

Gauging the cosmic microwave background

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(Received 15 August 2008; published 24 December 2008)

We provide a new derivation of the anisotropies of the cosmic microwave background (CMB), and find an exact expression that can be readily expanded perturbatively. Close attention is paid to gauge issues, with the motivation to examine the effect of super-Hubble modes on the CMB. We calculate a transfer function that encodes the behavior of the dipole, and examine its long-wavelength behavior. We show that contributions to the dipole from adiabatic super-Hubble modes are strongly suppressed, even in the presence of a cosmological constant, contrary to claims in the literature. We also introduce a naturally defined CMB monopole, which exhibits closely analogous long-wavelength behavior. We discuss the geometrical origin of this super-Hubble suppression, pointing out that it is a simple reflection of adiabaticity, and hence argue that it will occur regardless of the matter content.

DOI: [10.1103/PhysRevD.78.123529](https://doi.org/10.1103/PhysRevD.78.123529)

PACS numbers: 98.70.Vc, 98.80.Jk

I. INTRODUCTION

The anisotropies in the cosmic microwave background (CMB) reveal a great deal about our Universe, since they persist essentially unscathed from the epoch when fluctuations were well described by simple linear theory. The comparison of CMB observations with theory has become a mature subject, and has played an important role in forming our current understanding of the Universe (see, e.g., [1]).

Critical to that comparison is the accurate theoretical calculation of the anisotropies. Since the pioneering work of Sachs and Wolfe [2], the theoretical anisotropies have been refined and recalculated using different formalisms many times (see, e.g., [3–9]). Accurate calculations are now readily available via public code packages such as CAMB [10,11].

In the present work, we revisit the calculation of anisotropies. While our results may not lead to more accurate or efficient calculations, we hope that they will help to clarify some of the conceptual issues surrounding the calculations. In particular, our approach makes explicit the physical meaning of the various contributions to the anisotropies. Crucial to this is our use of the covariant approach to cosmology (see [12,13] for reviews), which is ideal for writing exact solutions and for physical clarity. We present a remarkably simple but exact expression for the anisotropy, which applies to arbitrary spacetimes and includes the effects of tensor as well as scalar perturbations and any line-of-sight integrated Sachs-Wolfe (ISW) effect. This general result can be readily expanded perturbatively, and here we turn to the metric formalism for computational efficiency and show that we recover previous results in the literature.

A main motivation for our work is in examining the behavior of the anisotropies due to super-Hubble fluctuations (we use the terms “super-Hubble”, “long-wavelength”, and “large-scale” interchangeably, to mean scales larger than the current Hubble or last scattering radius), where gauge issues are paramount. This question has been examined before in the context of the Grishchuck-Zel’dovich effect [14], which describes the large-angular-scale anisotropies that result from super-Hubble modes. In the context of a matter-dominated universe with adiabatic fluctuations, it was shown that the CMB dipole receives strongly suppressed contributions from long-wavelength modes. A claim was made in Ref. [15] that this suppression would not occur in models with cosmological constant, so that we could “see” very long-wavelength structure in the dipole. It was also found that in the presence of *isocurvature* perturbations, the suppression may not occur (see [16] and references therein).

To study this issue, we construct a transfer function that describes the scale dependence of contributions to the dipole. Working by analogy, we carefully define a CMB *monopole* perturbation, and find its transfer function. As is well known, such a monopole cannot be observable, but we show that its *variance* is well defined theoretically. Our definitions have simple interpretations: the dipole measures the departure of radiation and matter comoving worldlines, while the monopole measures how well radiation and matter constant-density hypersurfaces coincide. The usefulness of the monopole will be in examining its long-wavelength behavior, where it will help to clarify the dipole case.

We show that the contributions to both dipole and monopole vanish for large-scale sources, even in the presence of a cosmological constant. We close by pointing out that this is a direct consequence of adiabaticity, and hence that this result is expected to hold regardless of the matter content of the Universe, unless isocurvature modes are present.

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Another potential reason that a careful treatment of the monopole may be of interest involves the measurement of the mean CMB temperature and its relation to constraints on other cosmological parameters. The mean temperature is currently measured to a precision of a few parts in 10^4 [17]. It has been pointed out that the precision of this measurement could be improved by nearly 2 orders of magnitude with currently available technology [18]. Such a measurement would reach the naive cosmic variance limit of a part in 10^5 as suggested by the observed amplitude of fluctuations (see [19] for a related discussion). It would then become necessary to be very careful about exactly what information the mean temperature measurement is giving us, and about the nature of monopole fluctuations. Relevantly, recent studies have examined the importance of the mean temperature measurement to our ability to constrain the cosmological parameters [20,21].

Since the covariant approach to cosmology is essential to this work, we begin in Sec. II with a summary of the required formalism. Next, in Sec. III we present the derivation of the Sachs-Wolfe effect, beginning with an exact result before specializing to first order and recovering previous results. In Sec. IV, we present calculations of the dipole and monopole transfer functions, and we examine their long-wavelength behavior in Sec. V. Finally, we discuss our results in Sec. VI. The Appendices summarize relevant material in the metric formalism, and demonstrate both the gauge invariance and the gauge dependence of our results. We use signature $(-, +, +, +)$, and greek indices indicate four-tensors, while Latin indices indicate spatial three-tensors.

II. COVARIANT COSMOLOGY

This section will contain a brief summary of the elements of the covariant approach to cosmology (see, e.g., the reviews [12,13]) that will be needed in the following sections. (A collection of results from the metric-based approach to cosmological perturbations, which will also be needed, is presented in Appendix A.) Fundamental to the covariant approach is the notion of a *congruence* of worldlines, also known as a *threading* of the spacetime or sometimes as a choice of *frame*. This is a family of worldlines such that exactly one worldline passes through each event. A useful example is the congruence of comoving worldlines, when it is well defined. A timelike congruence is described by a vector field u^μ , tangent everywhere to the worldlines. We will assume that u^μ is normalized, $u^\mu u_\mu = -1$. At each event we can define the spatial projection tensor

$$h^\mu{}_\nu \equiv \delta^\mu{}_\nu + u^\mu u_\nu, \quad (1)$$

which projects orthogonal to u^μ . Hypersurfaces orthogonal everywhere to u^μ will exist if the twist of the congruence (defined below) vanishes [22], in which case $h_{\mu\nu}$ is the (Riemannian) metric tensor for those spatial hypersurfaces.

It is useful to describe the geometrical properties of the congruence by the covariant derivative $u_{\mu;\nu}$. By virtue of the normalization condition, this derivative satisfies

$$u_{\mu;\nu} u^\mu = 0. \quad (2)$$

The derivative can be decomposed into parts parallel to and orthogonal to u^μ in its second index using Eq. (1), giving

$$u_{\mu;\nu} = h^\rho{}_\nu u_{\mu;\rho} - u^\rho u_\nu u_{\mu;\rho}. \quad (3)$$

The temporal part can be written in terms of the acceleration of the worldlines, defined by

$$a_\mu \equiv u_{\mu;\rho} u^\rho. \quad (4)$$

The field a^μ measures the departure of the worldlines from geodesic. The spatial part $h^\rho{}_\nu u_{\mu;\rho}$ can be decomposed into trace, symmetric trace-free, and antisymmetric parts via

$$\theta \equiv h^{\rho\mu} u_{\mu;\rho} = u^\mu{}_{;\mu}, \quad (5)$$

$$\sigma_{\mu\nu} \equiv h^\rho{}_{(\nu} u_{\mu);\rho} - \frac{1}{3} \theta h_{\mu\nu}, \quad (6)$$

$$\omega_{\mu\nu} \equiv h^\rho{}_{[\nu} u_{\mu];\rho}. \quad (7)$$

The scalar θ and tensors $\sigma_{\mu\nu}$ and $\omega_{\mu\nu}$ measure the local rates of expansion, shear, and twist of the congruence, respectively. Combining Eqs. (3) to (7) we have

$$u_{\mu;\nu} = \frac{1}{3} \theta h_{\mu\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} - a_\mu u_\nu. \quad (8)$$

For the case of a homogeneous and isotropic Friedmann-Robertson-Walker (FRW) cosmology, if we choose u^μ to be comoving then $\sigma_{\mu\nu} = \omega_{\mu\nu} = a_\mu = 0$ and $\theta = 3H$, where $H \equiv \dot{a}/a$ is the Hubble rate, and a the scale factor.

We will also need two derivatives. For an arbitrary tensor X , the covariant time derivative is defined by $\dot{X} \equiv X_{;\mu} u^\mu$, and gives the proper time derivative along u^μ . The symbol D_μ represents the spatial (orthogonal to u^μ) covariant derivative defined using the spatial metric $h_{\mu\nu}$. For example,

$$D_\nu X^\mu \equiv h^\lambda{}_\nu h^\mu{}_\rho X^\rho{}_{;\lambda}, \quad (9)$$

for any tensor X^μ orthogonal to u^μ .

A congruence u^μ can be used to describe the matter content as observed locally by a family of observers, given the energy-momentum tensor $T_{\mu\nu}$. The energy density ρ , momentum density q_μ , pressure P , and anisotropic (shear) stress $\pi_{\mu\nu}$, as viewed locally by an observer with four-velocity u^μ , are defined by the projections

$$T_{\mu\nu} u^\mu u^\nu = \rho, \quad (10)$$

$$T_{\mu\nu} u^\mu h^\nu{}_\kappa = -q_\kappa, \quad (11)$$

$$T_{\mu\nu}h^\mu_\lambda h^\nu_\kappa = Ph_{\lambda\kappa} + \pi_{\lambda\kappa}, \quad (12)$$

where $\pi_{\mu\nu}$ is defined to be trace free.

III. SACHS-WOLFE EFFECT

In this section, we will provide a calculation of the anisotropies of the CMB radiation. The main goal will be to clarify the issues of gauge and frame choice, which will be critical to properly describing the dipole and monopole anisotropies in the next section. For this reason, it will not be necessary to consider here the finite thickness of the last scattering surface, or the effect of reionization on the anisotropies, which have negligible effect on the largest-scale anisotropies. Thus we assume tight coupling in the baryon-photon plasma before last scattering, followed by instantaneous recombination of matter and free streaming (i.e. unattenuated geodesic evolution) of the radiation. Apart from this approximation the calculation will be exact.

A. Exact expression

We are making the approximation that the CMB radiation is emitted abruptly when the local plasma temperature drops below some value T_E at which recombination occurs (E for “emission”), and thereafter travels freely. This temperature is defined with respect to the frame (or congruence) u^μ which is comoving with the plasma (i.e. for which the plasma momentum density vanishes). The spacelike hypersurface defined by the moment of recombination will be called the last scattering hypersurface, Σ_{LS} , while the intersection of Σ_{LS} with an observer’s past light cone defines the two-sphere commonly called the observer’s last scattering surface (LSS). Via the Stefan-Boltzmann law, Σ_{LS} must be a hypersurface of constant radiation energy density $\rho_{(\gamma)}$, with the density again defined with respect to the comoving congruence u^μ [23]. For adiabatic perturbations at last scattering and on large scales, Σ_{LS} will also be at constant matter energy density, and hence total energy density ρ . The definition of the last scattering hypersurface is critical here: note, in particular, that it does *not* contain colder and hotter regions which contribute to the anisotropies observed at later times, as some accounts state (see, e.g., [6,24,25]) [26]. Rather, it is a hypersurface of constant (comoving) temperature. Nevertheless, in a realistic cosmology, Σ_{LS} is still a perturbed surface, with generally nonvanishing intrinsic and extrinsic curvature and matter perturbations apart from $\rho_{(\gamma)}$, and these perturbations will source anisotropies. Also, because of these perturbations, the comoving congruence will not in general be orthogonal to Σ_{LS} .

