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Dispersive approach to chiral perturbation theory

M. Zdráhal[1,2,*] and J. Novotný[1,†]

[1]*Institute of Particle and Nuclear Physics, Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, CZ-180 00 Prague 8, Czech Republic*

[2]*Faculty of Physics, University of Vienna, Boltzmanngasse 5, A-1090 Vienna, Austria*

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We generalize the reconstruction theorem of Stern, Sazdjian, and Fuchs based on the dispersion relations to the case of the $(2 \rightarrow 2)$ scattering of all the pseudoscalar octet mesons (π, K, η). We formulate it in a general way and include also a discussion of the assumptions of the theorem. It is used to obtain the amplitudes of all such processes in the isospin limit to the one-loop order (and can be straightforwardly extended to two loops) independently on the particular power-counting scheme of the chiral perturbation theory in question. The results in this general form are presented.

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I. INTRODUCTION

In low-energy QCD the chiral perturbation theory (χPT, cf. [1]) has gained great prominence as a tool for description interactions of the lightest (pseudoscalar) mesons, π, K and η. Within this framework these mesons are understood as Goldstone bosons of the spontaneous breaking of the chiral symmetry $SU(3)_L \times SU(3)_R$ (appearing if all three lightest quarks were massless) down to $SU(3)_V$, gaining the masses through the explicit breaking of the symmetry due to the nonzero quark masses, whereas all the other hadrons (with mass at least of order $\Lambda \sim 1$ GeV) are included only effectively. Using the symmetry properties and the basic properties like analyticity, unitarity, and the crossing symmetry, the Lagrangian of this effective theory is constructed. It contains an infinite number of terms. Nevertheless, the Weinberg power-counting scheme assigns to a given diagram of this theory (and term of the Lagrangian) its importance by means of the chiral dimension $O(p^D)$ (see [2]), and so for computations to a given order in low energies it is sufficient to use only a finite number of them.

However, we can try to find the amplitudes of a process using the required properties directly with no need of the Lagrangian to be explicitly written (with all the advantages and disadvantages that this involves), which is the main goal of this paper. We extend the reconstruction theorem given by Stern, Sazdjian, and Fuchs in [3] and widely used in [4]. Their work was originally motivated by the problem[1] of possible smallness of the scalar condensate B_0 as the power-counting independent way to compute the amplitude of $\pi\pi$ scattering in the 2 flavor case. Even though the problem is not yet satisfactorily closed,[2] the method is interesting also in itself. The extension of this method to the 3 flavor case of the πK amplitude has been given by Ananthanarayan and Büttiker [7,8]. We have worked out a generalization to the processes of all the pseudo-Goldstone bosons (PGB) in the three flavor case.

In contrast to these previous papers, we have extracted from the theorem the entire information about the isospin structure and assumed just the needed crossing symmetry. The isospin properties are used afterwards separately for particular processes. Furthermore, the theorem is formulated formally generally by pointing out all the properties essential for the theorem. This should simplify the discussion of validity of the theorem and its eventual further extensions to more complicated cases. In the simple case of processes from the pure strong χPT (conserving the strong isospin), we have discussed the assumptions and showed their reasonability and concluded with the construction of their amplitudes to two loops (with results explicitly written just to the one loop) in a way independent of the particular power-counting scheme in use. In this aspect our work is thus also an extension of the work of Osborn [9] to higher orders.

The plan of the paper is as follows. In Sec. II we establish our notation; the fundamental content of the paper is Sec. III, where we reformulate the reconstruction theorem with its assumptions and give the proof of it. In Sec. IV we use the theorem (and Appendix A) to compute the amplitudes describing all the 2-PGB scattering processes to the $O(p^4)$ chiral order (explicitly written in the Appendix B). Finally, in Sec. V we provide a further discussion of the assumptions and discussion of the results. Appendix A summarizes all the needed amplitudes to describe the 4-PGB processes and lists a general parametrization of them to the $O(p^2)$ order. Appendix B gives the

*zdrahal@ipnp.troja.mff.cuni.cz

†novotny@ipnp.troja.mff.cuni.cz

[1]The standard power counting is based on the presumption that the value of B_0 is of order Λ. So there also appeared approaches extending the χPT to the case including the possibility of small B_0 too, which have a lesser predictive power connected with larger number of low-energy constants (LECs)—namely the generalized χPT [5].

[2]The only direct answer given by experiment is still the confirmation of the standard power counting in the 2 flavor case by the K_{l4} measurement at Brookhaven National Laboratory [6].

results of the application of the theorem to computation of amplitudes to the $O(p^4)$ order. In Appendix C we give relations between our parameters and renormalized LEC of the standard χPT [10] in a particular renormalization scheme [11]. The main points of the analysis of validity of dispersion relations and the regions of analyticity of amplitudes that are used in the theorem are briefly summarized in Appendix D.

II. NOTATION

We consider a scattering process of two-PGB of the type

$$A(p_A)B(p_B) \to C(p_C)D(p_D) \tag{1}$$

and write its amplitude according to[3]

$$\langle C(p_C)D(p_D)|_f A(p_A)B(p_B)\rangle_i = \delta_{if} + i(2\pi)^4\delta^{(4)}(p_C + p_D - p_A - p_B)A(s,t,u). \tag{2}$$

In the relations where we want to distinguish between more processes, we write explicitly $A_{i\to f}(s,t,u)$ or $A_{AB\to CD}(s,t,u)$.

Later on, we will use the crossing symmetry, so let us define the amplitude of the crossed processes. The amplitude of the direct process will be denoted by $S(s,t,u)$, i.e. $A_{AB\to CD}(s,t,u) = S(s,t,u)$. The amplitude of the T-crossed channel will be $A_{A\bar{C}\to\bar{B}D}(s,t,u) = f_T T(s,t,u)$, where the phase factor f_T is defined so that $T(s,t,u)$ fulfils the crossing relation $S(s,t,u) = T(t,s,u)$ and similarly for the U-crossed channel.

The transition from the CMS variables s and $\cos\theta$ to the Mandelstam variables s, t, u for the process (1) is possible using the relation

$$\cos\theta = \frac{s(t-u) + \Delta_{AB}\Delta_{CD}}{\lambda_{AB}^{1/2}(s)\lambda_{CD}^{1/2}(s)}, \tag{3}$$

where Δ_{ij} is a difference of the masses squared $\Delta_{ij} = m_i^2 - m_j^2$ and

$$\lambda_{ij}(x) = x^2 + m_i^4 + m_j^4 - 2x(m_i^2 + m_j^2) - 2m_i^2 m_j^2 \tag{4}$$

refers to the Källen's quadratic form (also called the triangle function).

The sum of squared masses of all the particles appearing in the process is denoted as $\mathcal{M}$, i.e.

$$s + t + u = m_A^2 + m_B^2 + m_C^2 + m_D^2 = \mathcal{M}. \tag{5}$$

[3] This process depends on two independent kinematical quantities only. It is convenient to choose the total energy squared s and the angle θ between the momenta p_A and p_C in the center of mass system (CMS) or two of three Mandelstam variables s, t, u. In order to express the crossing and Bose symmetry properties in a simple way in what follows, we keep writing the dependence on all three Mandelstam variables explicitly.

We will use the partial wave decomposition of the amplitudes in the form

$$A_{i\to f}(s,\cos\theta) = 32\pi\sum_l A_l^{i\to f}(s)(2l+1)P_l(\cos\theta), \tag{6}$$

where P_l are Legendre polynomials.

From the unitarity of S-matrix (and real analyticity of the physical amplitude as well as T-invariance) there follows the unitarity relations. Assuming that the only relevant intermediate states are those containing two particles α and β with masses m_α and m_β, its form for the partial wave of amplitudes reads

$$\mathrm{Im}A_l^{i\to f}(s) = \sum_{\alpha,\beta}\frac{2}{S_F}\frac{\lambda_{\alpha\beta}^{1/2}(s)}{s}A_l^{i\to(\alpha,\beta)}(s) \times [A_l^{f\to(\alpha,\beta)}(s)]^*\theta(s-(m_\alpha+m_\beta)^2). \tag{7}$$

Here, S_F denotes the symmetry factor ($S_F = 1$ if the two intermediate states are distinguishable and $S_F = 2$ if they are not).

Finally, we use the definition of the (signs of) meson fields given in the matrix

$$\phi(x) = \lambda_a\phi_a(x) = \begin{pmatrix} \pi^0 + \frac{1}{\sqrt{3}}\eta & -\sqrt{2}\pi^+ & -\sqrt{2}K^+ \\ \sqrt{2}\pi^- & -\pi^0 + \frac{1}{\sqrt{3}}\eta & -\sqrt{2}K^0 \\ \sqrt{2}K^- & -\sqrt{2}\bar{K}^0 & -\frac{2}{\sqrt{3}}\eta \end{pmatrix}. \tag{8}$$

The meson states are chosen so that the matrix element of mesons gets one minus sign for each charged particle in the final state.

III. RECONSTRUCTION THEOREM

A. Statement of the theorem

In this section we prove the following theorem: The amplitude $S(s,t;u)$ of a given process $A + B \to C + D$ fulfilling all the conditions from the next subsection can be reconstructed to (and excluding) $O(p^8)$ order just from the knowledge of imaginary parts of s and p partial waves of all crossed amplitudes and some polynomial

$$\begin{aligned} S(s,t;u) = {}& \Phi_0(s) + [s(t-u) + \Delta_{AB}\Delta_{CD}]\Phi_1(s) + \Psi_0(t) \\ & + [t(s-u) + \Delta_{AC}\Delta_{BD}]\Psi_1(t) + \Omega_0(u) \\ & + [u(t-s) + \Delta_{AD}\Delta_{BC}]\Omega_1(u) + R_4(s,t;u) \\ & + O(p^8), \end{aligned} \tag{9}$$

where $R_4(s,t;u)$ is a third-order polynomial in Mandelstam variables [obeying exactly the same symmetries as the original amplitude $S(s,t;u)$] and

$$\Phi_0(s) = 32s^3\int_\Sigma^{\Lambda^2}\frac{dx}{x^3}\frac{\mathrm{Im}S_0(x)}{x-s}, \tag{10}$$

$$\Phi_1(s) = 96s^3 \int_\Sigma^{\Lambda^2} \frac{dx}{x^3} \frac{1}{x-s} \mathrm{Im} \frac{S_1(x)}{\lambda_{AB}^{1/2}(x)\lambda_{CD}^{1/2}(x)}, \quad (11)$$

$$\Psi_0(t) = 32t^3 \int_\tau^{\Lambda^2} \frac{dx}{x^3} \frac{\mathrm{Im}T_0(x)}{x-t}, \quad (12)$$

$$\Psi_1(t) = 96t^3 \int_\tau^{\Lambda^2} \frac{dx}{x^3} \frac{1}{x-t} \mathrm{Im} \frac{T_1(x)}{\lambda_{AC}^{1/2}(x)\lambda_{BD}^{1/2}(x)}, \quad (13)$$

$$\Omega_0(u) = 32u^3 \int_\Upsilon^{\Lambda^2} \frac{dx}{x^3} \frac{\mathrm{Im}U_0(x)}{x-u}, \quad (14)$$

$$\Omega_1(u) = 96u^3 \int_\Upsilon^{\Lambda^2} \frac{dx}{x^3} \frac{1}{x-u} \mathrm{Im} \frac{U_1(x)}{\lambda_{AD}^{1/2}(x)\lambda_{BC}^{1/2}(x)}. \quad (15)$$

S_0 and S_1 are s and p partial waves of the process in question, similarly T_0 and T_1 and U_0 and U_1 the partial waves of the T and U crossed processes. Σ, τ, Υ are the minima of squared invariant mass of all possible intermediate states in the S, T, and U channels [see (18)]. Finally, Λ is the scale from the assumptions.

B. Assumptions of the theorem

We use this theorem in the (strong part of) chiral perturbation theory, but for better understanding of it, we are formulating all the assumptions in the general way and then discussing their validity in χPT separately. The assumptions of the theorem are:

(1) There exists a threshold Λ up to which we can regard the theory under consideration to be complete; this means, for example, that under this threshold there appears no particle of other types than those which are already explicitly included in the theory and all the influence of such extra particles is already taken into account effectively or negligible under the threshold. In chiral perturbation theory the threshold Λ is the threshold of production of non-PGB states, i.e. $\Lambda \sim 1$ GeV.

(2) There exists a well-behaved expansion of the amplitude in powers of p/Λ. In χPT it is enough to have any chiral expansion, not necessarily the standard Weinberg one. As is common in χPT, instead of $O((p/\Lambda)^n)$ we write just $O(p^n)$ for short.

(3) We can write a 3 times[4] subtracted dispersion relation for the amplitude $S(s, t; u)$ in the complex s plane for fixed value of u [in the form (16)]. This is the essential assumption of the proof—the theorem stands or falls by it. This assumption is also connected with the validity of the partial wave expansion (for the sake of simplicity, we assume that this expansion can be also analytically continued below the physical threshold where needed). The legitimacy of these assumptions in the case of χPT is discussed in Appendix D.

(4) We further assume that the amplitude considered as a function of a single Mandelstam variable (with the other fixed at some appropriate value) is analytic in some unempty open region. The regions where this and the previous assumptions are valid are discussed (and plotted) in Appendix D.

(5) The crossing relations given in the form of the previous section are valid, i.e. $S(s, t, u) = T(t, s, u) = U(u, t, s)$. That is a widely accepted assumption, whose validity for a general $2 \to 2$ process has been proven from axiomatic theory by Bross, Epstein, and Glaser [12].

(6) The absorptive parts of the $l \geq 2$ partial amplitudes are suppressed to the $O(p^8)$ order—thanks to that we can (up to this order) deal just with the first two partial waves in the theorem. In our case that follows from the (pseudo-)Goldstone boson character of the particles under consideration. In χPT the amplitudes behave dominantly as $O(p^2)$ and do not contain any bound state poles. The unitarity relation (7) then implies that its imaginary part behaves as $O(p^4)$, i.e. the amplitudes are dominantly real. According to the analyticity of the amplitude, its leading $O(p^2)$ part should be a polynomial in Mandelstam variables. Moreover, it has to be a polynomial of the first order; otherwise its coefficient would grow up as mass of the PGB went to zero, and that would contradict the finiteness of the S-matrix in the chiral limit with the external momenta fixed.[5] However, the first-order polynomial could not contribute to $l \geq 2$ partial waves, thus these partial waves should behave in the chiral limit as $O(p^4)$, thereby again using the unitarity relation, the imaginary parts of these partial waves are at least of the $O(p^8)$ order.

