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Born-Infeld gravity in Weitzenböck spacetime

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Using the teleparallel equivalent of general relativity formulated in Weitzenböck spacetime, we thoroughly explore a kind of Born-Infeld regular gravity leading to second order field equations for the vielbein components. We explicitly solve the equations of motion for two examples: the extended Bañados-Teitelboim-Zanelli black hole, which exists even if the cosmological constant is positive, and a cosmological model with matter, where the scale factor is well behaved, thus giving a singularity-free solution.

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I. INTRODUCTION: ULTRAVIOLET CORRECTIONS TO GENERAL RELATIVITY (GR)

In the last decade a wide variety of modified theories of gravity have been studied with the aim of solving or smoothing some puzzling features of conventional gravity and cosmology such as singularities, particle horizons, accelerated expansion of the Universe, etc. Many of these modified theories of gravity consist in the mere deformation of the current theory. In this case, one starts from a known Lagrangian $\mathcal{L} = eL$, where L is invariant and e is a density under general coordinate changes, and then the theory is deformed by replacing the Lagrangian by $\mathcal{L}_D = ef(L)$. It is expected that a suitable choice of the function f will heal the unwanted features of the original theory. To explain the method, let us consider an invariant Lagrangian $L = L(\phi^a, \phi^a_{,\mu}, \phi^a_{,\mu\nu}, \ldots, x^\mu)$ and a density e that does not depend on the derivatives of the fields ϕ^a: $e = e(\phi^a, x^\mu)$ (this is because ϕ^a will later become a field describing the geometry, and so the density e will be the square root of the determinant of the metric). Thus the Euler-Lagrange equations for the deformed Lagrangian $\mathcal{L}_D = ef(L)$ are

$$\begin{aligned} 0 &= \ldots - \partial_\mu \partial_\nu \left(\frac{\partial \mathcal{L}_D}{\partial \phi^a_{,\mu\nu}} \right) + \partial_\mu \left(\frac{\partial \mathcal{L}_D}{\partial \phi^a_{,\mu}} \right) - \frac{\partial \mathcal{L}_D}{\partial \phi^a} \\ &= \ldots - \partial_\mu \partial_\nu \left(f'(L) \frac{\partial \mathcal{L}}{\partial \phi^a_{,\mu\nu}} \right) + \partial_\mu \left(f'(L) \frac{\partial \mathcal{L}}{\partial \phi^a_{,\mu}} \right) \\ &\quad - f'(L) \frac{\partial \mathcal{L}}{\partial \phi^a} + (Lf'(L) - f(L)) \frac{\partial e}{\partial \phi^a}. \end{aligned} \tag{1}$$

If the deformed Lagrangian is intended to modify only the strong field (large L) regime, then f should satisfy

$$f(L) \simeq L + O(L^2), \tag{2}$$

i.e.,

$$f(0) = 0, \qquad f'(0) = 1. \tag{3}$$

In general, Eqs. (1) will have solutions differing from those coming from the original Lagrangian $\mathcal{L} = eL$. However, it should be noted that not all the solutions of the original theory get deformed by this procedure. In fact, let us consider solutions of the original theory such that $L = 0$. In this case, by substituting $L = 0$ in (1) and using (3), the result is that the last term vanishes. Moreover, since $f'(0) = 1$, then those solutions of the original theory having $L = 0$ also solve the Euler-Lagrange equations for the deformed Lagrangian $\mathcal{L}_D$. In particular, the invariant L for GR is the curvature scalar R associated with the Levi-Civita connection, which is null for all the (vacuum) solutions. Thus general relativity is a quite rigid theory, because its vacuum solutions remain as solutions for the (vacuum) field equations of deformed theories $\mathcal{L}_D \propto \sqrt{-g} f(R)$, with f satisfying conditions (3). This is a rather singular feature which is not shared by other field theories. For instance, in Maxwell electromagnetism it is $L \propto E^2 - B^2$, and only some vacuum solutions—mainly plane waves—make the Maxwell Lagrangian null.

Contrasting with other theories, the general relativity Lagrangian $L \propto R$ contains second derivatives of the metric. In spite of this feature, the Einstein equations are second order because the fourth order terms in the Euler-Lagrange equations cancel out (in other words, second derivatives in L appear to just contribute to a divergence term in the action). This property is lost in the deformed theory $\mathcal{L}_D \propto \sqrt{-g} f(R)$, whose dynamical equations become fourth order, as it follows from Eq. (1). This undesirable fact is usually relieved by splitting the metric in a new metric tensor times a conformal factor depending on a scalar field; the scalar field becomes a constant when the ($f' = 1$) general relativity case is retrieved. This procedure allows us to reformulate an $f(R)$ theory as a Brans-Dicke–like scalar-tensor theory of gravity having $\omega = 0$ (metric formalism [1,2]) or $\omega = -3/2$ (Palatini formalism [3–5]); thus the new metric turns out to be governed by second

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order equations, and the extra degrees of freedom are placed in a scalar field fulfilling a second order equation too. However, the scalar-tensor reformulation of $f(R)$ theories results in violations of the weak equivalence principle, since matter and gravity would couple not only through the (new) metric but also through the scalar field [6,7]. Incidentally, we mention that not all the $f(R)$'s appearing in the literature fulfill the conditions (3); see, for instance, the $f(R)$ used to build the spherically symmetric vacuum solution in Ref. [8], or the one proposed in Refs. [4,9] to explain the acceleration of the universe as an effect of modified gravity at the low curvature regime (which, if regarded as a Brans-Dicke–like theory, can be dismissed on the basis of well-established post-Newtonian constraints [2,5,10]; the Newtonian limit in the Palatini formalism is not retrieved either [11]).