In order to calculate the anisotropies observed at late times we must propagate the radiation along null geodesics from the observer’s LSS. Consider a light ray following a null geodesic \mathcal{O} with tangent vector v^μ that extends from a point of emission, E , on the observer’s LSS to a point of

reception, R , and also consider a timelike congruence u^μ defined in the vicinity of \mathcal{O} . We define the congruence in order to provide a frame with respect to which a local energy density (and hence temperature) can be expressed. Then we can decompose v^μ at each point on \mathcal{O} into parts parallel and orthogonal to u^μ according to

$$v^\mu = \gamma(u^\mu + n^\mu), \quad (13)$$

with $n^\mu u_\mu = 0$ and $n^\mu n_\mu = 1$. The spatial vector n^μ defines the spatial direction of propagation of the light ray at each point along \mathcal{O} . Figure 1 illustrates this geometry.

Since the radiation is emitted from E with a thermal spectrum, an observer at any point on \mathcal{O} , with four-velocity u^μ , will observe the radiation travelling along \mathcal{O} to also have a thermal spectrum with temperature

$$T \propto -u^\mu v_\mu = \gamma. \quad (14)$$

Because the temperature at emission, T_E , is defined with respect to the plasma frame and is constant on Σ_{LS} , the calculation will be simplest if we choose u^μ to be comoving with the plasma on the observer’s LSS. Similarly, it will be most natural to choose u^μ to be comoving at R , since any observer will presumably be composed of matter and comoving with it. Note that such an observer, comoving with the total (effectively matter) energy density at R , will generally not be comoving with the radiation, and so will observe a dipole anisotropy. As we will see, the congruence can be freely chosen in between E and R , although some choices will be computationally more efficient in practice. In Sec. III B 1, we will relax the constraint that u^μ be comoving at E and R .

Now we will derive an expression for the evolution of the “redshift parameter” γ along \mathcal{O} , which will tell us how the observed temperature evolves, using only the geodesic equation,

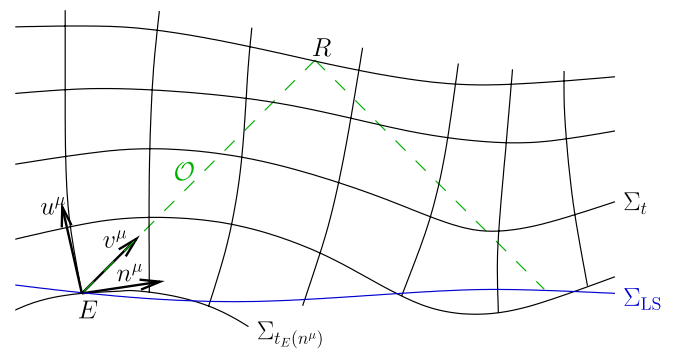


FIG. 1 (color online). A conformal spacetime diagram showing a light ray \mathcal{O} emitted from the LSS on Σ_{LS} at event E and received at R . A foliation Σ_t is indicated together with its orthogonal congruence u^μ , which is comoving on the LSS and at R . For clarity, orthogonal vectors are displayed as if the geometry was Euclidean.

$$v^\mu{}_{;\nu} v^\nu = 0. \quad (15)$$

Consider the quantity

$$H_n \equiv \frac{1}{\gamma^2} u_{\mu;\nu} v^\mu v^\nu = u_{\mu;\nu} n^\mu n^\nu + a_\mu n^\mu \quad (16)$$

$$= \frac{1}{3} \theta + \sigma_{\mu\nu} n^\mu n^\nu + a_\mu n^\mu, \quad (17)$$

which is related to the expansion rate of the congruence u^μ projected into the plane defined by u^μ and n^μ [27] (indeed H_n reduces to the familiar Hubble rate for the comoving congruence in a homogeneous and isotropic spacetime). Now, using Eqs. (14) and (15), we have

$$H_n = \frac{1}{\gamma^2} (u_\mu v^\mu)_{;\nu} v^\nu \quad (18)$$

$$= \frac{d(\gamma^{-1})}{d\lambda}, \quad (19)$$

where λ is an affine parameter along \mathcal{O} (note that this last expression is independent of the affine parameter chosen). The expression Eq. (19) gives the exact evolution of the redshift parameter γ along \mathcal{O} with respect to the congruence u^μ in an arbitrary spacetime. It describes the familiar behavior of increasing redshift (decreasing γ) in an expanding universe, with “expansion” now seen to mean precisely that $H_n > 0$.

We can in principle integrate Eq. (19) along \mathcal{O} to find the temperature $T_R(n^\mu)$ observed at R in direction $-n^\mu$, i.e. the CMB temperature sky map. With an appropriate choice of affine parameter [setting the proportionality constant equal to unity in Eq. (14)], we have [28]

$$T_R(n^\mu) = \left(T_E^{-1} + \int_{E(n^\mu)}^R H_n d\lambda \right)^{-1}. \quad (20)$$

However, in practice it would be very difficult to determine H_n as a function of affine parameter along each null geodesic in order to perform the integral in Eq. (20) for a particular spacetime. Instead, if we define a time coordinate t by foliating the spacetime into spacelike hypersurfaces of constant t , Σ_t , (see Fig. 1) we can transform the integral into one that is more tractable in terms of the new coordinate. The most convenient choice for the foliation is that which is everywhere orthogonal to the congruence u^μ [30].

Using Eq. (13), we can write

$$\frac{d(\gamma^{-1})}{d\lambda} \Big|_{v^\mu} = \gamma \frac{d(\gamma^{-1})}{d\tau} \Big|_{u^\mu} \quad (21)$$

$$= \frac{\gamma}{N} \frac{d(\gamma^{-1})}{dt} \Big|_{u^\mu} \quad (22)$$

$$= \frac{\gamma}{N} \frac{d(\gamma^{-1})}{dt} \Big|_{v^\mu}. \quad (23)$$

Here τ is proper time, N is the lapse function for the slicing Σ_t , and the subscript v^μ or u^μ indicates the direction in which the derivative is taken. In the intermediate steps, we have chosen to define γ away from \mathcal{O} to be constant along the Σ_t . Integrating Eq. (23) along \mathcal{O} and using Eq. (19) and the proportionality Eq. (14) between γ and temperature, we finally obtain

$$\frac{T_R(n^\mu)}{T_E} = \exp\left(\int_{t_R}^{t_E(n^\mu)} H_n N dt\right), \quad (24)$$

where $t_E(n^\mu)$ is the value of t at the point of emission E on the LSS corresponding to the observed direction $-n^\mu$ at R , and t_R is the time of observation. [Note that in general Σ_{LS} will not coincide with one of the slices Σ_t ; hence the dependence $t_E(n^\mu)$.] This remarkably simple expression is exact, and so is not restricted to linear, adiabatic, or scalar fluctuations, and it applies to all scales (subject of course to our basic assumption of abrupt recombination). In particular, Eq. (24) encapsulates in principle the acoustic peak structure of the CMB, any ISW contribution that may arise, as well as the effect of gravitational waves. This equation is a purely geometrical result, independent of any dynamical input such as stress-energy conservation or Einstein’s equations. To our knowledge Eq. (24) has not been written down before, although a related expression appears in Ref. [7], and a related linearized expression appears in Ref. [31].

Equation (24) tells us that the observed temperature in some direction on the sky is determined entirely by the integrated line-of-sight component of expansion (or number of “ e -folds”) along the null path from the LSS, of a congruence that is comoving with the plasma at the LSS and matches the observer’s four-velocity at R . Thus we can interpret the observed anisotropic CMB sky as a uniform temperature surface viewed through an anisotropically expanding universe: the observed hot and cold spots on the sky are simply “closer” and “farther”, respectively, from us, in terms of e -folds of expansion. Note, however, that we cannot view the redshifting as uniquely defined at intermediate points between E and R , since we are free to deform the congruence between those endpoints. Rather, it is only the total integral that is independent of the choice of congruence u^μ between the endpoints, when we fix the position and state of motion of the observer at R . For the special case of a homogeneous and isotropic FRW cosmology, and choosing the comoving congruence, for which $H_n = H$ and $N = 1$, we immediately recover from Eq. (24) the familiar result for the cosmological redshift

$$\frac{T_R}{T_E} = \exp\left(\int_{t_R}^{t_E} H dt\right) = \frac{a_E}{a_R}, \quad (25)$$

for scale factor a .

B. Linearized results

1. Arbitrary gauge

The result Eq. (24) is very general, but probably has limited direct use for calculating anisotropies. However, it can be straightforwardly expanded to linear (or even higher) order in perturbation theory, and such a linearized calculation will be very convenient, as linear theory captures very well the evolution of structure at early times and very large scales today. In doing this it will prove helpful to generalize the result to congruences u^μ that are noncomoving at E and R . With such threadings it will then be necessary to provide explicit boosts at the LSS and at the reception point R to compensate. Similarly, at linear order it will be simple to write the integral to the LSS in terms of an integral to some constant time slice, plus a contribution due to the linear temporal displacement to the actual LSS. The boost at the LSS will constitute what is often termed the ‘‘Doppler’’ or ‘‘dipole’’ contribution to the anisotropies, while the temporal displacement contributes to what is sometimes called the ‘‘monopole’’ contribution.

To calculate the boosts, consider the general noncomoving congruence u^μ and the direction \tilde{u}^μ comoving with the plasma at emission point E (see Fig. 2). If we temporarily construct scalar fields t and \tilde{t} in the vicinity of E such that $u^\mu = -\dot{t}^\mu$ and $\tilde{u}^\mu = -\dot{\tilde{t}}^\mu$, with $\tilde{t} \equiv t - \delta t_B$, then we have

$$\tilde{u}^\mu = u^\mu + \delta \dot{t}_B^\mu. \quad (26)$$

While this expression is exact, the ‘‘boost displacement’’ δt_B evaluated at linear order is simply the linear temporal displacement between hypersurfaces orthogonal to u^μ and \tilde{u}^μ , in the vicinity of E (see Fig. 2); this can be readily calculated in the metric formalism as the gauge transfor-

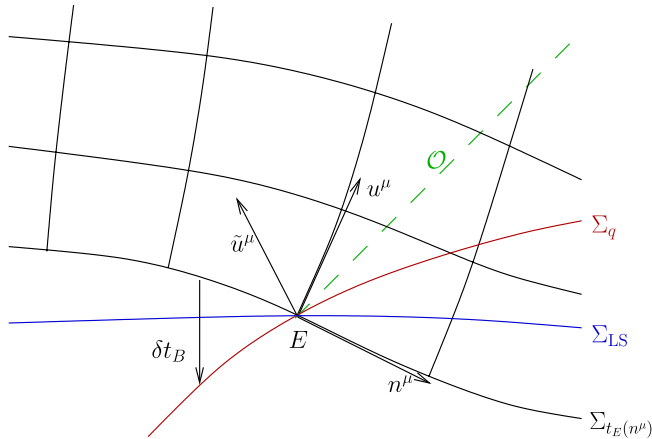


FIG. 2 (color online). A conformal spacetime diagram showing a light ray \mathcal{O} emitted from the LSS at E . The arbitrary foliation Σ_t is indicated together with its orthogonal congruence u^μ . Also, the hypersurface Σ_q orthogonal to the direction \tilde{u}^μ comoving with the plasma is indicated, together with the boost displacement δt_B , which takes $\Sigma_{tE}(n^\mu)$ into Σ_q .

mation required to take the gauge specified by the slicing orthogonal to u^μ into the plasma-comoving gauge. To calculate the change in observed temperature due to the boost, we require the quantity

$$\tilde{\gamma} \equiv -\tilde{u}^\mu v_\mu = \gamma(1 - n_\mu \delta \dot{t}_B^\mu), \quad (27)$$

which is valid at first order. To derive this expression we have used the relation $u_\mu \delta \dot{t}_B^\mu = \mathcal{O}(2)$, which follows from the normalization of the four-velocities. Note that Eq. (27) simply describes a local Lorentz transformation of photon energy. This expression can also be applied at the reception point R , although the direction \tilde{u}^μ is free in principle there. If we choose the observer to be comoving with matter then the displacement δt_B at R will be given by the gauge transformation required to take the gauge specified by u^μ into the comoving gauge. The freedom to choose \tilde{u}^μ at R only effects the dipole anisotropy at linear order, as shown in Appendix C.

