the well-defined objects (instantons) as the phase transition line at $T = T_c$ is crossed from above. We assume a certain Lagrangian for the low-temperature phase and show that under this assumption, instantons do not completely disappear from the system but rather dissociate into the instanton quarks,[1] the objects with fractional topological charges $\pm 1/N$ which become the dominant quasiparticles. At nonzero temperatures $T < T_c$, the instanton quarks carry, along with fractional topological charges, the fractional magnetic charges which makes them the perfect candidates to serve as the dynamical magnetic monopoles, the crucial element of the standard 't Hooft, and the Mandelstam picture for the confinement [3]; see the end of Sec. V for details. In Sec. VI we formulate our main results and discuss future directions.

II. HOLOGRAPHIC MODEL OF QCD

In this section we consider physics of the holographic model of large N QCD with $N_f \ll N$ flavors [5,6]. This section is the review of previous work; our main point here is the observation that θ dependence changes once we go from the confining to the deconfining phase. The holographic model of QCD is realized by placing N_f D8 − D̄8 pairs in the background created by N D4 branes. In the weak coupling regime the D4 branes span $x^0 \ldots x^4$ coordinates, while D8 − D̄8 branes are pointlike in the x^4 direction and span the rest of spacetime. One of the direc-

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[1]Instanton quarks originally appeared in 2d models. Namely, using an exact accounting and resummation of the n-instanton solutions in 2d, CP^{N-1} models, the original problem of a statistical instanton ensemble was mapped unto a 2d-Coulomb gas (CG) system of pseudoparticles with fractional topological charges $\sim 1/N$ [8]. This picture leads to the elegant explanation of the confinement phase and other important properties of the 2d, CP^{N-1} models [8]. Unfortunately, similar calculations in 4d gauge theories are proven to be much more difficult to carry out [9].

tions (denoted by x_4 below) along the D4 branes is compactified on a circle of radius R_4 (which sets the scale of the glueball masses) with antiperiodic boundary conditions for fermions. The value of the asymptotic separation between the D8 and D̄8 branes, denoted by L, is a parameter of the brane construction along with R_4, N, N_f and string coupling and length g_s and l_s (which will be set to unity in the rest of the paper). It will be convenient to introduce the five-dimensional 't Hooft coupling $\lambda_5 = g_s N l_s$ and its four-dimensional counterpart defined at the Kaluza-Klein scale $\lambda_4 = \lambda_5/R_4$. In the limit $\lambda_4 \ll 1$, $\Lambda_{QCD} \ll 1/R_4$ and hence the theory approximates QCD pretty well. String theory is solvable in the opposite regime, $\lambda_4 \gg 1$, where there is no clear separation between the QCD scale and the supergravity/Dirac-Born-Infeld dynamics. It is this regime that we consider below, in the hope of drawing some qualitative lessons.

For $\lambda_4 \gg 1$ the rules of gauge/string duality [10–12] instruct us to pass to the metric which is the product of the D4 branes backreaction:

$$ds^2 = \left(\frac{U}{R}\right)^{3/2}((dx_\mu)^2 + f(U)(dx^4)^2)$$
$$+ \left(\frac{U}{R}\right)^{-(3/2)}\left(\frac{dU^2}{f(U)} + U^2 d\Omega_4^2\right), \quad (1)$$

where $f(U) = 1 - U_K^3/U^3$ and the U coordinate is bounded from below by U_K. The (U, x^4) are the analogs of polar coordinates on the plane, which is ensured by the relation

$$2\pi R_4 = \frac{4\pi}{3}\left(\frac{R^3}{U_K}\right)^{1/2} = \frac{4\pi}{3}\left(\frac{\pi\lambda_5}{U_K}\right)^{1/2}, \quad (2)$$

where in the second equality we used the relation between the 't Hooft coupling and the curvature scale of the space (1).

As explained in [13], the inclusion of the θ angle in this model corresponds to having a nonvanishing integral of the Ramond-Ramond one-form over the x^4 circle

$$\int_{S^1} C_1 = \int_D F_2 = \theta \bmod 2\pi k; \qquad F_2 = dC_1, \quad (3)$$

where k is an integer number and the integral is over the S^1 parametrized by $x^4 \in [0, 2\pi R_4]$. In the first equality we used the fact that the (U, x^4) space has the disk topology and Stokes theorem. One can solve the equation of motion for F_2 without taking backreaction into account (which is justified as long as $N_f \ll N$) and substitute into the action; the result for the vacuum energy at small θ is [13]

$$E_{vac} \approx \frac{\chi_g}{2}\theta^2, \quad (4)$$

where $\chi_g \sim \mathcal{O}(1)$ is the topological susceptibility. The addition of fundamental matter results [6] in the effective Lagrangian consistent with Veneziano-Witten formula for

the η' mass[2]:

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}(\partial_\mu \eta')^2 + \frac{N^2}{2}\chi_g\left(\frac{\theta}{N} + \frac{1}{N}\frac{\eta'}{f_{\eta'}}\right)^2, \qquad (5)$$

where we included some numerical factors such as $\sqrt{2}$ and $\sqrt{N_f}$ into the definition of $f_{\eta'}$ to simplify notations in the following sections. This result is not significantly changed when the finite temperature is introduced, as long as the theory is in the confining phase and the topology of the space remains the same. Equation (4) is consistent with the fact that in the confining phase physics is expected to depend on θ via the combination θ/N [13],

$$E_{\text{vac}} = N^2 \min_k h\left(\frac{\theta + 2\pi k}{N}\right), \qquad (6)$$

where $h(x)$ is some function which satisfies $h(0) = h'(0) = 0$. Equation (6) can also be understood from the quantum field theory viewpoint for finite N as a result of summation over different branches in pure $SU(N)$ gluodynamics, see Sec. III of Ref. [18] where connection with approach [13] is discussed.

As we will see below, instantons are not well-defined objects in this phase. Indeed, this would contradict θ/N dependence since each instanton comes with an integer multiple of θ. In the holographic model this is resolved by identifying instantons with Euclidean D0 branes wrapping around the x^4 direction which tend to shrink to zero size and disappear [7].

At finite temperature the model exhibits confinement/deconfinement and chiral phase transitions [19,20]. Two possible metrics with Euclidean time t_E compactified on a circle with circumference β are (1) and its double analytic continuation,

$$ds^2 = \left(\frac{U}{R}\right)^{3/2}\left((dx_i)^2 + f_T(U)dt_E^2 + (dx^4)^2\right) + \left(\frac{U}{R}\right)^{-(3/2)}\left(\frac{dU^2}{f_T(U)} + U^2 d\Omega_4^2\right), \qquad (7)$$

where $f_T = 1 - U_T^3/U^3$ and $\beta = \frac{4\pi}{3}(\frac{R^3}{U_K})^{1/2}$. Since the two metrics are the same, the comparison of the free energies is simple: as soon as $U_T > U_K$ the black hole metric (7) becomes preferred. This corresponds to confinement/deconfinement transition at $T = 1/2\pi R_4$. The Polyakov loop, which is the order parameter for confinement, vanishes in the confining phase (1) and has a nonvanishing value in the deconfining phase (7). In the deconfining phase the x^4 circle does not shrink to zero size and Stokes theorem makes it possible to have vanishing F_2, which minimizes the energy [7]. That is, in this phase it is possible to have

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[2]Earlier works on the holographic derivation of the η' Lagrangian include [14,15]. Theta dependence in the holographic models has been also considered in [16,17].

$$C_1 = \frac{\theta}{2\pi}dx^4. \qquad (8)$$

This leads to $\chi_g = 0$ to order N^0; this is also consistent with the fact that instantons are well-defined objects in this phase and come with the factor of $e^{in\theta}$. In the holographic model this is again a consequence of the topology in the deconfined phase, where the D0 brane wrapping the x^4 circle cannot shrink to zero size and disappear. The factor of $e^{in\theta}$ in the D0 brane action follows from (8). Hence, we observe that the θ dependence is different in the confining and deconfining phases. We will also see that such a change in the behavior is also supported by analyzing instantons in field theory, see the next sections.

