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Uniqueness of Kerr space-time near null infinity

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We reexpress the Kerr metric in standard Bondi-Sachs coordinates near null infinity $\mathcal{I}^+$. Using the uniqueness result of the characteristic initial value problem, we prove the Kerr metric is the only asymptotically flat, stationary, axially symmetric, type-D solution of the vacuum Einstein equation. The Taylor series of Kerr space-time is expressed in terms of Bondi-Sachs coordinates, and the Newman-Penrose constants have been calculated.

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I. INTRODUCTION

Since the work by Bondi *et al.* [1], it is well known that the Bondi coordinates are a very natural choice when we want to describe the asymptotic behavior of a gravitational field near null infinity $\mathcal{I}^+$. Based on works by Penrose, Newman, and Unti [2,3], there is an elegant way to reexpress Bondi's work in the Newman-Penrose (N-P) formalism. This also gives us a general formalism to describe the asymptotic structure of general asymptotic flat space-time which is smooth enough near $\mathcal{I}^+$. Using the characteristic initial value (CIV) problem method, many authors [4–6] have shown the existence of null infinity in a general case and pointed out the degree of freedom of the gravitational field near null infinity. The CIV method has many advantages in dealing with the gravitational radiation problem. Recently, this method has been used in numerical relativity. Winicour and his colleagues developed the Cauchy-characteristic matching (CCM) method. They wanted to combine the CIV method and the standard Hamiltonian evolution method [7]. On the other hand, the Kerr solution is a very important exact solution of the Einstein equation both in the theoretical area and in application. An interesting question is whether the Kerr metric describes the space-time outside a stationary rotating star. For a long time, the Bondi coordinates of Kerr space-time were not very clear. For example, how does one describe the Kerr space-time in Unti-Newman's general formalism [3]? The uniqueness theorem [8,9] tells us that the Kerr solution is the only asymptotically flat, stationary, axially symmetric solution of the vacuum Einstein equation with a regular event horizon. From an application point of view, it is very difficult to get detailed information about the event horizon of a space-time because of the infinite redshift near the horizon. An interesting question is how to identify the Kerr solution based on information near null infinity. This is more practical in future gravitational experiments. This idea can also be understood from the Geroch conjecture [10]. Obviously, stationary and axially symmetric conditions are not enough because there are many asymptotically flat exact solutions of the Ernst equation. In the next section, it is found that such uncertainty comes from the homogeneous part of a control equation which comes from the Killing equation. The general solutions of that equation contain some free constants. These unknown constants are closely related to Geroch-Hansen multipole moments [11]. In order to identify the Kerr solution, we use Petrov classification [12] and show that this condition will help us to finally pick out the Kerr solution. Furthermore, N-P constants for Kerr space-time are also calculated as a by-product.

This paper is organized as following: In Sec. II, we prove a local uniqueness theorem of the Kerr solution based on the information near null infinity. This theorem also tells us the standard Unti-Newman expansion of the Kerr metric. The detailed expression of this extension is contained in Appendix A. Appendix B contains some spin-weight harmonics which are useful for our calculation.

II. MAIN THEOREM

Let (M, g) be an asymptotically flat space-time and (u, r, θ, φ) be the standard Bondi-Sachs (B-S) coordinates.

Theorem 1.—Suppose (M, g) is an asymptotically flat, stationary, axially symmetric, type-D, vacuum space-time in a neighborhood of null infinity; then it is isometric to Kerr space-time in the neighborhood of the Bondi coordinates.

In order to make the proof clear enough, we divide it into two subsections. In the first subsection, we calculate the Taylor series of general, stationary, axially symmetric space-time in Bondi coordinates. We find that all Taylor coefficients can be expressed in terms of $\{\Psi_0^k\}$ and their derivatives. Unknown functions $\{\Psi_0^k\}$ satisfy a linear inhomogeneous second order partial differential equation. We find that the general solution of such an equation has the form $\Psi_0^k = \tilde{\Psi}_0^k + D^k{}_2Y_{k+2,0}$, where $\tilde{\Psi}_0^k$ is the special solution which corresponds to the Kerr metric and D^k is a

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free constant; i.e. all axially symmetric and stationary solutions are characterized by the set of constants $\{D^k\}$. This set of constants is closely related to the famous Geroch-Hansen multipole. In Sec. II B, with the help of the type-D condition, we show that all $\{D^k\}$ vanish and the Kerr metric is the unique, stationary, axially symmetric, asymptotically flat type-D metric.

A. Taylor series of general, axially symmetric, vacuum, stationary space-time

Suppose (M, g) is a vacuum, stationary, axially symmetric space-time. t^a and ϕ^a are two commutative Killing vectors. Near $\mathcal{I}^+$, we use the standard B-S coordinates to do the standard asymptotic expansion. The detailed construction of these coordinates is well known and can be found in Refs. [2,3]. With this choice of coordinates, we can also choose a set of null tetrads $\{l^a, n^a, m^a, \bar{m}^a\}$, such that $l^a = (\frac{\partial}{\partial r})^a$, and these tetrads are parallel transported along l^a. Under such a choice of coordinates, $\phi^a = (\frac{\partial}{\partial \varphi})^a$. The timelike Killing vector t^a can be expressed in terms of a null tetrad as $t^a = Tl^a + Rn^a + A\bar{m}^a + \bar{A}m^a$. $[t^a, \phi^a] = 0$ means that T, R, A are independent of φ. The null tetrad components of the Killing equation for t^a are

$$-DR + (\varepsilon + \bar{\varepsilon})R + \bar{\kappa}A + \kappa\bar{A} = 0, \qquad -DT - (\varepsilon + \bar{\varepsilon})T - \pi A - \bar{\pi}\bar{A} - D'R + (\gamma + \bar{\gamma})R + \bar{\tau}A + \tau\bar{A} = 0,$$
$$-\kappa T + \bar{\pi}R + DA + (\bar{\varepsilon} - \varepsilon)A - \delta R + (\bar{\alpha} + \beta)R + \bar{\rho}A + \sigma\bar{A} = 0, \qquad -D'T - (\gamma + \bar{\gamma})T - \nu A - \bar{\nu}\bar{A} = 0,$$
$$-\tau T + \bar{\nu}R + D'A + (\bar{\gamma} - \gamma)A - \delta T - (\bar{\alpha} + \beta)T - \mu A - \bar{\lambda}\bar{A} = 0, \qquad -\sigma T + \bar{\lambda}R + \delta A + (\bar{\alpha} - \beta)A = 0,$$
$$-\rho T + \mu R + \delta\bar{A} - (\bar{\alpha} - \beta)\bar{A} - \bar{\rho}T + \bar{\mu}R + \bar{\delta}A - (\alpha - \bar{\beta})A = 0.$$