To calculate the temporal displacements at E and R , we can write the integral in the exact expression Eq. (24) as

$$\int_{\bar{t}_R}^{t_E(n^\mu)} H_n N dt = \int_{\bar{t}_R + \delta t_D(R)}^{\bar{t}_E + \delta t_D(E)} H_n N dt, \quad (28)$$

where \bar{t}_E and \bar{t}_R label particular slices $\Sigma_{\bar{t}}$, and can be considered the background emission and reception times. The displacement $\delta t_D(E)$ accounts for the separation between the background slice $\Sigma_{\bar{t}_E}$ and the true last scattering hypersurface Σ_{LS} , and is a function of n^μ . Similarly, the displacement $\delta t_D(R)$ accounts for the separation between the background slice $\Sigma_{\bar{t}_R}$ and the actual slice on which the reception point R is located. $\delta t_D(R)$ can be considered a function of position if we wish to evaluate the anisotropies at various reception points R . At linear order, we can then write

$$\int_{\bar{t}_R}^{t_E(n^\mu)} H_n N dt = \int_{\bar{t}_R}^{\bar{t}_E} H_n N dt + (\bar{H} \delta t_D) \Big|_R^E, \quad (29)$$

where $\bar{H}(t)$ is the zeroth order (background) Hubble rate and we have assumed that the lapse equals unity at zeroth order, so that the coordinate t is a perturbed proper time.

The displacement $\delta t_D(E)$, like the boost displacements, can be readily calculated in the metric formalism as the gauge transformation required to take the gauge specified by the slices Σ_t (orthogonal to u^μ) into the gauge specified by Σ_{LS} , i.e. the uniform radiation energy density gauge. The displacement $\delta t_D(R)$ is not fixed uniquely by any such physical prescription, but only affects the monopole anisotropy, as shown in Appendix C.

The temperature observed at R in direction $-n^\mu$ can now be written using the boost relation Eq. (27) as

$$\frac{T_R(n^\mu)}{T_E} = \frac{\tilde{\gamma}_R(n^\mu)}{\tilde{\gamma}_E} = \frac{\gamma_R(n^\mu)}{\gamma_E(n^\mu)} \left(1 - n_\mu \delta \dot{t}_B^\mu \Big|_E^R \right), \quad (30)$$

at linear order. The prefactor on the right-hand side of this equation is simply the redshift due to the expansion along

the congruence u^μ , so it is given by Eq. (24). Writing

$$H_n N = \bar{H} + \delta(H_n N) \quad (31)$$

and using Eq. (29), we have

$$\frac{\gamma_R(n^\mu)}{\gamma_E(n^\mu)} = \exp\left(\int_{t_R}^{t_E(n^\mu)} H_n N dt\right) \quad (32)$$

$$= \frac{\bar{T}_R}{T_E} \left(1 + \int_{\bar{t}_R}^{\bar{t}_E} \delta(H_n N) dt + (\bar{H} \delta t_D)\Big|_R^E\right) \quad (33)$$

at linear order, where we have defined the ‘‘background’’ observed temperature \bar{T}_R by

$$\bar{T}_R \equiv T_E \exp\left(\int_{\bar{t}_R}^{\bar{t}_E} \bar{H} dt\right). \quad (34)$$

Finally, combining Eqs. (30) and (33), and defining $\delta T(n^\mu) \equiv T_R(n^\mu) - \bar{T}_R$, we can write

$$\frac{\delta T(n^\mu)}{\bar{T}_R} = \int_{\bar{t}_R}^{\bar{t}_E} \delta(H_n N) dt + (\bar{H} \delta t_D + n_\mu \delta t_B^\mu)\Big|_R^E. \quad (35)$$

Equation (35) gives the observed temperature anisotropy at linear order in terms of the line-of-sight expansion perturbation $\delta(H_n N)$ in an arbitrary gauge (specified by the hypersurfaces orthogonal to u^μ) and the temporal displacements δt_B and δt_D required to transform from that arbitrary gauge to comoving and uniform density gauges. The only approximations involved in Eq. (35) are those of abrupt recombination and linearization. The terms $\bar{H} \delta t_D$ and $n_\mu \delta t_B^\mu$ evaluated at the LSS are sometimes called the monopole and Doppler contributions, respectively. The geometrical nature of the terms in Eq. (35) provides a clear and unambiguous interpretation of the anisotropy, without reliance on coordinate-dependent notions such as gravitational potentials (see Ref. [32] for a related discussion).

Equation (35) is in a form that makes it easy to evaluate the anisotropy using any gauge for the perturbations that we choose. First, for the line-of-sight integral, by linearizing the exact expression Eq. (17) we have

$$\delta(H_n N) = \frac{1}{3} \delta\theta + \sigma_{\mu\nu} n^\mu n^\nu + a_\mu n^\mu + \bar{H} \delta N. \quad (36)$$

The geometrical quantities in this expression can be written in terms of the metric perturbations using Eqs. (A2), (A5), (A6), and (A8), giving

$$\delta(H_n N) = -\dot{\psi} + \phi_{;\mu} n^\mu + \sigma_{;\mu\nu} n^\mu n^\nu + \frac{1}{2} \dot{H}_{\mu\nu} n^\mu n^\nu. \quad (37)$$

Here ψ is the curvature perturbation, ϕ is the lapse perturbation, σ is the shear scalar, and $H_{\mu\nu}$ is the tensor metric perturbation. Equation (37) is valid in arbitrary gauges, i.e. for arbitrary congruences u^μ , and it is now trivial to fix the gauge, as we will see in Sec. III B 2. Second, for the

boundary terms in Eq. (35), we can work out the required gauge transformations δt_D and δt_B using Eq. (A14) and (A15). At the emission point E those transformations are applied to the radiation quantities $\rho_{(\gamma)}$, $P_{(\gamma)}$, $\delta\rho_{(\gamma)}$, and $q_{(\gamma)}$, while at the observation point R the transformations are determined by the hypersurface and state of motion of the observer chosen, as we will see.

Since the quantity $T_R(n^\mu)$ is observable, the general expression Eq. (35) must be independent of the gauge or congruence chosen, if the point R and the four-velocity of the observer are held constant. This is demonstrated explicitly in Appendix B. However, if the observation point and four-velocity are allowed to transform with the gauge transformation, then the anisotropies will depend on the gauge, as shown in Appendix C. Expanding the anisotropy in terms of the spherical harmonics, $Y_{\ell m}(n^\mu)$, the multipole amplitudes are

$$a_{\ell m} \equiv \int \frac{\delta T(n^\mu)}{\bar{T}_R} Y_{\ell m}^*(n^\mu) d\Omega. \quad (38)$$

We show explicitly in Appendix C that only the dipole anisotropy a_{1m} and monopole perturbation a_{00} change in the latter case, at linear order.

2. Recovering previous results in zero-shear or longitudinal gauge

We will now illustrate the usefulness of the general linear expression for the anisotropies, Eq. (35), by calculating in a particular gauge the anisotropy due to both adiabatic scalar and tensor sources, recovering previous results. We require both the line-of-sight integral in that expression as well as the temporal displacements δt_D and δt_B for the boundary terms. If we choose the congruence u^μ such that the scalar-derived part of the shear $\sigma_{\mu\nu}$ vanishes at linear order, then the integrand Eq. (37) takes a particularly simple form. The frequently used longitudinal gauge has this property, giving

$$\delta(H_n N) = -\dot{\psi}_\sigma + \phi_{;\sigma}^\mu n_\mu + \frac{1}{2} \dot{H}_{\mu\nu} n^\mu n^\nu. \quad (39)$$

In these expressions, the subscript σ indicates a zero-shear or longitudinal gauge quantity, and the overdot indicates the proper time derivative in the direction of u^μ . Therefore, using the first order expression

$$\phi_{;\mu} n^\mu = \frac{d\phi}{dt}\Big|_{v^\mu} - \dot{\phi}, \quad (40)$$

we can write the integral in Eq. (35) as

$$\int_{\bar{t}_R}^{\bar{t}_E} \delta(H_n N) dt = \int_{\bar{t}_R}^{\bar{t}_E} \left(\dot{\psi}_\sigma + \dot{\phi}_\sigma - \frac{1}{2} \dot{H}_{\mu\nu} n^\mu n^\nu\right) dt + \phi_\sigma\Big|_R^E. \quad (41)$$

It is simple to calculate the displacements δt_D and δt_B for the case of large-scale adiabatic modes, for which the

uniform radiation energy density and uniform total energy density hypersurfaces coincide, and for which the plasma-comoving and total comoving directions coincide. (Additionally, these surfaces and directions coincide even on small scales for the artificial case of a cold dark matter free universe, in the tight coupling approximation.) Although Σ_{LS} is defined as a surface of constant $\rho_{(\gamma)}$, it will be easier to calculate the position of uniform total density surfaces. The temporal displacement that takes us from zero-shear to uniform total energy density gauge is, using Eq. (A14) and the linearized energy constraint equation Eq. (A16) for the total energy density perturbation $\delta\rho$,

$$\delta t_D = -\frac{\delta\rho_\sigma}{\dot{\rho}} = -\frac{3\bar{H}(\dot{\psi}_\sigma + \bar{H}\phi_\sigma) - \frac{1}{a^2}\nabla^2\psi_\sigma}{12\pi G\bar{H}(\rho + P)}. \quad (42)$$

This expression can be applied at point E on the LSS as well as at the reception point R if we choose to place R on a surface of uniform energy density. The boost displacement that transforms from zero-shear to comoving gauge is, using Eq. (A15) and the linearized momentum constraint equation Eq. (A17) for the total momentum density scalar q ,

$$\delta t_B = -\frac{\dot{\psi}_\sigma + \bar{H}\phi_\sigma}{4\pi G(\rho + P)}. \quad (43)$$

Again, this can be applied at both E and at R if we choose the observer to be comoving.

Equations (41)–(43) with Eq. (35) completely specify the anisotropies in terms of quantities in zero-shear or longitudinal gauge. In practice it is common to make approximations. If we assume that the anisotropic stress is negligible (as is the case for matter or Λ domination), then we have $\psi_\sigma = \phi_\sigma$ (see, e.g., [33]). As explained above, the displacements δt_D and δt_B at the observation point R only affect the observed monopole and dipole. Dropping these terms, and using the background energy constraint, the anisotropy for $\ell > 1$ then becomes

$$\begin{aligned} \frac{\delta T(n^\mu)}{\bar{T}_R} &= \int_{\bar{t}_E}^{\bar{t}_R} \left(2\dot{\psi}_\sigma - \frac{1}{2}\dot{H}_{\mu\nu}n^\mu n^\nu \right) dt + \frac{1}{3}\psi_\sigma \\ &\quad - \frac{2}{9} \left(\frac{3}{\bar{H}}\dot{\psi}_\sigma + \frac{\nabla^2}{a^2\bar{H}^2}\psi_\sigma \right) \\ &\quad - \frac{2}{3} \frac{1}{\bar{H}^2} (\dot{\psi}_\sigma + \bar{H}\psi_\sigma)_{;\mu} n^\mu. \end{aligned} \quad (44)$$

All quantities outside the integral are to be evaluated at the point on the LSS corresponding to viewing direction $-n^\mu$. This expression agrees precisely with Eq. (4.7) in Ref. [8], for the case of adiabatic perturbations, including even a term the authors of [8] describe as arising from ‘‘subtle gauge effects.’’