C. Proof of the theorem

One of our assumptions is that for the amplitude we can write the 3 times subtracted dispersion relation in the form

$$S(s, t; u) = P_3(s, t; u) + \frac{s^3}{\pi} \int_\Sigma^\infty \frac{dx}{x^3} \frac{\mathrm{Im}S(x, \mathcal{M} - x - u; u)}{x - s} + \frac{t^3}{\pi} \int_\tau^\infty \frac{dx}{x^3} \frac{\mathrm{Im}S(\mathcal{M} - x - u, x; u)}{x - t}, \quad (16)$$

[4]As discussed later on, the exact number of subtractions is not so relevant.

[5]We will see that the finiteness of the S-matrix in the chiral limit deserves to be regarded as one of the full-valued assumptions by itself.

where $P_n(s, t; u)$ is a polynomial of the $(n-1)$-th order in s and t.[6] Σ is the minimum of the squared invariant mass of all the possible intermediate states (α, β) in the S channel (or rather the lowest mass of the state with the same quantum numbers as the *in* and *out* state in the S channel),[7]

$$\Sigma = \min_{(\alpha,\beta)}(m_\alpha + m_\beta)^2, \tag{18}$$

and analogically τ for the T channel (later on we will also need the same for the U channel, what will be denoted by Υ).

Thanks to the crossing symmetry we can replace the $S(s, t; u)$ amplitude in the second integral with the $T(t, s; u)$. Further, since we assume that we know the complete theory only up to some threshold Λ, we have to split the dispersion integral into two parts—the low-energy ($x \le \Lambda^2$) and the high-energy ($x \ge \Lambda^2$) part. We consider the region where $s \ll \Lambda^2$, i.e. in the high-energy part, $s \ll x$, we have[8]

$$\frac{s^3}{\pi}\int_{\Lambda^2}^{\infty}\frac{dx}{x^3}\frac{\mathrm{Im}S(x, -x-u; u)}{x-s} = s^3 H_\Lambda(u) + O(p^8), \tag{19}$$

where

$$H_\Lambda(u) = \frac{1}{\pi}\int_{\Lambda^2}^{\infty}\frac{dx}{x^3}\frac{\mathrm{Im}S(x, \mathcal{M}-x-u; u)}{x} \tag{20}$$

is a function of u only. We can proceed similarly for the high-energy part of the integration of $T(s, t; u)$. Hence, these high-energy parts can be to the order $O(p^8)$ added to the subtraction polynomial now thereby of the third order.[9] Afterwards, the dispersion relation is of the form

$$S(s, t; u) = P_4(s, t; u) + \frac{s^3}{\pi}\int_{\Sigma}^{\Lambda^2}\frac{dx}{x^3} \times \frac{\mathrm{Im}S(x, \mathcal{M}-x-u; u)}{x-s} + \frac{t^3}{\pi}\int_{\tau}^{\Lambda^2}\frac{dx}{x^3} \times \frac{\mathrm{Im}T(x, \mathcal{M}-x-u; u)}{x-t} + O(p^8). \tag{21}$$

We decompose the (imaginary parts of) amplitudes into the partial waves

$$S(s, t; u) = 32\pi(S_0(s) + 3\cos\theta S_1(s) + S_{l\ge 2}(s, t; u)). \tag{22}$$

The terms $S_{l\ge 2}(s, t; u)$ incorporate contributions of all the higher ($l \ge 2$) partial waves and are suppressed to $O(p^8)$ according to our assumptions.

From the integrals of the P partial waves, we can extract functions depending just on u and include them in the polynomial, e.g.

$$\int_{\Sigma}^{\Lambda^2}\frac{dx}{x^3}\frac{x(s+t-x-u)}{x-s}\,\mathrm{Im}\frac{S_1(x)}{\lambda_{AB}^{1/2}(x)\lambda_{CD}^{1/2}(x)} = \int_{\Sigma}^{\Lambda^2}\frac{dx}{x^3}\frac{x(t-u)}{x-s}\,\mathrm{Im}\frac{S_1(x)}{\lambda_{AB}^{1/2}(x)\lambda_{CD}^{1/2}(x)} - \int_{\Sigma}^{\Lambda^2}\frac{dx}{x^2}\,\mathrm{Im}\frac{S_1(x)}{\lambda_{AB}^{1/2}(x)\lambda_{CD}^{1/2}(x)}. \tag{23}$$

Finally, we use this rearrangement of the fractions

$$\frac{x(t-u)}{x-s} = \frac{s(t-u)}{x-s} + t - u, \tag{24}$$

where the second term gives after integration a contribution in the form $t-u$ times a function of u only and therefore as a polynomial in t can also be included into the subtraction polynomial P_4.

After that, the amplitude reads

$$S(s, t; u) = P_4(s, t; u) + \Phi_0(s) + [s(t-u) + \Delta_{AB}\Delta_{CD}]\Phi_1(s) + \Psi_0(t) + [t(s-u) + \Delta_{AC}\Delta_{BD}]\Psi_1(t) + O(p^8), \tag{25}$$

where $\Phi_0(s)$, $\Phi_1(s)$, $\Psi_0(t)$, and $\Psi_1(t)$ are given by (10)–(13).

In order to implement the $s \leftrightarrow u$ crossing symmetry, we add and subtract the following polynomial in s and t (with coefficients depending on u)

$$\Omega_0(u) + [u(t-s) + \Delta_{AD}\Delta_{BC}]\Omega_1(u), \tag{26}$$

with $\Omega_0(u)$ and $\Omega_1(u)$ given by (14) and (15) and we include the subtraction of this polynomial in the polynomial $P_4(s, t; u)$ and thereby get the amplitude in the symmetric form (9).

Until now, we have proved the whole theorem except for the u-dependence of the polynomial $P_4(s, t; u)$. However,

[6]The coefficients of this polynomial depend on u. Thanks to the relation for the sum of the Mandelstam variables, such polynomials could be written in the form

$$P_n(s, t; u) = \alpha(u) + \beta(u)(s-t) + \cdots + \omega(u)(s-t)^{n-1}. \tag{17}$$

[7]In this relation we anticipate again that we will consider the problems under conditions implying relevance of the two-particle intermediate states only.

[8]The fact that the remainder $\sum_{n\ge 4} s^n R_n(u)$ is of order at least $O(p^8)$ follows again from the finiteness of the amplitude in the chiral limit (the importance of this assumption stems from here) and the crossing symmetry.

[9]The consequence of that is also that we can formally extend the validity of the original theory even beyond threshold Λ and compute the dispersion integral up to infinity, pretending that the original theory is really complete (naturally with the appropriate modification of the third-order polynomial). Similar arguments would appear even if we introduced the cutoff Λ by other reasons, e.g. by reason of numerical integration.

we could write similar relations also for the amplitudes $T(s,t;u)$ and $U(s,t;u)$ with the polynomials in (s,u) and the coefficients depending on variable t. Thanks to the crossing symmetry and to the symmetric form of the rest of the formula, these polynomials should obey the same crossing symmetry. From this and from the analyticity of the amplitudes in a neighborhood of u for some fixed s in our kinematic region, we can conclude that $P_4(s,t;u)$ are polynomials in respect to all Mandelstam variables.

IV. APPLICATION OF THE THEOREM TO THE 4-MESON PROCESSES TO THE $O(p^4)$ ORDER

As we have already outlined, in the case of $2\to 2$ scattering processes of pseudoscalar octet mesons, we can use the theorem and the unitarity relation to a self-consistent construction of all those amplitudes to (and excluding) $O(p^8)$ order. We need just a parametrization of $O(p^2)$ amplitudes as its input.

In the following we will consider the strong isospin conservation limit,[10] where Ward identities imply that there are just 7 independent processes. From them the amplitudes of all the possible physical processes can be determined as summarized in Appendix A. Since we want to keep the independence on the specific power-counting scheme, we have parametrized the $O(p^2)$ amplitudes in the most general form obeying their symmetries (crossing, Bose, and isospin)—given also in Appendix A. Inserting them into the unitarity relations (7) (assuming that we have amplitudes of all the needed intermediate processes), we get the imaginary parts to $O(p^4)$ order, which can be installed into the theorem and we receive thereby the amplitudes in that order. With the same procedure (though more technically involved), we can proceed with the second iteration and obtain thereby the amplitudes to (and excluding) $O(p^8)$ order. For the sake of simplicity, in this paper we confine ourselves to the first iteration and thus to the results to $O(p^4)$ order.

It remains to show that we already have the amplitudes of all the relevant intermediate processes, i.e. the self-consistency of this procedure. We are concerned with the processes to (and excluding) $O(p^8)$ chiral order and (in accordance with the first assumption of the theorem) in the region bellow the appearance of all non-Goldstone particles and thereby away from poles and cuts of such intermediate states. Therefore, their effect is included just in the polynomial of the reconstruction theorem and in the higher orders. The intermediate states with an odd number of Goldstone bosons are forbidden by the even intrinsic parity of the effective Lagrangian—because the effect of the axial anomaly enters at the $O(p^4)$ order (Wess, Zumino, and Witten [13,14]), such intermediate states induce the contribution of the order at least $O(p^8)$. Furthermore, the contribution of the states with more than two Goldstone bosons is also suppressed to $O(p^8)$ order since the n-Goldstone-boson invariant phase space scales like p^{2n-4} and amplitudes with an arbitrary number of external Goldstone boson legs behave dominantly as $O(p^2)$. Consequently, the contribution of such $(n>2)$ intermediate states to the imaginary part of the amplitude is according to the unitarity relation at least $O(p^8)$. Thus, we can consider the two-Goldstone-boson intermediate states only.

Further simplification in the $O(p^4)$ order computation appears. Using again the arguments of finiteness of S-matrix in the chiral limit with the external momenta fixed, the coefficients of the polynomial in the theorem should be at least $O(p^0)$, and so the polynomial in the $O(p^4)$ amplitude is maximally of the second order in Mandelstam variables.

A. The application schematically

The $O(p^2)$ amplitudes can be written [cf. with (17), and Appendix A with use of (5)] in the form[11]

$$A=\frac{1}{F_\pi^2}(\alpha(s)+\beta(t-u)). \tag{27}$$

The decomposition into the partial waves can be obtained using (3) and (22)

$$A_0=\frac{1}{32\pi}\frac{1}{F_\pi^2}\left(\alpha(s)-\beta\frac{\Delta_{AB}\Delta_{CD}}{s}\right), \tag{28}$$

$$A_1=\frac{1}{32\pi}\frac{1}{3F_\pi^2}\beta\frac{\lambda_{AB}^{1/2}(s)\lambda_{CD}^{1/2}(s)}{s}. \tag{29}$$

The (right) discontinuity of our function S_0 and S_1 of the theorem, given by the unitarity relation, is therefore in the $O(p^4)$ case very simple:

$$\begin{aligned}\mathrm{Im}S_0(s)=\sum_k\frac{2}{S_{F_k}}\frac{\lambda_k^{1/2}(s)}{s}\left(\frac{1}{32\pi}\right)^2\frac{1}{F_\pi^4}\\ \times\left[\left(\alpha_{ik}(s)-\beta_{ik}\frac{\Delta_{AB}\Delta_k}{s}\right)\right.\\ \times\left.\left(\alpha_{kf}(s)-\beta_{kf}\frac{\Delta_k\Delta_{CD}}{s}\right)\right]\theta(s-s_k^t),\end{aligned} \tag{30}$$

[10]Recall that in this case the validity of the required dispersion relations can be proven directly from the axiomatic theory as discussed in Appendix D.

[11]Let us remind that in the $O(p^2)$ case $\alpha(s)=\alpha_0+\alpha_1 s$, i.e. it is just the first-order polynomial in s.

$$\begin{aligned}\mathrm{Im}\, S_1(s) &= \sum_k \frac{2}{S_{F_k}} \frac{\lambda_k^{1/2}(s)}{s}\left(\frac{1}{32\pi}\right)^2 \frac{1}{9F_\pi^4}\\ &\times\left[\beta_{ik}\frac{\lambda_{AB}^{1/2}(s)\lambda_k^{1/2}(s)}{s}\beta_{kf}\frac{\lambda_k^{1/2}(s)\lambda_{CD}^{1/2}(s)}{s}\right]\theta(s-s_k^t),\end{aligned} \tag{31}$$

where the sums go over all the possible (two-PGB) intermediate states k with its symmetry factor S_{F_k}, and all the objects with lower index k are the quantities relating to this intermediate state; similarly indices i and f refer to initial and final state. Finally s_k^t is a threshold of this intermediate state.

Thanks to the behavior of square root of the triangle function, $\lambda^{1/2}(s) \to s$ for $s \to \infty$ and the fact that $\alpha(s)$ is the polynomial maximally of the first order in s, we know that not all the terms (in the brackets) need all the subtractions to give finite integrals (10)–(15) (e.g. the parts with negative power of s in the polynomial in the brackets do not even need any subtraction). Any subtraction redundant in this aspect then gives just a polynomial, which can be included into the polynomial $R_4(s, t, t)$ of the theorem (9) and into the contributions of higher orders. Furthermore, if the integrals rising on both sides of rearrangement (24) have a good mathematical sense, we can pull out the polynomial in front of the integral again with possible change of the polynomial $R_4(s, t, t)$. Therefore, applying the reconstruction theorem, we can write Φ_0 and Φ_1 in the minimal[12] form [except for the polynomial included to $R_4(s, t, t)$]:

$$\begin{aligned}\Phi_0(s) &= \frac{1}{16\pi^2}\frac{1}{F_\pi^4}\sum_k\frac{1}{S_{F_k}}\bigg(\alpha_{ik}(s)\alpha_{kf}(s)s\int_{s_k^t}^\infty\frac{dx}{x}\frac{1}{x-s}\\ &\times\frac{\lambda_k^{1/2}(x)}{x}+\beta_{ik}\beta_{kf}\Delta_k^2\Delta_{AB}\Delta_{CD}\int_{s_k^t}^\infty\frac{dx}{x^2}\frac{1}{x-s}\\ &\times\frac{\lambda_k^{1/2}(x)}{x}-(\alpha_{kf}(s)\beta_{ik}\Delta_{AB}\\ &+\alpha_{ik}(s)\beta_{kf}\Delta_{CD})\Delta_k\int_{s_k^t}^\infty\frac{dx}{x}\frac{1}{x-s}\frac{\lambda_k^{1/2}(x)}{x}\bigg),\end{aligned} \tag{32}$$

$$\begin{aligned}\Phi_1(s) &= \frac{1}{16\pi^2}\frac{1}{3F_\pi^4}\sum_k\frac{1}{S_{F_k}}\beta_{ik}\beta_{kf}\bigg((s-2\Sigma_k)\int_{s_k^t}^\infty\frac{dx}{x}\\ &\times\frac{1}{x-s}\frac{\lambda_k^{1/2}(x)}{x}+\Delta_k^2\int_{s_k^t}^\infty\frac{dx}{x^2}\frac{1}{x-s}\frac{\lambda_k^{1/2}(x)}{x}\bigg).\end{aligned} \tag{33}$$

[12] With the minimal number of subtractions needed to get a finite nonpolynomial part, i.e. here with one or two subtractions.