Using the Bondi gauge $\kappa = \varepsilon = \pi = 0$, $\rho = \bar{\rho}$, $\tau = \bar{\alpha} + \beta$, we have

$$-DR = 0, \tag{1}$$

$$-DT - D'R + (\gamma + \bar{\gamma})R + \bar{\tau}A + \tau\bar{A} = 0, \tag{2}$$

$$DA - \delta R + \tau R + \bar{\rho}A + \sigma\bar{A} = 0, \tag{3}$$

$$-D'T - (\gamma + \bar{\gamma})T - \nu A - \bar{\nu}\bar{A} = 0, \tag{4}$$

$$-\tau T + \bar{\nu}R + D'A + (\bar{\gamma} - \gamma)A - \delta T - \tau T - \mu A - \bar{\lambda}\bar{A} = 0, \tag{5}$$

$$-\sigma T + \bar{\lambda}R + \delta A + (\bar{\alpha} - \beta)A = 0, \tag{6}$$

$$-\rho T + \mu R + \delta\bar{A} - (\bar{\alpha} - \beta)\bar{A} - \bar{\rho}T + \bar{\mu}R + \bar{\delta}A - (\alpha - \bar{\beta})A = 0. \tag{7}$$

Here we use the standard notation of [2,3,12]. Differential operators in the above equations are defined as

$$D := \frac{\partial}{\partial r}, \qquad D' := \frac{\partial}{\partial u} + U\frac{\partial}{\partial r} + X\frac{\partial}{\partial \zeta} + \bar{X}\frac{\partial}{\partial \bar{\zeta}},$$
$$\delta := \omega\frac{\partial}{\partial r} + \xi^3\frac{\partial}{\partial \zeta} + \xi^4\frac{\partial}{\partial \bar{\zeta}}, \qquad \zeta = e^{i\varphi}\cot\frac{\theta}{2}. \tag{8}$$

It is well known that stationary solutions to Einstein's vacuum field equations are analytic [13]. Moreover, it is also known that asymptotically flat, stationary, vacuum solutions are not only analytic, but even admit an analytic conformal extension through null infinity [14,15]. Keeping this result in mind, all geometric quantities (the coordinate components of the null tetrad, N-P coefficients, components of the timelike Killing vector, and components of Weyl curvature) can be expressed in terms of a power series of $\frac{1}{r}$; for example,

$$T = T^0 + \frac{T^1}{r} + \cdots, \qquad A = A^0 + \frac{A^1}{r} + \cdots. \tag{9}$$

Some lower order Taylor coefficients of the components of the tetrad, N-P coefficients, and components of the Weyl tensor (up to third order) can be found in Sec. 9.8 of [2].

First of all, let us consider the function R. Equation (1) and the axially symmetric condition tell us that $R = R(u, \theta)$. With the formal expansion of the null tetrad and N-P coefficients [2,3], the zeroth order of Eq. (2) gives

$$\frac{\partial R}{\partial u} = 0, \tag{10}$$

so $R = R(\theta, \phi)$. In order to get more information about R, a higher order of the Killing equation is needed. The first order of Eqs. (3), (5), and (7) is

$$\delta_0 R + A^0 = 0, \tag{11}$$

$$-\bar{\Psi}_3^0 R + \dot{A}^1 - \delta_0 T^0 + \tfrac{1}{2}A^0 - \dot{\sigma}^0\bar{A}^0 = 0, \tag{12}$$

$$2T^0 - R = 0, \tag{13}$$

where $\delta_0 = \frac{(1+\zeta\bar{\zeta})}{\sqrt{2}}\frac{\partial}{\partial\bar{\zeta}}$ and the overdot means ∂_u. Because the space-time is stationary, there is no Bondi flux. This implies $\dot{\sigma}^0 = 0$ [2]; then the N-P equations tell us that this leads to $\Psi_3^0 = 0$. Combining this condition with (11)–(13), we get

$$\dot{A}^1 + A^0 = 0. \tag{14}$$

The second order of Eq. (3) is

$$2A^1 = \frac{\zeta(1+\zeta\bar{\zeta})}{\sqrt{2}\bar{\zeta}}\sigma^0\frac{\partial R}{\partial \zeta} + \sigma^0\bar{A}^0. \quad (15)$$

It has been shown that the right-hand side of the above equation is independent of u, so $\dot{A}^1 = 0$. This implies $A^0 = 0$ and R is a constant. So we can take $R = 1$ without loss of generality.

With $R = 1$, the Killing equations become

$$-DT + (\gamma + \bar{\gamma}) + \bar{\tau}A + \tau\bar{A} = 0, \quad (16)$$

$$DA + \tau + \bar{\rho}A + \sigma\bar{A} = 0, \quad (17)$$

$$-D'T - (\gamma + \bar{\gamma})T - \nu A - \bar{\nu}\bar{A} = 0, \quad (18)$$

$$-\tau T + \bar{\nu} + D'A + (\bar{\gamma} - \gamma)A - \delta T - \tau T - \mu A - \bar{\lambda}\bar{A} = 0, \quad (19)$$

$$-\sigma T + \bar{\lambda} + \delta A + (\bar{\alpha} - \beta)A = 0, \quad (20)$$

$$-\rho T + \mu + \delta\bar{A} - (\bar{\alpha} - \beta)\bar{A} - \bar{\rho}T + \bar{\mu} + \bar{\delta}A - (\alpha - \bar{\beta})A = 0. \quad (21)$$

With Eq. (9), the nontrivial zero order Killing equation is

$$\dot{T}^0 = 0. \quad (22)$$

The nontrivial first order Killing equations are

$$-A^0 = 0, \quad (23)$$

$$-\dot{T}^1 + \Psi_3^0 A^0 + \bar{\Psi}_3^0\bar{A}^0 = 0, \quad (24)$$

$$-\bar{\Psi}_3^0 + \dot{A}^1 + \tfrac{1}{2}A^0 - \dot{\bar{\sigma}}^0 A^0 = 0, \quad (25)$$

$$\dot{\bar{\sigma}}^0 + \delta_0 A^0 + 2\bar{\alpha}^0 A^0 = 0, \quad (26)$$

$$2T^0 - 1 = 0, \quad (27)$$

which implies $\dot{T}^1 = 0$ and $\dot{A}^1 = 0$. The same order N-P equations also give $\Psi_3^0 = 0$ and $\Psi_4^0 = 0$.