Another comment we would like to make is the existence of the phase where the glue is deconfined but chiral symmetry is broken. While it is not necessarily true that such a phase exists in QCD (after all, the holographic model contains two variable parameters, as opposed to Λ_{QCD}), we discuss a field theoretic model with this property to illuminate the topological charges of the relevant constituents in the confining (Sec. V) and the deconfining phases (Sec. IV).

III. LARGE N QCD AT $T > T_c$

In this section we estimate the value of T_c where the instanton expansion breaks down and the θ dependence presumably experiences a sharp change. In the regime $T > T_c$ the θ dependence is determined by the dilute instanton gas approximation (see below for the discussion of the assumptions that are necessary for this statement to hold). Applicability of the instanton expansion (small density) implies that the θ parameter enters partition function in a very simple way $\sim \exp(-i\theta)$. The instanton contribution to the η' mass $\sim \exp(-\gamma(T)N)$ is exponentially suppressed for any small (but finite) positive $\gamma > 0$. In contrast: at arbitrary small and negative $\gamma < 0$ the instanton expansion obviously breaks down. The θ behavior presumably drastically changes at T_c determined by

$$\gamma(T = T_c) = 0 \Rightarrow T_c = c\Lambda_{\text{QCD}}. \qquad (9)$$

A. Instantons at $T > T_c$ with $\langle 0|\bar{\psi}\psi|0\rangle \neq 0$

In the following we will be interested in the instanton density in the dilute gas regime at $T > T_c$. We assume that the nonvanishing chiral condensate $\langle 0|\bar{\psi}\psi|0\rangle \neq 0$ exists in this region.

The instanton-induced effective action for N_f massless fermions can be easily constructed. In particular, for $N_f = 2$ flavors, u, d the corresponding expression takes the following form [21–24]:

$$L_{\text{inst}} = e^{-i\theta} \int d\rho \, n(\rho)(\tfrac{4}{3}\pi^2\rho^3)^2 \{(\bar{u}_R u_L)(\bar{d}_R d_L)$$

$$+ \tfrac{3}{32}[(\bar{u}_R\lambda^a u_L)(\bar{d}_R\lambda^a d_L) - \tfrac{3}{4}(\bar{u}_R\sigma_{\mu\nu}\lambda^a u_L)$$

$$\times (\bar{d}_R\sigma_{\mu\nu}\lambda^a d_L)]\} + \text{H.c.} \qquad (10)$$

We wish to study this problem at nonzero temperature and small chemical potential μ (to be discussed later in the text) for $T > T_c$, and we use the standard formula for the instanton density at two-loop order [25]

$$n(\rho) = C_N(\beta_I(\rho))^{2N}\rho^{-5}\exp[-\beta_{II}(\rho)]$$

$$\times \exp[-(N_f\mu^2 + \tfrac{1}{3}(2N + N_f)\pi^2 T^2)\rho^2], \quad (11)$$

where

$$C_N = \frac{0.466e^{-1.679N}1.34^{N_f}}{(N-1)!(N-2)!},$$

$$\beta_I(\rho) = -b\log(\rho\Lambda_{\text{QCD}}),$$

$$\beta_{II}(\rho) = \beta_I(\rho) + \frac{b'}{2b}\log\left(\frac{2\beta_I(\rho)}{b}\right),$$

$$b = \frac{11}{3}N - \frac{2}{3}N_f,$$

$$b' = \frac{34}{3}N^2 - \frac{13}{3}N_f N + \frac{N_f}{N}.$$

This formula contains, of course, the standard instanton classical action $\exp(-8\pi^2/g^2(\rho)) \sim \exp[-\beta_I(\rho)]$ which, however, is hidden as it is expressed in terms of Λ_{QCD} rather than in terms of coupling constant $g^2(\rho)$. By taking the average of Eq. (10) over the state with nonzero vacuum expectation value for the chiral condensate $\langle 0|\bar{\psi}\psi|0\rangle \neq 0$, one finds the following expression for the instanton-induced interaction, defined as $V_{\text{inst}}(\varphi) \equiv -\langle 0|L_{\text{inst}}(\varphi)|0\rangle$:

$$V_{\text{inst}}(\varphi) = -[\langle 0|\bar{\psi}\psi|0\rangle^{N_f}\cos(\varphi - \theta)] \cdot \int d\rho \, n(\rho)$$

$$\times \left(\frac{4}{3}\pi^2\rho^3\right)^{N_f}\left(\frac{1}{2}\right)^{N_f-1}$$

$$= -a \cdot \Lambda_{\text{QCD}}^4\cos(\varphi - \theta), \qquad (12)$$

where we introduced a small dimensionless parameter $a \ll 1$ which essentially governs the relevant physics. We assumed factorization for the chiral condensates in the large N limit in deriving (12). Hence the square bracket in Eq. (10) vanishes. We also assumed that the condensates for all flavors are equal, $\langle 0|\bar{u}_R u_L|0\rangle = \langle 0|\bar{d}_R d_L|0\rangle = \ldots = \frac{1}{2}\langle 0|\bar{\psi}\psi|0\rangle$. Finally, we introduced a singlet phase of the chiral condensate $\varphi(x)$ which we identify with the $\eta'(x)$ field in the standard way, $\bar{\psi}_R\psi_L \sim e^{i\varphi(x)/N_f}$. As expected, the η' field $\varphi(x)$ enters the Lagrangian in unique combi-

nation with θ as $(\varphi - \theta)$ which is a consequence of the anomalous Ward identity.[3]

The mass of the φ field in the chiral limit is determined by the instanton density in this phase and is expressed in terms of parameter a,

$$L = -\tfrac{1}{2}f_{\eta'}^2(\vec{\nabla}\varphi)^2 + m_{\eta'}^2 f_{\eta'}^2\cos(\varphi - \theta),$$

$$m_{\eta'}^2 f_{\eta'}^2 \equiv a \cdot \Lambda_{\text{QCD}}^4, \qquad (13)$$

where $f_{\eta'}$ is defined in the standard way as a normalization of the φ field, $\eta'(x) \equiv f_{\eta'}\varphi(x)$, and in general $f_{\eta'}(T)$ depends on temperature T (though it will not be explicitly shown later in the text). We also keep only the lowest Matsubara frequency for the φ field in the environment with $T \neq 0$ which ensures the validity of the static approximation for all interactions involving φ. This effective Lagrangian is, by definition, a Wilson type Lagrangian for the light η' field which is valid as long as the η' field is light,

$$m_{\eta'} \sim \sqrt{a}\Lambda_{\text{QCD}} \ll \cdot\Lambda_{\text{QCD}}. \qquad (14)$$

In the large N limit parameter $a \sim e^{-\gamma N}$ is exponentially suppressed[4] for temperatures above T_c, $a \ll 1$, and the instanton expansion converges. For $T < T_c$ the instanton expansion makes no sense (breaks down) and the expansion parameter becomes large $a \gg 1$. We assume that θ dependence sharply changes at $T = T_c$. We estimate the value of T_c by equalizing $\gamma = 0$ according to Eq. (9); see below.[5] In deriving the low-energy effective Lagrangian

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[3]In the presence of the massless chiral fermions the θ dependence as is known goes away in full QCD. To avoid any confusion later in the text we remark here that our discussions of θ dependence in this paper deal exclusively with the dynamics of gluons when light fermion degrees of freedom are frozen such that essentially we analyze the θ dependence in gluodynamics rather than in full QCD. In different words, we assume a quenched approximation for $N_f \ll N$. Precisely the θ dependence in quenched approximation plays a crucial role in the understanding of the dynamics of strongly interacting systems.