In addition, in the P wave function Φ_1 we have used $\lambda_k(x) = x^2 - 2x\Sigma_k + \Delta_k^2$, where Σ_k is the sum of the second powers of masses of the two particles in the intermediate state k.

In the integrals we recognize the once and twice subtracted dispersion integrals from Appendix B [(B1) and (B2)], i.e.

$$\begin{aligned}\Phi_0(s) &= \frac{1}{F_\pi^4}\sum_k\frac{1}{S_{F_k}}\bigg(\beta_{ik}\beta_{kf}\frac{\Delta_k^2}{s^2}\Delta_{AB}\Delta_{CD}\bar{\bar{J}}_k(s)\\ &+\Big(\alpha_{ik}(s)\alpha_{kf}(s)-(\alpha_{kf}(s)\beta_{ik}\Delta_{AB}\\ &+\alpha_{ik}(s)\beta_{kf}\Delta_{CD})\frac{k}{s}\Big)\bar{J}_k(s)\bigg),\end{aligned} \tag{34}$$

$$\begin{aligned}\Phi_1(s) &= \frac{1}{3F_\pi^4}\sum_k\frac{1}{S_{F_k}}\beta_{ik}\beta_{kf}\bigg(\Big(1-2\frac{\Sigma_k}{s}\Big)\bar{J}_k(s)\\ &+\frac{\Delta_k^2}{s^2}\bar{\bar{J}}_k(s)\bigg).\end{aligned} \tag{35}$$

Using the change of the polynomial for the last time (in the replacement of $\frac{\bar{\bar{J}}(s)}{s}$ with $\frac{\bar{J}(s)}{s}$), we have the result

$$\begin{aligned}&\Phi_0(s)+[s(t-u)+\Delta_{AB}\Delta_{CD}]\Phi_1(s)\\ &=\frac{1}{3F_\pi^4}\sum_k\frac{1}{S_{F_k}}\bigg(4\Delta_{AB}\Delta_{CD}\beta_{ik}\beta_{kf}\Delta_k^2\frac{\bar{\bar{J}}_k(s)}{s^2}\\ &+(3\alpha_{ik}(s)\alpha_{kf}(s)+\beta_{ik}\beta_{kf}((t-u)(s-2\Sigma_k)\\ &+\Delta_{AB}\Delta_{CD}))\bar{J}_k(s)+(\beta_{ik}\beta_{kf}((t-u)\Delta_k^2\\ &-2\Delta_{AB}\Delta_{CD}\Sigma_k)-3(\alpha_{kf}(s)\beta_{ik}\Delta_{AB}\\ &+\alpha_{ik}(s)\beta_{kf}\Delta_{CD})\Delta_k)\frac{\bar{J}_k(s)}{s}\bigg).\end{aligned} \tag{36}$$

We get similar results also for the T- and U-crossed parts.[13] Final results of this procedure for all the amplitudes are given in Appendix B.

V. SUMMARY AND DISCUSSIONS

In this paper we have worked out a generalization of the dispersive analysis from [3] to the processes involving all the light octet mesons. We have formulated the reconstruction theorem for their amplitudes free of the particular isospin structure of the processes (which is used afterwards for the individual processes separately) and have provided (an outline of) its proof. We have also attempted to point out exactly the properties of the theory which are needed

[13] If we carry out the same extraction of polynomial also for them, we get again these results by the change $s \leftrightarrow t$ respectively $s \leftrightarrow u$ and appropriate permutation in $ABCD$ and then the symmetries of the polynomial of theorem remain the same.

for its proof mainly due to better understanding of it as well as of its further generalization on more complicated cases (as below). In our particular case we have succeeded in justifying most of them practically from the first principles of axiomatic quantum field theory—at least in the regions from Fig. 1.

The amplitudes of 4-octet-meson processes (on condition of the strong isospin conservation) form a self-consistent system with respect to the theorem and the unitarity relations, thanks to the fact that we can simply construct these amplitudes to (and excluding[14]) $O(p^8)$ order [in this work explicitly computed only to $O(p^4)$ order] independently on the specific power-counting model in use. The amplitudes for the particular model can be obtained with the specific choice of the parameters. For example, we can compare the nonpolynomial part of our results (with the standard choice of $O(p^2)$ constants from Appendix A) with that part of the standard chiral perturbation results (given by [11])—we see an agreement except for the different sign convention.[15] We can also use their result to get a relation between our $O(p^4)$ polynomial parameters and low-energy constants (LECs) of the standard chiral perturbation theory in the particular renormalization scheme used therein just by comparison of the polynomial parts, as is discussed and given in Appendix C.[16] It is worth mentioning that by subtracting the center of the Mandelstam triangle in the second-order polynomial part of our results, the parameters of the first-order polynomial, i.e. α's, β's, and γ's, do not depend in the subtraction scheme used by [11] in $O(p^4)$ order on the LECs of operators with four derivatives from $\mathcal{L}_4$ in [10], i.e. L_1^r, L_2^r, and L_3.

Some of the computed amplitudes ($KK \to$ anything, processes involving η's) have not been computed in such a general form yet. However, this newness is connected with one problem of the assumptions of the theorem. We have assumed the existence of a threshold separating the processes containing just those 8 mesons from the processes in which particles of other types (e.g. resonances) appear, and then we focus on a kinematical range far below that threshold. However, in the case of the processes with the exception of $\pi\pi \to \pi\pi$ and $\pi K \to \pi K$, this range is very narrow or does not even exist (e.g. in the process $KK \to KK$ with a kinematical threshold of 990 MeV there can appear the ρ resonance with its mass of 770 MeV). The significance of our result thereby decreases. Nevertheless, beside the elegance of our construction due to its intrinsic self-consistency, our results can become useful for a check of complicated models or simulations, where one can separate (or completely turn off) the effects of these resonances. [The advantage is that even in the most general case in (to and excluding) $O(p^8)$ order, there is a small total number ($=47$) of parameters, coupling different processes together.]

To that end it would be naturally preferable to perform also the second iteration and get thereby the whole information achievable from the reconstruction theorem. As has been already mentioned, the completion of the second iteration is conceptually straightforward; however, there appear certain technical complications which are connected with the computation of partial waves of amplitudes, where we need to compute integrals of the type $\int t^n \bar{J}(t)dt$ and then to "dispersively integrate" them. Because of the appearance of the three different masses, the results are much more complicated than in the 2 flavor $\pi\pi$ case (with only one mass) and so will be presented with all the technical details in a separate paper.

The careful reader has surely noticed that in the relation (21) or (11) and the other similar (in opposition to the verbal formulation of the theorem), we have emphasized that the required imaginary part is not of the partial wave but of the partial wave divided by $\lambda_{AB}^{1/2}\lambda_{CD}^{1/2}$, i.e. in principle in the higher order there could appear a problem under the physical threshold of the amplitudes where $\lambda_{AB}^{1/2}\lambda_{CD}^{1/2}$ could be imaginary. There presents itself a question whether we can obtain this quantity also from the unitarity relations. But we should remind that the origin of the relation (7) where the imaginary part of the partial wave times the relevant Legendre polynomial also occurs. So the only question is whether we can believe the analytical continuation of the partial wave decomposition and of the unitarity relation under the physical threshold. If we do so, then the unitarity relation gives exactly what we need to the theorem, i.e. what we call the imaginary part of partial wave for simplicity.

The further important question is of the number of subtractions in the dispersive relations. Jin and Martin [15] have shown that thanks to the Froissart bound [16] it suffices to consider just two subtractions. However, the Froissart bound deals with the complete theory (full QCD), and in an effective theory one does not know (or more exactly one does not deal with) what is above Λ, and so the situation can occur that more than two subtractions are needed. In other words Froissart tells us two subtractions are sufficient if we supply the dispersive integral above Λ

[14]Since we also want to include the power countings where the contributions of odd chiral orders appear (like generalized chiral power counting), we write [instead of valid to $O(p^6)$ order (what will be case if odd orders do not appear) or to $O(p^7)$ order (where the odd orders are included)] just valid to and excluding $O(p^8)$ order to avoid misunderstanding. We should also remind that in this odd chiral order power counting, the one-loop result is not $O(p4)$ but $O(p^5)$—we can get such a result by simple modification of the results presented keeping the $\beta_{\pi\eta}$ and γ_{KK} parameters also in the intermediate processes.

[15]When comparing, one should pay attention to their caption of the 4-kaon process $\bar{K}^0K^0 \to K^+K^-$, which gives us a false illusion about the value of the Mandelstam variables, whereas the right value would be better evoked by the caption $\bar{K}^0K^0 \to K^-K^+$.

[16]If we thereby obtained $O(p^4)$ parameters in the unitarity part, we would be already counting a part of the $O(p^6)$ corrections.

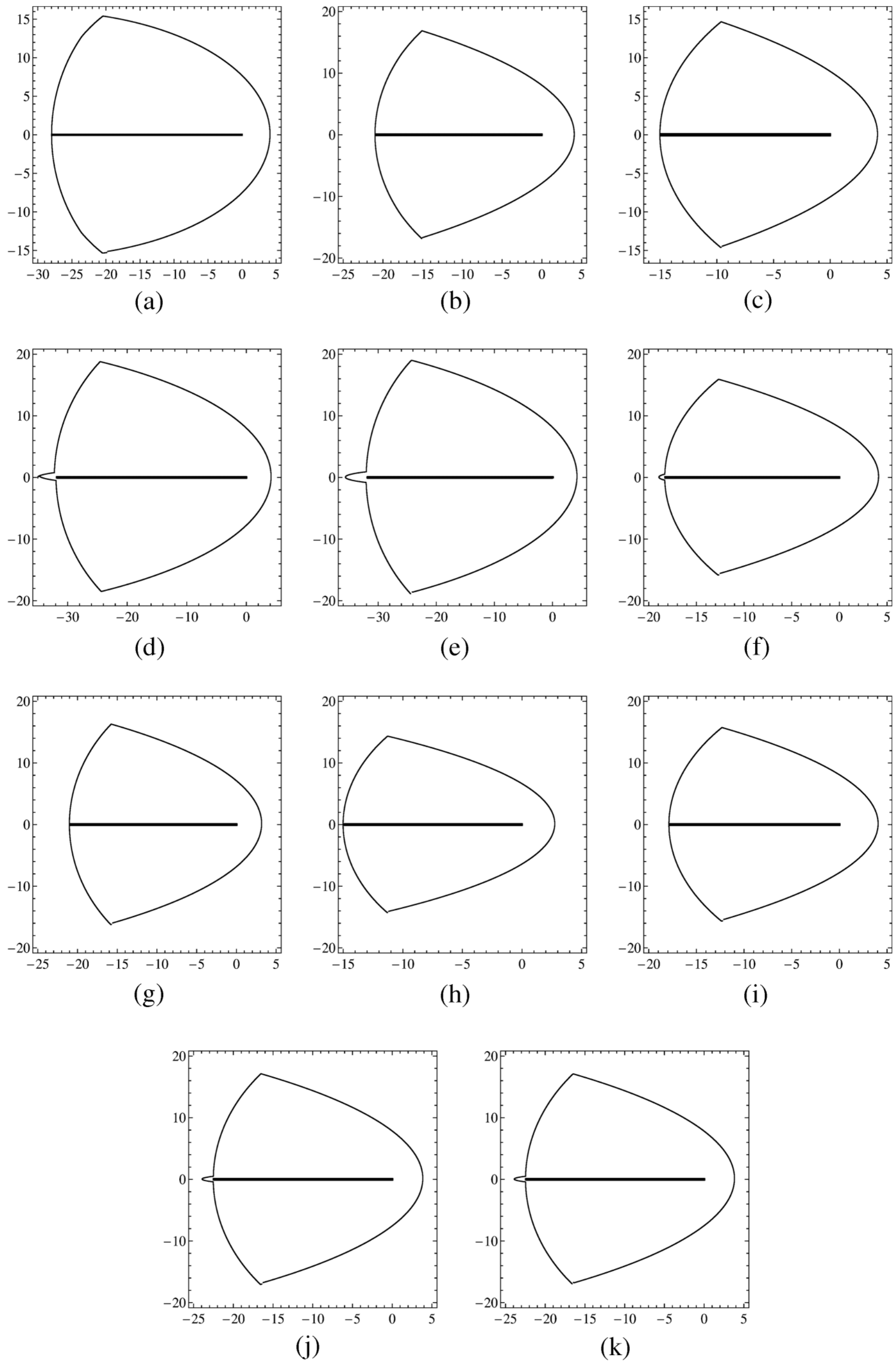

FIG. 1. The regions of validity of fixed u dispersion relations (complex u-plane) in the multiples of m_π^2 and therefore also regions where the assumptions of our theorem are valid. With the bold lines, the intervals of validity coming from the use of just the Lehmann theory (ellipses) are depicted. (a) $\pi\pi \to \pi\pi$; (b) $KK \to KK$; (c) $\eta\eta \to \eta\eta$; (d) $\pi K \to K\pi$; (e) $\pi\eta \to \eta\pi$; (f) $K\eta \to \eta K$; (g) $\pi\pi \to KK$ and $\pi K \to K\pi$; (h) $\pi\pi \to \eta\eta$ and $\pi\eta \to \pi\eta$; (i) $KK \to \eta\eta$ and $K\eta \to K\eta$; (j) $KK \to \pi\eta$ and $K\pi \to K\eta$; and (k) $K\pi \to \eta K$.

with the complete theory. How to deal with this unknown, we have already seen near the relation (19)—for n subtractions we get a series of infinite order beginning with the n-th power of Mandelstam variable. Fortunately, we have shown that the terms with more than fourth power are at least of $O(p^8)$ order. And so, if we assume the finiteness of the S-matrix in the chiral limit (and the other assumptions used there), we end up with the third-order polynomial and the remainder of $O(p^8)$ order no matter with how many subtraction we have begun.