The second order Killing equations are

$$T^1 - \tfrac{1}{2}(\Psi_2^0 + \bar{\Psi}_2^0) = 0, \quad (28)$$

$$-2A^1 = 0, \quad (29)$$

$$\dot{T}^2 = 0, \quad (30)$$

$$\tfrac{1}{2}\eth\Psi_2^0 + \dot{A}^2 - \delta_0 T^1 = 0, \quad (31)$$

$$\tfrac{1}{2}\sigma^0 = 0, \quad (32)$$

$$2T^1 - \sigma^0\dot{\bar{\sigma}}^0 - \bar{\sigma}^0\dot{\sigma}^0 - \Psi_2^0 - \bar{\Psi}_2^0 = 0, \quad (33)$$

where $\eth$ is the spin-weight operator [2] and is defined as $\eth f := (\delta_0 + 2s\bar{\alpha}^0)f$, $\alpha^0 = -\frac{\cot\theta}{2\sqrt{2}}$. From these equations, we know $\sigma^0 = 0$, $A^1 = 0$, $\dot{T}^2 = 0$, $T^1 = \frac{1}{2}(\Psi_2^0 + \bar{\Psi}_2^0)$, $\Psi_2^0 = \bar{\Psi}_2^0$, $\dot{\Psi}_2^0 = 0$, $\dot{A}^2 = -\frac{1}{2}\eth\Psi_2^0 + \frac{1}{2}\delta_0(\Psi_2^0 + \bar{\Psi}_2^0)$. It is worth pointing out that the result $\dot{\sigma}^0 = 0$ tells us that the Bondi coordinates, which are chosen as in [3], are associated with the "good cut" of stationary space-time, i.e. $\sigma^0 = 0$, so the freedom of supertranslation has been removed.

The third order Killing equations are

$$2T^2 + \tfrac{1}{3}(\bar{\eth}\Psi_1^0 + \eth\bar{\Psi}_1^0) = 0, \quad (34)$$

$$-3A^2 - \tfrac{1}{2}\Psi_1^0 = 0, \quad (35)$$

$$\dot{T}^3 = 0, \quad (36)$$

$$\tfrac{1}{2}\Psi_1^0 + \bar{\nu}^3 + \dot{A}^3 + \tfrac{3}{2}A^2 - \delta_0 T^2 = 0, \quad (37)$$

$$\delta_0 A^2 + 2\bar{\alpha}^0 A^2 = 0, \quad (38)$$

$$2T^2 + \tfrac{1}{2}\bar{\eth}\Psi_1^0 + \tfrac{1}{2}\eth\bar{\Psi}_1^0 + \delta_0\bar{A}^2 - 2\bar{\alpha}^0\bar{A}^2 + \bar{\delta}_0 A^2 - 2\alpha^0 A^2 = 0. \quad (39)$$

Equations (35) and (38) imply

$$\eth\Psi_1^0 = 0. \quad (40)$$

The spin weight of Ψ_1^0 is 1, so it is a linear combination of spin-weight harmonics $\{{}_1Y_{l,m}\}$. The axially symmetric condition implies $m = 0$. The behaviors of the spin-weight harmonics under the action of the operators $\eth$ and $\bar{\eth}$ are [2]

$$\eth_s Y_{lm} = -\sqrt{\frac{(l+s+1)(l-s)}{2}}\,{}_{s+1}Y_{lm},$$
$$\bar{\eth}_s Y_{lm} = \sqrt{\frac{(l-s+1)(l+s)}{2}}\,{}_{s-1}Y_{lm}, \qquad {}_0Y_{lm} = Y_{lm}. \quad (41)$$

So we get $\Psi_1^0 = c(u)\,{}_1Y_{1,0} = c(u)\sin\theta$. Detailed calculations of N-P equations of the same order also give

$$\nu^3 = -\frac{1}{12}\bar{\Psi}_1^0 - \frac{1}{6}\bar{\eth}^2\Psi_1^0, \qquad \Psi_3^1 = 0,$$
$$\Psi_3^2 = \frac{1}{2}\bar{\eth}^2\Psi_1^0, \qquad \Psi_4^1 = \Psi_4^2 = \Psi_4^3 = 0, \quad (42)$$
$$\Psi_4^4 = -\frac{1}{4}\bar{\eth}\Psi_3^3.$$

Equation (34) and $\dot{T}^2 = 0$ give

$$\dot{c}\,\bar{\eth}\sin\theta + \dot{\bar{c}}\,\eth\sin\theta = 2(\dot{c} + \dot{\bar{c}})\cos\theta = 0, \quad (43)$$

which implies $\dot{c} + \dot{\bar{c}} = 0$. The first order Bianchi identities tell us $\dot{\Psi}_1^0 - \eth\Psi_2^0 = 0$, which implies $\Psi_2^0 = -\dot{c}\cos\theta + C_2$; however, it is well known that $\bar{\Psi}_2^0 = \Psi_2^0$ for the stationary case [2,3], so $\dot{c} = 0$. This means $\Psi_1^0 = C_1 \sin\theta$ and $\Psi_2^0 = C_2$. The Komar integral shows that $-C_2$ is just the Bondi mass of the space-time. Additionally, Eq. (37) tells us that $\dot{A}^3 = 0$.