[4]See also [26] for earlier discussions on the subject.

[5]It is conceivable that the phase transition and sudden change in θ behavior occur at the same point T_c for any finite N, and not only for $N = \infty$. This assumption allows us to make some reasonable estimate for T_c for finite N. By obvious reasons, an estimate of T_c at finite N suffers from some inherent uncertainties. Indeed, T_c in this case is determined by an approximate condition $a \sim 1$ in contrast with precise Eq. (9) valid for the $N = \infty$ case. The condition $a \ll 1$ implies that the η' field is much lighter than all other degrees of freedom in the system in the chiral limit and condition (14) is satisfied. It is clear that this condition can be always satisfied for sufficiently large N where the parameter a is exponentially small at $T > T_c$. When T becomes close to T_c from above, parameter a increases and becomes order of unity at some point. This is precisely the region where instanton approximation breaks down. Therefore, according to our logic, the θ dependence may sharply change here. We identify this point where $a \sim 1$ with the point of the phase transition T_c. Of course we do not know the precise coefficient here (magnitude of a could be, for example, 3 instead of 1), but the extracting of a large power in such an estimate, $T_c \sim \Lambda_{\text{QCD}} \cdot a^{-(3/11N)}$ should not produce a large error for the estimation of T_c, even for the physically relevant case $N = 3$.

for the η' field we should, in principle, use the exact formula for the instanton density and not (11) which is only valid in the two-loop approximation. We assume that the perturbative corrections for $T \sim T_c$, although large, do not drastically change the physics. Then we will see that for any $T > T_c$ the dilute instanton approximation is valid, since the average distance between the instantons is parametrically larger then their size, see Eq. (19) below. To reiterate, we do not know how to do an honest instanton calculation in the close vicinity of T_c, but we assume that the perturbative expansion around the instanton field configuration can still be performed and would yield $a \sim e^{-\gamma(T)N}$, where $\gamma(T)$ is a monotonic function vanishing at $T = T_c$. Then, for $T > T_c$ the dilute instanton gas approximation is good, for $T < T_c$ it is no longer valid, while $T = T_c$ describes the phase transition point with drastic changes in θ behavior.

We should also note that one can estimate $T_c(\mu)$ for nonzero chemical potential $\mu \neq 0$ as long as the chiral condensate does not drastically vary with μ, which we assume to be the case at least for sufficiently small μ. It allows us to estimate not only a single point T_c on the phase diagram but the entire phase transition line $T_c(\mu)$ for sufficiently small $\mu \ll T_c$.

B. Numerical estimates

First, we estimate the critical temperature T_c by solving Eq. (9) and calculating coefficient c using the expression for the instanton density (11). As the first approximation (which greatly simplifies computations) we neglect all $\log(\rho \Lambda_{\text{QCD}})$ factors in evaluating the $\int d\rho$ integral. In this case the integral can be computed analytically and the limit $N \to \infty$ can be easily evaluated. The result for the instanton contribution takes the following form (as expected):

$$V_{\text{inst}}(\varphi) \sim e^{-\gamma N} \cos(\varphi - \theta),$$
$$\gamma = \left[\frac{11}{3} \ln\left(\frac{\pi T}{\Lambda_{\text{QCD}}}\right) - 1.86\right], \qquad (15)$$

where we neglected all powers N^p in front of $e^{-\gamma N}$ (as it does not have any impact on the computation of T_c at $N = \infty$) and used the standard Stirling formula

$$\Gamma(N + 1) = \sqrt{2\pi N} N^N e^{-N}\left(1 + \frac{1}{12N} + O\left(\frac{1}{N^2}\right)\right) \quad (16)$$

to evaluate the $N \to \infty$ limit.

There are three main reasons for a generic structure (15) to emerge. First of all, there is an exponentially large "T-independent" contribution, expressed as $e^{+1.86N}$ in Eq. (15). This term basically describes the entropy of the configuration such as the number of embedding $SU(2)$ into $SU(N)$, etc. Second, there is a "T-dependent" contribution to $V_{\text{inst}}(\varphi)$ which comes from the $\int n(\rho)d\rho$ integration (11). It is proportional to

$$\left(\frac{\Lambda_{\text{QCD}}}{\pi T}\right)^{(11/3)N} = \exp\left[-\frac{11}{3} N \cdot \ln\left(\frac{\pi T}{\Lambda_{\text{QCD}}}\right)\right]. \quad (17)$$

Finally, all fermion related contributions such as a chiral condensate or nonvanishing mass term enter the instanton density as follows: $\sim \langle 0|\bar{\psi}\psi|0\rangle^{N_f} \sim e^{N \cdot (\kappa \ln|\langle 0|\bar{\psi}\psi|0\rangle|)}$. For $\kappa \equiv \frac{N_f}{N} \to 0$ this term obviously leads to a subleading effect $1/N$ in comparison with two main terms in the exponent (15). Therefore, such terms can be neglected as they do not change any estimates at $N = \infty$. It is in accordance with the general arguments suggesting that the fundamental fermions cannot change the dynamics of the relevant gluon configurations as long as $N_f \ll N$. Indeed, the formula for $\gamma(T)$ for pure gluodynamics is given by the same expression (15) as it should be for $N_f \ll N$. If the chiral condensate vanishes, one can replace it by a small but nonzero value for the quark's mass to proceed with our calculations. It would not alter Eq. (15). Therefore our estimate below (18) is not affected whether the chiral condensate develops or not. In different words, we essentially study a pure gluodynamics. Our treatment of the problem is the equivalent of a quenched approximation for $N_f \ll N$. The fermion fields in the present study play an auxiliary (not a dynamical) role as a probe of the topological charges relevant for the phase transition as will be discussed in the next sections. When a number of fermions increases and $N_f \sim N$ we cannot proceed with the estimations as we have done above. In this case we do anticipate a strong dependence on fermion properties such as the quark's masses and the chiral condensate as argued in a recent paper [27]. Lattice simulations also suggest that for physical values for the quark's masses one should expect a smooth crossover rather than a first order phase transition, see e.g. [28,29]. We remind the reader that for pure gluodynamics which is the main subject of the present paper for all $N \geq 3$ and a small number of flavors $N_f \ll N$ one should expect the first order phase transition.

From discussions above it should be obvious that there will be always a point T_c where two leading contributions with exponential e^N dependence cancel each other. As a result, at $N \to \infty$ for $T > T_c$ the instanton gas is dilute with density $e^{-\gamma N}$, $\gamma > 0$ which ensures a nice $\cos\theta$ dependence, while for $T < T_c$ the expansion breaks down, and θ dependence must sharply change at $T < T_c$. We identify such sharp changes with first order phase transition.