The problems inherent in the formulation of an $f(R)$ theory can be avoided by starting from an alternative theory of gravity whose Lagrangian only contains first derivatives of the dynamical variables. In a recent article [12] we have proposed to deform the teleparallel equivalent of general relativity (TEGR) [13]. As currently formulated, the TEGR is an alternative formulation of general relativity. Although the dynamical object of the theory is not the metric but the vielbein $e^a_\mu(x)$, the teleparallel action is invariant under local Lorentz transformations $\Lambda^{a'}_a(x)$ of the vielbein,

$$e^a_\mu(x) \longrightarrow e^{a'}_\mu(x) = \Lambda^{a'}_a(x) e^a_\mu(x), \tag{4}$$

which do not change the metric

$$g_{\mu\nu}(x) = \eta_{ab} e^a_\mu(x) e^b_\nu(x), \tag{5}$$

where $\eta_{ab} = \mathrm{diag}(1, -1, -1, \ldots)$. Since the TEGR action is not sensitive to some of the degrees of freedom of the vielbein, the theory can be driven to be equivalent to general relativity for the metric (5) [14,15]. The teleparallel Lagrangian is built from the torsion associated with the Weitzenböck connection [16],

$$\overset{W}{\Gamma}{}^\lambda_{\mu\nu} = e^\lambda_a \partial_\nu e^a_\mu = -e^a_\mu \partial_\nu e^\lambda_a, \tag{6}$$

where e^λ_a stands for the vielbein inverse matrix:

$$e^\mu_a e^b_\mu = \delta^b_a, \qquad e^\mu_a e^a_\nu = \delta^\mu_\nu. \tag{7}$$

The Weitzenböck connection has zero Riemannian curvature $\overset{W}{R}$, but non-null torsion:

$$T^\lambda{}_{\mu\nu} = \overset{W}{\Gamma}{}^\lambda_{\nu\mu} - \overset{W}{\Gamma}{}^\lambda_{\mu\nu} = e^\lambda_a(\partial_\mu e^a_\nu - \partial_\nu e^a_\mu). \tag{8}$$

The structure of the torsion tensor resembles the one of the electromagnetic field tensor. Moreover, like the Maxwell Lagrangian, the teleparallel Lagrangian density is quadratic in this tensor. In fact, the TEGR Lagrangian with a cosmological constant Λ is [17]

$$\mathcal{L}_{\mathbf{T}}[e^a_\mu(x)] = \frac{e}{16\pi G}(\mathbb{S}\cdot\mathbb{T} - 2\Lambda), \tag{9}$$

where e is the determinant of the matrix e^a_μ (which is equal to $\sqrt{-g}$), $\mathbb{S}\cdot\mathbb{T} \doteq S_\rho{}^{\mu\nu} T^\rho{}_{\mu\nu}$, and $S_\rho{}^{\mu\nu}$ is defined as

$$S_\rho{}^{\mu\nu} = -\tfrac{1}{4}(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}) + \tfrac{1}{2}\delta^\mu_\rho T^{\theta\nu}{}_\theta - \tfrac{1}{2}\delta^\nu_\rho T^{\theta\mu}{}_\theta. \tag{10}$$

While the Einstein-Hilbert Lagrangian depends on second derivatives of the metric, the teleparallel Lagrangian is built with just first derivatives of the vielbein, which makes the study of its deformation more attractive, in the sense that the field equations of the deformed theory will remain second order equations. The Euler-Lagrange equations for the Lagrangian $\mathcal{L}_{\mathbf{T}} + \mathcal{L}_{\mathrm{matter}}$ are

$$\partial_\sigma(e e^\lambda_a S_\lambda{}^{\nu\sigma}) - e e^\lambda_a S_\eta{}^{\mu\nu} T^\eta{}_{\mu\lambda} + \tfrac{1}{4} e e^\nu_a(\mathbb{S}\cdot\mathbb{T} - 2\Lambda) = 4\pi G e e^\lambda_a T_\lambda{}^\nu, \tag{11}$$

where $T_\lambda{}^\nu$ is the matter energy-momentum tensor. By contracting Eq. (11) with the inverse vielbein e^a_ν, one obtains for the vacuum solutions

$$4e^{-1} e^a_\nu \partial_\sigma(e e^\lambda_a S_\lambda{}^{\nu\sigma}) + (n-4)\mathbb{S}\cdot\mathbb{T} = 2n\Lambda, \tag{12}$$

where n is the spacetime dimension. In contrast to general relativity, where the Einstein equations compel R to vanish in vacuum (or to be a constant when the cosmological constant is included), Eq. (12) does not compel the invariant $\mathbb{S}\cdot\mathbb{T}$ to be null or constant for vacuum solutions, which raises the hope that a deformed teleparallelism could be useful to smooth singularities of vacuum general relativity solutions.