Making further simplifications, we have $\dot{\psi}_\sigma = 0$ at last scattering if the pressure vanishes exactly there. If we consider anisotropies on the largest angular scales, sourced by modes with comoving wavenumber $k \ll a\bar{H}$, we can

drop all the gradient terms in Eq. (44). The result is

$$\frac{\delta T(n^\mu)}{\bar{T}_R} = \int_{\bar{t}_E}^{\bar{t}_R} \left(2\dot{\psi}_\sigma - \frac{1}{2}\dot{H}_{\mu\nu}n^\mu n^\nu \right) dt + \frac{1}{3}\psi_\sigma(E), \quad (45)$$

in agreement with the well-known result for the Sachs-Wolfe effect due to large-scale scalar and tensor sources. Note that this result includes a part that is evaluated at the boundary E , which comes from both the temporal displacement $\delta t_D(E)$ and the integral in Eq. (35). The remainder of that integral, which cannot be placed at the boundary, appears in Eq. (45) as a contribution that is due to physical metric fluctuations along the line of sight. The scalar part of this contribution is known as the integrated Sachs-Wolfe effect. During matter domination $\dot{\psi}_\sigma = 0$, so all scalar effects of the perturbed expansion along the line of sight can be placed at the boundary.

IV. TRANSFER FUNCTIONS

A. Dipole

Next we will use the results derived so far to perform a careful calculation of the CMB dipole due to scalar sources. More precisely we will calculate the dipole power, or the variance in the dipole anisotropy a_{1m} , namely,

$$C_1 \equiv \langle |a_{1m}|^2 \rangle - \langle a_{1m} \rangle^2, \quad (46)$$

over realizations of the assumed Gaussian random primordial fluctuations. (The independence of C_1 on m will follow from statistical isotropy.) For the dipole the choice of frame for the observer at R is critical, since a boost at the observation point changes the dipole according to Eq. (C12). For example, if we choose the observer’s frame to be comoving with radiation, so that the radiation momentum density $q_{(\gamma)}^\mu$ vanishes, then trivially the observed dipole vanishes. Indeed, combining Eqs. (A12) and (C12) we have

$$\frac{1}{3} \sum_m |a_{1m}|^2 = \frac{\pi}{4\rho_{(\gamma)}^2} q_{(\gamma)}^\mu q_{\mu}^{(\gamma)}, \quad (47)$$

so that the magnitude of the observed dipole in any frame is proportional to the radiation flux observed in that frame. We will adopt the most natural and physically best-motivated choice, namely, the frame comoving with matter (essentially the total comoving frame at late times) for the calculation of C_1 . With this choice of frame, Eq. (47) has the simple interpretation that the observed dipole is a measure of how well worldlines comoving with radiation and matter coincide.

We begin with the general linear result, Eq. (35). All of the parts of this equation were carefully calculated in zero-shear gauge for a comoving observer in Sec. III B 2. We can ignore the monopole contribution $\delta t_D(R)$ since it only affects the $\ell = 0$ mode. The displacements $\delta t_D(E)$ and δt_B were calculated in Eqs. (42) and (43). Combining these

results with Eq. (41) for the line-of-sight integral, and ignoring anisotropic stress at all times (so $\psi_\sigma = \phi_\sigma$), assuming matter domination at last scattering (so $\dot{\psi}_\sigma(E) = 0$), and finally ignoring the term $\frac{1}{a^2}\nabla^2\psi_\sigma(E)$, we have

$$\begin{aligned} \frac{\delta T(n^\mu)}{\bar{T}_R} &= \frac{1}{3}\psi_\sigma(E) - \frac{2}{3}\frac{n_\mu\psi_\sigma^{i\mu}(E)}{\bar{H}_E} + \left(\frac{5}{3}g_R^{-1} - 1\right) \\ &\times \frac{n_\mu\psi_\sigma^{i\mu}(R)}{\bar{H}_R} + 2\int_E^R \dot{\psi}_\sigma dt. \end{aligned} \quad (48)$$

Here we have used Eq. (A21) for the growth function $g_R \equiv g(t_R)$ and the relation Eq. (A24) to simplify the expression. The approximation of matter domination at last scattering (which implies zero anisotropic stress) results in errors in C_ℓ for small ℓ on the order of 10% [34]. Neglecting the term $\frac{1}{a^2}\nabla^2\psi_\sigma(E)$ is entirely justified considering that Eq. (42) for $\delta t_D(E)$ employed the approximation that the uniform radiation energy density and uniform total energy density hypersurfaces coincide, which is only valid on large scales (scales that were super-Hubble at last scattering). The contribution to the dipole from last scattering will be dominated by these large scales, as we will see.

To calculate the variance of the dipole, it will be helpful to expand the function $\psi_\sigma(E)$ in spherical harmonics as

$$\begin{aligned} \psi_\sigma(E) &= -\frac{3}{5}\sqrt{\frac{2}{\pi}} \\ &\times \int dkkT(k)\sum_{\ell m}\mathcal{R}_{\ell m}^{\text{pr}}(k)j_\ell(kr_{\text{LS}})Y_{\ell m}(-n^\mu). \end{aligned} \quad (49)$$

Here j_ℓ is the spherical Bessel function of the first kind, r_{LS} is the comoving radius of the LSS, k is the comoving wave number, and $T(k)$ is the transfer function defined in Eq. (A19). With the aim of expressing C_1 in terms of the primordial spectrum $\mathcal{P}_{\mathcal{R}}(k)$, we have used Eq. (A22) to write ψ_σ in terms of the primordial comoving curvature perturbation \mathcal{R}^{pr} . Similarly we will need the expansion of the Doppler contributions at E and R ,

$$\begin{aligned} n_\mu\psi_\sigma^{i\mu}(E) &= \frac{3}{5}\sqrt{\frac{2}{\pi}}\frac{1}{a_E} \\ &\times \int dkk^2T(k)\sum_{\ell m}\mathcal{R}_{\ell m}^{\text{pr}}(k)j'_\ell(kr_{\text{LS}})Y_{\ell m}(-n^\mu), \end{aligned} \quad (50)$$

and

$$\begin{aligned} n_\mu\psi_\sigma^{i\mu}(R) &= \lim_{r\rightarrow 0}\frac{3}{5}\sqrt{\frac{2}{\pi}}\frac{g_R}{a_R} \\ &\times \int dkk^2T(k)\sum_{\ell m}\mathcal{R}_{\ell m}^{\text{pr}}(k)j'_\ell(kr)Y_{\ell m}(-n^\mu) \end{aligned} \quad (51)$$

$$= \frac{1}{5}\sqrt{\frac{2}{\pi}}\frac{g_R}{a_R}\int dkk^2T(k)\sum_m\mathcal{R}_{1m}^{\text{pr}}(k)Y_{1m}(-n^\mu), \quad (52)$$

where the prime indicates differentiation with respect to the argument, and we have used the relation $j'_\ell(0) = \delta_\ell^1/3$. Inserting these expressions into Eq. (48), we can evaluate the dipole anisotropy using Eq. (38) (with $\ell = 1$) and the orthonormality of the $Y_{\ell m}$. This gives

$$a_{1m} = -\frac{1}{5}\sqrt{\frac{2}{\pi}}\int dkk\mathcal{R}_{1m}^{\text{pr}}(k)T(k)T_1(k), \quad (53)$$

where $T_1(k)$ is a new transfer function, called the *dipole transfer function*, and defined by

$$\begin{aligned} T_1(k) &\equiv j_1(kr_{\text{LS}}) + \frac{2k}{a_E\bar{H}_E}j'_1(kr_{\text{LS}}) + \left(g_R - \frac{5}{3}\right)\frac{k}{a_R\bar{H}_R} \\ &+ 6\int_E^R \dot{g}(t)j_1[kr(t)]dt. \end{aligned} \quad (54)$$

Here $r(t)$ is the comoving radial coordinate of the light ray as a function of the time coordinate. Finally, using Eq. (46) and the statistical relation Eq. (A25) for the primordial power spectrum $\mathcal{P}_{\mathcal{R}}(k)$, we find for the dipole power

$$C_1 = \frac{4\pi}{25}\int \frac{dk}{k}\mathcal{P}_{\mathcal{R}}(k)T^2(k)T_1^2(k). \quad (55)$$

Note the third term in Eq. (54), which is proportional to k and dominates on small scales today. It might appear that this term would lead to an ultraviolet divergence for the dipole, but Eq. (55) is rendered finite by the function $T(k)$, which decays like k^{-2} on small scales. This results in a dipole amplitude $\sqrt{C_1} \sim 10^{-3}$ for standard cosmological parameters.

B. Monopole

The monopole temperature perturbation is the $\ell = 0$ component of Eq. (38), i.e.

$$a_{00} = \frac{1}{\sqrt{4\pi}}\int \frac{\delta T(n^\mu)}{\bar{T}_R}d\Omega, \quad (56)$$

and its variance over realizations of the primordial fluctuations will be called C_0 . As discussed above Eq. (B14), the monopole so defined is ambiguous in that different choices of the background times \bar{t}_E and \bar{t}_R lead to different a_{00} . This well-known freedom is equivalent to our inability to uniquely separate a background CMB temperature from the observed mean temperature, which would be required to define a monopole perturbation. Hence the monopole perturbation in Eq. (56) cannot be observable.

Nevertheless, it is still possible to sensibly define a monopole and consider its theoretical properties. Recall that for the dipole it was necessary to fix the observer's frame by a physical prescription (namely that it be comoving with matter). Analogously, to define a monopole the key point is that we must fix, by a physical prescription, the

spacelike hypersurface on which we place the observer. Clearly there is freedom in how we choose this slice. Recall that an observer chosen to comove with radiation observes no dipole. The analogous situation with the monopole is an observer placed on a uniform radiation energy density hypersurface, for which the calculated C_0 must vanish. Analogously to Eq. (47) for the dipole, we can write

$$|a_{00}|^2 = \frac{\pi}{4\rho_{(\gamma)}^2} (\delta\rho_{(\gamma)})^2, \quad (57)$$

where $\delta\rho_{(\gamma)}$ is the radiation energy density perturbation on the same slicing used to define the monopole a_{00} . Therefore the monopole defined with respect to any slicing is proportional to the radiation density perturbation for the same slicing.

The simplest and most natural choice of slice on which to place the observer is that of uniform matter (essentially total) energy density. This choice is simplest because it requires knowledge of just the local density, ρ , which acts as a clock. It is natural because, as we will see, it exhibits close analogy with the dipole. Comoving slices could also be used, although their construction as hypersurfaces orthogonal to the comoving worldlines is more elaborate [35]. Thus, while the dipole defined above is a measure of the degree to which comoving radiation and matter worldlines coincide, the monopole defined here has the simple interpretation as a measure of how well hypersurfaces of uniform radiation and matter energy density coincide [36]. The arbitrariness in a_{00} mentioned above (due to the freedom to choose the background times) only amounts to a constant shift to a_{00} , so its variance is unchanged. This is why, although the monopole perturbation a_{00} is ambiguous and unobservable for any single observer, the *variance* C_0 is still well defined theoretically (and could, in principle, be approximated through observations [37]).

It should be clear immediately that there is a serious problem with attempting to define the variance of the CMB temperature on a uniform matter density slice. Namely, matter has of course entered the nonlinear stage on small scales, and hence hypersurfaces of constant matter density cannot actually be defined. Nevertheless, smoothing over small scales can recover meaningful linear results, at the expense of further ambiguity in the form of the smoothing scale. This issue arises because in calculating the monopole as defined here, we will see that the dominant contribution will be given by the variance $\langle(\delta\rho/\rho)^2\rangle$ evaluated at R , which is dominated by small scales (where the slicing is irrelevant). The important practical aspect of this brief examination of the monopole will actually be in understanding the effects of long-wavelength sources, for which the difficulties with small scales do not arise.