From the possible extensions of the theorem we find especially interesting two of them, the extension to the analysis of the meson form factors and the computation of amplitudes of $K \to 3\pi$ decays (with appearance of cusps). The latter of them is the subject we are recently working on [17,18].

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APPENDIX A: $O(p^2)$ AMPLITUDES AND THE SYMMETRY PROPERTIES OF AMPLITUDES OF THE PHYSICAL MESONS

From the isospin Ward identities it follows that all the physical amplitudes could be expressed in terms of the 7 independent amplitudes. In our convention:

(1) $\eta\eta \to \eta\eta$

$$A(\eta\eta \to \eta\eta) = A_{\eta\eta}(s,t;u). \tag{A1}$$

(2) $\pi\eta \to \pi\eta$

$$A(\pi^\pm\eta \to \pi^\pm\eta) = A_{\pi\eta}(s,t;u), \tag{A2}$$

$$A(\pi^0\eta \to \pi^0\eta) = A_{\pi\eta}(s,t;u). \tag{A3}$$

(3) $\pi\pi \to \pi\pi$

$$A(\pi^+\pi^- \to \pi^0\pi^0) = -A_{\pi\pi}(s,t;u), \tag{A4}$$

$$A(\pi^+\pi^- \to \pi^+\pi^-) = A_{\pi\pi}(s,t;u) + A_{\pi\pi}(t,s;u), \tag{A5}$$

$$A(\pi^0\pi^0 \to \pi^0\pi^0) = A_{\pi\pi}(s,t;u) + A_{\pi\pi}(t,s;u) + A_{\pi\pi}(u,t;s). \tag{A6}$$

(4) $KK \to \eta\eta$

$$A(\bar{K}^0K^0 \to \eta\eta) = -A(K^-K^+ \to \eta\eta) = A_{\eta K}(s,t;u). \tag{A7}$$

(5) $KK \to \pi\eta$

$$A(K^-K^+ \to \pi^0\eta) = -A_{\eta\pi K}(s,t;u), \tag{A8}$$

$$A(\bar{K}^0K^0 \to \pi^0\eta) = -A_{\eta\pi K}(s,t;u), \tag{A9}$$

$$A(K^-K^0 \to \pi^-\eta) = -\sqrt{2}A_{\eta\pi K}(s,t;u), \tag{A10}$$

$$A(\bar{K}^0K^+ \to \pi^+\eta) = -\sqrt{2}A_{\eta\pi K}(s,t;u). \tag{A11}$$

(6) $\pi\pi \to KK$

$$A(\pi^0\pi^0 \to \bar{K}^0K^0) = -A(\pi^0\pi^0 \to K^-K^+) = A_{\pi K}(s,t;u), \tag{A12}$$

$$A(\pi^-\pi^0 \to K^-K^0) = \sqrt{2}B_{\pi K}(s,t;u), \tag{A13}$$

$$A(\pi^+\pi^0 \to \bar{K}^0K^+) = -\sqrt{2}B_{\pi K}(s,t;u), \tag{A14}$$

$$A(\pi^-\pi^+ \to K^-K^+) = A_{\pi K}(s,t;u) + B_{\pi K}(s,t;u), \tag{A15}$$

$$A(\pi^-\pi^+ \to \bar{K}^0K^0) = -A_{\pi K}(s,t;u) + B_{\pi K}(s,t;u). \tag{A16}$$

For these processes there are two important amplitudes—the symmetric one $A_{\pi K}(s,t,u)$ and the antisymmetric one $B_{\pi K}(s,t,u)$. In fact, we can consider the only independent amplitude as (A15) and then extract these amplitudes as the symmetric and the antisymmetric part of it in respect to the exchange of t and u.

(7) $KK \to KK$

$$A(K^-K^+ \to \bar{K}^0K^0) = -A_{KK}(s,t;u), \tag{A17}$$

$$A(K^-K^+ \to K^-K^+) = A(\bar{K}^0K^0 \to \bar{K}^0K^0) = A_{KK}(s,t;u) + A_{KK}(t,s;u). \tag{A18}$$

A. The most general form of the $O(p^2)$ [$O(p^3)$] amplitudes

The $O(p^2)$ amplitudes could be constructed as the most general invariant amplitudes satisfying the crossing, Bose, and isospin symmetries and using the fact that they should be polynomials of the first order in the Mandelstam variables obeying (5). [In fact, it is the form of the $O(p^3)$ amplitudes, because this should hold also for them.] This form is independent of the power-counting used:

$$A_{\eta\eta} = -\frac{M_\pi^2}{3F_\pi^2}\alpha_{\eta\eta}\left(1 - \frac{4M_\eta^2}{M_\pi^2}\right), \tag{A19}$$

$$A_{\pi\eta} = \frac{1}{3F_\pi^2}(\beta_{\pi\eta}(3t - 2M_\eta^2 - 2M_\pi^2) + \alpha_{\pi\eta}M_\pi^2), \tag{A20}$$

$$A_{\pi\pi} = \frac{1}{3F_\pi^2}(\beta_{\pi\pi}(3s - 4M_\pi^2) + \alpha_{\pi\pi}M_\pi^2), \tag{A21}$$

$$A_{\eta K} = \frac{1}{4F_\pi^2}\Big[\beta_{\eta K}(3s - 2M_K^2 - 2M_\eta^2) + \alpha_{\eta K}\Big(2M_\eta^2 - \frac{2}{3}M_K^2\Big)\Big], \tag{A22}$$

$$A_{\eta\pi K} = \frac{1}{4\sqrt{3}F_\pi^2}[\beta_{\eta\pi K}(3s - 2M_K^2 - M_\pi^2 - M_\eta^2) - (2M_K^2 - M_\pi^2 - M_\eta^2 + \alpha_{\eta\pi K}M_\pi^2)], \tag{A23}$$

$$A_{\pi K} = \frac{1}{12F_\pi^2}[\beta_{\pi K}(3s - 2M_K^2 - 2M_\pi^2) + 2(M_K - M_\pi)^2 + 4\alpha_{\pi K}M_\pi M_K], \tag{A24}$$

$$B_{\pi K} = \frac{1}{4F_\pi^2}\gamma_{\pi K}(t - u), \tag{A25}$$

$$A_{KK} = \frac{1}{6F_\pi^2}(\beta_{KK}(4M_K^2 - 3u) + 3\gamma_{KK}(s - t) + 2\alpha_{KK}M_K^2). \tag{A26}$$

After a deeper analysis one can show that in $O(p^2)$ order the amplitude $A_{\pi\eta}$ is t-independent, so the constant $\beta_{\pi\eta}$ is of the $O(p^3)$ order and similarly for γ_{KK}, which is also equal to zero in the $O(p^2)$ case in an arbitrary power counting. So, generally

$$\beta_{\pi\eta} \sim O(p^1) \quad \text{and} \quad \gamma_{KK} \sim O(p^1). \tag{A27}$$

The particular power countings differ in the particular values of the (13 + 2) constants in those relations. For example, the standard power counting gives [to $O(p^3)$]

$$\alpha_{\eta\eta}^{st} = \alpha_{\pi\eta}^{st} = \alpha_{\pi\pi}^{st} = \alpha_{\eta K}^{st} = \alpha_{\pi K}^{st} = \alpha_{KK}^{st} = 1, \tag{A28}$$

$$\beta_{\pi\pi}^{st} = \beta_{\eta K}^{st} = \beta_{\eta\pi K}^{st} = \beta_{\pi K}^{st} = \gamma_{\pi K}^{st} = \beta_{KK}^{st} = 1, \tag{A29}$$

$$\alpha_{\eta\pi K}^{st} = 0, \tag{A30}$$

$$\beta_{\pi\eta}^{st} = \gamma_{KK}^{st} = 0. \tag{A31}$$

APPENDIX B: RESULTS

In the results we have denoted the once subtracted integral

$$\bar{J}_{PQ}(s) = \frac{s}{16\pi^2}\int_\Sigma^\infty \frac{dx}{x}\frac{1}{x - s}\frac{\lambda_{PQ}^{1/2}(x)}{x} \tag{B1}$$

in the S channel (analogically for the other crossed channels) and similarly the twice subtracted integral

$$\bar{\bar{J}}_{PQ}(s) = \frac{s^2}{16\pi^2}\int_\Sigma^\infty \frac{dx}{x^2}\frac{1}{x - s}\frac{\lambda_{PQ}^{1/2}(x)}{x}. \tag{B2}$$

The amplitudes using the first iteration of the reconstruction procedure read:

(i) $\eta\eta \to \eta\eta$

$$A_{\eta\eta} = \frac{1}{3F_\pi^2}\alpha_{\eta\eta}(4M_\eta^2 - M_\pi^2) + \frac{1}{3F_\pi^4}\delta_{\eta\eta}(s^2 + t^2 + u^2 - 4M_\eta^4) + \Big[\frac{1}{6F_\pi^4}(Z_{\eta\eta}^{\eta\eta}\bar{J}_{\eta\eta}(s) + Z_{\eta\eta}^{\pi\pi}\bar{J}_{\pi\pi}(s) + Z_{\eta\eta}^{KK}(s)\bar{J}_{KK}(s))\Big] + [s \leftrightarrow t] + [s \leftrightarrow u] + O(p^5). \tag{B3}$$

$$Z_{\eta\eta}^{\eta\eta} = \frac{1}{3}\alpha_{\eta\eta}^2(M_\pi^2 - 4M_\eta^2)^2, \tag{B4}$$

$$Z_{\eta\eta}^{\pi\pi} = \alpha_{\pi\eta}^2 M_\pi^4, \tag{B5}$$

$$Z^{KK}_{\eta\eta}(s) = \frac{3}{4}\left(3\beta_{\eta K}s - 2\left(\beta_{\eta K} + \frac{1}{3}\alpha_{\eta K}\right)M_K^2 + 2(\alpha_{\eta K} - \beta_{\eta K})M_\eta^2\right)^2. \tag{B6}$$

(ii) $\pi^0\eta \to \pi^0\eta$

$$\begin{aligned} A_{\pi\eta} = {} & \frac{1}{3F_\pi^2}(\beta_{\pi\eta}(3t - 2M_\eta^2 - 2M_\pi^2) + M_\pi^2\alpha_{\pi\eta}) + \frac{1}{3F_\pi^4}(\delta_{\pi\eta}((s - M_\pi^2 - M_\eta^2)^2 + s \leftrightarrow u) \\ & + \varepsilon_{\pi\eta}(t - 2M_\pi^2)(t - 2M_\eta^2)) + \frac{1}{72F_\pi^4}(Z^{KK}_{\pi\eta}(t)\bar{J}_{KK}(t) + Z^{\eta\eta}_{\pi\eta}\bar{J}_{\eta\eta}(t) + Z^{\pi\pi}_{\pi\eta}(t)\bar{J}_{\pi\pi}(t)) \\ & + \left[\frac{1}{9F_\pi^4}(Y^{\pi\eta}_{\pi\eta}\bar{J}_{\pi\eta}(s) + Y^{KK}_{\pi\eta}(s)\bar{J}_{KK}(s))\right] + [s \leftrightarrow u] + O(p^5). \end{aligned} \tag{B7}$$

$$\begin{aligned} Z^{KK}_{\pi\eta}(t) = {} & (9\beta_{\eta K}t - 2(3\beta_{\eta K} + \alpha_{\eta K})M_K^2 + 6(\alpha_{\eta K} - \beta_{\eta K})M_\eta^2)(3\beta_{\pi K}t \\ & + 2(1 - \beta_{\pi K})(M_K^2 + M_\pi^2) + 4(\alpha_{\pi K} - 1)M_\pi M_K), \end{aligned} \tag{B8}$$

$$Z^{\eta\eta}_{\pi\eta} = -4\alpha_{\pi\eta}\alpha_{\eta\eta}M_\pi^2(M_\pi^2 - 4M_\eta^2), \tag{B9}$$

$$Z^{\pi\pi}_{\pi\eta}(t) = 4\alpha_{\pi\eta}M_\pi^2(6\beta_{\pi\pi}t + (5\alpha_{\pi\pi} - 8\beta_{\pi\pi})M_\pi^2), \tag{B10}$$

$$Y^{\pi\eta}_{\pi\eta} = \alpha_{\pi\eta}^2 M_\pi^4, \tag{B11}$$

$$Y^{KK}_{\pi\eta}(s) = \frac{3}{8}(3\beta_{\eta\pi K}s - 2(1 + \beta_{\eta\pi K})M_K^2 + (1 - \alpha_{\eta\pi K} - \beta_{\eta\pi K})M_\pi^2 + (1 - \beta_{\eta\pi K})M_\eta^2)^2. \tag{B12}$$