As explained above, the critical temperature is determined by the condition $\gamma = 0$. Numerically, at one loop level approximation, it happens at

$$\gamma = \left[\frac{11}{3} \ln\left(\frac{\pi T_c}{\Lambda_{\text{QCD}}}\right) - 1.86\right] = 0 \Rightarrow T_c(N = \infty)$$
$$\simeq 0.53\Lambda_{\text{QCD}}, \qquad (18)$$

where Λ_{QCD} is defined in the Pauli-Villars scheme. A few remarks are in order.

(a) Our computations are carried out in the regime where the instanton density $\sim \exp(-\gamma N)$ is parametrically suppressed at $N = \infty$. From Eq. (15) one can obtain the following expression for instanton density in the vicinity of $T > T_c$:

$$a \sim \cos(\varphi - \theta) \cdot e^{-\alpha N((T-T_c)/T_c)},$$
$$1 \gg \left(\frac{T - T_c}{T_c}\right) \gg 1/N, \tag{19}$$

where α is a numerical coefficient of order one. Such a behavior does imply that the dilute gas approximation is justified even in close vicinity of T_c as long as $\frac{T-T_c}{T_c} \gg \frac{1}{N}$. In this case the diluteness parameter remains small[6] even in the close vicinity of T_c. Therefore, the θ dependence, which is sensitive to the topological fluctuations only, remains unaffected all the way down to the temperatures very close to the phase transition point, $T = T_c + O(1/N)$. We cannot rule out, of course, the possibility that the perturbative corrections may change our numerical estimate for T_c. However, we expect that a qualitative picture of the phase transition advocated in this paper remains unaffected as a result of these corrections in the dilute gas regime.

(b) In our estimate for T_c we neglected $(\log \rho \Lambda_{\text{QCD}})^k$ in the evaluating of the $\int d\rho$ integral. One can easily take into account the corresponding contribution by noticing that $\int d\rho$ is saturating at $\rho \simeq (\pi T)^{-1}$. The corresponding correction changes our estimate (18) very slightly, and it will be ignored in what follows. Numerical smallness of the correction is due to the strong cancellation between the second loop contribution in the exponent (term proportional to b'/b) and the first loop contribution in the preexponent in Eq. (11).

(c) The transition to a different scheme leads to very large changes in the instanton density. For example, the transition to the so-called MS scheme is achieved by replacing $e^{-1.679N}$ in the expression for C_N, see Eq. (11), as follows $e^{-1.679N} \rightarrow e^{(-1.679+3.721)N}$ with a number of other changes, see e.g. [23]. The corresponding results would be expressed in terms of $\Lambda_{\text{QCD}}^{\text{MS}}$, where MS stands for the MS scheme, to be distinguished from Λ_{QCD} which is defined in the Pauli-Villars scheme and will be used throughout this paper. We shall not elaborate on these numerical issues in the present work.

(d) Unfortunately, we cannot compare our calculations with the precise lattice results [30] for the ratio $T_c/\sqrt{\sigma}$ at large N as we compute T_c in the deconfined phase where the string tension σ vanishes.

(e) As expected, the result (18) does not depend on a number of flavors N_f nor does it depend on the magnitude of the chiral condensate in the $N = \infty$ limit as our treatment of the problem corresponds essentially pure Yang-Mills (YM) computations.

(f) For finite but large $N \gg N_f$ the corresponding numerical estimates for T_c can also be given. It can be estimated from the condition $a \sim 1$. However, numerical estimates in this case would depend on the value of the $U_A(1)$ condensate $a \sim \langle 0|(\bar{\psi}\psi)^{N_f}|0\rangle$ which is not well known for $T > T_c$. Therefore, we shall not discuss the corresponding numerical estimates in the present work.

(g) A similar procedure for the estimation of the critical chemical potential μ_c for confinement-deconfinement phase transition at finite N, N_f at $T \sim 0$ has been previously used in Ref. [31] where the analogous arguments on drastic changes of θ at $\mu = \mu_c$ have been presented, see also a review paper [32].

(h) Once T_c is fixed one can compute the entire line of the phase transition $T_c(\mu)$ for relatively small $\mu \ll T_c$ for large but finite $N \gg N_f$. Indeed, in the weak coupling regime at $T > T_c$ the μ dependence of the instanton density is determined by a simple insertion $\sim \exp[-N_f \mu^2 \rho^2]$ in the expression for the density (11). In the leading loop order $T_c(\mu)$ varies as follows:

$$T_c(\mu) = T_c(\mu = 0)\left[1 - \frac{3N_f \mu^2}{4N\pi^2 T_c^2(\mu = 0)}\right], \tag{20}$$
$$\mu \ll \pi T_c, \qquad N_f \ll N.$$

As expected, μ dependence goes away in the large N limit in agreement with general large N arguments [33]. This formula is in excellent agreement with numerical computations [34–36] which show very little changes of the critical temperature T_c with μ for sufficiently small chemical potential. In particular, even for the case $N_f = 2$, $N = 3$ where the expression (20) is not expected to give a good numerical estimate, it still works amazingly well even for $N = 3$. Indeed, the result quoted in [34] can be written as

$$T_c(\mu)^{\text{lat}} = T_c(\mu = 0)^{\text{lat}}\left[1 - 0.500(38)\right.$$
$$\left. \times \frac{\mu^2}{\pi^2 T_c^2(\mu = 0)^{\text{lat}}}\right],$$
$$N_f = 2, \qquad N = 3.$$

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[6]This should be contrasted with the standard requirement for finite N when the condition $a \sim (\Lambda_{\text{QCD}}/T)^b \ll 1$ can be only achieved when the temperature is very large, $T \gg \Lambda_{\text{QCD}}$. For large N the condition $a \ll 1$ is satisfied as long as $\frac{T-T_c}{T_c} \gg \frac{1}{N}$, as can be seen from Eq. (19).

It should be compared with our theoretical prediction (20) for this case

$$T_c(\mu)^{\text{th}} = T_c(\mu = 0)^{\text{th}}\left[1 - \frac{1}{2}\frac{\mu^2}{\pi^2 T_c^2(\mu = 0)^{\text{th}}}\right].$$

(i) It is natural to expect that the phase transition line $T_c(\mu)$ at $\mu \ll T_c$ from (20) connects with the phase transition line at very large $\mu_c \sim \sqrt{N}$ as estimated in a recent paper [27],

$$\mu_c(T) \simeq \mu_c(T = 0)\left[1 - \frac{N\pi^2 T^2}{3N_f \mu_c^2(T = 0)}\right], \quad (21)$$

$$\sqrt{N}T \ll \mu_c, \qquad N_f \ll N,$$

where $\mu_c(T = 0) \simeq 1.4 \cdot \Lambda_{\text{QCD}}\sqrt{\frac{N}{N_f}}$ at $N_f \ll N$ [27]. This expectation is motivated by the observation that the nature for the phase transition along the entire line is one and the same: it is the drastic changes of θ dependence when the phase transition line is crossed. Therefore, we believe that the entire line is the first order phase transition as long as $N_f \ll N$.

IV. DUAL REPRESENTATION

The main goal of this section is to present the low-energy effective Lagrangian for the η' field (13) in the dual form. The η' field will play a crucial role in the following two sections. As we shall see in a moment the η' field is a perfect probe of the glue configurations. This field will help us to investigate the topological charges of the constituents in both phases, and therefore it will help us to interpret the nature of the phase transition whose critical temperature T_c was computed in the previous section. In Sec. II we discussed a holographic model with nonvanishing chiral condensate. Here we consider a field theoretic model with this property.