II. BORN-INFELD GRAVITY

Born-Infeld (BI) electrodynamics [18] has experienced a renewed interest in recent years due to its close connection with string theory, particularly because of its capability to describe the electromagnetic fields of D-branes [19,20]. Inspired by these fruitful properties, together with the ability of the BI program concerning the cure of singularities, we shall study a teleparallel theory of gravity deformed *à la* Born-Infeld. In a rather different approach, this subject has received some attention in the past [21–26], where several deformations *à la* Born-Infeld combining higher order invariants constructed with the curvature in a Riemannian context were tried. All these constructions, however, lead to troublesome fourth order field equations for the metric. As a matter of fact, explicit solutions in four dimensions within these frameworks were never found [27]. In a different direction, BI actions were explored more recently in Refs. [28,29] using the Palatini formalism, where the metric and the connection are taken as independent entities. In turn, along the lines of [12], we will work with the Lagrangian

$$\mathcal{L}_{\mathbf{BI}}[e^a_\mu(x)] = \frac{\lambda e}{16\pi G}\left[\sqrt{1+\frac{2(\mathbb{S}\cdot\mathbb{T}-2\Lambda)}{\lambda}}-1\right], \quad (13)$$

where λ is a constant that controls the scale at which the deformed solutions depart from the original ones: Lagrangian (13) tends to (9) when $\lambda \to \infty$. According to Eq. (1), the Euler-Lagrange equations become

$$\begin{aligned}\partial_\sigma&[(1+2\lambda^{-1}(\mathbb{S}\cdot\mathbb{T}-2\Lambda))^{-1/2} e e^\lambda_a S_\lambda{}^{\nu\sigma}]\\ &- (1+2\lambda^{-1}(\mathbb{S}\cdot\mathbb{T}-2\Lambda))^{-1/2} e e^\lambda_a S_\eta{}^{\mu\nu} T^\eta{}_{\mu\lambda}\\ &+ \frac{\lambda}{4} e e^\nu_a[(1+2\lambda^{-1}(\mathbb{S}\cdot\mathbb{T}-2\Lambda))^{1/2}-1]\\ &= 4\pi G e e^\lambda_a T_\lambda{}^\nu. \end{aligned} \quad (14)$$

In order to explore the aptitude of deformed teleparallelism to modify solutions of general relativity, we will try two types of examples: the Bañados-Teitelboim-Zanelli (BTZ) black hole and an n-dimensional cosmological model with matter. In the first example, both GR and teleparallel Lagrangians turn out to be constant for the chosen solution, and thus the deformation is limited to a shift of the cosmological constant. In spite of this, teleparallelism exhibits a better aptitude to deform the solution because it allows for a BTZ solution even for a positive cosmological constant. The strength of modified teleparallelism is, however, revealed in solutions with sources, where modified teleparallelism is able to control the growth of the Hubble parameter by avoiding the universe reaching a singularity in a finite time.

A. Extended BTZ black hole

The BTZ black hole is a vacuum solution for general relativity with a negative cosmological constant Λ in 2 + 1 dimensions [30]. The spinning BTZ metric is

$$\begin{aligned} ds^2 = &\left(-M-\Lambda r^2+\frac{J^2}{4r^2}\right)dt^2 - \left(-M-\Lambda r^2+\frac{J^2}{4r^2}\right)^{-1}dr^2\\ &- r^2\left(-\frac{J}{2r^2}dt+d\phi\right)^2, \end{aligned} \quad (15)$$

where M and J are integration constants related to the mass and the angular momentum, respectively. For $\Lambda = -\ell^{-2}$, $M > 0$, and $|J| \leq M\ell$, this metric has the structure of a rotating black hole. The BTZ black hole displays event horizons (the place where the lapse function vanishes) at [31]

$$r^\pm = \ell\left[\frac{M}{2} \pm \frac{M}{2}\sqrt{1-\left(\frac{J}{M\ell}\right)^2}\right]^{1/2}, \quad (16)$$

and the ergosphere (the place where g_{tt} vanishes) at

$$r^{\rm erg} = \ell M^{1/2} > r^+ > r^-. \quad (17)$$

The extremal case $|J| = M\ell$ corresponds to $r^+ = r^- = r_{\rm erg}/\sqrt{2}$. A suitable dreibein field for the metric (15) is given by

$$\begin{aligned} e^0 &= \left(-M-\Lambda r^2+\frac{J^2}{4r^2}\right)^{1/2}dt,\\ e^1 &= \left(-M-\Lambda r^2+\frac{J^2}{4r^2}\right)^{-1/2}dr,\\ e^2 &= -\frac{J}{2r}dt + r d\phi. \end{aligned} \quad (18)$$