(iii) $\pi^+\pi^- \to \pi^0\pi^0$

$$\begin{aligned} A_{\pi\pi} = {} & \frac{1}{3F_\pi^2}(\beta_{\pi\pi}(3s - 4M_\pi^2) + \alpha_{\pi\pi}M_\pi^2) + \frac{1}{F_\pi^4}(\delta_{\pi\pi}(s - 2M_\pi^2)^2 + \varepsilon_{\pi\pi}((t - 2M_\pi^2)^2 + (u - 2M_\pi^2)^2)) \\ & + \frac{1}{72F_\pi^4}(Z^{\pi\pi}_{\pi\pi}(s)\bar{J}_{\pi\pi}(s) + Z^{\eta\eta}_{\pi\pi}\bar{J}_{\eta\eta}(s) + Z^{KK}_{\pi\pi}(s)\bar{J}_{KK}(s)) \\ & + \left[\frac{1}{72F_\pi^4}(Y^{KK}_{\pi\pi}(t,u)\bar{J}_{KK}(t) + Y^{\pi\pi}_{\pi\pi}(t,u)\bar{J}_{\pi\pi}(t))\right] + [t \leftrightarrow u] + O(p^5). \end{aligned} \tag{B13}$$

$$Z^{\pi\pi}_{\pi\pi}(s) = 4(3\beta_{\pi\pi}s + (7\alpha_{\pi\pi} - 4\beta_{\pi\pi})M_\pi^2)(3\beta_{\pi\pi}s + (\alpha_{\pi\pi} - 4\beta_{\pi\pi})M_\pi^2), \tag{B14}$$

$$Z^{\eta\eta}_{\pi\pi} = 4\alpha_{\pi\eta}^2 M_\pi^4, \tag{B15}$$

$$Z^{KK}_{\pi\pi}(s) = (3\beta_{\pi K}s + 2(1 - \beta_{\pi K})(M_K^2 + M_\pi^2) + 4(\alpha_{\pi K} - 1)M_\pi M_K)^2, \tag{B16}$$

$$Y^{KK}_{\pi\pi}(t,u) = 3\gamma_{\pi K}^2(t - 4M_K^2)(4M_\pi^2 - t - 2u), \tag{B17}$$

$$Y^{\pi\pi}_{\pi\pi}(t,u) = 4(3\beta_{\pi\pi}^2 t(t - u) + 6\beta_{\pi\pi}(2\beta_{\pi\pi}u - \alpha_{\pi\pi}t)M_\pi^2 + 2(\alpha_{\pi\pi}^2 + 4\beta_{\pi\pi}(\alpha_{\pi\pi} - 2\beta_{\pi\pi}))M_\pi^4). \tag{B18}$$

(iv) $\bar{K}^0K^0 \to \eta\eta$

$$A_{\eta K} = \frac{1}{4F_\pi^2}\left(\beta_{\eta K}(3s - 2M_K^2 - 2M_\eta^2) + \alpha_{\eta K}\left(2M_\eta^2 - \frac{2}{3}M_K^2\right)\right) + \frac{1}{4F_\pi^4}(\delta_{\eta K}(s - 2M_\eta^2)(s - 2M_K^2) + \varepsilon_{\eta K}((t - M_\eta^2 - M_K^2)^2 + (u - M_\eta^2 - M_K^2)^2)) + \frac{1}{24F_\pi^4}(Z_{\eta K}^{\eta\eta}(s)\bar{J}_{\eta\eta}(s) + Z_{\eta K}^{\pi\pi}(s)\bar{J}_{\pi\pi}(s) + Z_{\eta K}^{KK}(s)\bar{J}_{KK}(s)) + \left[\frac{1}{32F_\pi^4}(Y_{\eta K}^{\eta K}(t,u)\bar{J}_{\eta K}(t) + Y_{\eta K}^{K\pi}(t,u)\bar{J}_{K\pi}(t)) - \frac{3}{32F_\pi^4}\frac{1}{t}(X_{\eta K}^{\eta K}(u)\bar{J}_{\eta K}(t) + X_{\eta K}^{K\pi}(u)\bar{J}_{K\pi}(t)) + \frac{3}{16F_\pi^4}\frac{1}{t^2}\Delta_{K\eta}^2(W_{\eta K}^{\eta K}\bar{\bar{J}}_{\eta K}(t) + W_{\eta K}^{K\pi}\bar{\bar{J}}_{K\pi}(t))\right] + [t \leftrightarrow u] + O(p^5). \quad \text{(B19)}$$

$$W_{\eta K}^{\eta K} = \Delta_{K\eta}^2\beta_{\eta K}^2, \quad \text{(B20)}$$

$$W_{\eta K}^{K\pi} = \Delta_{K\pi}^2\beta_{\eta\pi K}^2, \quad \text{(B21)}$$

$$X_{\eta K}^{\eta K}(u) = \left(\beta_{\eta K}u + 2\left(\beta_{\eta K} - \frac{2}{3}\alpha_{\eta K}\right)M_K^2 + 2(\beta_{\eta K} + 2\alpha_{\eta K})M_\eta^2\right)\beta_{\eta K}\Delta_{K\eta}^2, \quad \text{(B22)}$$

$$X_{\eta K}^{K\pi}(u) = \beta_{\eta\pi K}(\beta_{\eta\pi K}(M_K^2 - M_\pi^2)^2u + 2(\beta_{\eta\pi K} - 2)M_K^6 - 2(2\beta_{\eta\pi K} - 3)M_K^4M_\eta^2 + 2(3 + \beta_{\eta\pi K} - \alpha_{\eta\pi K})M_K^4M_\pi^2 - 2(\beta_{\eta\pi K} + 1 - \alpha_{\eta\pi K})M_\pi^4M_K^2 - 2M_\eta^4M_K^2 + 2(\alpha_{\eta\pi K} - 4)M_K^2M_\eta^2M_\pi^2 - 2(\alpha_{\eta\pi K} - 1)M_\pi^4M_\eta^2 + 2(\beta_{\eta\pi K} + 1)M_\eta^4M_\pi^2), \quad \text{(B23)}$$

$$Z_{\eta K}^{\eta\eta}(s) = \left(3\beta_{\eta K}s - 2\left(\beta_{\eta K} + \frac{1}{3}\alpha_{\eta K}\right)M_K^2 - 2(\beta_{\eta K} - \alpha_{\eta K})M_\eta^2\right)\alpha_{\eta\eta}(4M_\eta^2 - M_\pi^2), \quad \text{(B24)}$$

$$Z_{\eta K}^{\pi\pi}(s) = \alpha_{\pi\eta}M_\pi^2(3\beta_{\pi K}s + 2(1 - \beta_{\pi K})(M_K^2 + M_\pi^2) + 4(\alpha_{\pi K} - 1)M_\pi M_K), \quad \text{(B25)}$$

$$Z_{\eta K}^{KK}(s) = 3\left(3\beta_{\eta K}s - 2\left(\beta_{\eta K} + \frac{1}{3}\alpha_{\eta K}\right)M_K^2 - 2(\beta_{\eta K} - \alpha_{\eta K})M_\eta^2\right)\left(\frac{3}{2}\beta_{KK}s + 2(\alpha_{KK} - \beta_{KK})M_K^2\right), \quad \text{(B26)}$$

$$Y_{\eta K}^{\eta K}(t,u) = \frac{1}{9}(27\beta_{\eta K}^2t(t-u) - 6(8\alpha_{\eta K}^2 - 8\alpha_{\eta K}\beta_{\eta K} + 39\beta_{\eta K}^2)M_K^2M_\eta^2 + 9(8\alpha_{\eta K}^2 + 8\alpha_{\eta K}\beta_{\eta K} + 5\beta_{\eta K}^2)M_\eta^4 + (8\alpha_{\eta K}^2 - 24\alpha_{\eta K}\beta_{\eta K} + 45\beta_{\eta K}^2)M_K^4 + 36\alpha_{\eta K}\beta_{\eta K}(M_K^2 - 3M_\eta^2)t + 54\beta_{\eta K}^2(M_K^2 + M_\eta^2)u), \quad \text{(B27)}$$

$$Y_{\eta K}^{K\pi}(t,u) = 3\beta_{\eta\pi K}^2t(t-u) + (5\beta_{\eta\pi K}^2 - 8\beta_{\eta\pi K} + 8)M_K^4 + 2(\beta_{\eta\pi K}^2 + \beta_{\eta\pi K} + 1)M_\eta^4 - 8(2\beta_{\eta\pi K}^2 + 1)M_K^2M_\eta^2 + (2\beta_{\eta\pi K}(1 - \alpha_{\eta\pi K}) + 2(1 - \alpha_{\eta\pi K})^2 - \beta_{\eta\pi K}^2)M_\pi^4 + 2(2(\beta_{\eta\pi K}^2 + \beta_{\eta\pi K} + 1 - \alpha_{\eta\pi K}) - \alpha_{\eta\pi K}\beta_{\eta\pi K})M_\eta^2M_\pi^2 - 2(5\beta_{\eta\pi K}^2 + 2\alpha_{\eta\pi K}\beta_{\eta\pi K} + 4(1 - \alpha_{\eta\pi K}))M_K^2M_\pi^2 - 6\beta_{\eta\pi K}(M_\eta^2 - 2M_K^2 - (\alpha_{\eta\pi K} - 1)M_\pi^2)t + 6\beta_{\eta\pi K}^2(M_K^2 + M_\pi^2)u. \quad \text{(B28)}$$

(v) $\bar{K}^0K^0 \to \pi^0\eta$

$$A_{\eta\pi K} = \frac{1}{4\sqrt{3}F_\pi^2}V_{\eta\pi K}(s) + \frac{1}{4\sqrt{3}F_\pi^4}(\delta_{\eta\pi K}(s-2M_K^2)(s-M_\pi^2-M_\eta^2) + \varepsilon_{\eta\pi K}((t-M_K^2-M_\pi^2)(t-M_K^2-M_\eta^2) + t\leftrightarrow u)) + \frac{1}{24\sqrt{3}F_\pi^4}(Z^{\eta\pi}_{\eta\pi K}(s)\bar{J}_{\eta\pi}(s) + Z^{KK}_{\eta\pi K}\bar{J}_{KK}(s))V_{\eta\pi K}(s) + \Big[-\frac{1}{96\sqrt{3}F_\pi^4}(Y^{\eta K}_{\eta\pi K}(t,u)\bar{J}_{\eta K}(t) + Y^{K\pi}_{\eta\pi K}(t,u)\bar{J}_{K\pi}(t)) - \frac{1}{32\sqrt{3}F_\pi^4}\frac{1}{t}(X^{\eta K}_{\eta\pi K}(u)\bar{J}_{\eta K}(t) + X^{K\pi}_{\eta\pi K}(u)\bar{J}_{K\pi}(t)) + \frac{1}{16\sqrt{3}F_\pi^4}\frac{1}{t^2}U_{\eta\pi K}(W^{\eta K}_{\eta\pi K}\bar{\bar{J}}_{\eta K}(t) + W^{K\pi}_{\eta\pi K}\bar{\bar{J}}_{K\pi}(t))\Big] + [t\leftrightarrow u] + O(p^5). \tag{B29}$$

$$U_{\eta\pi K} = \beta_{\eta\pi K}\Delta_{K\eta}\Delta_{K\pi}, \tag{B30}$$

$$V_{\eta\pi K}(s) = \beta_{\eta\pi K}(3s - 2M_K^2 - M_\pi^2 - M_\eta^2) - (2M_K^2 - M_\pi^2 - M_\eta^2 + \alpha_{\eta\pi K}M_\pi^2), \tag{B31}$$

$$Z^{\eta\pi}_{\eta\pi K} = 2\alpha_{\pi\eta}M_\pi^2, \tag{B32}$$

$$Z^{KK}_{\eta\pi K}(s) = \frac{3}{2}\beta_{KK}s + 2(\alpha_{KK} - \beta_{KK})M_K^2, \tag{B33}$$

$$Y^{\eta K}_{\eta\pi K}(t,u) = 9\beta_{\eta\pi K}\beta_{\eta K}(t(u-t) - 2(M_\eta^2 + M_K^2)u) - 3(\beta_{\eta\pi K}+2)(2\alpha_{\eta K} + \beta_{\eta K})M_\eta^4 + (3\beta_{\eta K}(4-5\beta_{\eta\pi K}) + 4\alpha_{\eta K}(\beta_{\eta\pi K} - 2))M_K^4 + 2(\alpha_{\eta K}(2 - 2\alpha_{\eta\pi K} + \beta_{\eta\pi K}) + 3\beta_{\eta K}(\alpha_{\eta\pi K} - 1 + 4\beta_{\eta\pi K}))M_K^2M_\pi^2 + (\alpha_{\eta K}(28 - 10\beta_{\eta\pi K}) + 6\beta_{\eta K}(9\beta_{\eta\pi K}+1))M_K^2M_\eta^2 + 6((\alpha_{\eta\pi K} - 1)(2\alpha_{\eta K} + \beta_{\eta K}) - \beta_{\eta\pi K}(\alpha_{\eta K} + 2\beta_{\eta K}))M_\eta^2M_\pi^2 - 3t(2(3\beta_{\eta K} + \alpha_{\eta K}\beta_{\eta\pi K})M_K^2 - 3(\beta_{\eta K} + 2\alpha_{\eta K}\beta_{\eta\pi K})M_\eta^2 + 3\beta_{\eta K}(\alpha_{\eta\pi K} - 1)M_\pi^2), \tag{B34}$$

$$Y^{K\pi}_{\eta\pi K}(t,u) = 3\beta_{\eta\pi K}t((10\gamma_{\pi K} - \beta_{\pi K})t + (\beta_{\pi K} + 2\gamma_{\pi K})u) + 3(\beta_{\pi K} - 2\gamma_{\pi K}(3 + 2\beta_{\eta\pi K}))tM_\eta^2 - 6\beta_{\eta\pi K}(\beta_{\pi K} + 2\gamma_{\pi K}) \times (M_K^2 + M_\pi^2)u - 6(2\gamma_{\pi K}(4\beta_{\eta\pi K} - 3) + \beta_{\pi K} - \beta_{\eta\pi K})tM_K^2 - 3(2\beta_{\eta\pi K}(6\gamma_{\pi K} - 1) + (1 - \alpha_{\eta\pi K}) \times (6\gamma_{\pi K} - \beta_{\pi K}))tM_\pi^2 + (2 + \beta_{\eta\pi K} - 2\alpha_{\eta\pi K})(6\gamma_{\pi K} - \beta_{\pi K} - 2)M_\pi^4 + (\beta_{\pi K}(4 - 5\beta_{\eta\pi K}) + 2\gamma_{\pi K}(21\beta_{\eta\pi K} - 12) + 8 - 4\beta_{\eta\pi K})M_K^4 + 2(\beta_{\eta\pi K}(4\beta_{\pi K} - 6\gamma_{\pi K} - 1) - \beta_{\pi K} + 6\gamma_{\pi K} - 2)M_K^2M_\eta^2 - 2(\beta_{\pi K}(2\beta_{\eta\pi K} + 1) - 6\gamma_{\pi K}(3\beta_{\eta\pi K} + 1) + 2 + \beta_{\eta\pi K})M_\pi^2M_\eta^2 + 4(\alpha_{\pi K} - 1)(V_{\eta\pi K}(t) + 3(2M_K^2 - M_\pi^2 - M_\eta^2 + \alpha_{\eta\pi K}M_\pi^2))M_KM_\pi + 2(3\beta_{\eta\pi K}(3\beta_{\pi K} + 4\gamma_{\pi K} - 1) + (\alpha_{\eta\pi K} + 1)(\beta_{\pi K} - 6\gamma_{\pi K} + 2))M_K^2M_\pi^2, \tag{B35}$$