A. Coulomb gas representation: formal derivation

The effective low-energy dense-QCD Lagrangian (13) is the sine-Gordon (SG) Lagrangian. Many of the special properties of the SG theory apply. One of these properties is the admittance of a Coulomb gas (CG) representation for the partition function. Although this is a four-dimensional theory at nonzero temperature T (rather than two dimensional, where all known exact results regarding the SG model were discussed) and questions about renormalizability of the theory may come to mind, there are no such issues here since the effective action is a low-energy one. Following the usual procedure for mapping a statistical CG model into the field theoretic SG model, the CG picture that arises from the effective low-energy QCD action, Eq. (13), will be derived in this section. The statistical model contains some charges which appear due to the presence of cosine interaction in the field theory model.

The physical meaning of these charges will be illuminated in the next section by analyzing the corresponding measure of the statistical ensemble.

The mapping between the SG theory and its CG representation is well known. All we need to do is to reverse the derivation of the SG functional representation of the CG in Ref. [1]. The partition function corresponding to the Lagrangian (13) is given by[7]

$$Z = \int \mathcal{D}\varphi\, e^{-\int d^3x \int_0^\beta d\tau L_E}$$

$$= \int \mathcal{D}\varphi\, e^{-(1/2T)f_{\eta'}^2 \int d^3x(\vec{\nabla}\varphi)^2} e^{\lambda \int d^3x \cos(\varphi(x)-\theta)}, \quad (22)$$

where we introduced fugacity for the CG ensemble to be defined as

$$\lambda \equiv \left(\frac{\Lambda_{\text{QCD}}}{T}\right)a\Lambda_{\text{QCD}}^3, \quad (23)$$

L_E is the Euclidean space Lagrangian. Leaving alone the integration over $\varphi(x)$ for a moment, we expand the last exponent in Eq. (22), represent the cosine as a sum of two exponents, and perform the binomial expansion:

$$e^{\lambda \int d^3x \cos(\varphi(x)-\theta)} = \sum_{M=0}^\infty \frac{(\lambda/2)^M}{M!}$$

$$\times \left(\int d^3x \sum_{Q=\pm 1} e^{iQ(\varphi(x)-\theta)}\right)^M$$

$$= \sum_{M_\pm=0}^\infty \frac{(\lambda/2)^M}{M_+!M_-!}$$

$$\times \int d^3x_1 \ldots \int d^3x_M e^{i\sum_{a=0}^M Q_a(\varphi(x_a)-\theta)}.$$

(24)

The last sum is over all possible sets of M_+ positive and M_- negative charges $Q_a = \pm 1$. The last line in Eq. (24) is a classical partition function of an ideal gas of $M = M_+ + M_-$ identical (except for charge) particles of charges $+1$ or -1 placed in an external potential given by $i(\theta - \varphi(x))$. It is easy to see that (for a constant or slowly varying potential) the average number of these particles per unit of 3-volume $\langle M\rangle/V_3$, i.e., the density, is equal to λ. Thus making λ small one can make the gas arbitrarily dilute, which is precisely the physical meaning of the fugacity. From this observation, one can immediately see that the average distance between charges Q_a is $\lambda^{-1/3}$.

While θ can indeed be viewed as an external potential for the gas (24), $\varphi(x)$ is a dynamical variable, since it fluctuates as signified by the path integration in (22). For

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[7]To be precise, the path integral in Eq. (22) should be understood as an integral over *low-momentum* modes of φ only. The upper limit of the momentum of φ is the ultraviolet cutoff of the effective Lagrangian (13), which should be taken as some scale smaller than T. Only tree graphs contribute to Z so there is no dependence on the precise value of the cutoff.

each term in (24) the path integral is Gaussian and can be easily taken:

$$\int \mathcal{D}\varphi \, e^{-(1/2T)f_{\eta'}^2 \int d^3x(\vec{\nabla}\varphi)^2} e^{i\sum_{a=0}^{M} Q_a(\varphi(x_a)-\theta)}$$

$$= e^{-i\theta \sum_{a=0}^{M} Q_a} e^{-(T/f_{\eta'}^2)\sum_{a>b=0}^{M} Q_a G(x_a-x_b)Q_b}. \quad (25)$$

We see that, for a given configuration of charges Q_a, $-i\varphi(x)$ is the Coulomb potential created by such a distribution.[8] The function $G(x)$ is the solution of the three-dimensional Poisson equation with a point source (the inverse of $-\vec{\nabla}^2$):

$$G(x_a - x_b) = \frac{1}{4\pi|\vec{x}_a - \vec{x}_b|}. \quad (26)$$

Thus we obtain the dual CG representation for the partition function (22):

$$Z = \sum_{M_{\pm}=0}^{\infty} \frac{(\lambda/2)^M}{M_+! M_-!} \int d^3x_1 \ldots \int d^3x_M$$

$$\times e^{-i\theta\sum_{a=0}^{M} Q_a} e^{-(T/f_{\eta'}^2)\sum_{a>b=0}^{M} Q_a G(x_a-x_b)Q_b}. \quad (27)$$

The two representations of the partition function (22) and (27) are equivalent.

Note that the physical meaning of λ is the fugacity of the system with charges Q and it is proportional to a which is small in the regime under discussion. There are several important features of the action (27) which should be noted. First, the total Q charge of the configuration, Q_T, appears together with θ, which we kept as a free parameter, see (27). Such a dependence will play an important role in the following identification of Q charges as the topological charges, see below. The θ dependence in CG representation (27) gives an overall phase factor for each configuration. Finally, our dimensional parameters λ and $\vec{x}_M$ come into the expression (27) in the combination λd^3x which is nothing but the coefficient in front of the instanton contribution to the effective action $S_{\text{inst}} = \int_0^{\beta} d\tau \int d^3x V_{\text{inst}}(\varphi) = -\int d^3x \lambda \cos(\varphi - \theta)$ at nonzero temperature T, see Eq. (12) and the definition of λ (23).

B. Physical interpretation

The charges Q_a were originally introduced in a rather formal manner so that the QCD effective low-energy Lagrangian can be written in the dual CG form (27). However, now the physical interpretation of these charges

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[8]One notices that the term $a = b$ in the double sum (25) is dropped. This is the self-interaction of each charge. It would renormalize the fugacity λ by a factor $\exp(-G(0)/(f_{\eta'}^2))$. This factor should be dropped as it represents the contribution of very short wavelength fluctuations of φ. Such fluctuations have to be cut off at the scale $1/T$. The self-energy of the charges comes from much smaller scales which are already calculated and contained in a.

becomes clear: since $Q_{\text{net}} \equiv \sum_a Q_a$ is the total charge and it appears in the action multiplied by θ [see Eq. (27)], one concludes that Q_{net} *is* the total topological charge of a given configuration. Indeed, in QCD the θ parameter appears in the Lagrangian only in the combination with the topological charge density $-i\theta G_{\mu\nu}\tilde{G}_{\mu\nu}/(32\pi^2)$. It is also quite obvious that each charge Q_a in a given configuration should be identified with an integer topological charge well localized at the point x_a. This, by definition, corresponds to a small instanton positioned at x_a (to be precise, "caloron" at temperature $T \neq 0$ which has topological charge $Q = 1$ and action $8\pi^2/g^2$ independent of temperature, see [24,25] for review). To support this identification we note that every particle with charge Q_a brings along a factor of fugacity $\lambda \sim a$ (23) which contains the classical one-instanton suppression factor $\exp(-8\pi^2/g^2(\rho))$ in the density of instantons (12) if one restores the instanton density in terms of the coupling constant $\exp(-8\pi^2/g^2(\rho))$ rather than directly in terms of Λ_{QCD} which is used in Eq. (11) and which is more convenient for numerical estimates.