This dreibein satisfies Eq. (11) for a vanishing energy-momentum tensor, and $\eta_{ab}e^a e^b$ reproduces the interval (15). Let us investigate how this solution is affected by a deformation of the theory. In order to understand the changes that the dreibein (18) has to undergo for becoming a solution of the deformed equations (14), let us note that the invariant $\mathbb{S}\cdot\mathbb{T}$ turns out to be constant for the dreibein (18): its value is -2Λ. Although $\mathcal{L}_{\mathbf{T}}$ is not zero, a vacuum solution like (18) which renders $\mathcal{L}_{\mathbf{T}}$ *constant* is very close to a vacuum solution of the deformed theory. In fact, let us modify solution (18) by replacing Λ with a new constant $\tilde\Lambda$. Then $\mathbb{S}\cdot\mathbb{T} = -2\tilde\Lambda$, so Eq. (14) turns out to be

$$\begin{aligned}\partial_\sigma&(e e^\lambda_a S_\lambda{}^{\nu\sigma}) - e e^\lambda_a S_\eta{}^{\mu\nu}T^\eta{}_{\mu\lambda} + \tfrac{1}{4} e e^\nu_a[\mathbb{S}\cdot\mathbb{T}\\ &- 2(2\Lambda+\tilde\Lambda) + \lambda - \lambda(1-4\lambda^{-1}(\Lambda+\tilde\Lambda))^{1/2}] = 0. \end{aligned} \quad (19)$$

Since the solution we are trying solves the teleparallel vacuum equation (11) for $\Lambda = \tilde\Lambda$, then it will solve Eq. (19) if $\tilde\Lambda$ is chosen such that

$$-2(2\Lambda+\tilde\Lambda) + \lambda - \lambda(1-4\lambda^{-1}(\Lambda+\tilde\Lambda))^{1/2} = -2\tilde\Lambda, \quad (20)$$

i.e.,

$$\tilde\Lambda = \Lambda(1-\epsilon), \qquad \epsilon = 4\Lambda/\lambda. \quad (21)$$

This solution represents a black hole if the effective cosmological constant $\tilde\Lambda$ is negative. Summarizing, the BTZ dreibein for the deformed gravity theory described by Lagrangian (13) is

$$\begin{aligned} e^0 &= \left(-M-\Lambda(1-\epsilon)r^2+\frac{J^2}{4r^2}\right)^{1/2}dt,\\ e^1 &= \left(-M-\Lambda(1-\epsilon)r^2+\frac{J^2}{4r^2}\right)^{-1/2}dr,\\ e^2 &= -\frac{J}{2r}dt + r d\phi, \end{aligned} \quad (22)$$

and the metric is

$$ds^2 = \left(-M - \Lambda(1-\epsilon)r^2 + \frac{J^2}{4r^2}\right)dt^2 - \left(-M - \Lambda(1-\epsilon)r^2 + \frac{J^2}{4r^2}\right)^{-1} dr^2 - r^2\left(-\frac{J}{2r^2}dt + d\phi\right)^2. \tag{23}$$

The dreibein (22) or the metric (23) genuinely differs from (15) and (18); these two metrics are not connected by a coordinate change because the invariant $\mathbb{S}\cdot\mathbb{T}$ is different for each one (also R is different). For a negative effective cosmological constant $\tilde{\Lambda} = -\tilde{\ell}^{-2}$, the solution is a rotating BTZ black hole. Thus, even for $\Lambda > 0$ the BI Lagrangian (13) allows for BTZ rotating black holes; specifically, the metric (23) is a rotating black hole for $\Lambda < 0$ and $\epsilon < 1$, but also for $\Lambda > 0$ and $\epsilon > 1$. The horizons are placed at

$$r_{\rm BI}^{\pm} = \frac{r_{\rm BI}^{\rm erg}}{2}\left[1 \pm \sqrt{1 + \Lambda(1-\epsilon)\left(\frac{J}{M}\right)^2}\right]^{1/2}, \qquad r_{\rm BI}^{\rm erg} = \sqrt{\frac{M}{-\Lambda(1-\epsilon)}}. \tag{24}$$

Let us compare the lengths of the horizons for the solutions (15) and (22) corresponding to fixed values of Λ, M, and J. Since the horizons are circles, we will study the ratio $r_{\rm BI}^{\pm}/r^{\pm}$. For $\tilde{\Lambda} < 0$, three ranges of the parameter ϵ can be distinguished in this comparison:

(i) *Type I*: $\epsilon < 0$ ($\Lambda < 0$, $\lambda > 0$). This type results in $r_{\rm BI}^{\rm erg}/r^{\rm erg} < 1$, $r_{\rm BI}^{+}/r^{+} < 1$, and $r_{\rm BI}^{-}/r^{-} > 1$; then the horizons approach each other as a consequence of the deformation.

(ii) *Type II*: $\epsilon > 1$ ($\Lambda > 0$). There is no black hole for the GR counterpart.

(iii) *Type III*: $0 < \epsilon < 1$ ($\Lambda < 0$, $\lambda < 0$). This type results in $r_{\rm BI}^{\rm erg}/r^{\rm erg} > 1$, $r_{\rm BI}^{+}/r^{+} > 1$, and $r_{\rm BI}^{-}/r^{-} < 1$; in this case the horizons move away from each other as a consequence of the deformation. However, the case $\lambda < 0$ will be rejected in the next section since it produces physically unacceptable solutions in cosmology.