The fourth order Killing equations are

$$3T^3 + (\gamma^4 + \bar{\gamma}^4) = 0, \tag{44}$$

$$4A^3 = \tfrac{1}{3}\bar{\eth}\Psi_0^0, \tag{45}$$

$$\dot{T}^4 + \tfrac{1}{3}(\bar{\eth}\Psi_1^0 + \eth\bar{\Psi}_1^0)(\Psi_2^0 + \bar{\Psi}_2^0) = 0, \tag{46}$$

$$\tfrac{1}{2}\Psi_1^0 T^1 - \tfrac{1}{3}\bar{\eth}\Psi_0^0 + \bar{\nu}^4 + \dot{A}^4 + (\Psi_2^0 + \bar{\Psi}_2^0)A^2 + \tfrac{3}{2}A^3 - \delta_0 T^3 + \Psi_2^0 A^2 + \tfrac{1}{2}A^3 = 0, \tag{47}$$

$$\tfrac{1}{4}\Psi_0^0 + \bar{\lambda}^4 + \eth A^3 = 0, \tag{48}$$

$$2T^3 + \mu^4 + \bar{\mu}^4 + \eth\bar{A}^3 + \bar{\eth}A^3 = 0, \tag{49}$$

where $\gamma^4 = -\frac{1}{12}(\alpha^0\bar{\eth}\Psi_0^0 - \bar{\alpha}^0\eth\bar{\Psi}_0^0) - \frac{1}{8}\bar{\eth}^2\Psi_0^0$, $\lambda^4 = -\frac{1}{12}\bar{\Psi}_0^0$, $\mu^4 = -\frac{1}{3}\Psi_2^2 = -\frac{1}{6}\bar{\eth}^2\Psi_0^0$, $\nu^4 = \frac{1}{24}\times(\eth\bar{\Psi}_0^0 + \bar{\eth}^3\Psi_0^0)$, $\Psi_3^3 = -\frac{1}{2}\bar{\Psi}_1^0\Psi_2^0 - \frac{1}{6}\bar{\eth}^3\Psi_0^0$. (These results are obtained for N-P equations of the same order.)

The spin weight of Ψ_0 is 2, so Eqs. (45) and (48) imply

$$\Psi_0^0 = D^5(u)\sin^2\theta. \tag{50}$$

Equations (37) and (45) eliminate the time dependence of $D^5(u)$, i.e. $\Psi_0^0 = D^5\sin^2\theta$. Equation (47) is

$$\dot{A}^4 = 0. \tag{51}$$

The fifth order Killing equations are

$$4T^4 + (\gamma^5 + \bar{\gamma}^5) - \tfrac{1}{2}\bar{\Psi}_1^0 A^2 - \tfrac{1}{2}\Psi_1^0\bar{A}^2 = 0, \tag{52}$$

$$A^4 = \tfrac{1}{5}\tau^5, \tag{53}$$

$$-\tfrac{1}{2}\sigma^5 + \bar{\lambda}^5 + \tfrac{1}{2}\Psi_0^0 T^1 + \tfrac{3}{2}\Psi_1^0 A^2 + \eth A^4 = 0, \tag{54}$$

$$-2\rho^5 + 2T^4 + (\mu^5 + \bar{\mu}^5) + \tfrac{3}{2}\Psi_1^0\bar{A}^2 + \tfrac{3}{2}\bar{\Psi}_1^0 A^2 + \eth\bar{A}^4 + \bar{\eth}A^4 = 0, \tag{55}$$

where

$$\rho^5 = 0, \qquad \mu^5 = -\frac{1}{4}\Psi_2^3, \qquad \sigma^5 = -\frac{1}{3}\Psi_0^1,$$

$$\lambda^5 = -\frac{1}{8}\bar{\Psi}_0^0\Psi_2^0 - \frac{1}{24}\bar{\Psi}_0^1,$$

$$\gamma^5 = -\frac{1}{40}(\alpha^0\bar{\eth}\Psi_0^1 - \bar{\alpha}^0\eth\bar{\Psi}_0^1) + \frac{1}{12}|\Psi_1^0|^2 - \frac{1}{30}\bar{\eth}^2\Psi_0^1,$$

$$\tau^5 = \frac{1}{8}\bar{\eth}\Psi_0^1, \qquad \Psi_2^3 = -\frac{2}{3}|\Psi_1^0|^2 + \frac{1}{6}\bar{\eth}^2\Psi_0^1,$$

$$\Psi_1^2 = -\frac{1}{2}\bar{\eth}\Psi_0^1.$$

Equations (46) and (52) and the Bianchi identities imply $\Psi_1^0 = iC_1\sin\theta$, $C_1 \in \mathbf{R}$, where C_1 is the Komar angular momentum. Equations (53) and (54) give

$$\eth\bar{\eth}\Psi_0^1 + 5\Psi_0^1 = 10(\Psi_1^0)^2 - 15\Psi_0^0\Psi_2^0. \tag{56}$$

The homogeneous part of the above equation is

$$\eth\bar{\eth}\Psi_0^1 + 5\Psi_0^1 = 0. \tag{57}$$

Because the spin weight of Ψ_0^1 is 2, it is a linear combination of $\{{}_2Y_{l,0}\}$. Using Eq. (41), the homogeneous equation is

$$(-l^2 - l + 12)\,{}_2Y_{l,0} = 0, \tag{58}$$

which gives $l = 3$. The general solution of Eq. (54) is

$$\Psi_0^1 = \left(\frac{10}{3}(C_1)^2 - 5C_2D^5\right)\sin^2\theta + D^6\,{}_2Y_{3,0}. \tag{59}$$

(Bianchi identities insure $\dot{\Psi}_0^1 = 0$.) By definition [2], the nonzero N-P constant for stationary axially symmetric space-time is

$$G_0 = \frac{10}{3}(C_1)^2 - 5C_2D^5. \tag{60}$$

Until now, we have obtained the series expression of tetrad components up to fourth order, N-P coefficients up to fifth order, and Weyl components up to sixth order. To prove this theorem, all Taylor coefficients of all geometric quantities are needed. We use the inductive method to solve this problem order by order.