This identification is also supported by the following observation: every extra particle with charge Q_a brings an additional weight $e^{-i\theta Q_a}$ to the partition function. This is certainly the most distinguishable feature of the nonzero topological charge configuration.

The following hierarchy of scales exists in such an instanton ensemble for temperatures slightly higher than T_c. The typical size of the instantons $\rho \sim T^{-1} \sim \Lambda_{\text{QCD}}^{-1}$. The average distance between the instantons $\bar{r} = \lambda^{-1/3} = \Lambda_{\text{QCD}}^{-1} a^{-1/3}$ is much larger than both the average size of the instantons and the cutoff T^{-1}. The largest scale is the Debye screening length in the Coulomb gas, $r_D = \Lambda_{\text{QCD}}^{-1} a^{-1/2}$. This coincides with the static correlation length of the φ field, which is precisely η' mass. It is important that the Debye screening length r_D is parametrically larger than the average distance between the instantons $\bar{r}$, therefore a large number of instantons can be accommodated within the volume determined by the Debye screening length r_D which justifies our Coulomb gas interpretation, at least in the large N limit. In short,

$$(\text{size}, \rho) \ll (\text{distance}, \bar{r}) \ll (\text{Debye}, r_D),$$

$$\frac{1}{T} \ll \frac{1}{\Lambda_{\text{QCD}}\sqrt[3]{a}} \ll \frac{1}{\Lambda_{\text{QCD}}\sqrt{a}}. \quad (28)$$

Because of this hierarchy, ensured by small $a \ll 1$, we acquire analytical control. In reality, of course, $a \sim (\frac{\Lambda_{\text{QCD}}}{T})^N \sim e^{-N} \ll 1$ is parametrically very small only at very large N while $(\frac{\Lambda_{\text{QCD}}}{T}) \leq 1$ could be very close to 1 from below. It implies that at $N = 3$ all scales could be numerically very close to each other.

It is also quite interesting that, although the starting low-energy effective Lagrangian contains only a colorless field φ, we have ended up with a representation of the partition

function in which objects carrying color (instantons, their interactions and distributions) can be studied. In particular, from the discussion above, one can immediately deduce that II and $I\bar{I}$ interactions are exactly the same up to a sign and are Coulomb-like at large distances.

This looks highly nontrivial since it has long been known that at the semiclassical level an instanton interacts only with anti-instantons but not with another instanton carrying a topological charge of the same sign. As we demonstrated above it is not true any more at the quantum level in the presence of the η' field. Indeed, what we have found is that the interactions between dressed (as opposed to bare) instantons and anti-instantons after one takes into account their classical and quantum interactions, after integration over all their possible sizes and color orientations, and after accounting for the interaction with the background chiral condensate must become very simple at large distances as explicitly described by Eq. (27). It is impressive how the problem which looks so complicated in terms of the original bare (anti)instantons becomes so simple in terms of the dressed (anti)instantons when all integrations over all possible sizes, color orientations, and interactions with background fields are properly accounted for.

Such a simplification of the interactions is of course due to the presence of an almost massless pseudo-Goldstone boson η' which couples to the topological charge. When the instanton gas becomes very dilute all semiclassical interactions (due to zero modes) cannot contribute much, since they fall off with distance faster then the Coulomb interaction mediated by η'. On the other hand, when the instanton density increases when T is getting smaller, the Coulomb interaction becomes more screened and, as the Debye length becomes comparable to the interinstanton distances, we lose analytical control. Based on this picture one can estimate the critical temperature T_c where this transition must happen. It corresponds to the same condition $a \sim 1$ discussed previously in Sec. III.

We collect here the most important results of the present section based on CG representation (27):

(a) Since $Q_{\text{net}} \equiv \sum_a Q_a$ is the total charge and it appears in the action multiplied by the parameter θ, one concludes that Q_{net} *is the total topological charge of a given configuration.*

(b) Each charge Q_a in a given configuration should be identified with an integer topological charge $Q_a = \pm 1$ well localized at the point x_a. This, by definition, corresponds to a small instanton (caloron at $T \neq 0$) positioned at x_a.

(c) While the starting low-energy effective Lagrangian contains only a colorless field φ we have ended up with a representation of the partition function in which objects carrying color (the instantons) can be studied.

(d) In particular, II and $I\bar{I}$ interactions (at very large distances) are exactly the same up to a sign, order

g^0, and are Coulomb-like. This is in contrast with semiclassical expressions when II interaction is zero and $I\bar{I}$ interaction is order $1/g^2$.

(e) The very complicated picture of the bare II and $I\bar{I}$ interactions becomes very simple for dressed instantons/anti-instantons when all integrations over all possible sizes, color orientations, and interactions with background fields are properly accounted for.

(f) As expected, the ensemble of small $\rho \sim T^{-1}$ instantons cannot produce confinement because small instantons cannot produce a correlation at arbitrary large distances which is a crucial feature of the confinement. This is in accord with the fact that there is no confinement in the high temperature phase.

(g) Physical interpretation of the CG representation (27) is simple. The η' field being a dynamical field couples to the topological charge Q exactly as the θ parameter does due to the specific combination $(\varphi(x) - \theta)$ which appears in the low-energy Lagrangian. In the dual language the η' mass emerges as a result of Debye screening in the plasma of topologically charged instantons (interacting via η' Coulomb exchange) similar to the well-known effect of generating the photon's mass in the ionized plasma due to the Coulomb interaction of charged particles. In our case, instead of a conventional vector photon we are dealing with a pseudo scalar η' field which receives its mass through the interaction with topological charges Q. Uncovering this picture (which allows us to measure the topological charges of constituents) was the main motivation for introducing the chiral condensate into the theory.

(h) We should also remark here that a similar picture for the instanton interactions occurs at large chemical potential $\mu \gg \Lambda_{\text{QCD}}$ in the deconfined, so-called color superconducting phase [37]. In the present case $T > T_c$ the weak coupling regime (small instanton density) is governed by the small parameter $a \sim \exp(-\gamma N) \ll 1$ while at the large $\mu \gg \Lambda_{\text{QCD}}$, $N = 3$ case the corresponding small factor is $(\Lambda_{\text{QCD}}/\mu)^b \ll 1$ [37].

V. SMALL $T < T_c$: CONFINED PHASE. SPECULATIONS

In this section we want to speculate on the fate of the instantons when we cross the phase transition line at $T = T_c$. To be more precise: we want to see if any traces of well-defined instantons discussed above can be recovered. The instanton expansion is not justified in the strong coupling regime $T < T_c$ where the expansion coefficient becomes of order one, $a \sim 1$.

Therefore, we do not even attempt to use instanton calculus or any other semiclassical computations in the

present section. Instead, we present a few indirect arguments supporting the picture that the instantons do not completely disappear from the system when we cross the phase transition line from above, but rather dissociate into the instanton quarks [8,9], the self-dual objects with fractional topological charges $\pm 1/N$ which become the dominant quasiparticles. The arguments are not based on the semiclassical calculations, but rather on the analysis of the low-energy Lagrangian written in the dual form similar to CG representation discussed in the previous section. Our proposal about the fate of instantons at $T < T_c$ originally derived in Ref. [38] and to be reviewed for completeness in this section should be considered as one of the many possible outcomes. In this sense this section is very speculative in nature in contrast with previous sections where the weak coupling regime is justified at large N and precise statements can be made.