This example shows the strategy to be followed to obtain deformed solutions when one starts from a Lagrangian having a "cosmological constant" term like the one in Eq. (9), i.e. $\mathcal{L} \propto e(L - 2\Lambda)$. If a given (vacuum) solution makes L a (Λ-dependent) constant, then replace Λ in the solution by a new constant $\tilde{\Lambda}$ and substitute the so built solution in the modified field equation. Using that L is constant, Eq. (1) becomes

$$\ldots - \partial_\mu\partial_\nu\left(e\frac{\partial L}{\partial \phi^a_{,\mu\nu}}\right) + \partial_\mu\left(e\frac{\partial L}{\partial \phi^a_{,\mu}}\right) - e\frac{\partial L}{\partial \phi^a} + \left(L - \frac{f(L-2\Lambda)}{f'(L-2\Lambda)}\right)\frac{\partial e}{\partial \phi^a} = 0. \tag{25}$$

The proposed solution now solves the undeformed Euler-Lagrange equations for the cosmological constant $\tilde{\Lambda}$. Therefore $\tilde{\Lambda}$ should be chosen in such a way that

$$L(\tilde{\Lambda}) - \frac{f(L(\tilde{\Lambda}) - 2\Lambda)}{f'(L(\tilde{\Lambda}) - 2\Lambda)} = 2\tilde{\Lambda}, \tag{26}$$

where $L(\tilde{\Lambda})$ is the Lagrangian evaluated on the proposed solution. This means that the deformation replaces the role of the cosmological constant in the solution for a new parameter depending also on the scale λ. Teleparallelism *à la* Born-Infeld [Lagrangian (13)] uses the function f,

$$f_{\mathbf{BI}}(x) = \lambda\sqrt{1 + \frac{2x}{\lambda}} - \lambda, \tag{27}$$

so, writing Eq. (26) for the function (27), one gets Eq. (20) and the solution (21).

Einstein equations with a cosmological constant imply $L = -R = 2\Lambda n/(n-2)$ for any vacuum solution in n spacetime dimensions. Therefore, vacuum solutions for theories $f(-R - 2\Lambda)$ can be straightforwardly obtained from general relativity vacuum solutions by shifting Λ to be

$$\frac{2n}{n-2}\tilde{\Lambda} - \frac{f(2\tilde{\Lambda}[n/(n-2)] - 2\Lambda)}{f'(2\tilde{\Lambda}[n/(n-2)] - 2\Lambda)} = 2\tilde{\Lambda}. \tag{28}$$

Contrasting with the teleparallel equation (12), the vacuum solutions of GR share the same value of $L(\Lambda)$. Thus, the effective cosmological constant (28) for modified GR is the same for all vacuum solutions.

Just for comparing with the Born-Infeld modified teleparallelism result (21), let us compute the modified GR solution (28) for the Born-Infeld deformation (27) in $n = 3$ dimensions. The result is

$$\tilde{\Lambda} = \frac{\Lambda}{2}\left[1 - \frac{1}{4\epsilon}(1 - \sqrt{1 + 8\epsilon})\right]. \tag{29}$$

Thus the Born-Infeld deformation for GR is well defined if $\epsilon > -1/8$, and the bracketed quantity in Eq. (29) is positive definite. This means that the effective cosmological constant keeps the sign of Λ. Therefore, in GR modified *à la* Born-Infeld, the BTZ black hole only exists for $\Lambda < 0$, which is different from the result obtained for modified teleparallelism.

B. Regular cosmology

The inability of the Einstein-Hilbert Lagrangian to allow for high energy deformed solutions not only embraces the vacuum solutions but also any GR ($\Lambda = 0$) solution satisfying $R = 0$. This assertion remains valid even if there are sources. For instance, the Friedmann-Robertson-Walker (FRW) solution for a radiation fluid cannot be smoothly deformed, because the energy-momentum tensor is traceless and so it is $R = 0$. In contrast, the teleparallel Lagrangian (9) does not vanish in this case; thus teleparallelism allows for a smooth deformation of such kinds of solutions [12].

Let us study the deformation (13) in the context of a spatially flat FRW geometry in the presence of a homogeneous and isotropic fluid. Then the source is represented by the stress-energy tensor $T^{\mu}{}_{\nu} = \mathrm{diag}(\rho(t), -p(t), -p(t), \ldots)$ in the comoving reference frame. The teleparallel equations can be solved by considering the vielbein

$$e^a_\mu = \mathrm{diag}(1, a(t), a(t), \ldots), \qquad e = a^{n-1}, \tag{30}$$

leading to the metric $g_{\mu\nu} = \mathrm{diag}(1, -a(t)^2, -a(t)^2, \ldots)$. In this case the only non-null components of $\mathbb{S}$ and $\mathbb{T}$ are

$$\begin{aligned} S_{\alpha 0\alpha} &= -S_{\alpha\alpha 0} = -\tfrac{1}{2}(n-2)a(t)\dot{a}(t), \\ T_{\alpha 0\alpha} &= -T_{\alpha\alpha 0} = a(t)\dot{a}(t), \qquad \alpha \neq 0. \end{aligned} \tag{31}$$

Thus $\mathbb{S}\cdot\mathbb{T} = -(n-1)(n-2)H^2$, $H = \dot{a}(t)/a(t)$ being the Hubble parameter, which is not null or constant whenever a source is present. The first term in Eq. (14) for the indices $a = 0 = \nu$ is null; then the *initial value* equation for the modified FRW cosmology is

$$\frac{1-\epsilon}{\left(1-\epsilon-2(n-1)(n-2)\frac{H^2}{\lambda}\right)^{1/2}} - 1 = \frac{16\pi G}{\lambda}\rho. \tag{32}$$