Suppose we know the Taylor coefficients of the tetrad components up to $(k-3)$th order, the Taylor coefficients of connections up to $(k-2)$th order, and the Taylor coefficients of the Weyl curvature components up to $(k-1)$th order. The $(k-1)$th order of the Killing equations (17) and (20) is

$$-(k-1)A^{k-2} + \tau^{k-1} = \cdots, \tag{61}$$

$$\cdots + \bar{\lambda}^{k-1} + \eth A^{k-2} = 0, \tag{62}$$

where "$\cdots$" means terms which only contain lower order coefficients. Based on the induction hypothesis, those terms are known. In order to solve these equations, we need the coefficients λ^{k-1} and τ^{k-1}. From the N-P equations,

$$D\Psi_1 - \bar{\delta}\Psi_0 = -4\alpha\Psi_0 + 4\rho\Psi_1 \Rightarrow \Psi_1^k = -\frac{1}{(k-4)}\bar{\eth}\Psi_0^k + \cdots,$$

$$D\sigma = 2\rho\sigma + \Psi_0 \Rightarrow \sigma^{k-1} = -\frac{1}{(k-3)}\Psi_0^k + \cdots,$$

$$D\lambda = \rho\lambda + \bar{\sigma}\mu \Rightarrow \lambda^{k-1} = \frac{1}{2(k-2)}\bar{\sigma}^{k-1} + \cdots,$$

$$D\tau = \tau\rho + \bar{\tau}\sigma + \Psi_1 \Rightarrow \tau^{k-1} = -\frac{1}{(k-2)}\Psi_1^k + \cdots. \tag{63}$$

Combining Eqs. (61)–(63), we get

$$\eth\bar{\eth}\Psi_0^k + \frac{(k+4)(k+1)}{2}\Psi_0^k = \cdots. \tag{64}$$

The homogeneous part of the above equation is

$$\eth\bar{\eth}\hat{\Psi}_0^k + \frac{(k+4)(k+1)}{2}\hat{\Psi}_0^k = 0. \tag{65}$$

Because of Eq. (41) and the axially symmetric condition, the general solution should be

$$\Psi_0^k = \tilde{\Psi}_0^k + D^k{}_2Y_{k+2,0}, \tag{66}$$

where $\tilde{\Psi}^k$ is a special solution of Eq. (64) and D^k is a constant. Obviously, the Kerr solution satisfies all conditions of our theorem, so $\tilde{\Psi}_0^k$ must exist. The concrete form of $\tilde{\Psi}_0^k$ can also be obtained by direct calculation. One can express the "$\cdots$" terms in Eq. (64) as a linear combination of spin-weight harmonics $\{{}_2Y_{l,0}\}$. The inductive method insures that the maximal value of l in that expression will be finite for any given order; then we can get $\tilde{\Psi}_0^k$ by comparing coefficients between both sides of this equation. With the general solution of Ψ_0^k, Eq. (63) will give τ^{k-1}, σ^{k-1}, λ^{k-1}, and Ψ_1^k. Furthermore, Cartan structure equations and Bianchi equations will help us to get other coefficients,

$$\begin{aligned}
&D\rho = \rho^2 + |\sigma|^2 \Rightarrow -(k-3)\rho^{k-1} = \cdots,\\
&D\alpha = \alpha\rho + \beta\bar{\sigma} \Rightarrow -(k-2)\alpha^{k-1} = \cdots,\\
&D\beta = \beta\rho + \alpha\sigma + \Psi_1 \Rightarrow -(k-2)\beta^{k-1} = \Psi_1^k + \cdots,\\
&D\Psi_2 - \bar{\delta}\Psi_1 = 3\rho\Psi_2 - 2\alpha\Psi_1 - \lambda\Psi_0 \Rightarrow -(k-3)\Psi_2^k = \bar{\eth}\Psi_1^k + \cdots,\\
&D\Psi_3 - \bar{\delta}\Psi_2 = 2\rho\Psi_3 - 2\lambda\Psi_1 \Rightarrow -(k-2)\Psi_3^k = \bar{\eth}\Psi_2^k + \cdots,\\
&D\Psi_4 - \bar{\delta}\Psi_3 = \rho\Psi_4 + 2\alpha\Psi_3 - 3\lambda\Psi_2 \Rightarrow -(k-1)\Psi_4^k = \bar{\eth}\Psi_3^k + \cdots,\\
&D\gamma = \tau\alpha + \bar{\tau}\beta + \Psi_2 \Rightarrow -(k-1)\gamma^{k-1} = \Psi_2^k + \alpha_0\tau^{k-1} - \bar{\alpha}_0\bar{\tau}^{k-1} + \cdots,\\
&D\mu = \mu\rho + \lambda\sigma + \Psi_2 \Rightarrow -(k-2)\mu^{k-1} = \frac{1}{2}\rho^{k-1} + \Psi_2^k + \cdots,\\
&D\nu = \tau\lambda + \bar{\tau}\mu + \Psi_3 \Rightarrow -(k-1)\nu^{k-1} = \frac{1}{2}\bar{\tau}^{k-1} + \Psi_3^k + \cdots,\\
&D\xi^3 = \rho\xi^3 + \sigma\bar{\xi}^4 \Rightarrow -(k-3)\xi^3_{k-2} = \cdots, \qquad D\xi^4 = \rho\xi^4 + \sigma\bar{\xi}^3 \Rightarrow -(k-3)\xi^4_{k-2} = \cdots,\\
&D\omega = \rho\omega + \sigma\omega - (\bar{\alpha} + \beta) \Rightarrow -(k-3)\omega^{k-2} = -\bar{\alpha}^{k-1} - \beta^{k-1} + \cdots,\\
&DX = (\bar{\alpha} + \beta)\xi^3 + (\alpha + \bar{\beta})\bar{\xi}^4 \Rightarrow -(k-2)X^{k-2} = \frac{1 + |\zeta|^2}{\sqrt{2}}(\alpha^{k-1} + \bar{\beta}^{k-1}) + \cdots,\\
&DU = (\bar{\alpha} + \beta)\bar{\omega} + (\alpha + \bar{\beta})\omega - \gamma - \bar{\gamma} \Rightarrow -(k-2)U^{k-2} = -\gamma^{k-1} - \bar{\gamma}^{k-1} + \cdots.
\end{aligned} \tag{67}$$

From the above results, we find that we can express all $(k-2)$th order coefficients of tetrad components, $(k-1)$th order coefficients of connection components, and kth order coefficients of Weyl curvature in terms of Ψ_0^k, $\eth$ derivatives of Ψ_0^k and lower order coefficients which we have known. The form of Ψ_0^k is given in Eq. (66). This means we can get all those coefficients for any given order.