We start from the chiral Lagrangian and keep only the singlet η' field. We assume the following expression for the effective Lagrangian for the φ field which has a specific SG form,

$$L_\varphi = \frac{1}{2} f_{\eta'}^2 (\partial_\mu \varphi)^2 + E_{\text{vac}} \cos\left(\frac{\varphi - \theta}{N}\right), \quad f_{\eta'}^2 m_{\eta'}^2 = \frac{E_{\text{vac}}}{N^2},$$
(29)

where $E_{\text{vac}} \sim N^2$ is the vacuum energy of the ground state in the chiral limit, expressed in terms of the gluon condensate,

$$E_{\text{vac}} = \tfrac{1}{4}\langle 0|\Theta_\mu^\mu|0\rangle = \langle 0|b\alpha_s/(32\pi)G^2|0\rangle \sim N^2, \quad (30)$$

where we use the standard expression for the conformal anomaly of the energy-momentum tensor, Θ_μ^μ. The expression (29) of course satisfies the standard requirement crucial for the resolution of the $U_A(1)$ problem: the vacuum energy in gluodynamics depends on θ through the combination θ/N. It also has a very specific SG structure for the singlet combination corresponding to the following behavior of the $(2k)^{\text{th}}$ derivative of the vacuum energy in pure gluodynamics [39],

$$\frac{\partial^{2k} E_{\text{vac}}(\theta)}{\partial \theta^{2k}} \Big|_{\theta=0} \sim \int \prod_{i=1}^{2k} dx_i \langle Q(x_1)\dots Q(x_{2k})\rangle \sim \left(\frac{i}{N}\right)^{2k},$$

$$\text{where} \quad Q \equiv \frac{g^2}{32\pi^2} G_{\mu\nu}\tilde{G}_{\mu\nu}. \tag{31}$$

The same structure was also advocated in [40] from a different perspective. We shall not discuss any additional arguments supporting such Sine-Gordon structure referring to the original papers.[9] This is precisely the place where the term " speculation" from the title of this section enters our analysis. One should also note that the combination

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[9]One more additional argument supporting SG structure $\sim \cos(\frac{\theta}{N})$ in pure gluodynamics will be given later in the text.

$$\chi_g = \frac{E_{\text{vac}}}{N^2} = \frac{\partial^2 E_{\text{vac}}(\theta)}{\partial \theta^2} \Big|_{\theta=0}$$

is nothing but topological susceptibility χ_g for gluodynamics in the large N limit.

Now we want to represent the low-energy Lagrangian (29) in the dual form (CG representation) to see if any traces from the instantons discussed at $T > T_c$ can be recognized. The effective Lagrangian is obviously the color singlet object. Therefore, all color dynamics cannot be recovered by this method. However, the topological charge is the color singlet operator which is coupled to θ. The θ parameter is not a dynamical field in QCD, however the η' field is, and it always enters the dynamics in combination $(\theta - \varphi)$. Let us repeat again that this was the main reason to introduce the chiral condensate into the system: it allows to study the dynamics of the topological charges. Therefore, in principle, the analysis of the η' field gives the information about the topological charges of the constituents. We use the trick (SG-CG mapping) below to attempt to answer the following question: what kind of constituents can provide the low-energy behavior (29), [39]?

We use the technique developed in the previous section and represent SG action in the dual form. Technically, it goes as follows: Eq. (29) replaces expression (13) discussed previously. As in (13) the Sine-Gordon effective field theory (29) can be represented in terms of a classical statistical ensemble (CG representation) similar to (27) with the replacements $\lambda \to E_{\text{vac}}$, $d^3x \to d^4x$ as we assume zero temperature $T = 0$ in this phase. By repeating all previous steps we arrive at the following expression:

$$Z = \sum_{M=0}^{\infty} \frac{(\frac{E_{\text{vac}}}{2})^M}{M!} \int d^4x_1 \dots \int d^4x_M$$

$$\cdot \sum_{Q_a=\pm 1/N} \int D\varphi \, e^{-(1/2)f_{\eta'}^2 \int d^4x (\partial_\mu \varphi)^2}$$

$$\cdot (e^{i\sum_{a=1}^{M} Q_a[\varphi(x_a)-\theta]}). \tag{32}$$

The functional integral is trivial to perform and one arrives at the dual CG action,

$$Z = \sum_{M_\pm=0}^{\infty} \frac{(\frac{E_{vac}}{2})^M}{M_+! M_-!} \int d^4x_1 \dots \int d^4x_M$$

$$\cdot e^{-i\theta \sum_{(a=0, Q_a=\pm 1/N)}^{M} Q_a}$$

$$\cdot e^{-(1/f_{\eta'}^2) \sum_{(a>b=0, Q_a=\pm 1/N)} Q_a G(x_a-x_b)Q_b}, \tag{33}$$

where $G(x_a - x_b)$ is the 4d Green's function,

$$G(x_a - x_b) = \frac{1}{4\pi^2(x_a - x_b)^2}. \tag{34}$$

The fundamental difference in comparison with the previous case (27) is that while the total charge is integer, the individual charges are *fractional* $\pm 1/N$. This is a direct consequence of the θ/N dependence in the underlying

effective Lagrangian (29) before integrating out φ fields, see Eq. (32).

A few remarks on the physical interpretation of the CG representation (33) of theory (29) are in order:

(a) As before, one can identify $Q_{net} \equiv \sum_a Q_a$ with the total topological charge of the given configuration.

(b) Because of the 2π periodicity of the theory, only configurations which contain an integer topological number contribute to the partition function. Therefore, the number of particles for each given configuration Q_i with charges $\sim 1/N$ must be proportional to N.

(c) Therefore, the number of integrations over d^4x_i in CG representation exactly equals $4Nk$, where k is an integer. This number $4Nk$ exactly corresponds to the number of zero modes in the k-instanton background. This is the basis for the conjecture [38] that at low energies (large distances) the fractionally charged species, $Q_i = \pm 1/N$, are the *instanton quarks* suspected long ago [8].

(d) For the gauge group G the number of integrations would be equal to $4kC_2(G)$ where $C_2(G)$ is the quadratic Casimir of the gauge group (θ dependence in physical observables comes in the combination $\frac{\theta}{C_2(G)}$). This number $4kC_2(G)$ exactly corresponds to the number of zero modes in the k-instanton background for gauge group G.

(e) We do not use the weak coupling regime or instanton calculus anywhere in our arguments. Still, we recover the moduli space which we identify with strongly interacting instantons in the confinement phase of the theory.

(f) The role of the fugacity for this statistical ensemble plays $E_{vac} \sim N^2$. Therefore, an average distance between constituents is of order $\bar{r} \sim E_{vac}^{-1/4} \sim \Lambda_{QCD}^{-1} N^{-1/2}$ which suggests that the system is very dense. It obviously implies that the instanton expansion makes no sense in this regime as all terms are equally important, which is in huge contrast with hierarchy from the previous case at $T > T_c$ (28).

(g) The Debye screening length $r_D \sim m_{\eta'}^{-1} \sim \Lambda_{QCD}^{-1}\sqrt{N} \gg \bar{r}$ is large. It means that the number of constituents participating in the screening is of order $(r_D/\bar{r})^4 \sim N^4$.