The isotropy of the proposed solution makes Eqs. (14) equal for spatial indexes $a = \nu$; they are

$$\begin{aligned} &(1-\epsilon)\left(2(n-2)q\frac{H^2}{\lambda} + 2n(n-2)\frac{H^2}{\lambda} - 1 + \epsilon\right) \\ &\quad \times\left(1-\epsilon-2(n-1)(n-2)\frac{H^2}{\lambda}\right)^{-3/2} + 1 = \frac{16\pi G}{\lambda}p \end{aligned} \tag{33}$$

In the last expression $q = -a\dot{a}^{-2}\ddot{a} = -(1+\dot{H}H^{-2})$ is the deceleration parameter. Equations (32) and (33) lead to the energy-momentum conservation. In fact, by differentiating the initial value equation (32) with respect to time, and combining this result with Eq. (33), one gets

$$\begin{aligned} &\frac{d}{dt}(\rho a^{n-1}) = -p\frac{d}{dt}a^{n-1} \quad \text{or} \\ &\dot{\rho} + (n-1)(\rho+p)H = 0. \end{aligned} \tag{34}$$

For a barotropic fluid satisfying the state equation $p = \omega(n)\rho$, the conservation law (34) leads to the behavior

$$\rho(t) = \rho_o\left(\frac{a_o}{a(t)}\right)^{(n-1)(1+\omega)}. \tag{35}$$

Equations (32) and (33) in vacuum ($\rho = p = 0$) have the solution $H = \pm H_o$, $q = -1$, for the constant $H_o^2 = 2\Lambda(1-\epsilon)/[(n-1)(n-2)]$. In this case the result is the de Sitter metric for the effective cosmological constant $\tilde{\Lambda} = \Lambda(1-\epsilon)$. The similarity with the shift of the previous section comes from the fact that the invariant is $\mathbb{S}\cdot\mathbb{T} = -2\tilde{\Lambda}$ in both cases.

In the presence of a barotropic fluid, the system (32) and (35) can be rewritten by using the variable

$$y = \ln\left[\left(\frac{a}{a_o}\right)^{(n-1)(1+\omega)}\right] \Rightarrow \dot{y} = (n-1)(1+\omega)H. \tag{36}$$

Thus the dynamics of the spatially flat FRW universe in Born-Infeld teleparallelism is described by the equation

$$\begin{aligned} &\frac{2(n-2)}{(n-1)(1+\omega)^2}\dot{y}^2 + \frac{\lambda(1-\epsilon)^2}{(1+16\pi G\rho_o\lambda^{-1}\exp(-y))^2} \\ &\quad = (1-\epsilon)\lambda, \end{aligned} \tag{37}$$

whose GR ($\lambda \to \infty$) limit is

$$\frac{2(n-2)}{(n-1)(1+\omega)^2}\dot{y}^2 - 32\pi G\rho_o\exp(-y) = 4\Lambda. \tag{38}$$

The variable y is monotonically increasing with the scale factor $a(t)$. So the behavior of the scale factor can be read directly from the "energy conservation" equations (37) and (38). As known, the effective potential for a spatially flat FRW universe in GR expands forever for $\Lambda > 0$ and recollapses for $\Lambda < 0$. On the other hand, the Born-Infeld teleparallel potential for $\lambda > 0$ is an increasing function, vanishing for $y \to -\infty$ ($a \to 0$) and going to $\lambda(1-\epsilon)^2$ for $y \to \infty$. Since the energy level in Eq. (37) is $\lambda(1-\epsilon)$, then (I) the universe recollapses if $1-\epsilon > 1$ (i.e. $\Lambda < 0$), or (II) it expands forever if $0 < 1-\epsilon < 1$ (i.e. $0 < \Lambda < \lambda/4$). Although this behavior does not seem to differ considerably from the GR one, it should be emphasized that the main difference lies in the behavior of the Hubble parameter when $y \to -\infty$: while H diverges in GR, in Born-Infeld teleparallelism H goes to the constant value

$$H^2 \to \frac{(1-\epsilon)\lambda}{2(n-1)(n-2)} = \frac{\lambda - 4\Lambda}{2(n-1)(n-2)}. \tag{39}$$

For $\lambda < 0$ and $\epsilon \neq 1$ the effective potential becomes an infinite well. This is an unphysical feature, since it leads H to diverge for $16\pi G\rho = |\lambda|$ [see Eq. (32)]. Therefore we will only consider the case $\lambda > 0$.