B. Uniqueness of the Kerr solution

In the above subsection, we have obtained the Taylor series of a general, stationary, axially symmetric metric. From Eq. (66), we can see that the freedom in each order of the Taylor coefficients is just the constant D^k ($k \geq 5$). These arbitrary constants should be closely related to the famous Geroch-Hansen multipole moments [10,11,16,17]. What we want to do in this section is pick out the Kerr solution from those possible solutions; i.e. we need to fix the value of $\{D^k\}$. In order to do that, we consider the Petrov classification [12]. It is well known that the Kerr solution belongs to the type-D class; i.e. its Weyl curvature satisfies [12]

$$K = \Psi_1(\Psi_4)^2 - 3\Psi_4\Psi_3\Psi_2 + 2(\Psi_3)^3 = 0. \tag{68}$$

Writing down the 12th order coefficient of the above equation, we get

$$-3\Psi_4^4\Psi_3^2\Psi_2^0 + 2(\Psi_3^2)^3 = 0. \tag{69}$$

In the previous section, we obtained

$$\begin{aligned} &\Psi_2^0 = C_2, \qquad \Psi_1^0 = iC_{11}Y_{1,0}, \qquad \Psi_3^2 = \tfrac{1}{2}\bar{\eth}^2\Psi_1^0,\\ &\Psi_0^0 = D^5{}_2Y_{2,0}, \qquad \Psi_3^3 = -\tfrac{1}{2}\bar{\Psi}_1^0\Psi_2^0 - \tfrac{1}{6}\bar{\eth}^3\Psi_0^0,\\ &\Psi_4^4 = -\tfrac{1}{4}\bar{\eth}\Psi_3^3. \end{aligned} \tag{70}$$

This constant is fixed in the following way: from Komar integrals $M = \frac{1}{8\pi}\int_{S_\infty} *dt$ and $J = Ma = \frac{1}{16\pi}\int_{S_\infty} *d\phi$, we know $\Psi_2^0 = -M$, $\Psi_1^0 = 3iMa\sqrt{\frac{4\pi}{3}}{}_1Y_{1,0}$. If we submit these into Eq. (70), then we get

$$\begin{aligned} \Psi_3^2 &= \frac{3iMa}{2}\sqrt{\frac{4\pi}{3}}{}_{-1}Y_{1,0},\\ \Psi_3^3 &= \left(\frac{3iM^2a}{2}\sqrt{\frac{4\pi}{3}} - \frac{1}{\sqrt{2}}D^5\right){}_{-1}Y_{2,0},\\ \Psi_4^4 &= \left(-\frac{i\sqrt{6\pi}M^2a}{4} + \frac{D^5}{4}\right){}_{-2}Y_{2,0}. \end{aligned} \tag{71}$$

Submitting the above result into Eq. (69), we find

$$\begin{aligned} 0 &= 3M\left(-\frac{i\sqrt{6\pi}M^2a}{4} + \frac{D^5}{4}\right){}_{-2}Y_{2,0} + 2\left[\frac{3iMa}{2}\sqrt{\frac{4\pi}{3}}{}_{-1}Y_{1,0}\right]^2\\ &= \left[3M\left(-\frac{i\sqrt{6\pi}M^2a}{4} + \frac{D^5}{4}\right) - \frac{3M^2a^2}{4}\sqrt{\frac{16\pi}{5}}\right]{}_{-2}Y_{2,0}. \end{aligned} \tag{72}$$

Solving the simple linear algebraic equation $[3M(-\frac{i\sqrt{6\pi}M^2a}{4} + \frac{D^5}{4}) - \frac{3M^2a^2}{4}\sqrt{\frac{16\pi}{5}}] = 0$, we can fix the value of D^5 as

$$D^5 = Ma^2\sqrt{\frac{16\pi}{5}} + i\sqrt{6\pi}M^2a. \tag{73}$$

Submitting the above result into Eq. (60), it is easy to check that the N-P constant of Kerr space-time is zero, which has been obtained in [18,19].

In order to fix the general D^k's, we also use the inductive method and fix them order by order. Suppose that we know D^k up to order n. To get the value of D^{n+1}, we consider the $(n+8)$th coefficient of Eq. (68); a long but direct calculation shows that it should be

$$-3\Psi_4^{n+1}\Psi_3^2\Psi_2^0 + \cdots = 0. \tag{74}$$

Here "$\cdots$" also mean terms which only contain lower order coefficients. From Eqs. (63), (66), and (67), we know that

$$\begin{aligned} \Psi_0^{n+1} &= \tilde{\Psi}_0^{n+1} + D^{n+1}{}_2Y_{n+3,0},\\ \Psi_1^{n+1} &= -\frac{1}{(n-3)}\bar{\eth}\Psi_0^{n+1} + \cdots,\\ \Psi_2^{n+1} &= -\frac{1}{n-2}\bar{\eth}\Psi_1^{n+1} + \cdots,\\ \Psi_3^{n+1} &= -\frac{1}{n-1}\bar{\eth}\Psi_2^{n+1} + \cdots,\\ \Psi_4^{n+1} &= -\frac{1}{n}\bar{\eth}\Psi_3^{n+1} + \cdots, \end{aligned} \tag{75}$$

where $\tilde{\Psi}_0^{n+1}$ is the special solution of Eq. (64) which corresponds to the Kerr solution. We know that the Kerr solution is type-D; i.e. Eq. (74) holds for $\tilde{\Psi}_0^k$, so Eq. (74) can be written as

$$\frac{-3\Psi_3^2\Psi_2^0}{(n-3)(n-2)(n-1)n}\bar{\eth}^4\tilde{\Psi}_0^{n+1} + \cdots = 0. \tag{76}$$

If the general Ψ_0^{n+1} in Eq. (75) also satisfies Eq. (74), we have

$$\begin{aligned} &\frac{-3\Psi_3^2\Psi_2^0}{(n-3)(n-2)(n-1)n}\bar{\eth}^4\tilde{\Psi}_0^{n+1}\\ &\quad + D^{n+1}\frac{-3\Psi_3^2\Psi_2^0}{(n-3)(n-2)(n-1)n}\bar{\eth}^4{}_2Y_{n+3,0} + \cdots = 0. \end{aligned} \tag{77}$$

Because terms in "$\cdots$" only contain lower order coefficients, they remain unchanged when we change $\tilde{\Psi}_0^{n+1}$ to the general Ψ_0^{n+1}. Obviously, $\frac{-3\Psi_3^2\Psi_2^0}{(n-3)(n-2)(n-1)n}\bar{\eth}^4{}_2Y_{n+3,0}$ is a nonzero function for any n, so the general solution of Ψ_0^{n+1} satisfies Eq. (74), meaning $D^{n+1} = 0$ and $\Psi_0^{n+1} = \tilde{\Psi}_0^{n+1}$. This tells us that the Kerr solution is the only solution which satisfies all requirements of our theorem.