Proceeding as with the dipole above, the relevant contributions to the general linear result Eq. (35) give

$$\begin{aligned} \frac{\delta T(n^\mu)}{\bar{T}_R} &= \frac{1}{3} \psi_\sigma(E) - \frac{2}{3} \frac{n_\mu \psi_\sigma^\mu(E)}{\bar{H}_E} + \left(\frac{5}{3} g_R^{-1} - 2\right) \psi_\sigma(R) \\ &\quad - \frac{2}{9\Omega_m} \frac{\nabla^2}{a_R^2 \bar{H}_R^2} \psi_\sigma(R) + 2 \int_E^R \dot{\psi}_\sigma dt. \end{aligned} \quad (58)$$

We can ignore the Doppler contribution δt_B at R since it only affects the $\ell = 1$ mode, and the same approximations have been applied here as for the dipole Eq. (48).

To calculate the variance of the monopole we will need, in addition to Eqs. (49) and (50), the monopole term evaluated at the reception point R ,

$$\psi_\sigma(R) = -\frac{3}{5} \sqrt{\frac{2}{\pi}} g_R \int dk k T(k) \sum_{\ell m} \mathcal{R}_{\ell m}^{\text{pr}}(k) j_\ell(0) Y_{\ell m}(n^\mu) \quad (59)$$

$$= -\frac{3}{5} \sqrt{\frac{2}{\pi}} g_R \int dk k T(k) \mathcal{R}_{00}^{\text{pr}}(k) Y_{00}, \quad (60)$$

using the relation $j_\ell(0) = \delta_\ell^0$. Proceeding as for the dipole case, we find

$$a_{00} = -\frac{1}{5} \sqrt{\frac{2}{\pi}} \int dk k \mathcal{R}_{00}^{\text{pr}}(k) T(k) T_0(k), \quad (61)$$

where $T_0(k)$ is the monopole transfer function defined by

$$\begin{aligned} T_0(k) &\equiv j_0(kr_{\text{LS}}) + \frac{2k}{a_E \bar{H}_E} j_0'(kr_{\text{LS}}) + 5 - 6g_R \\ &\quad + \frac{2}{3} \frac{g_R}{\Omega_m} \frac{k^2}{a_R^2 \bar{H}_R^2} + 6 \int_E^R \dot{g}(t) j_0[kr(t)] dt. \end{aligned} \quad (62)$$

Finally, using the statistical relation Eq. (A25) to evaluate the variance we find

$$C_0 = \frac{4\pi}{25} \int \frac{dk}{k} \mathcal{P}_{\mathcal{R}}(k) T^2(k) T_0^2(k). \quad (63)$$

Note the presence of the Laplacian term ($\propto k^2$) in $T_0(k)$, which will dominate on small scales, and implies that by calculating C_0 we are essentially calculating the (effectively gauge independent) matter variance $\langle(\delta\rho/\rho)^2\rangle$ at the observation point. In this monopole case the resulting divergence is too strong to be saved by the transfer function $T(k)$. Therefore we expect the variance C_0 to diverge on small scales, which is simply a reflection of the nonlinear nature of matter fluctuations on small scales today, as predicted above. That is, the monopole variance as we have defined it cannot be quantified. Again, the importance of Eq. (62) will lie in its long-wavelength behavior.

V. LONG-WAVELENGTH BEHAVIOR

Recall that all of our approximations have been good on very large scales. In this section, we examine the long-wavelength limit of the $T_1(k)$ and $T_0(k)$ transfer functions. The dipole transfer function for a comoving observer,

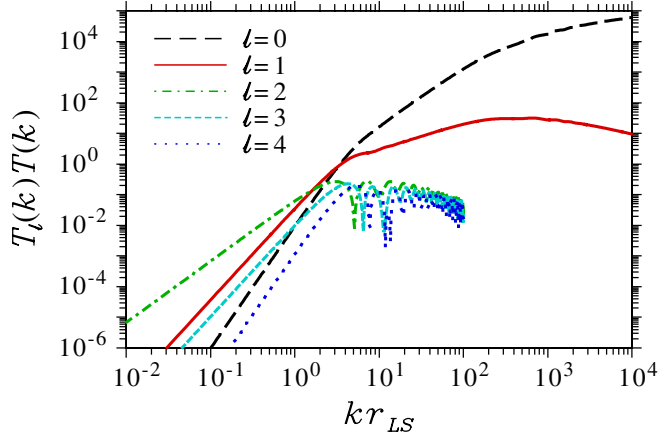


FIG. 3 (color online). Dipole transfer function $T_1(k)T(k)$ for a comoving observer (solid curve). For comparison, the monopole transfer functions for an observer on a uniform energy density slice, $T_0(k)T(k)$, and the transfer functions for $\ell = 2, 3$, and 4 are also shown. Absolute values are plotted. The scales $kr_{LS} = 10^2$ and $kr_{LS} = 1$ correspond roughly to the Hubble scales at last scattering and today, respectively.

Eq. (54), is plotted in Fig. 3, together with the monopole function for an observer on a uniform energy density slice, Eq. (62), and the transfer functions for $\ell = 2, 3$, and 4. The transfer functions for $\ell > 1$ can be calculated from Eq. (48) in exactly the same way as for the dipole, with the result

$$T_\ell(k) = j_\ell(kr_{LS}) + \frac{2k}{a_E \bar{H}_E} j'_\ell(kr_{LS}) + 6 \int_E^R \dot{g}(t) j_\ell[kr(t)] dt. \quad (64)$$

Note that the large-scale approximations involved in Eq. (48) imply that this expression is only valid for scales that are super-Hubble at last scattering. [The transfer functions $T_1(k)$ and $T_0(k)$ are valid for small scales, since for large kr_{LS} the second-to-last terms in both Eqs. (54) and (62) dominate. These terms are generated *locally* at the observation point R .] A value of $\Omega_\Lambda = 0.77$ today was used for all of these calculations.

Using asymptotic forms for the Bessel functions [38], we can show from Eq. (64) that $T_\ell(k)$ should decay like k^ℓ as $kr_{LS} \rightarrow 0$, for $\ell > 1$. This is verified in Fig. 3. However, the figure also shows that $T_1(k)$ does not decay like k for small k ; instead it decays like k^3 , which is *faster* than the decay rate of $T_2(k)$.

To examine the behavior of the transfer function $T_1(k)$ in the limit $k \rightarrow 0$, we can use the small-argument approximations to the Bessel functions [38] to give

$$T_1(k) = \left[\left(g_R - \frac{5}{3} \right) \frac{1}{a_R \bar{H}_R} - \frac{5}{3} \eta_R + 2 \int_0^{\eta_R} g(\eta) d\eta \right] k + \mathcal{O}(kr_{LS})^3, \quad (65)$$

where the conformal time η , defined via $d\eta = dt/a$, has

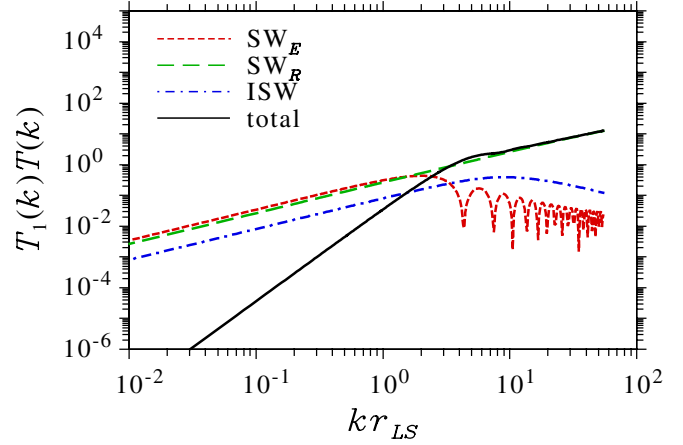


FIG. 4 (color online). Dipole transfer function $T_1(k)T(k)$ for a comoving observer. Absolute values of individual contributions from Sachs-Wolfe terms evaluated at emission, E , and at the observation point, R , as well as the line-of-sight ISW contribution, are indicated.

been used to simplify the expression. In writing Eq. (65), we have used the approximation $g(t_E) = 1$, although for our numerical calculations we have evaluated Eq. (54) without approximation. It is not obvious from Eq. (65) that the $\mathcal{O}(k)$ term, in square brackets, vanishes. However, the numerical calculation of Fig. 3 shows that the $\mathcal{O}(k)$ term does indeed vanish, as pointed out above.

This is illustrated in greater detail in Fig. 4. There we have plotted separately the first two terms in Eq. (54), which originate at the LSS (called SW_E in the plot), the third term in Eq. (54), which comes from the observation point R (SW_R), and the line-of-sight ISW term. We can see that each component separately does scale like k on large scales, in particular, the LSS term SW_E scales like the predicted k^ℓ , but the sum demonstrates that the $\mathcal{O}(k)$ terms all cancel. It is the local contribution, SW_R , which is not present for $\ell > 1$, which enables the cancellation. (Of course the individual SW and ISW components are not separately observable.) It is possible to show analytically the $\mathcal{O}(k^3)$ dependence in the special case of a cosmological-constant free Einstein-de Sitter universe. Then Eq. (65) becomes

$$T_1(k) = -\frac{k^3}{30} (r_{LS}^3 + 3r_{LS}^2 \eta_{LS}) + \mathcal{O}(k^5). \quad (66)$$

Figure 4 also illustrates that the source of the small-scale increase in $T_1(k)$ is due to the local contribution at R , as explained above. The dipole transfer function, Eq. (54), suggests a divergence in C_1 like a power of k on small scales, although this is moderated by the transfer function $T(k)$. The result is that the observed dipole amplitude is a factor $\sim 10^2$ larger than the other multipoles. Figure 3 actually illustrates this directly: Eq. (55) generalizes to

$$C_\ell = \frac{4\pi}{25} \int \frac{dk}{k} \mathcal{P}_{\mathcal{R}}(k) T^2(k) T_\ell^2(k) \quad (67)$$

for all $\ell \geq 0$. Therefore the expected multipole amplitude is simply proportional to the area under the appropriate curve in Fig. 3 [assuming a nearly scale-invariant primordial spectrum $\mathcal{P}_{\mathcal{R}}(k)$]. Geometrically, the comoving matter and radiation worldlines coincide on large scales, and begin to diverge strongly on small scales. The contribution to the dipole from last scattering, SW_E , will vary smoothly if the observer's position is varied, whereas the local contribution, SW_R , will vary greatly.

These results indicate that the dipole defined with respect to the comoving frame receives strongly suppressed contributions from super-Hubble modes. This was in fact noted some time ago [15] in relation to the Grishchuk-Zel'dovich effect [14], but in the context of a matter-dominated universe. In fact, it was claimed in [15] that this cancellation would not persist in the presence of a cosmological constant, so that super-Hubble modes would have an observable imprint on the CMB, a claim which we have now demonstrated to be incorrect.