(h) According to Eq. (29) the number of instanton quarks in the spacetime box of size Λ_{QCD} should be N^2 as an average distance between constituents is $\bar{r} \sim N^{-1/2}$. Each instanton contains N instanton quarks, hence the density of instantons should be of order $N\Lambda_{QCD}^4$.[10] It is consistent with observation from holography, Sec. II, that any finite number of instantons will disappear from the system.

A. The relation to other studies

As we mentioned above our arguments in the present section look extremely speculative as they are not based on instanton calculus or any other dynamical calculations which include color degrees of freedom. Still, by analyzing θ dependence in the deconfining phase we infer (indirectly) that some fractionally charged degrees of freedom emerge at low $T < T_c$. We presented arguments suggesting that the corresponding fractionally charged constituents are the colored instanton quarks. We should remark here that the fractionally charged constituents have been discussed in the literature in a number of papers previously. In particular, there seems to be a close relation between instanton quarks and the "periodic instantons" [41–43], center vortices and nexuses with fractional fluxes $1/N$, see e.g. [44] and references therein. We shall not discuss the corresponding connections in detail in the present paper by referring to the original literature and the recent review by one of the authors [32] where some comments on the corresponding connections have been made. In the present work we want to make a few comments on two recent papers [45,46] where the picture similar to the one presented in this work is advocated.

We start with [45]. In that work the authors consider a specifically deformed $SU(N)$ gluodynamics at $T \neq 0$. It has been shown that such a deformation supports a reliable analysis in the weak coupling regime in the confining phase. The results of the corresponding calculations imply that the relevant degrees of freedom in the confined phase are the self-dual magnetic monopoles with action $\frac{8\pi^2}{g^2N}$ and topological charges $Q = \pm 1/N$ which are precisely the features of the instanton quarks discussed above.

In contrast, the starting point of Ref. [46] is semiclassical calculations in the background of calorons [41] where the weak coupling regime cannot be guaranteed. Although the calculations are semiclassical in nature, and therefore, cannot be trusted in the strong coupling regime, still, the corresponding analysis shows how well localized instantons with integer topological charges at $T > T_c$ may dissociate into the fractional constituents at $T < T_c$ and become the key players in the confining phase. This is precisely the picture we are advocating in the present work based on the analysis of sharp θ changes at $N = \infty$. It is impressive how complicated semiclassical calculations carried out in [46] lead to the expression for the vacuum energy $E_{vac} \cos(\frac{\theta}{N})$ advocated in [40] using a completely different technique.[11] Our technique does not allow us to make any dynamical calculations in this phase as all color degrees of freedom have been integrated out in the course of obtaining (29). In other words, we cannot study the dynamics of fractionally charged constituents in con-

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[10]In [38] it was conjectured that these constituents (instanton quarks) are the driving force for the confinement.

[11]Such a SG structure was a crucial element for recovering the fractional topological charges $Q = \pm 1/N$ in the confining phase using η' as a probe, see Sec. V and original discussions in [38].

trast with papers [45,46].[12] However, the fact that the constituents carry fractional topological charge $1/N$ can be recovered in our approach because the color-singlet η' field enters the effective Lagrangian as $\cos(\frac{\theta-\varphi}{N})$ and serves as a perfect probe of the topological charges of the constituents. One should also emphasize that the procedure of the recovering of the fractional topological charge $1/N$ (which has been used here) is not based on the weak coupling expansion.

Our final comment in this subsection is as follows. The main ingredient in the holographic picture discussed in Sec. II was D0 brane wrapping around x^4 which behaves differently in confined and deconfined phases, and correspondingly leads to a different θ behavior on opposite sides of the phase transition line. A similar picture was also observed in Ref. [47] where the authors studied the D2 brane in confined and deconfined phases to arrive at the same conclusion on sharp changes in θ behavior. The topological objects (sensitive to θ) were identified as magnetic strings in Ref. [47]. These objects apparently have been observed in the lattice simulations [48].

VI. CONCLUSION

We explore the consequences of the assumption that in the large N QCD and QCD-like theories confinement-deconfinement phase transition takes place at the temperature where the dilute instanton calculation breaks down, and θ dependence drastically changes. This assumption is supported by holographic and field theoretic arguments. At very high temperatures, $T \gg \Lambda_{\text{QCD}}$ instantons are well localized configurations with a typical size $\rho \sim T^{-1} \ll \Lambda_{\text{QCD}}^{-1}$. As the temperature is getting lower the instanton size becomes of order $\rho \sim \Lambda_{\text{QCD}}^{-1}$, however, provided the perturbative corrections in the instanton background do not significantly change the picture, the instanton density remains dilute. Instanton expansion breaks down at T_c, and for $T < T_c$, the strong interacting regime and confinement are realized. Instantons are no longer well localized configurations for $T < T_c$, but rather, in the picture of Sec. V, they are represented by N instanton quarks which can propagate far away from each other. The presence of the light field η' in the model is important for this picture and a specific Lagrangian is assumed in Sec. V. The mass of η' in both phases in the dual picture can be thought of as the Debye screening mass generated by the Coulomb interaction of the topological charges.

We have made a number of assumptions in the field theoretic analysis to arrive at the conclusion that the θ

dependence changes sharply at some value of T_c. We also used the holographic model to argue that this transition coincides with confinement-deconfinement phase transition. This conclusion is supported by the lattice simulations [30,49–52].

The value of the critical temperature as a function of (sufficiently small) chemical potential $T_c(\mu)$ is estimated in Sec. III. The obtained expression is in excellent agreement with numerical computations [34–36]. Finally, we presented the arguments that this line of the phase transition (which is the first order for large N and $N_f \ll N$) continuously transforms into the line $\mu_c(T)$ studied at large μ in Ref. [27]. The argument is based on the observation that the physical nature of the phase transition along the entire line is the same: it is the drastic changes in θ behavior when the phase transition line is crossed.

It would be very interesting to see if Coulomb law between instantons can be understood holographically in the deconfining phase. As we mentioned in the text it is quite nontrivial that at large distances in the presence of the η' field the interaction between instantons and anti-instantons is the same (up to sign) as the interaction between two instantons.

Finally, we would like to make a short comment on the relevance of the present analysis to real QCD with $N_f = N = 3$. The main subject of the present paper is pure gluodynamics at large N, and therefore one cannot immediately apply the results of the present analysis to the real QCD with $N_f = N = 3$. However, we do expect that our results can be and should be compared with the lattice simulations for pure gluodynamics for $N \geq 3$ and for QCD with $N_f \ll N$ when the first order phase transition is expected, see [30,49–52]. For the case $N_f \sim N$ our technique is not applicable as we explained in Sec. III B. In this case one should expect that the properties of the phase transition is very sensitive to the details of the fermion matter fields. It a subject of a separate analysis which will not be discussed here. However, we expect that the picture of the phase transition as a transition between plasma phase (when the instanton quarks are in plasma state at $T < T_c$) and molecular phase (when the instanton quarks form a small instanton at $T > T_c$) qualitatively describes real QCD with $N = 3$ as the confinement in non-Abelian gauge theories is determined by the dynamics of gluon (not quark) degrees of freedom.

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[12]In particular, we do not see a beautiful picture of a multi-component color Coulomb plasma with nearest-neighbor inter-actions in the Dynkin space advocated in [45,46]. Still, we do see the color-singlet Coulomb interaction of the fractionally charged $\pm 1/N$ constituents due to η' at very large distances where color is already confined.

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