Remark: In the above subsection, we proved our theorem under the requirement that the whole space-time is type D. In Sec. II B, we showed that the freedom of vacuum, stationary, axially symmetric space-time is just a set of constants $\{D^k\}$. The reason we need the condition of type D is to fix the value of $\{D^k\}$. If the type-D condition holds at several points in the Bondi neighborhood and those points are not zero points of $\frac{-3\Psi_3^2\Psi_2^0}{(n-3)(n-2)(n-1)n}\bar{\eth}^4{}_2Y_{n+3,0}$, then it is easy to see that $\{D^k\}$ should be zero; i.e. if the type-D condition holds at several points, it will imply that this condition holds in the whole neighborhood. This feature may be a practical method from the experiment prospective. That means only several points need to be checked for the type-D condition to see whether the space-time around us is Kerr space-time. This may be a useful property for future gravitational experiments, such as "mapping space-time"[20].

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APPENDIX A

The asymptotic extensions of Kerr space-time in B-S coordinates are as follows:

(1) The null tetrad,

$$l^a = \frac{\partial}{\partial r},$$
$$n^a = \frac{\partial}{\partial u} + \left[-\frac{1}{2} + \frac{M}{r} - \frac{Ma^2}{2r^3}(3\cos^2\theta - 1) + O(r^{-4})\right]\frac{\partial}{\partial r} + \left[\frac{iMa}{2r^3}\cot\frac{\theta}{2} + O(r^{-4})\right]\frac{\partial}{\partial \zeta} + \left[-\frac{iMa}{2r^3}\cot\frac{\theta}{2} + O(r^{-4})\right]\frac{\partial}{\partial \bar{\zeta}},$$
$$m^a = \left[-\frac{3iMa}{2\sqrt{2}r^2}\sin\theta + \frac{Ma^2}{\sqrt{2}r^3}\sin^2\theta + O(r^{-4})\right]\frac{\partial}{\partial r} + O(r^{-4})\frac{\partial}{\partial \zeta} + \left[\frac{(1+\zeta\bar{\zeta})}{\sqrt{2}r} + O(r^{-4})\right]\frac{\partial}{\partial \bar{\zeta}}. \tag{A1}$$

(2) N-P coefficients,

$$\rho = -\frac{1}{r} + O(r^{-5}), \qquad \sigma = -\frac{3Ma^2\sin\theta}{2r^4} + O(r^{-5}), \qquad \alpha = -\frac{\cot\theta}{2\sqrt{2}r} + \frac{3Ma^2\sin\theta\cos\theta}{2\sqrt{2}r^4} + O(r^{-5}),$$
$$\beta = \frac{\cot\theta}{2\sqrt{2}r} - \frac{3iMa\sin\theta}{2\sqrt{2}r^3} + \frac{33Ma^2\sin\theta\cos\theta}{2\sqrt{2}r^4} + O(r^{-5}), \qquad \tau = -\frac{3iMa\sin\theta}{2\sqrt{2}r^3} + \frac{18Ma^2\sin\theta\cos\theta}{\sqrt{2}r^4} + O(r^{-5}),$$
$$\lambda = -\frac{Ma^2\sin^2\theta}{4r^4} + O(r^{-5}), \qquad \mu = -\frac{1}{2r} + \frac{M}{r^2} + \frac{3iMa\cos\theta}{2r^3} - \frac{Ma^2(3\cos^2\theta - 1)}{r^4} + O(r^{-5}),$$
$$\gamma = \frac{M}{2r^2} + \frac{(2\sqrt{2}-1)3iMa\cos\theta}{\sqrt{2}r^3} - \frac{3Ma^2(3\cos^2\theta - 1)}{4r^4} + O(r^{-5}), \qquad \nu = \frac{3iMa\sin\theta}{4\sqrt{2}r^3} - \frac{Ma^2\sin\theta\cos\theta}{\sqrt{2}r^4} + O(r^{-5}). \tag{A2}$$

(3) Weyl curvature,

$$\Psi_0 = \frac{3Ma^2\sin^2\theta}{r^5} + O(r^{-6}), \qquad \Psi_1 = \frac{3iMa\sin\theta}{\sqrt{2}r^4} - \frac{12Ma^2\sin\theta\cos\theta}{\sqrt{2}r^5} + O(r^{-6}),$$
$$\Psi_2 = -\frac{M}{r^3} - \frac{3iMa\cos\theta}{4r^4} + \frac{3Ma^2(3\cos^2\theta - 1)}{r^5} + O(r^{-6}), \tag{A3}$$
$$\Psi_3 = -\frac{3iMa\sin\theta}{2\sqrt{2}r^4} + \left[-\frac{3i}{2\sqrt{2}}M^2a\sin\theta + \frac{6i}{\sqrt{2}}Ma^2\sin\theta\cos\theta\right]r^{-5} + O(r^{-6}), \qquad \Psi_4 = \frac{3Ma^2\sin^2\theta}{4r^5} + O(r^{-6}).$$