$$X^{\eta K}_{\eta\pi K}(u) = \Delta_{\eta K}(3\beta_{\eta K}\beta_{\eta\pi K}(M_\eta^2 - M_K^2)u + 3\beta_{\eta K}M_\eta^4 + (2\alpha_{\eta K}\beta_{\eta\pi K} + 6\beta_{\eta K}(1 - \beta_{\eta\pi K}))M_K^4 + 3(\beta_{\eta K}(1 - \alpha_{\eta\pi K}) + 2\beta_{\eta\pi K}(\alpha_{\eta K} + \beta_{\eta K}))M_\eta^2M_\pi^2 - 3(3\beta_{\eta K} + 2\beta_{\eta\pi K}(\beta_{\eta K} + \alpha_{\eta K}))M_K^2M_\eta^2 + (3\beta_{\eta K}(\alpha_{\eta\pi K} - 1) + 2\beta_{\eta\pi K}(3\beta_{\eta K} - \alpha_{\eta K}))M_K^2M_\pi^2), \tag{B36}$$

$$X^{K\pi}_{\eta\pi K}(u) = \Delta_{\pi K}(\beta_{\eta\pi K}(\beta_{\pi K} + 2\gamma_{\pi K})(M_\pi^2 - M_K^2)u + 4\beta_{\eta\pi K}(1 - \alpha_{\pi K})(M_K^2 - M_\eta^2)M_KM_\pi + 2\beta_{\eta\pi K}(2\gamma_{\pi K} - \beta_{\pi K} - 1) \times (M_K^4 - M_K^2M_\eta^2 - M_\pi^2M_\eta^2 + M_K^2M_\pi^2) + (\beta_{\pi K} + 2\gamma_{\pi K})(1 - \alpha_{\eta\pi K})M_\pi^4 + (\beta_{\pi K} + 2\gamma_{\pi K})(2M_K^4 - M_K^2M_\eta^2 + M_\pi^2M_\eta^2 + (\alpha_{\eta\pi K} - 3)M_K^2M_\pi^2)), \tag{B37}$$

$$W^{\eta K}_{\eta\pi K} = 3\Delta_{K\eta}^2\beta_{\eta K}, \tag{B38}$$

$$W^{K\pi}_{\eta\pi K} = \Delta^2_{K\pi}(\beta_{\pi K} + 2\gamma_{\pi K}). \tag{B39}$$

(vi) $\pi^0\pi^0 \to \bar{K}^0 K^0$

$$\begin{aligned} A_{\pi K} = {} & \frac{1}{12F_\pi^2}(\beta_{\pi K}(3s - 2M_K^2 - 2M_\pi^2) + 2(M_K - M_\pi)^2 + 4\alpha_{\pi K}M_\pi M_K) + \frac{1}{12F_\pi^4}(\delta_{\pi K}(s - 2M_\pi^2)(s - 2M_K^2) \\ & + \varepsilon_{\pi K}((t - M_K^2 - M_\pi^2)^2 + (u - M_K^2 - M_\pi^2)^2)) + \frac{1}{72F_\pi^4}(Z^{\pi\pi}_{\pi K}(s)\bar{J}_{\pi\pi}(s) + Z^{\eta\eta}_{\pi K}(s)\bar{J}_{\eta\eta}(s) + Z^{KK}_{\pi K}(s)\bar{J}_{KK}(s)) \\ & + \bigg[\frac{1}{96F_\pi^4}(Y^{\eta K}_{\pi K}(t,u)\bar{J}_{\eta K}(t) + Y^{\pi K}_{\pi K}(t,u)\bar{J}_{\pi K}(t)) + \frac{1}{32F_\pi^4}\frac{1}{t}(X^{\eta K}_{\pi K}(u)\bar{J}_{\eta K}(t) + X^{\pi K}_{\pi K}(u)\bar{J}_{\pi K}(t)) \\ & + \frac{1}{16F_\pi^4}\frac{1}{t^2}K\pi^2(W^{\eta K}_{\pi K}\bar{\bar{J}}_{\eta K}(t) + W^{\pi K}_{\pi K}\bar{\bar{J}}_{\pi K}(t))\bigg] + [t \leftrightarrow u] + O(p^5). \end{aligned} \tag{B40}$$

$$Z^{\pi\pi}_{\pi K}(s) = (3\beta_{\pi K}s + 2(1 - \beta_{\pi K})(M_K^2 + M_\pi^2) + 4(\alpha_{\pi K} - 1)M_\pi M_K)(6\beta_{\pi\pi}s + (5\alpha_{\pi\pi} - 8\beta_{\pi\pi})M_\pi^2), \tag{B41}$$

$$Z^{\eta\eta}_{\pi K}(s) = 3\alpha_{\pi\eta}M_\pi^2\bigg(3\beta_{\eta K}s - 2\bigg(\beta_{\eta K} + \frac{1}{3}\alpha_{\eta K}\bigg)M_K^2 + 2(\alpha_{\eta K} - \beta_{\eta K})M_\eta^2\bigg), \tag{B42}$$

$$Z^{KK}_{\pi K}(s) = 3(3\beta_{\pi K}s + 2(1 - \beta_{\pi K})(M_K^2 + M_\pi^2) + 4(\alpha_{\pi K} - 1)M_\pi M_K)\bigg(\frac{3}{2}\beta_{KK}s + 2(\alpha_{KK} - \beta_{KK})M_K^2\bigg), \tag{B43}$$

$$\begin{aligned} Y^{\eta K}_{\pi K}(t,u) = {} & 3\beta_{\eta K}^2 t(t - u) - 6\beta_{\eta\pi K}(M_\eta^2 - 2M_K^2 + (1 - \alpha_{\eta\pi K})M_\pi^2)t + 6\beta_{\eta\pi K}^2(M_\eta^2 + M_K^2)u + (5\beta_{\eta\pi K}^2 \\ & + 8(1 - \beta_{\eta\pi K}))M_K^4 + (2 + 2\beta_{\eta\pi K} - \beta_{\eta\pi K}^2)M_\eta^4 - 2(5\beta_{\eta\pi K}^2 + 4)M_K^2M_\eta^2 + 2(\alpha_{\eta\pi K}(\alpha_{\eta\pi K} - \beta_{\eta\pi K} - 2) \\ & + \beta_{\eta\pi K}^2 + \beta_{\eta\pi K} + 1)M_\pi^4 - 4(\beta_{\eta\pi K}(4\beta_{\eta\pi K} + \alpha_{\eta\pi K}) + 2(1 - \alpha_{\eta\pi K}))M_K^2M_\pi^2 + 2(2(\beta_{\eta\pi K}^2 + \beta_{\eta\pi K} + 1) \\ & - \alpha_{\eta\pi K}(\beta_{\eta\pi K} + 2))M_\pi^2M_\eta^2, \end{aligned} \tag{B44}$$

$$\begin{aligned} Y^{\pi K}_{\pi K}(t,u) = {} & \frac{1}{3}(3t((\beta_{\pi K}^2 + 26\gamma_{\pi K}^2)t - (\beta_{\pi K}^2 + 2\gamma_{\pi K}^2)u) + 6(\beta_{\pi K}^2 + 2\gamma_{\pi K}^2)(M_K^2 + M_\pi^2)u - 12((8\gamma_{\pi K}^2 + \beta_{\pi K})(M_K^2 + M_\pi^2) \\ & + 2\beta_{\pi K}(\alpha_{\pi K} - 1)M_KM_\pi)t + 2(16\alpha_{\pi K}(\alpha_{\pi K} - 2) + \beta_{\pi K}(8 - 13\beta_{\pi K}) + 6(4 + 13\gamma_{\pi K}^2))M_K^2M_\pi^2 \\ & + (5\beta_{\pi K}^2 + 8(\beta_{\pi K} + 1) - 30\gamma_{\pi K}^2)(M_K^4 + M_\pi^4) + 16(\alpha_{\pi K} - 1)(\beta_{\pi K} + 2)M_KM_\pi(M_K^2 + M_\pi^2)), \end{aligned} \tag{B45}$$

$$\begin{aligned} X^{\eta K}_{\pi K}(u) = {} & \beta_{\eta\pi K}(-\beta_{\eta\pi K}\Delta^2_{K\eta}u + 2(2 - \beta_{\eta\pi K})M_K^6 + 2M_\eta^4((\beta_{\eta\pi K} + 1)M_K^2 - M_\pi^2) + 2M_K^2M_\pi^2M_\eta^2(4 - \alpha_{\eta\pi K}) \\ & + 2M_K^4((\alpha_{\eta\pi K} + 2\beta_{\eta\pi K} - 3)M_\pi^2 - (\beta_{\eta\pi K} + 3)M_\eta^2) + 2M_\pi^4((\alpha_{\eta\pi K} - \beta_{\eta\pi K} - 1)M_\eta^2 + (1 - \alpha_{\eta\pi K})M_K^2)), \end{aligned} \tag{B46}$$

$$X^{\pi K}_{\pi K}(u) = -\frac{1}{3}\Delta^2_{K\pi}((\beta_{\pi K}^2 + 2\gamma_{\pi K}^2)u + 8(\alpha_{\pi K} - 1)\beta_{\pi K}M_KM_\pi + 2(\beta_{\pi K}^2 + 2\beta_{\pi K} - 6\gamma_{\pi K}^2)(M_K^2 + M_\pi^2)), \tag{B47}$$

$$W^{\eta K}_{\pi K} = \Delta^2_{K\eta}\beta^2_{\eta\pi K}, \tag{B48}$$

$$W^{\pi K}_{\pi K} = \frac{1}{3}\Delta^2_{K\pi}(\beta^2_{\pi K} + 2\gamma^2_{\pi K}). \tag{B49}$$

(vii) $\pi^-\pi^0 \to K^-K^0$

$$B_{\pi K} = \frac{1}{4F_\pi^2}\gamma_{\pi K}(t-u) + \frac{1}{4F_\pi^4}\varphi_{\pi K}s(t-u) + \frac{1}{48F_\pi^4}\gamma_{\pi K}(t-u)(2\bar{J}_{\pi\pi}(s)\beta_{\pi\pi}(s-4M_\pi^2) + \bar{J}_{KK}(s)\beta_{KK}(s-4M_K^2)) + \bigg[\frac{1}{96F_\pi^4}(Y_{\pi Kch}^{\eta K}(t,u)\bar{J}_{\eta K}(t) + Y_{\pi Kch}^{\pi K}(t,u)\bar{J}_{\pi K}(t)) + \frac{1}{32F_\pi^4}\frac{1}{t}(X_{\pi Kch}^{\eta K}(u)\bar{J}_{\eta K}(t) + X_{\pi Kch}^{\pi K}(u)\bar{J}_{\pi K}(t)) + \frac{1}{16F_\pi^4}\frac{1}{t^2}\Delta_{K\pi}^2(W_{\pi Kch}^{\eta K}\bar{\bar{J}}_{\eta K}(t) + W_{\pi Kch}^{\pi K}\bar{\bar{J}}_{\pi K}(t))\bigg] - [t\leftrightarrow u] + O(p^5). \tag{B50}$$

$$Y_{\pi Kch}^{\eta K}(t,u) = Y_{\pi K}^{\eta K}(t,u), \tag{B51}$$

$$Y_{\pi Kch}^{\pi K}(t,u) = -\gamma_{\pi K}(t((10\beta_{\pi K} - 13\gamma_{\pi K})t + (2\beta_{\pi K} + \gamma_{\pi K})u) - 2(2\beta_{\pi K} + \gamma_{\pi K})(M_K^2 + M_\pi^2)u - 4((4(\beta_{\pi K} - \gamma_{\pi K}) + 3)(M_K^2 + M_\pi^2) + 6(\alpha_{\pi K} - 1)M_K M_\pi)t + (14\beta_{\pi K} + 5\gamma_{\pi K} + 8)(M_K^4 + M_\pi^4) + 2(2\beta_{\pi K} - 13\gamma_{\pi K} + 8)M_K^2 M_\pi^2 + 16(\alpha_{\pi K} - 1)M_K M_\pi(M_K^2 + M_\pi^2)), \tag{B52}$$

$$X_{\pi Kch}^{\eta K}(u) = X_{\pi K}^{\eta K}(u), \tag{B53}$$

$$X_{\pi Kch}^{\pi K}(u) = -\frac{1}{3}\gamma_{\pi K}(M_K^2 - M_\pi^2)^2((2\beta_{\pi K} + \gamma_{\pi K})u + 8(\alpha_{\pi K} - 1)M_K M_\pi + 2(2 - 2\beta_{\pi K} - 3\gamma_{\pi K})(M_K^2 + M_\pi^2)), \tag{B54}$$

$$W_{\pi Kch}^{\eta K} = W_{\pi K}^{\eta K}, \tag{B55}$$

$$W_{\pi Kch}^{\pi K} = \frac{1}{3}\gamma_{\pi K}\Delta_{K\pi}^2(2\beta_{\pi K} + \gamma_{\pi K}). \tag{B56}$$