As we did for the dipole, we can examine the behavior of the monopole transfer function $T_0(k)$ in the limit $k \rightarrow 0$, giving

$$T_0(k) = \left[-\frac{1}{6} r_{LS}^2 - \frac{2r_{LS}}{3a_E \bar{H}_E} - \frac{2}{3a_E^2 \bar{H}_E^2} + \frac{2g_R}{3\Omega_{m,R} a_R^2 \bar{H}_R^2} - \int_E^R \dot{g}(t) r^2(t) dt \right] k^2 + \mathcal{O}(k^4), \quad (68)$$

where Ω_m is the standard matter density parameter. In this monopole case, we have thus shown analytically that the leading-order terms in the transfer function, Eq. (62), which go like k^0 , do cancel in the presence of a cosmological constant. However, Fig. 3 shows that an even

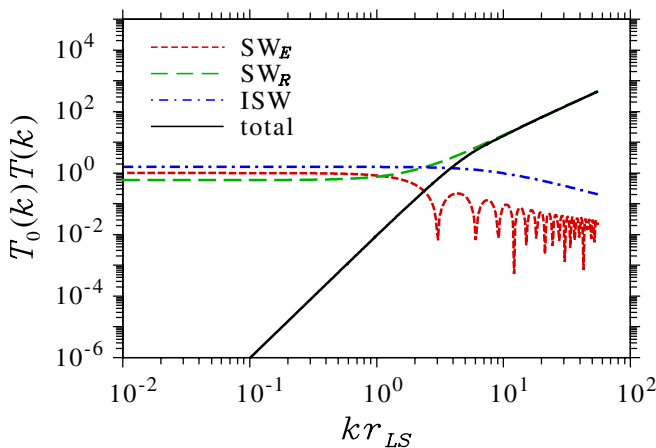


FIG. 5 (color online). Monopole transfer function $T_0(k)T(k)$, for an observer on a uniform energy density slice. Absolute values of individual contributions from Sachs-Wolfe terms evaluated at emission, E , and at the observation point, R , as well as the line-of-sight ISW contribution, are indicated.

stronger result holds in the monopole case: the $\mathcal{O}(k^2)$ terms in Eq. (68) actually cancel as well. Again, this is shown in greater detail in Fig. 5, where it is apparent that an exquisite cancellation occurs in the individual components, which scale like k^0 on large scales, to give the $\mathcal{O}(k^4)$ total transfer function on large scales. Figure 5 also illustrates the local source, SW_R , of the strong small-scale divergence discussed above. Geometrically, the hypersurfaces of uniform matter and radiation density coincide on large scales, and begin to very strongly diverge on small scales. This divergence makes it impossible to quantify the total power C_0 with our definition of the monopole.

VI. DISCUSSION

Our result in Sec. V that the dipole and monopole receive suppressed contributions from large scales, even in the presence of a dominant cosmological constant, strongly suggests that the cancellations involved are not accidental. To understand the origin of this behavior, consider first the monopole case. Physically, the suppression as $k \rightarrow 0$ in the monopole transfer function shown in Figs. 3 and 5 means that surfaces of uniform total and radiation energy density coincide on the largest scales, according to our definition of the monopole in Sec. IV B. But this is just the statement of adiabaticity: an adiabatic matter-radiation fluid is characterized by the condition

$$\frac{\delta\rho}{\dot{\rho}} = \frac{\delta\rho_{(\gamma)}}{\dot{\rho}_{(\gamma)}}, \quad (69)$$

on the largest scales. Therefore, Eq. (A14) shows that the same gauge transformation takes us to both constant total matter and constant radiation hypersurfaces; in other words, those surfaces must coincide. It is important to point out that adiabatic long-wavelength modes remain adiabatic under evolution [33], and indeed the comoving curvature perturbation \mathcal{R} remains constant in time (see, e.g., [39]). This trivial evolution on super-Hubble scales means that when constant matter and radiation density surfaces coincide on large scales at last scattering, due to adiabatic initial conditions, they must also coincide today.

An analogous situation holds for the dipole. In this case, the adiabaticity condition implies that, on large scales, the radiation and total matter comoving worldlines coincide (i.e. there is no ‘‘peculiar velocity’’ isocurvature mode between the two components). According to our definition of the dipole in Sec. IV A, this is simply the statement that the dipole is suppressed on large scales.

This leads us to the important conclusion that this insensitivity to long-wavelength sources must apply regardless of the matter content of the Universe, as long as adiabaticity holds. Therefore the suppression we found for the specific case of cosmological-constant (or Einstein-de Sitter) universes must in fact occur in general. Note that this may have relevance to very recent discussions regarding a potential power asymmetry in the CMB

[40]. The exquisite cancellations visible at large scales in Figs. 4 and 5 between the SW and ISW components illustrate a previously unrecognized relation between the two, which is enforced by the condition of adiabaticity.

In brief, the dipole and monopole as defined here are just measures of departures from the adiabaticity condition, Eq. (69), which generally occur on small scales. Are these results sensitive to the definitions of dipole and monopole used? As long as the dipole and monopole are defined *physically*, i.e. in relation to locally measureable quantities, then the results must hold. An important example which does *not* satisfy this criterion is the zero-shear, or longitudinal gauge, since it is not defined in terms of local, observable matter quantities. If the dipole (or monopole) is defined with respect to a zero-shear frame, then it may appear from theoretical calculations that the dipole *is* sensitive to long-wavelength modes. However, this sensitivity cannot be observable, since zero-shear frames cannot be *uniquely* constructed locally. (A linear boost or “tilt” of a zero-shear frame is still a zero-shear frame.) Thus great care must be taken when considering behavior on large scales with longitudinal gauge.

Indeed, another way to understand this result is to realize that, in the limit $k/(aH) \rightarrow 0$, an adiabatic perturbation mode locally becomes essentially pure gauge, and can be removed within any *sub-Hubble* region by a simple boost coordinate transformation. Such a mode is indistinguishable locally from a homogeneous background, and hence cannot have any observational consequences such as a dipole anisotropy [41].

Finally, we note that when the condition of adiabaticity is relaxed, then our conclusions no longer hold [16]. In the presence of isocurvature perturbations, it is possible that a *physically* defined dipole be sensitive to super-Hubble modes, since the extra freedom allows for a relative tilt between comoving matter and radiation comoving worldlines on large scales.

While we have emphasized here the consequences of adiabaticity, it is hoped that other applications will follow from the exact formalism for CMB anisotropies that we have developed. One possibility is the evaluation of anisotropies, in particular, the ISW effect, in void models of acceleration [42] (see [43] for a brief review), which have not yet been confronted with observations at the perturbative level [44]. Note however that the present approach is not limited to calculating CMB anisotropies, and that more generally it is applicable to calculating redshifts in arbitrary spacetimes. Potential uses include calculating the redshift-luminosity distance relation in perturbed spacetimes (see, e.g., [45]).

ACKNOWLEDGMENTS

This research was supported by the Natural Sciences and Engineering Research Council of Canada. We thank Tom Waterhouse for many useful discussions.

Note added.—When this work was essentially complete, a related paper appeared [50], which appears to support our conclusion that long-wavelength perturbations cannot effect the CMB dipole.

APPENDIX A: METRIC-BASED APPROACH

In this appendix, we will collect together several elements of the metric-based approach to cosmological perturbations that will be useful in describing the CMB anisotropies at linear order (see, e.g., [33] for a review of metric-based perturbation theory). In the metric approach a set of coordinates are defined in the spacetime by a foliation into spacelike hypersurfaces of constant t , Σ_t , and a threading into timelike worldlines of constant spatial coordinates x^i , where Latin indices run from 1 to 3. The gauge freedom of perturbation theory is related to our freedom to choose such a slicing and threading. However, it can be shown (see [46,47]) that at linear order, *physical* perturbations in FRW backgrounds are gauge invariant under changes in the threading. Therefore the gravitational dynamics can be entirely expressed in terms of spatially gauge-invariant quantities. The reason for this invariance is simply the homogeneity of the background spacetime, and its importance is that it means that there is effectively just a single degree of gauge freedom on FRW backgrounds, namely, the temporal position of Σ_t at each event. Thus, to simplify expressions to follow, we will choose the congruence of coordinate threads (with tangents u^μ) to be orthogonal to the Σ_t , so that the shift vector (metric component g_{0i}) vanishes. These spatial coordinates are comoving at zeroth order, but may depart from comoving at first order. With this choice, there is a direct correspondence between the gauge freedom (i.e. the freedom to choose the slicing) in the metric formalism, and the freedom to choose the congruence in the covariant formalism. Evaluating the metric using Eq. (1) in the chosen coordinates we have

$$g_{00} = -N^2, \quad g_{ij} = h_{ij}, \quad (\text{A1})$$

where N is the lapse function for the slicing. We consider scalar and tensor perturbations only, as vectors are ordinarily thought to be cosmologically irrelevant.

The spacetime is completely described by the lapse function, the intrinsic curvature of the spatial metric $h_{\mu\nu}$, and the extrinsic curvature of the Σ_t . We define the lapse perturbation ϕ through

$$N \equiv 1 + \phi. \quad (\text{A2})$$

The only part of the intrinsic curvature of $h_{\mu\nu}$ that we will need is the perturbed Ricci scalar, $\delta^{(3)}R$, for the spatial slices Σ_t , which defines the curvature perturbation ψ via the relation

$$\delta^{(3)}R \equiv \frac{4}{a^2} \nabla^2 \psi. \quad (\text{A3})$$

Here $\nabla^2/a^2 \equiv D_\mu D^\mu$, for background scale factor a , so ∇^2 is the comoving Laplacian. The extrinsic curvature of the slicing is specified by the expansion and shear of the normal congruence to the Σ_t , and is related to the spatial metric through [22]

$$\frac{1}{3}\theta h_{ij} + \sigma_{ij} = \frac{1}{2}\dot{h}_{ij}, \quad (\text{A4})$$

where the overdot represents the proper time derivative along u^μ . At linear order the trace of this equation gives for the expansion perturbation

$$\delta\theta = -3\bar{H}\phi - 3\dot{\psi} + \frac{1}{a^2}\nabla^2\sigma, \quad (\text{A5})$$

where \bar{H} is the background Hubble rate. The shear scalar σ describes the scalar-derived part of the shear, with

$$\sigma_{ij} = D_i D_j \sigma - \frac{1}{3a^2}\nabla^2\sigma h_{ij} + \frac{1}{2}\dot{H}_{ij}, \quad (\text{A6})$$

where H_{ij} is the transverse and traceless (tensor) part of h_{ij} , and we have ignored the vector-derived part of the shear. One further important quantity is the acceleration of the worldlines normal to the Σ_t , Eq. (4). It is related to the lapse through the exact expression

$$a_\mu = \frac{1}{N}D_\mu N, \quad (\text{A7})$$

which at linear order becomes

$$a_\mu = D_\mu \phi. \quad (\text{A8})$$

The various quantities defined above can be related to the explicit component form of the metric at linear order (ignoring the vector part),

$$\delta g_{00} = -2\phi, \quad \delta g_{ij} = a^2(-2\psi\gamma_{ij} + 2E_{,ij} + H_{ij}), \quad (\text{A9})$$

where γ_{ij} is the background spatial metric, and the trace-free scalar part of Eq. (A4) gives $\sigma = a^2\dot{E}$.

Once the slicing Σ_t is specified, i.e. the time coordinate is chosen, then the perturbation in any exact quantity $X(x^i, t)$ is fixed via

$$\delta X(x^i, t) = X(x^i, t) - \bar{X}(t), \quad (\text{A10})$$

where $\bar{X}(t)$ is the homogeneous background value. Therefore our freedom to vary the Σ_t (equivalently to vary the orthogonal congruence u^μ) results in an inherent ambiguity in our ability to specify any perturbation. This temporal gauge freedom can be used to simplify calculations by choosing an appropriate congruence, as we see with the Sachs-Wolfe calculation in Sec. III B 2. It is straightforward to calculate the change in any perturbation under a gauge transformation $t \rightarrow t - \delta t$ (see, e.g., [47]). A few results that we will need are, at linear order,

$$\delta\rho \rightarrow \delta\rho + \dot{\rho}\delta t, \quad (\text{A11})$$

$$q \rightarrow q - (\rho + P)\delta t, \quad (\text{A12})$$

$$\psi \rightarrow \psi - \bar{H}\delta t, \quad (\text{A13})$$

where q is the scalar from which the momentum density q^μ is derived, through $q^\mu = D^\mu q$. These results imply that the gauge transformation required to go from an arbitrary initial gauge to uniform energy density gauge, defined by $\delta\rho = 0$, is

$$\delta t = -\frac{\delta\rho}{\dot{\rho}}, \quad (\text{A14})$$

and the transformation that takes an arbitrary initial gauge into comoving gauge, defined by $q = 0$, is

$$\delta t = \frac{q}{\rho + P}. \quad (\text{A15})$$

Unless otherwise stated, all expressions in this work will be presented in an unspecified gauge, i.e. they will apply to arbitrary gauges.