APPENDIX B

Some spin-weight harmonics are as follows:

$$Y_{0,0}=\frac{1}{\sqrt{4\pi}};\qquad {}_1Y_{1,1}=\sqrt{\frac{3}{16\pi}}(\cos\theta+1)e^{i\phi},\qquad {}_1Y_{1,0}=\sqrt{\frac{3}{8\pi}}\sin\theta,\qquad {}_1Y_{1,-1}=\sqrt{\frac{3}{16\pi}}(1-\cos\theta)e^{-i\phi};$$

$$Y_{1,1}=-\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi},\qquad Y_{1,0}=\sqrt{\frac{3}{4\pi}}\cos\theta,\qquad Y_{1,-1}=\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi};\qquad {}_{-1}Y_{1,1}=\sqrt{\frac{3}{16\pi}}(1-\cos\theta)e^{i\phi},$$

$${}_{-1}Y_{1,0}=-\sqrt{\frac{3}{8\pi}}\sin\theta,\qquad {}_{-1}Y_{1,-1}=\sqrt{\frac{3}{16\pi}}(1+\cos\theta)e^{-i\phi};\qquad {}_2Y_{2,2}=3\sqrt{\frac{5}{96\pi}}(1+\cos\theta)^2e^{2i\phi},$$

$${}_2Y_{2,1}=3\sqrt{\frac{5}{24\pi}}\sin\theta(1+\cos\theta)e^{i\phi},\qquad {}_2Y_{2,0}=3\sqrt{\frac{5}{16\pi}}\sin^2\theta,\qquad {}_2Y_{2,-1}=3\sqrt{\frac{5}{24\pi}}\sin\theta(1-\cos\theta)e^{-i\phi},$$

$${}_2Y_{2,-2}=3\sqrt{\frac{5}{96\pi}}(1-\cos\theta)^2e^{-2i\phi};\qquad {}_1Y_{2,2}=-3\sqrt{\frac{5}{24\pi}}\sin\theta(1+\cos\theta)e^{2i\phi},$$

$${}_1Y_{2,1}=3\sqrt{\frac{5}{24\pi}}(2\cos\theta-1)(1+\cos\theta)e^{i\phi},\qquad {}_1Y_{2,0}=3\sqrt{\frac{5}{4\pi}}\sin\theta\cos\theta,$$

$${}_1Y_{2,-1}=3\sqrt{\frac{5}{24\pi}}(2\cos\theta+1)(1-\cos\theta)e^{-i\phi},\qquad {}_1Y_{2,-2}=3\sqrt{\frac{5}{24\pi}}\sin\theta(1-\cos\theta)e^{-2i\phi};\qquad Y_{2,2}=3\sqrt{\frac{5}{16\pi}}\sin^2\theta e^{2i\phi},$$

$$Y_{2,1}=-6\sqrt{\frac{5}{16\pi}}\sin\theta\cos\theta e^{i\phi},\ Y_{2,0}=\sqrt{\frac{5}{24\pi}}(3\cos^2\theta-1),\qquad Y_{2,-1}=6\sqrt{\frac{5}{16\pi}}\sin\theta\cos\theta e^{-i\phi},$$

$$Y_{2,-2}=3\sqrt{\frac{5}{16\pi}}\sin^2\theta e^{-2i\phi};\qquad {}_{-1}Y_{2,2}=-3\sqrt{\frac{5}{24\pi}}\sin\theta(1-\cos\theta)e^{2i\phi},$$

$${}_{-1}Y_{2,1}=3\sqrt{\frac{5}{24\pi}}(2\cos\theta+1)(1-\cos\theta)e^{i\phi},\qquad {}_{-1}Y_{2,0}=-\sqrt{\frac{5}{4\pi}}\sin\theta\cos\theta,$$

$${}_{-1}Y_{2,-1}=3\sqrt{\frac{5}{24\pi}}(2\cos\theta-1)(1+\cos\theta)e^{-i\phi},\qquad {}_{-1}Y_{2,-2}=3\sqrt{\frac{5}{24\pi}}\sin\theta(1+\cos\theta)e^{-2i\phi};$$

$${}_{-2}Y_{2,2}=3\sqrt{\frac{5}{96\pi}}(1-\cos\theta)^2e^{2i\phi},\qquad {}_{-2}Y_{2,1}=-3\sqrt{\frac{5}{24\pi}}\sin\theta(1-\cos\theta)e^{i\phi},\qquad {}_{-2}Y_{2,0}=3\sqrt{\frac{5}{16\pi}}\sin^2\theta,$$

$${}_{-2}Y_{2,-1}=-3\sqrt{\frac{5}{24\pi}}\sin\theta(1+\cos\theta)e^{-i\phi},\qquad {}_{-2}Y_{2,-2}=3\sqrt{\frac{5}{96\pi}}(1+\cos\theta)^2e^{-2i\phi}.$$

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[1] H. Bondi, M. G. J. van der Burg, and A. W. K. Metzner, Proc. R. Soc. A **269**, 21 (1962).
[2] R. Penrose and R. Rindler, *Spinors and Space-Time* (Cambridge University Press, Cambridge, England, 1986), Vols. I and II.
[3] E. T. Newman and T. W. J. Unti, J. Math. Phys. (N.Y.) **3**, 891 (1962).
[4] H. Friedrich, Proc. R. Soc. A **375**, 169 (1981); **378**, 401 (1981).
[5] J. Kánnár, Proc. R. Soc. A **452**, 945 (1996).
[6] F. Nicoló, arXiv:gr-qc/0603118.
[7] J. Winicour, Living Rev. Relativity **8**, 10 (2005), http://www.livingreviews.org/irr-2005-10.
[8] D. C. Robinson, Phys. Rev. Lett. **34**, 905 (1975).
[9] M. Heusler, *Black Hole Uniqueness Theorems* (Cambridge University Press, Cambridge, England, 1996).
[10] R. Geroch, J. Math. Phys. (N.Y.) **11**, 1955 (1970); **11**, 2580 (1970).
[11] H. Friedrich, Ann. Inst. Henri Poincaré, A **8**, 817 (2007).
[12] D. Kramer, H. Stephani, E. Herlt, and M. MacCallum, *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England, 1980).
[13] H. Müller zum Hagen, Proc. Cambridge Philos. Soc. **68**, 199 (1970).
[14] T. Damour and B. Schmidt, J. Math. Phys. (N.Y.) **31**, 2441 (1990).

[15] S. Dain, Classical Quantum Gravity **18**, 4329 (2001).
[16] R. Hansen, J. Math. Phys. (N.Y.) **15**, 46 (1974).
[17] P. K. Kundu, J. Math. Phys. (N.Y.) **29**, 1866 (1988).
[18] X. Wu and Y. Shang, Classical Quantum Gravity **24**, 679 (2007).
[19] S. Bai *et al.*, Phys. Rev. D **75**, 044003 (2007).
[20] K. Glampedakis and S. Babak, Classical Quantum Gravity **22**, S605 (2005).