(viii) $K^- K^+ \to \bar{K}^0 K^0$

$$A_{KK} = \frac{1}{6F_\pi^2}(\beta_{KK}(4M_K^2 - 3u) + 3\gamma_{KK}(s-t) + 2\alpha_{KK}M_K^2) + \frac{1}{6F_\pi^4}(\delta_{KK}(s-2M_K^2)^2 + \varepsilon_{KK}(t-2M_K^2)^2 + \varphi_{KK}(u-2M_K^2)^2) + \frac{1}{288F_\pi^4}(Z_{KK}^{KK}(s,t)\bar{J}_{KK}(s) + Z_{KK}^{\pi\eta}(s)\bar{J}_{\pi\eta}(s) + Z_{KK}^{\eta\eta}(s)\bar{J}_{\eta\eta}(s) + Z_{KK}^{\pi\pi}(s,t)\bar{J}_{\pi\pi}(s)) + \frac{1}{72F_\pi^4}(Y_{KK}^{KK}(s,t)\bar{J}_{KK}(t) + Y_{KK}^{\pi\eta}(t)\bar{J}_{\pi\eta}(t) + Y_{KK}^{\pi\pi}(s,t)\bar{J}_{\pi\pi}(t)) + \frac{1}{36F_\pi^4}\bar{J}_{KK}(u)(3\beta_{KK}u - 2(\alpha_{KK} + 2\beta_{KK})M_K^2)^2 + O(p^5). \tag{B57}$$

$$Z_{KK}^{KK}(s,t) = 8((3\beta_{KK}s + 4(\alpha_{KK} - \beta_{KK})M_K^2)^2 + 3\beta_{KK}^2(s - 4M_K^2)(s + 2t - 4M_K^2)), \tag{B58}$$

$$Z_{KK}^{\pi\eta}(s) = -6(3\beta_{\eta\pi K}s - 2(1 + \beta_{\eta\pi K})M_K^2 + (1 - \alpha_{\eta\pi K} - \beta_{\eta\pi K})M_\pi^2 + (1 - \beta_{\eta\pi K})M_\eta^2)^2, \tag{B59}$$

$$Z_{KK}^{\eta\eta}(s) = (9\beta_{\eta K}s - 2(3\beta_{\eta K} + \alpha_{\eta K})M_K^2 + 6(\alpha_{\eta K} - \beta_{\eta K})M_\eta^2)^2, \tag{B60}$$

$$Z_{KK}^{\pi\pi}(s,t) = 3(s((9\beta_{\pi K}^2 - 2\gamma_{\pi K}^2)s - 4\gamma_{\pi K}^2 t) + 4(1 - \beta_{\pi K})^2(M_K^4 + M_\pi^4) + 4s(3\beta_{\pi K}(1 - \beta_{\pi K}) + 2\gamma_{\pi K}^2)(M_K^2 + M_\pi^2) + 8(2\gamma_{\pi K}^2 t + ((1 - \beta_{\pi K})^2 + 2(1 - \alpha_{\pi K})^2 - 4\gamma_{\pi K}^2)M_K^2)M_\pi^2 + 8(\alpha_{\pi K} - 1)(3\beta_{\pi K}s + 2(1 - \beta_{\pi K})(M_K^2 + M_\pi^2))M_\pi M_K), \tag{B61}$$

$$Y_{KK}^{KK}(s,t) = \frac{1}{16} Z_{KK}^{KK}(t,s), \tag{B62}$$

$$Y_{KK}^{\pi\eta}(t) = 3(3\beta_{\eta\pi K}t - 2(1+\beta_{\eta\pi K})M_K^2 + (1-\alpha_{\eta\pi K}-\beta_{\eta\pi K})M_\pi^2 + (1-\beta_{\eta\pi K})M_\eta^2)^2, \tag{B63}$$

$$Y_{KK}^{\pi\pi}(s,t) = 3\gamma_{\pi K}^2(2s+t-4M_K^2)(t-4M_\pi^2). \tag{B64}$$

APPENDIX C: STANDARD CHIRAL PERTURBATION THEORY $O(p^4)$ VALUES OF THE POLYNOMIAL PARAMETERS

In this appendix we give the values of our polynomial parameters which reproduce the $O(p^4)$ results of the standard chiral perturbation theory. To that end we have used the computation of [11], which contains all the considered amplitudes and its advantage is also that their results are in the unitary form.

The form of our results is in terms of physical observables and thus has to be scale-independent and the same for all the possible regularization schemes. The only thing which can change by an eventual change of the scheme is the relation between the parameters of our parametrization and the (renormalized) constants of the Lagrangian theory (LECs). By the change of the scale, the values of the LEC change, but their combinations giving the value of our parameters remain scale-independent—what can be another test of the results obtained from the Lagrangian theory. A further difference between (Lagrangian theory) results of different authors can rise from a different choice of the way they parametrize the $O(p^2)$ constants (bare masses and decay constants) using the physical parameters, e.g. in [11] they expand F_K and F_η decay constants in terms of F_π, L_4^r, and L_5^r.

In [11] they have used the Gell-Mann-Okubo relation (GMO) to get their results more simplified. In the standard chiral power counting the GMO formula has correction of the $O(p^4)$ order and thus we could also use it to simplify our results in some places, where it would give only correction of the $O(p^6)$ order at least. However, to let these relations be closely connected to the results [11], we have not done it and the only place where we refer to the GMO formula [and its $O(p^4)$ order correction] is in those places where we want to emphasize the validity of the $O(p^2)$ values of the parameters from Appendix A. Nevertheless, the use of

$$\Delta_{\mathrm{GMO}}(M_\eta^2 - M_\pi^2) = 4M_K^2 - M_\pi^2 - 3M_\eta^2, \tag{C1}$$

which is in the standard power counting of the $O(p^4)$ order, can also be understood just as a (more complicated) notation of the right-hand side of this definition.

Other objects appearing in the relations are the chiral logarithms, given in accordance with [11] by

$$\mu_i = \frac{M_i^2}{32\pi^2 F_\pi^2}\log\frac{M_i^2}{\mu^2} \quad \text{with} \quad i = \pi, K, \eta. \tag{C2}$$

(As has been already stated, its dependence on the scale μ is compensated on the right-hand side of the following relations by the scale-dependence of LEC L_i^r as listed in [10].)

By the comparison of [11] with our results from the previous appendix we get the following relations:

(i) $\eta\eta$

$$\delta_{\eta\eta} = 12(2L_1^r + 2L_2^r + L_3) - \frac{27F_\pi^2\mu_K}{4M_K^2} - \frac{27}{128\pi^2}, \tag{C3}$$

$$\begin{aligned} F_\pi^2(\alpha_{\eta\eta}-1)(M_\pi^2-4M_\eta^2) = {} & -\frac{4}{3}F_\pi^2\Delta_{\mathrm{GMO}}(M_\eta^2-M_\pi^2) + 96M_\eta^4(L_4^r-3L_6^r) + 8L_5^r(3M_\pi^4-10M_\eta^2M_\pi^2+13M_\eta^4) \\ & - 192L_7(M_\pi^4-3M_\eta^2M_\pi^2+2M_\eta^4) - 48L_8^r(2M_\pi^4-6M_\eta^2M_\pi^2+7M_\eta^4) \\ & + \frac{(7M_\pi^4-24M_\pi^2M_\eta^2-54M_\eta^4)\mu_K F_\pi^2}{3M_K^2} + \frac{(48M_\eta^2-7M_\pi^2)\mu_\pi F_\pi^2}{3} \\ & + \frac{(M_\pi^4-8M_\eta^2M_\pi^2+24M_\eta^4)\mu_\eta F_\pi^2}{M_\eta^2} + \frac{80M_K^4-176M_\eta^2M_K^2+103M_\eta^4}{32\pi^2}. \end{aligned} \tag{C4}$$

(ii) $\pi\eta$

$$\varepsilon_{\pi\eta} = 24L_1^r + 4L_3 - \frac{9F_\pi^2\mu_K}{4M_K^2} - \frac{9}{128\pi^2}, \tag{C5}$$

$$\delta_{\pi\eta} = 12L_2^r + 4L_3 - \frac{9F_\pi^2\mu_K}{4M_K^2} - \frac{9}{128\pi^2}, \tag{C6}$$

$$F_\pi^2\beta_{\pi\eta} = 8L_4^r(M_\pi^2 + M_\eta^2) - \frac{M_\pi^2 + M_\eta^2}{32\pi^2} - \frac{(M_\pi^2 + 3M_\eta^2)F_\pi^2\mu_K}{3M_K^2} - \frac{2F_\pi^2\mu_\pi}{3}, \tag{C7}$$

$$\begin{aligned} F_\pi^2(\alpha_{\pi\eta} - 1)M_\pi^2 = {} & 16L_4^r(M_\pi^4 - 4M_\eta^2M_\pi^2 + M_\eta^4) + 16M_\eta^2M_\pi^2(-L_5^r + 6L_6^r) + 96L_7(M_\pi^2 - M_\eta^2)M_\pi^2 \\ & + 48L_8^rM_\pi^4 - \frac{(5M_\pi^4 - 28M_\eta^2M_\pi^2 + 6M_\eta^4)F_\pi^2\mu_K}{3M_K^2} + \frac{(-17M_\pi^4 + 9M_\eta^2M_\pi^2 + 4M_\eta^4)F_\pi^2\mu_\pi}{3(M_\pi^2 - M_\eta^2)} \\ & + \frac{(M_\pi^4 - M_\eta^2M_\pi^2 + 4M_\eta^4)M_\pi^2F_\pi^2\mu_\eta}{3M_\eta^2(M_\pi^2 - M_\eta^2)} - \frac{8M_\pi^4 - 11M_\eta^2M_\pi^2 + 6M_\eta^4}{96\pi^2}. \end{aligned} \tag{C8}$$

(iii) $\pi\pi$

$$\varepsilon_{\pi\pi} = 4L_2^r + \frac{1}{12}F_\pi^2\left(-\frac{\mu_K}{M_K^2} - \frac{8\mu_\pi}{M_\pi^2}\right) - \frac{7}{384\pi^2}, \tag{C9}$$

$$\delta_{\pi\pi} = 8L_1^r + 4L_3 + \frac{1}{12}F_\pi^2\left(-\frac{\mu_K}{M_K^2} - \frac{8\mu_\pi}{M_\pi^2}\right) - \frac{13}{384\pi^2}, \tag{C10}$$

$$F_\pi^2(\beta_{\pi\pi} - 1) = 8(2L_4^r + L_5^r)M_\pi^2 - \frac{F_\pi^2\mu_KM_\pi^2}{M_K^2} - \frac{5M_\pi^2}{32\pi^2} - 4F_\pi^2\mu_\pi, \tag{C11}$$

$$F_\pi^2(\alpha_{\pi\pi} - 1)M_\pi^2 = -16(2L_4^r + L_5^r - 6L_6^r - 3L_8^r)M_\pi^4 - \frac{F_\pi^2\mu_\eta M_\pi^4}{3M_\eta^2} - \frac{F_\pi^2\mu_KM_\pi^4}{M_K^2} - \frac{7M_\pi^4}{96\pi^2} - F_\pi^2\mu_\pi M_\pi^2. \tag{C12}$$

(iv) ηK

$$\varepsilon_{\eta K} = 16L_2^r + \frac{4}{3}L_3 - \frac{F_\pi^2(2\mu_K + \mu_\pi + 3\mu_\eta)}{2(M_K^2 - M_\eta^2)} + \frac{1}{64\pi^2}, \tag{C13}$$

$$\delta_{\eta K} = 32L_1^r + \frac{40}{3}L_3 + \frac{(9M_\eta^2 - 8M_K^2)F_\pi^2\mu_K}{2M_K^2(M_K^2 - M_\eta^2)} + \frac{F_\pi^2(\mu_\pi - 3\mu_\eta)}{4(M_K^2 - M_\eta^2)} - \frac{11}{64\pi^2}, \tag{C14}$$

$$\begin{aligned} F_\pi^2(\beta_{\eta K} - 1) = {} & \frac{32}{3}(M_K^2 + M_\eta^2)L_4^r + 8L_5^rM_\pi^2 - \frac{M_\eta^2(7M_K^2 - 9M_\eta^2)F_\pi^2\mu_K}{3M_K^2(M_K^2 - M_\eta^2)} + \frac{(13M_\eta^2 - 11M_K^2)F_\pi^2\mu_\pi}{3(M_K^2 - M_\eta^2)} \\ & + \frac{(4M_K^4 - 15M_\eta^2M_K^2 + 7M_\eta^4)F_\pi^2\mu_\eta}{3M_\eta^2(M_K^2 - M_\eta^2)} - \frac{17M_\eta^2 + 37M_K^2}{192\pi^2}, \end{aligned} \tag{C15}$$

$$F_\pi^2(\alpha_{\eta K}-1)(3M_\eta^2-M_K^2)=F_\pi^2\Delta_{\rm GMO}(M_\eta^2-M_\pi^2)+32L_4^r(M_K^4-4M_\eta^2M_K^2+M_\eta^4)+192L_6^rM_K^2M_\eta^2$$
$$+4L_5^r(-3M_\pi^4+6M_K^2M_\pi^2+M_\eta^2M_\pi^2-12M_\eta^4)$$
$$+48L_7(M_\pi^4-4M_\eta^2M_\pi^2+3M_\eta^4)+24L_8^r(M_\pi^4-3M_\eta^2M_\pi^2+6M_\eta^4)$$
$$-\frac{(14M_K^6-84M_\eta^2M_K^4+102M_\eta^4M_K^2-27M_\eta^6)F_\pi^2\mu_K}{3(M_K^2-M_\eta^2)}$$
$$-\frac{(M_K^4+12M_\eta^2M_K^2-14M_\eta^4)F_\pi^2\mu_\pi}{2(M_K^2-M_\eta^2)}$$
$$+\frac{(8M_K^6-41M_\eta^2M_K^4+104M_\eta^4M_K^2-78M_\eta^6)F_\pi^2\mu_\eta}{6M_\eta^2(M_K^2-M_\eta^2)}$$
$$-\frac{31M_K^4-55M_\eta^2M_K^2+39M_\eta^4}{96\pi^2}. \tag{C16}$$