We will also need the Einstein constraint equations in order to relate the matter perturbations to the metric perturbations. Projecting the Einstein equations twice along u^μ and linearizing, we find the energy constraint,

$$3\bar{H}(\dot{\psi} + \bar{H}\phi) - \frac{1}{a^2}\nabla^2(\psi + \bar{H}\sigma) = -4\pi G\delta\rho. \quad (\text{A16})$$

Similarly, projecting once with u^μ and once with $h^\mu{}_\nu$ gives the linearized (scalar) momentum constraint

$$\dot{\psi} + \bar{H}\phi = -4\pi Gq. \quad (\text{A17})$$

It is conventional to express the primordial power spectrum in terms of the comoving gauge curvature perturbation, usually denoted \mathcal{R} , since \mathcal{R} is constant on large (super-Hubble) scales for adiabatic modes and hence its value at late times can be trivially related to the predictions of an inflationary model (see, e.g., [39]). However, we will see that it is simplest to perform the linear Sachs-Wolfe calculation in terms of the zero-shear gauge curvature perturbation, ψ_σ , where we denote zero-shear gauge perturbations by the subscript σ . For matter domination, it is simple to relate ψ_σ to \mathcal{R} by performing a gauge transformation from zero-shear to comoving gauge; using Eqs. (A12) and (A13) gives

$$\psi_\sigma = -\frac{3}{5}\mathcal{R}. \quad (\text{A18})$$

In terms of Fourier modes at comoving wavevector \mathbf{k} , on large scales $\mathcal{R}(\mathbf{k}, t) = \mathcal{R}^{\text{pr}}(\mathbf{k})$, where $\mathcal{R}^{\text{pr}}(\mathbf{k})$ is the constant primordial value of $\mathcal{R}(\mathbf{k})$, i.e. the value at early times sufficiently later than Hubble exit during inflation. However, on small scales \mathcal{R} decays during radiation domination. The total decay incurred through to matter domination is described by a linear transfer function $T(k)$, defined by

$$\mathcal{R}(\mathbf{k}, t_m) = T(k)\mathcal{R}^{\text{pr}}(\mathbf{k}), \quad (\text{A19})$$

where $k \equiv |\mathbf{k}|$ and t_m is some time during matter domination. ($T(k)$ will be distinguished from the temperature T by the presence of its argument, k .) The transfer function $T(k)$ approaches unity at small k , and decays roughly like k^{-2} for $k \gtrsim k_{\text{eq}}$, where k_{eq} is the wave number which enters the Hubble radius at matter-radiation equality (see, e.g., [39]). $T(k)$ can be calculated accurately numerically, e.g. using packages such as CAMB [10,11]. Combining Eqs. (A18) and (A19) gives

$$\psi_\sigma(\mathbf{k}, t_m) = -\frac{3}{5}T(k)\mathcal{R}^{\text{pr}}(\mathbf{k}). \quad (\text{A20})$$

The perturbation ψ_σ is constant during matter domination, but it decays once a cosmological constant becomes important. This decay is independent of scale and can be described by a function $g(t)$ via

$$\psi_\sigma(\mathbf{k}, t) \equiv g(t)\psi_\sigma(\mathbf{k}, t_m), \quad (\text{A21})$$

which gives

$$\psi_\sigma(\mathbf{k}, t) = -\frac{3}{5}g(t)T(k)\mathcal{R}^{\text{pr}}(\mathbf{k}). \quad (\text{A22})$$

The function $g(t)$ approaches unity and zero at early and late times, respectively, and solving the linearized dynamical Einstein equation for ψ_σ gives

$$g(t) = \frac{5}{2} \frac{\Omega_{m,0} \bar{H}_0^2 \bar{H}}{a} \int^t \frac{dt'}{a^2 \bar{H}^2}. \quad (\text{A23})$$

Here \bar{H}_0 is the background Hubble rate today, and $\Omega_{m,0}$ is the ratio of matter to total energy density today. Using this last expression we can derive the useful relation

$$\frac{\dot{g}}{\bar{H}} + g = \Omega_m \left(\frac{5}{2} - \frac{3}{2}g \right). \quad (\text{A24})$$

The statistics of the assumed Gaussian random primordial fluctuations are completely described by the power spectrum $\mathcal{P}_{\mathcal{R}}(k)$, defined by

$$\langle \mathcal{R}^{\text{pr}}(\mathbf{k}) \mathcal{R}^{\text{pr}*}(\mathbf{k}') \rangle = 2\pi^2 \delta^3(\mathbf{k} - \mathbf{k}') \frac{\mathcal{P}_{\mathcal{R}}(k)}{k^3}. \quad (\text{A25})$$

With this definition $\mathcal{P}_{\mathcal{R}}(k)$ is dimensionless, and it is constant for a scale-invariant spectrum.

APPENDIX B: GAUGE INVARIANCE OF ANISOTROPIES

The calculated temperature anisotropy cannot depend on the coordinate choice (in the metric framework) or congruence choice (in the covariant framework), since the anisotropy is directly observable. A number of workers have demonstrated this explicitly in the past in the metric formalism (see, e.g., [9,31]). Nevertheless, it will be useful to demonstrate the result explicitly using the present co-

variant framework, as it will help to illuminate the issues involved.

We wish to demonstrate explicitly that the expression Eq. (35) for the anisotropy observed at R , namely,

$$\frac{\delta T(n^\mu)}{\bar{T}_R} = \int_{\bar{t}_R}^{\bar{t}_E} \delta(H_n N) dt + (\bar{H} \delta t_D + n_\mu \delta t_B^\mu) \Big|_R^E, \quad (\text{B1})$$

is invariant under arbitrary linear gauge transformations, if the point R and the four-velocity of the observer are held constant. Such a transformation changes the hypersurfaces of constant time according to

$$t \rightarrow t - \delta t, \quad (\text{B2})$$

for small temporal shift $\delta t = \delta t(x^\mu)$. Equivalently, it changes the congruence orthogonal to the slicing by the spatial gradient of δt ,

$$u^\mu \rightarrow u^\mu + D^\mu \delta t, \quad (\text{B3})$$

at linear order. Under this change in u^μ , each term in the integrand in Eq. (B1) will change, and the change in the slices $\Sigma_{\bar{t}_E}$ and $\Sigma_{\bar{t}_R}$ implied by Eq. (B2) means that the displacements δt_D and δt_B to the last scattering hypersurface Σ_{LS} and to the reception point R will also change.

Straightforward but lengthy calculations using the definitions Eqs. (4)–(6) give the following transformations under Eq. (B3), to first order:

$$\theta \rightarrow \theta + D^2 \delta t + 3\dot{\bar{H}} \delta t, \quad (\text{B4})$$

$$\sigma_{\mu\nu} n^\mu n^\nu \rightarrow \sigma_{\mu\nu} n^\mu n^\nu + n^\mu n^\nu D_\mu D_\nu \delta t - \frac{1}{3} D^2 \delta t, \quad (\text{B5})$$

$$a_\mu n^\mu \rightarrow a_\mu n^\mu + n^\mu (\delta t)_{;\mu}, \quad (\text{B6})$$

where $D^2 \equiv D^\mu D_\mu$ is the physical Laplacian. Note that, by definition, after the gauge transformation (B2) all quantities in the integrand in Eq. (B1) are to be evaluated at the new event temporally displaced by δt from the original event. This makes no difference at linear order to quantities that vanish at zeroth order (such as $\sigma_{\mu\nu}$ and a_μ), but accounts for the term $3\dot{\bar{H}} \delta t$ in Eq. (B4). Next, considering that the quantity $N dt$ is the proper time interval along u^μ between hypersurfaces separated by coordinate time interval dt , we can easily derive the linear transformation law for the lapse perturbation,

$$\delta N \rightarrow \delta N + \dot{\delta t}. \quad (\text{B7})$$

Combining Eqs. (B4)–(B7) with Eq. (36), we have the linear transformation of the integrand of Eq. (B1),

$$\delta(H_n N) \rightarrow \delta(H_n N) + (\bar{H} \delta t)^\cdot + n^\mu n^\nu D_\mu D_\nu \delta t + n^\mu (\delta t)_{;\mu} \quad (\text{B8})$$

$$= \delta(H_n N) + \frac{d}{dt} (\bar{H} \delta t + n_\mu \delta t^\mu) \Big|_{v^\mu}. \quad (\text{B9})$$

This last line follows from the previous line by straightforward algebra, and contains a coordinate time derivative along the null geodesic \mathcal{O} . Therefore the integral in Eq. (B1) transforms according to

$$\int_{\bar{t}_R}^{\bar{t}_E} \delta(H_n N) dt \rightarrow \int_{\bar{t}_R}^{\bar{t}_E} \delta(H_n N) dt + (\bar{H} \delta t + n_\mu \delta t^\mu)|_R^E. \quad (\text{B10})$$

Now all we need are the transformations for the boundary terms in the expression for the temperature anisotropy, Eq. (B1). By Eq. (B2), the temporal displacement δt_{disp} between any hypersurface of constant t and some fixed hypersurface must transform like

$$\delta t_{\text{disp}} \rightarrow \delta t_{\text{disp}} - \delta t. \quad (\text{B11})$$

Applying this expression to $\delta t_{\text{disp}} = \delta t_D$ and $\delta t_{\text{disp}} = \delta t_B$ gives for the transformation of the boundary terms

$$\begin{aligned} (\bar{H} \delta t_D + n_\mu \delta t_B^\mu)|_R^E &\rightarrow (\bar{H} \delta t_D + n_\mu \delta t_B^\mu)|_R^E \\ &\quad - (\bar{H} \delta t + n_\mu \delta t^\mu)|_R^E. \end{aligned} \quad (\text{B12})$$

Combining Eqs. (B10) and (B12) we finally find

$$\frac{\delta T(n^\mu)}{\bar{T}_R} \rightarrow \frac{\delta T(n^\mu)}{\bar{T}_R}, \quad (\text{B13})$$

so that the temperature anisotropy is invariant under linear transformations of the congruence u^μ , or equivalently the slicing Σ_t , used in the calculation, if the point of observation and the four-velocity of the observer are held constant. This of course was to be expected since the anisotropies are observable.

Note that we are free to vary the ‘‘background times’’ \bar{t}_R and \bar{t}_E at first order. Through Eq. (34) this freedom simply shifts the temperature perturbation $\delta T(n^\mu)$ by an irrelevant constant. Indeed, we can use this freedom to fix $\delta T(n^\mu)$ such that its mean over the whole sky vanishes for some particular observation point R ,

$$\int \delta T(n^\mu) d\Omega = 0, \quad (\text{B14})$$

so that \bar{T}_R coincides with the mean temperature over the sky. Some authors consider it important to make this choice (see, e.g., [7]).

APPENDIX C: GAUGE DEPENDENCE OF ANISOTROPIES

After demonstrating in Appendix B the gauge *invariance* of the anisotropies described by Eq. (B1) for fixed observation point R and observer four-velocity, we will now show how the anisotropies *do* depend on the gauge, when R and the four-velocity are allowed to transform. At linear order, we will see that such transformations will only effect the monopole and dipole anisotropies, as is well known.