The dependence on time of the scale factor can be obtained from the initial value Eq. (32) or, equivalently, from (37). We will use the variable

$$\mathrm{y} = \frac{\lambda}{16\pi G\rho_o}\left(\frac{a(t)}{a_o}\right)^{(n-1)(1+\omega)}. \tag{40}$$

In this way, the initial value equation takes the form

$$\dot{\mathrm{y}} = \pm\mathcal{A}\frac{\mathrm{y}}{1+\mathrm{y}}\sqrt{1+2\mathrm{y}+\epsilon\mathrm{y}^2}, \tag{41}$$

with $\mathcal{A} = (1+\omega)\sqrt{\frac{\lambda(1-\epsilon)(n-1)}{2(n-2)}}$ a non-null constant. The solution of (41) can be obtained in a closed but implicit form by direct integration and depends on the sign and range of the parameter ϵ. Concretely, we have two types of solutions:

(i) *Type I*: $\epsilon < 0$ ($\Lambda < 0$) (the universe recollapses),

$$\pm \mathcal{A} t \pm c = \mathcal{F}(y) - \frac{1}{(-\epsilon)^{1/2}} \arcsin\left[\frac{1 + \epsilon y}{\sqrt{1-\epsilon}}\right]. \tag{42}$$

(ii) *Type II*: $0 < \epsilon < 1$ ($\Lambda > 0$) (the universe expands forever),

$$\mathcal{A} t + c = \mathcal{F}(y) + \frac{1}{\sqrt{\epsilon}} \times \ln\left[\frac{1 + \epsilon y}{\sqrt{\epsilon}} + \sqrt{1 + 2y + \epsilon y^2}\right], \tag{43}$$

where the function $\mathcal{F}(y)$ is, in both cases,

$$\mathcal{F}(y) = \ln\left[\frac{y}{1 + y + \sqrt{1 + 2y + \epsilon y^2}}\right]. \tag{44}$$

In Eq. (42) the sign $\pm$ corresponds to the expanding and the collapsing branches, respectively. Both branches can be joined at $t = 0$ by choosing the integration constant c to equalize the right member for the maximum scale factor. According to Eq. (41) the maximum scale factor ($\dot{y} = 0$) is

$$y_{max} = \frac{1 + \sqrt{1-\epsilon}}{-\epsilon}, \tag{45}$$

and thus

$$c = -\ln\left[1 - \frac{\epsilon}{1 + \sqrt{1-\epsilon}}\right] + \frac{\pi/2}{(-\epsilon)^{1/2}}. \tag{46}$$

Figure 1 shows the *type I* recollapsing case. The scale factor as a function of time is depicted for a radiation fluid in three dimensions, i.e. $\omega = 1/2$ and $n = 3$. Besides, we set $16\pi G\rho_o = 1$ and $\Lambda = -1$. The upper, middle, and lower (dashed) curves correspond to $\epsilon = -1$ ($\lambda = 4$), $\epsilon = -0.1$ ($\lambda = 40$), and GR ($\lambda \rightarrow \infty$), respectively. Note that the GR scale factor exists only for $-1 \leq t \leq 1$, whereas it exists for all values of time in the BI case.

Physically more relevant, at least in four dimensions, is the *type II* case where the cosmological constant is positive. In this case, Eq. (43) says that the late time behavior ($y \rightarrow \infty$) of the scale factor is $a(t) \propto \exp[\sqrt{\frac{2\Lambda(1-\epsilon)}{(n-1)(n-2)}}t]$, while the initial stage is described by $a(t) \propto \exp[\sqrt{\frac{\lambda(1-\epsilon)}{2(n-1)(n-2)}}t]$ [see Eq. (39)]. Thus, the universe evolves from an inflationary stage, driven by the (vacuum-like) energy $\lambda(1 - \epsilon)$, to another exponential epoch ruled by the vacuum energy $\Lambda(1 - \epsilon)$ (a similar de Sitter-de Sitter evolution was obtained in a quite different approach in Ref. [32]). Since ϵ should be very small in order that the theory does not appreciably differ from GR for most of the history of the universe (see a lower bound for λ in Ref. [12]), one concludes that in four dimensions the scale factor evolves in time as

$$a(t \rightarrow -\infty) \propto e^{\sqrt{(\lambda/12)}t} \rightsquigarrow a(t \rightarrow \infty) \propto e^{\sqrt{(\Lambda/3)}t}. \tag{47}$$

Finally, the limiting case $\epsilon = 1$ corresponds to the scale factor being constant, as follows from Eq. (37).

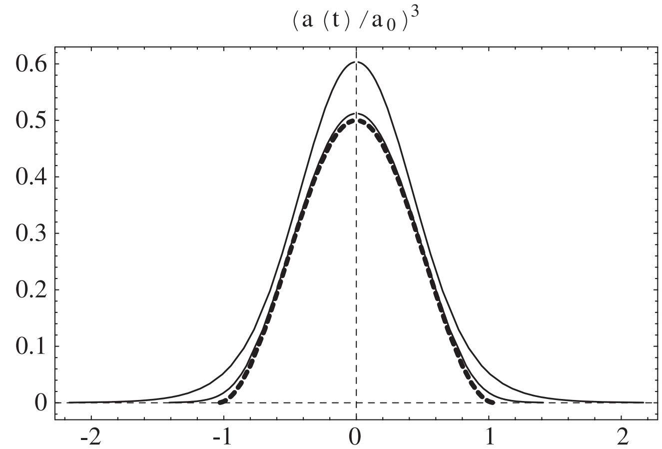

FIG. 1. Behavior of the nondimensional cubed scale factor $(a(t)/a_0)^3$ as emerges from (42) for $\omega = 1/2$, $\Lambda = -1$ in $n = 3$ dimensions. The upper curve is for $\epsilon = -1$, the middle curve is for $\epsilon = -0.1$, and the dashed one corresponds to GR.