(v) $\eta\pi K$

$$\varepsilon_{\eta\pi K}=-4L_3+\frac{6\mu_KF_\pi^2}{M_K^2-M_\pi^2}-\frac{3\mu_\pi F_\pi^2}{2(M_K^2-M_\pi^2)}+\frac{3\mu_\eta F_\pi^2}{2(M_K^2-M_\eta^2)}+\frac{1}{64\pi^2}, \tag{C17}$$

$$\delta_{\eta\pi K}=8L_3+\frac{3(M_\pi^2-2M_K^2)\mu_KF_\pi^2}{2M_K^2(M_K^2-M_\pi^2)}-\frac{3\mu_\pi F_\pi^2}{4(M_K^2-M_\pi^2)}-\frac{3\mu_\eta F_\pi^2}{4(M_K^2-M_\eta^2)}-\frac{5}{64\pi^2}, \tag{C18}$$

$$F_\pi^2(\beta_{\eta\pi K}-1)=\frac{4}{3}L_3\Delta_{\rm GMO}(M_\eta^2-M_\pi^2)+8M_\pi^2L_5^r$$
$$+\frac{(-32M_K^4+2M_K^2(2M_\pi^2-3M_\eta^2)+3M_\pi^2(M_\pi^2+M_\eta^2))F_\pi^2\mu_K}{6(M_K^2-M_\pi^2)}$$
$$+\frac{(-40M_K^2+9M_\eta^2+33M_\pi^2)F_\pi^2\mu_\pi}{12(M_K^2-M_\pi^2)}-\frac{(M_\pi^2+5M_\eta^2)F_\pi^2\mu_\eta}{4(M_K^2-M_\eta^2)}$$
$$+\frac{-57M_K^2+4M_\pi^2+27M_\eta^2}{192\pi^2}, \tag{C19}$$

$$F_\pi^2\alpha_{\eta\pi K}M_\pi^2=-\frac{F_\pi^2}{3}\Delta_{\rm GMO}(M_\eta^2-M_\pi^2)+\frac{8}{3}L_3(8M_K^2+M_\pi^2+M_\eta^2)\Delta_{\rm GMO}(M_\eta^2-M_\pi^2)$$
$$+4(M_\pi^4-4M_K^2M_\pi^2+4M_\eta^2M_\pi^2-M_\eta^4)L_5^r+24(M_\eta^4-M_\pi^4)(2L_7+L_8^r)$$
$$+\frac{(40M_K^6+8(M_\eta^2-10M_\pi^2)M_K^4-3M_\pi^2(M_\pi^2+M_\eta^2)^2+2M_K^2(17M_\pi^4+25M_\pi^2M_\eta^2-6M_\eta^4))F_\pi^2\mu_K}{6(M_K^2-M_\pi^2)}$$
$$+\frac{(-70M_K^4+2(22M_\pi^2+19M_\eta^2)M_K^2-19M_\pi^4+9M_\eta^4-12M_\pi^2M_\eta^2)F_\pi^2\mu_\pi}{6(M_K^2-M_\pi^2)}$$
$$+\frac{(14M_K^4-M_\pi^4+21M_\eta^4+6M_\pi^2M_\eta^2-2M_K^2(M_\pi^2+18M_\eta^2))F_\pi^2\mu_\eta}{2(M_K^2-M_\eta^2)}$$
$$+\frac{-408M_K^4+(73M_\pi^2+387M_\eta^2)M_K^2+8M_\pi^4-63M_\eta^4-M_\pi^2M_\eta^2}{192\pi^2}. \tag{C20}$$

(vi) πK

$$\varphi_{\pi K} = -4L_3 + \frac{(M_\pi^2 - 4M_K^2)F_\pi^2\mu_K}{6M_K^2(M_K^2 - M_\pi^2)} + \frac{(M_\pi^2 - 4M_K^2)F_\pi^2\mu_\pi}{12M_\pi^2(M_K^2 - M_\pi^2)} - \frac{F_\pi^2\mu_\eta}{M_\pi^2 - M_\eta^2}, \tag{C21}$$

$$\varepsilon_{\pi K} = 12(4L_2^r + L_3) - \frac{3F_\pi^2\mu_K}{M_K^2 - M_\pi^2} + \frac{15F_\pi^2\mu_\pi}{2(M_K^2 - M_\pi^2)} + \frac{6F_\pi^2\mu_\eta}{M_\pi^2 - M_\eta^2} + \frac{1}{64\pi^2}, \tag{C22}$$

$$\delta_{\pi K} = 24(4L_1^r + L_3) + \frac{3(3M_\pi^2 - 4M_K^2)F_\pi^2\mu_K}{2M_K^2(M_K^2 - M_\pi^2)} + \frac{3(7M_\pi^2 - 8M_K^2)F_\pi^2\mu_\pi}{4M_\pi^2(M_K^2 - M_\pi^2)} - \frac{3F_\pi^2\mu_\eta}{M_\pi^2 - M_\eta^2} - \frac{23}{64\pi^2}, \tag{C23}$$

$$F_\pi^2(\gamma_{\pi K} - 1) = 8M_\pi^2 L_5^r - \frac{2M_K^2F_\pi^2\mu_K}{M_K^2 - M_\pi^2} - \frac{(M_K^2 - 5M_\pi^2)F_\pi^2\mu_\pi}{2(M_K^2 - M_\pi^2)} + \frac{(2M_K^2 + M_\pi^2 - 3M_\eta^2)F_\pi^2\mu_\eta}{M_\pi^2 - M_\eta^2} - \frac{5M_K^2 + M_\pi^2}{192\pi^2}, \tag{C24}$$

$$\begin{aligned} F_\pi^2(\beta_{\pi K} - 1) = {} & 32(M_K^2 + M_\pi^2)L_4^r + 8L_5^r M_\pi^2 - \frac{15M_\pi^2 + 19M_K^2}{64\pi^2F_\pi^2} + \frac{(3M_\pi^4 + 3M_K^2M_\pi^2 - 8M_K^4)F_\pi^2\mu_K}{M_K^2(M_K^2 - M_\pi^2)} \\ & - \frac{(3M_\pi^4 - 3M_K^2M_\pi^2 - 4M_K^4)F_\pi^2\mu_\pi}{2M_\pi^2(M_K^2 - M_\pi^2)} - \frac{(2M_\pi^4 + 9M_\eta^4 + (4M_K^2 + M_\pi^2)M_\eta^2)F_\pi^2\mu_\eta}{2M_\eta^2(M_\pi^2 - M_\eta^2)}, \end{aligned} \tag{C25}$$

$$\begin{aligned} F_\pi^2(\alpha_{\pi K} - 1)M_\pi M_K = {} & 16L_4^r(M_K^4 - 4M_\pi^2M_K^2 + M_\pi^4) + 4L_5^r M_\pi^2(M_\pi^2 - 5M_K^2) + 48M_K^2M_\pi^2(2L_6^r + L_8^r) \\ & - \frac{21M_K^4 - 25M_\pi^2M_K^2 + 21M_\pi^4}{192\pi^2} + \frac{(2M_K^6 - 4M_K^4M_\pi^2 + 6M_\pi^4M_K^2 - 3M_\pi^6)F_\pi^2\mu_K}{2M_K^2(M_K^2 - M_\pi^2)} \\ & - \frac{(8M_K^6 - 13M_K^4M_\pi^2 + 12M_\pi^4M_K^2 - 6M_\pi^6)F_\pi^2\mu_\pi}{4M_\pi^2(M_K^2 - M_\pi^2)} \\ & - \frac{(17M_\pi^6 - 24M_K^2M_\pi^4 + 3(27M_\eta^2 + 7M_\pi^2 - 36M_K^2)M_\eta^4 + (24M_K^4 + 12M_\pi^2M_K^2 + M_\pi^4)M_\eta^2)F_\pi^2\mu_\eta}{24M_\eta^2(M_\pi^2 - M_\eta^2)}. \end{aligned} \tag{C26}$$

(vii) KK

$$\varphi_{KK} = 24L_2^r - \frac{17F_\pi^2\mu_K}{4M_K^2} - \frac{F_\pi^2\mu_\pi}{4M_\pi^2} - \frac{7}{64\pi^2}, \tag{C27}$$

$$\varepsilon_{KK} = 12(2L_2^r + L_3) - \frac{F_\pi^2\mu_K}{4M_K^2} - \frac{F_\pi^2}{8}\left(\frac{2}{M_\pi^2} - \frac{27}{M_K^2 - M_\pi^2}\right)\mu_\pi + \frac{9F_\pi^2\mu_\eta}{2(M_\pi^2 - M_\eta^2)} - \frac{1}{64\pi^2}, \tag{C28}$$

$$\delta_{KK} = 12(4L_1^r + L_3) - \frac{11F_\pi^2\mu_K}{4M_K^2} - \frac{F_\pi^2}{16}\left(\frac{10}{M_\pi^2} + \frac{27}{M_K^2 - M_\pi^2}\right)\mu_\pi - \frac{9}{8}F_\pi^2\left(\frac{3}{M_\eta^2} + \frac{2}{M_\pi^2 - M_\eta^2}\right)\mu_\eta - \frac{29}{128\pi^2}, \tag{C29}$$

$$\begin{aligned} F_\pi^2\gamma_{KK} = {} & 16M_K^2L_4^r - \frac{7M_K^2}{64\pi^2} - \frac{3F_\pi^2\mu_K}{2} - \frac{3M_K^2(2M_K^2 + M_\pi^2)F_\pi^2\mu_\pi}{8M_\pi^2(M_K^2 - M_\pi^2)} \\ & + \frac{(M_\pi^4 - 3M_\eta^4 + 2M_\pi^2M_\eta^2 + 3(M_\eta^2 - 3M_\pi^2)M_K^2)F_\pi^2\mu_\eta}{4M_\eta^2(M_\pi^2 - M_\eta^2)}, \end{aligned} \tag{C30}$$

$$\begin{aligned} F_\pi^2(\beta_{KK} - 1) = {} & 16M_K^2L_4^r + 8L_5^rM_\pi^2 - \frac{9M_K^2}{64\pi^2} - \frac{3F_\pi^2\mu_K}{2} - \frac{(6M_K^4 + 11M_\pi^2M_K^2 - 20M_\pi^4)F_\pi^2\mu_\pi}{8M_\pi^2(M_K^2 - M_\pi^2)} \\ & + \frac{(M_\pi^4 - 9M_\eta^4 + 8M_\pi^2M_\eta^2 + M_K^2(11M_\eta^2 - 9M_\pi^2))F_\pi^2\mu_\eta}{4M_\eta^2(M_\pi^2 - M_\eta^2)}, \end{aligned} \tag{C31}$$

$$F_\pi^2(\alpha_{KK}-1)M_K^2 = 16M_K^4(-2L_4^r+6L_6^r+3L_8^r)+8L_5^rM_K^2(M_\pi^2-3M_K^2)-\frac{7M_K^4}{192\pi^2}-\frac{3M_K^2F_\pi^2\mu_K}{2} - \frac{M_K^2(6M_K^4+17M_\pi^2M_K^2-20M_\pi^4)F_\pi^2\mu_\pi}{8M_\pi^2(M_K^2-M_\pi^2)} - \frac{(M_\pi^6+2M_\eta^2M_\pi^4-3M_\eta^4M_\pi^2+3M_K^4(9M_\pi^2-7M_\eta^2)-6M_K^2(2M_\pi^4+5M_\eta^2M_\pi^2-7M_\eta^4))F_\pi^2\mu_\eta}{12M_\eta^2(M_\eta^2-M_\pi^2)}. \tag{C32}$$

In [11] they believe $K^+K^-\to K^+K^-$ to be independent on $K^+K^-\to K^0\bar{K}^0$, but we know that isospin structure (Fierz-like identities) and the crossing symmetry dictate the relation between these two processes given by (A18) together with (A17). Therefore, we have used the values of our parameters obtained from their $K^+K^-\to K^0\bar{K}^0$ amplitude to explicitly check their $K^+K^-\to K^+K^-$ result T_{ch}.

APPENDIX D: SHORT COMMENT ON THE ASSUMPTIONS OF THE ANALYTICITY OF THE AMPLITUDE AND THE DISPERSION RELATIONS

The most important assumptions of the theorem—the existence of dispersion relations and the analyticity of the amplitude and its absorptive parts—are results of a complicated theory of analytic properties of scattering amplitudes, which is even older than QCD itself. A good and still valid summary of it is the article by Sommer [19].

We will not address this theory in more detail, just summarizing results interesting for us (details can be found in [19,20]).

From the principles of axiomatic field theory (even without use of the unitarity of S-matrix), Lehmann has proven that the amplitude with s fixed at some physical value is holomorphic in some finite region of the u-plane (if s is above the physical threshold, this region is the so-called small Lehmann ellipse). Absorptive parts of amplitudes [$\mathrm{Im}A(s,u)$ for $s\geq\Sigma$ and similarly for t] are holomorphic in the large Lehmann ellipses in the u-plane (depending on s) and there this absorptive part has also a convergent partial wave decomposition. Further, the N-times subtracted u-fixed dispersion relations can be proven on the intersection of those large Lehmann ellipses for $s\geq\Sigma$ and $t\geq\tau$ (if not empty). Since its semiminor axis tends to zero for $s\to\infty$, this intersection is just an interval on the negative real u-axis.

Taking into account the unitarity, Martin and others have succeeded in enlarging the validity of the dispersion relations into the circle $|u|<R$ with some fixed radius R.

There exist further methods to enlarge the region of validity as well as the analyticity domain. Using the ones given in [19], we end up with the results of Table IV there. We are interested in their specific application for our processes. If we assume the isospin conservation and that all the mesons π, K, η are thereby stable (and forget about resonances), we get the validity of fixed u dispersion relations within the regions depicted in Fig. 1 (the intersections of Lehmann ellipses are marked by bold lines there). These regions are glued ellipses and have been obtained analytically in [20] using procedures from [19]. Let us remind that these regions are only the minimal domains where the dispersive relations are valid, proven directly from the axiomatic theory. Using further methods, we could also extend these regions—e.g. we have not taken into account the further specific crossing (and Bose) symmetries of some of the amplitudes. Finally, let us once more emphasize that the method described in this appendix is ineffective if we allow e.g. the $\eta\to3\pi$ decay and such cases have to be proven e.g. using the unphysical mass of η and then analytically continued in it.

Another assumption of our theorem was the existence of a point s where the amplitude is analytic with respect to all the values of variable u for which the theorem should be valid—but in all the cases one can show that there exists a value s for which the small Lehmann ellipse is larger than the regions from the Fig. 1 and so we can conclude that both the assumptions of the theorem are valid within these regions.

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