Consider again the gauge transformation

$$t \rightarrow t - \delta t, \quad (\text{C1})$$

with corresponding change in the orthogonal congruence

$$u^\mu \rightarrow u^\mu + D^\mu \delta t. \quad (\text{C2})$$

Let us now evaluate the anisotropies using the general linear expression, Eq. (B1), but moving the reception point R according to Eq. (C1), and boosting the observer four-velocity according to Eq. (C2). Since we move R , the corresponding emission point E must also move. If R moves to the future, then the corresponding LSS will increase in diameter, and E will move radially outwards. We can schematically indicate the contributions to the change in the anisotropies under Eq. (C1) by

$$\begin{aligned} \delta \frac{\delta T(n^\mu)}{\bar{T}_R} &= \frac{\partial}{\partial E} \frac{\delta T(n^\mu)}{\bar{T}_R} \delta E + \frac{\partial}{\partial R} \frac{\delta T(n^\mu)}{\bar{T}_R} \delta R \\ &\quad + \frac{\partial}{\partial u^\mu} \frac{\delta T(n^\mu)}{\bar{T}_R} \delta u^\mu. \end{aligned} \quad (\text{C3})$$

The first term in Eq. (C3) arises due to the change in diameter of the LSS (and the entire past light cone). As the LSS moves, it samples different perturbation modes, so the observed anisotropies change. This effect is greatest for the structures at the smallest scales (comoving and angular), and was calculated in detail in [48]. There it was shown that this contribution is of order

$$\frac{\partial}{\partial E} \frac{\delta T(n^\mu)}{\bar{T}_R} \delta E \sim \frac{\bar{H}(t_R) \delta t}{\theta} \frac{\delta T(n^\mu)}{\bar{T}_R}, \quad (\text{C4})$$

where θ is the angular scale of the feature in question. Since δt and $\delta T(n^\mu)$ are both first order quantities, this effect can be considered to be second order, and so will not be considered further here. However, for large changes in observation time, substantial changes to the anisotropies will be observed [48].

To calculate the second and third terms in Eq. (C3), note first that by displacing the observation point by δt and the observation four-velocity by $D^\mu \delta t$, the displacements δt_D and δt_B at R do not change under Eq. (C1):

$$\delta t_D(R) \rightarrow \delta t_D(R), \quad (\text{C5})$$

$$\delta t_B(R) \rightarrow \delta t_B(R). \quad (\text{C6})$$

The displacements at the emission point E still transform according to Eq. (B12),

$$\begin{aligned} (\bar{H} \delta t_D + n_\mu \delta t_B^\mu)|_E &\rightarrow (\bar{H} \delta t_D + n_\mu \delta t_B^\mu)|_E \\ &\quad - (\bar{H} \delta t + n_\mu \delta t^\mu)|_E. \end{aligned} \quad (\text{C7})$$

Combining Eqs. (C5)–(C7) with the transformation Eq. (B10) for the integral, we find that the anisotropies described by Eq. (B1) transform according to

$$\frac{\delta T(n^\mu)}{\bar{T}_R} \rightarrow \frac{\delta T(n^\mu)}{\bar{T}_R} - (\bar{H}\delta t + n_\mu \delta t^\mu) \Big|_R. \quad (\text{C8})$$

To illuminate the nature of this change in the anisotropies, we can use the multipole expansion of the anisotropy, Eq. (38). If we align the polar axis along $D_\mu \delta t$, we have

$$\bar{H}\delta t = \sqrt{4\pi}\bar{H}\delta t Y_{00}(n^\mu), \quad (\text{C9})$$

$$n_\mu \delta t^\mu = \sqrt{\frac{4\pi}{3}}|\delta u^\mu|Y_{10}(n^\mu), \quad (\text{C10})$$

where $\delta u^\mu \equiv D^\mu \delta t$. Combining these expressions with Eq. (C8), the multipole expansion (38) gives

$$a_{00} \rightarrow a_{00} - \sqrt{4\pi}\bar{H}\delta t, \quad (\text{C11})$$

$$a_{10} \rightarrow a_{10} - \sqrt{\frac{4\pi}{3}}|\delta u^\mu|, \quad (\text{C12})$$

and all other multipoles are invariant under the transformation. That is, only the monopole and dipole change. Therefore the calculation of the higher multipoles is forgiving with respect to the care taken regarding gauge. However, in Sec. IV, where we calculate the dipole and monopole, we must be completely explicit about the specification of the frame in which we evaluate the dipole and the hypersurface on which we evaluate the monopole.

To close this discussion of gauge dependence, we note that going beyond first order, a boost at the observation point R transfers power to *all* multipoles, and also distorts anisotropies through aberration [49].

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- [1] E. Komatsu *et al.* (WMAP), arXiv:0803.0547.
- [2] R. K. Sachs and A. M. Wolfe, *Astrophys. J.* **147**, 73 (1967).
- [3] M. Panek, *Phys. Rev. D* **34**, 416 (1986).
- [4] J. C. R. Magueijo, *Phys. Rev. D* **47**, R353 (1993).
- [5] H. Russ, M. Soffel, C. Xu, and P. K. S. Dunsby, *Phys. Rev. D* **48**, 4552 (1993).
- [6] M. J. White and W. Hu, *Astron. Astrophys.* **321**, 8 (1997).
- [7] P. K. S. Dunsby, *Classical Quantum Gravity* **14**, 3391 (1997).
- [8] A. Challinor and A. Lasenby, *Phys. Rev. D* **58**, 023001 (1998).
- [9] J.-c. Hwang and H. Noh, *Phys. Rev. D* **59**, 067302 (1999).
- [10] A. Lewis, A. Challinor, and A. Lasenby, *Astrophys. J.* **538**, 473 (2000).
- [11] Information on CAMB is available at <http://camb.info/>.
- [12] G. F. R. Ellis and H. van Elst, in *Theoretical and Observational Cosmology*, edited by M. Lachieze-Rey, NATO Science Series (Kluwer Academic Publishers, Dordrecht, 1999), p. 1.
- [13] C. G. Tsagas, A. Challinor, and R. Maartens, *Phys. Rep.* **465**, 61 (2008).
- [14] L. P. Grishchuk and I. B. Zeldovich, *Sov. Astron.* **22**, 125 (1978).
- [15] M. S. Turner, *Phys. Rev. D* **44**, 3737 (1991).
- [16] D. Langlois and T. Piran, *Phys. Rev. D* **53**, 2908 (1996).
- [17] J. C. Mather *et al.*, *Astrophys. J.* **512**, 511 (1999).
- [18] D. J. Fixsen and J. C. Mather, *Astrophys. J.* **581**, 817 (2002).
- [19] T. P. Waterhouse and J. P. Zibin, arXiv:0804.1771.
- [20] J. Chluba and R. A. Sunyaev, *Astron. Astrophys.* **478**, L27 (2008).
- [21] J. Hamann and Y. Y. Y. Wong, *J. Cosmol. Astropart. Phys.* **03** (2008) 025.
- [22] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
- [23] The energy density at some event, as seen by an observer with four-velocity u^μ , will change at second order under a first order change of u^μ .
- [24] M. White, D. Scott, and J. Silk, *Annu. Rev. Astron. Astrophys.* **32**, 319 (1994).
- [25] N. Bartolo, S. Matarrese, and A. Riotto, *J. Cosmol. Astropart. Phys.* **08** (2005) 010.
- [26] Temperature perturbations *do* generally exist along an arbitrarily chosen background time slice which is a linear perturbation from the physically defined Σ_{LS} , and this appears to be the source of the confusion.
- [27] In the case that the congruence u^μ is hypersurface orthogonal, the quantity H_n is also simply related to the extrinsic curvature $K_{\mu\nu}$ of the orthogonal slicing through $H_n = K_{\mu\nu}n^\mu n^\nu$.
- [28] There is a subtlety here regarding the mapping of directions n^μ at R to points $E(n^\mu)$ on the LSS. If gravitational lensing is important, there is the possibility that a single point on the LSS is mapped to multiple directions n^μ at R . Regardless, Eq. (20) can always be applied. A further subtlety here is the implicit assumption that the radiation released at E is isotropic in the comoving frame. In principle, quadrupole and higher multipole anisotropies can exist at E , but our assumption of tight coupling in the plasma before last scattering implies that all such anisotropies are suppressed due to Thomson scattering (see, e.g., [29]).
- [29] A. Challinor, arXiv:astro-ph/0403344.
- [30] This will not be possible in general if the congruence has nonvanishing twist $\omega_{\mu\nu}$ [22]. In this case a nonorthogonal slicing can be used, or a different orthogonal slicing could be chosen for each geodesic \mathcal{O} , since we only require the congruence and slicing to exist in the neighborhood of each \mathcal{O} . At linear order, however, vanishing vector modes imply vanishing twist.
- [31] W. R. Stoeger, C.-M. Xu, G. F. R. Ellis, and M. Katz,

- Astrophys. J. **445**, 17 (1995).
- [32] J. Hwang, T. Padmanabhan, O. Lahav, and H. Noh, Phys. Rev. D **65**, 043005 (2002).
- [33] V. F. Mukhanov, H. A. Feldman, and R. H. Brandenberger, Phys. Rep. **215**, 203 (1992).
- [34] W. Hu and N. Sugiyama, Astrophys. J. **444**, 489 (1995).
- [35] The construction of the comoving hypersurfaces in a thought experiment to attempt an estimate of the monopole variance would involve exchanging light signals to synchronize clocks, whereas uniform matter density slices can be defined by simple local measurements of ρ .
- [36] While we have had a single observation location R in mind in defining the monopole, with the variance taken over the primordial ensemble, we could, employing the ergodic property of Gaussian random fields, consider the *spatial* variance taken over a slice in a single realization. Once the slices are specified, the spatial gradient of the monopole is determined. This fixes the monopole up to a (time-dependent) constant, which is equivalent to the freedom to vary the background times \bar{t}_E and \bar{t}_R . Again, the monopole would measure how well the uniform matter and radiation slices coincide.
- [37] We can imagine an experiment that could in principle estimate C_0 by receiving information about the total and radiation energy densities on a slice inside the past light cone.
- [38] *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, edited by M. Abramowitz and I. A. Stegun (U.S. Department of Commerce, Washington, D.C., 1972).
- [39] A. R. Liddle and D. H. Lyth, *Cosmological Inflation and Large-Scale Structure* (Cambridge University Press, Cambridge, 2000).
- [40] A. L. Erickcek, M. Kamionkowski, and S. M. Carroll, arXiv:0806.0377 [Phys. Rev. D (to be published)].
- [41] W. Unruh, arXiv:astro-ph/9802323.
- [42] M.-N. Celerier, Astron. Astrophys. **353**, 63 (2000).
- [43] K. Enqvist, Gen. Relativ. Gravit. **40**, 451 (2008).
- [44] J. P. Zibin, Phys. Rev. D **78**, 043504 (2008).
- [45] L. Hui and P. B. Greene, Phys. Rev. D **73**, 123526 (2006).
- [46] J. M. Bardeen, in *Cosmology and Particle Physics*, edited by L.-Z. Fang and A. Zee (Gordon and Breach, New York, 1988), p. 1.
- [47] J.-c. Hwang, Astrophys. J. **375**, 443 (1991).
- [48] J. P. Zibin, A. Moss, and D. Scott, Phys. Rev. D **76**, 123010 (2007).
- [49] A. Challinor and F. van Leeuwen, Phys. Rev. D **65**, 103001 (2002).
- [50] A. L. Erickcek, S. M. Carroll, and M. Kamionkowski, Phys. Rev. D **78**, 083012 (2008).