III. CONCLUDING COMMENTS

In spite of having different causal structures, locally the BTZ black hole is the anti-de Sitter spacetime [31], i.e. the maximally symmetric solution with a negative cosmological constant. When evaluated on maximally symmetric solutions, both R and $\mathbb{S} \cdot \mathbb{T}$ Lagrangians are equal to a constant that is independent of the integration constants: it only depends on Λ. So, in both theories, general relativity and teleparallelism, the deformation of these maximally symmetric solutions only amounts to a shifting of the cosmological constant. The shifting is controlled by the nondimensional parameter $\epsilon = 4\Lambda/\lambda$, where λ is a Born-Infeld–like constant going to infinity when the deformed theory approaches the original one. The cosmological constant Λ and the (shifted) effective cosmological constant $\tilde{\Lambda}$ can have opposite signs in deformed teleparallelism, so the anti-de Sitter solution can solve the deformed teleparallel equations even for a positive cosmological constant.

On the other hand, we have studied the deformation of nonvacuum cosmological solutions. Although the parameter ϵ takes part in Eq. (41), its presence does not alter the nondeformed result that spatially flat Friedmann-Robertson-Walker nonvacuum solutions expand for $\Lambda > 0$ and recollapse for $\Lambda < 0$. Instead, the deformed theory smooths the initial singularities, which is the effect pursued by Born-Infeld deformations. In fact, the Hubble parameter goes to a constant when the scale factor a vanishes [see Eq. (39)]. This value is also the maximum value that can be attained by H [see Eq. (32)].

The BI approach (13) generates regular solutions. In the cosmological setting this is so, not only because the scale factor is always different from zero, but because the geometrical invariants (both in Riemannian and Weitzenböck spacetimes) are bounded for all finite proper times. In fact, each invariant in Weitzenböck spacetime that is quadratic in the torsion tensor has to be proportional to H^2 in the setting (30) [see Eq. (31)]. On the other hand, the Riemannian invariants for the metric $g_{\mu\nu} = \mathrm{diag}(1, -a(t)^2, -a(t)^2, \ldots)$ can be cast in the polynomial form $\mathcal{P} = (H, \dot{H})$. For instance, in four dimensions, the scalar curvature is $R = 6(2H^2 + \dot{H})$, the squared Ricci scalar $R^2_{\mu\nu} = R^{\mu\nu}R_{\mu\nu}$ is $R^2_{\mu\nu} = 12(3H^4 + 3H^2\dot{H} + \dot{H}^2)$, and the Kretschmann invariant $K = R^{\alpha}{}_{\beta\gamma\delta}R_{\alpha}{}^{\beta\gamma\delta}$ reads $K = 12(2H^4 + 2H^2\dot{H} + \dot{H}^2)$. All these invariants are well behaved due to the saturation value (39) that the Hubble parameter reaches as $a(t) \to 0$. Regarding this matter, the time derivative of Eq. (41) combined with the definition given in Eq. (40) shows that

$$\dot{H} = -\alpha \frac{\mathrm{y}^2}{(1 + \mathrm{y})^3}, \tag{48}$$

where $\alpha = \lambda(1 + \omega)(1 - \epsilon)^2/2(n - 2)$ is a non-null constant. By setting $n = 4$, $\omega = 1/3$, and $\epsilon = 0$, one finds the following expressions for the invariants:

$$\begin{aligned} R &= \lambda\left[\frac{1 + 3\mathrm{y}}{(1 + \mathrm{y})^3}\right], \\ R^2_{\mu\nu} &= \frac{\lambda^2}{12}\left[\frac{3 + 18\mathrm{y} + 27\mathrm{y}^2 + 4\mathrm{y}^4}{(1 + \mathrm{y})^6}\right], \\ K &= \frac{\lambda^2}{6}\left[\frac{1 + 6\mathrm{y} + 9\mathrm{y}^2 + 4\mathrm{y}^4}{(1 + \mathrm{y})^6}\right]. \end{aligned} \tag{49}$$

This means that the BI parameter λ not only bounds the dynamics of $H(t)$ and characterizes the minimum density for having inflation [12], but also establishes a maximum attainable curvature.

One might wonder if the BI framework considered here can be viewed as a particular case of a more general determinantal Born-Infeld action for gravity. Indeed, following the BI action more closely, one could try the n-dimensional determinantal action

$$I_{\mathbf{BIG}} = \lambda \int d^n x\left[\sqrt{\det(g_{\mu\nu} + \lambda^{-1}F_{\mu\nu})} - \sqrt{\det(g_{\mu\nu})}\right], \tag{50}$$

where $F_{\mu\nu}$ is quadratic in the Weitzenböck torsion: $F_{\mu\nu} = AS_{\mu\lambda\rho}T_{\nu}{}^{\lambda\rho} + BS_{\lambda\mu\rho}T^{\lambda}{}_{\nu}{}^{\rho}$, A and B being nondimensional constants. Such a combination ensures the correct GR limit since both constituents of $F_{\mu\nu}$ have trace $\mathbb{S} \cdot \mathbb{T}$. Besides, the dynamical equations coming from (50) will still be of second order in the vielbein derivatives. For the choice $2A + B = 0$, the action (50) reproduces the solutions considered in the last section, though the equivalence to the scheme (13) for other solutions is not clear yet [33]. The complete characterization of the theory (50) for the whole parameter space (A, B) will be a matter for future developments.

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