Determination of the chiral couplings L_{10} and C_{87} from semileptonic τ decays

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(Received 14 October 2008; published 15 December 2008)

Using recent precise hadronic τ-decay data on the $V - A$ spectral function, and general properties of QCD such as analyticity, the operator product expansion, and chiral perturbation theory, we get accurate values for the QCD chiral order parameters $L_{10}^r(M_\rho)$ and $C_{87}^r(M_\rho)$. These two low-energy constants appear at order p^4 and p^6, respectively, in the chiral perturbation theory expansion of the $V - A$ correlator. At order p^4 we obtain $L_{10}^r(M_\rho) = -(5.22 \pm 0.06) \times 10^{-3}$. Including in the analysis the two-loop (order p^6) contributions, we get $L_{10}^r(M_\rho) = -(4.06 \pm 0.39) \times 10^{-3}$ and $C_{87}^r(M_\rho) = (4.89 \pm 0.19) \times 10^{-3}$ GeV^{-2}. In the SU(2) chiral effective theory, the corresponding low-energy coupling takes the value $\bar{l}_5 = 13.30 \pm 0.11$ at order p^4, and $\bar{l}_5 = 12.24 \pm 0.21$ at order p^6.

DOI: 10.1103/PhysRevD.78.116012 PACS numbers: 13.35.Dx, 11.55.Hx, 12.39.Fe

I. INTRODUCTION

The precise hadronic τ-decay data provided in Refs. [1–6] are a very important source of information, both on perturbative and nonperturbative QCD parameters. The theoretical analysis of the inclusive τ decay width into hadrons allows one to perform an accurate determination of the QCD coupling $\alpha_s(M_\tau)$ [7–11], which becomes the most precise determination of $\alpha_s(M_Z)$ after QCD running. In this case, nonperturbative QCD effects parametrized by power corrections are strongly suppressed. Another example of the use of hadronic τ-decay data is the study of SU(3)-breaking corrections to the strangeness-changing two-point functions [12–16]. The separate measurement of the $|\Delta S| = 0$ and $|\Delta S| = 1$ tau decay widths provides accurate determinations of fundamental parameters of the standard model, such as the strange quark mass and the Cabibbo-Kobayashi-Maskawa quark mixing $|V_{us}|$ [16].

Very important phenomenological hadronic matrix elements and nonperturbative QCD quantities can also be obtained from τ-decay data. Of special interest is the difference of the vector and axial-vector spectral functions, because in the chiral limit the corresponding $V - A$ correlator is exactly zero in perturbation theory. The τ-decay measurement of the $V - A$ spectral function has been used to perform [17–19] phenomenological tests of the so-called Weinberg sum rules (WSRs) [20], to compute the electromagnetic mass difference between the charged and neutral pions [18], and to determine several QCD vacuum condensates [21,22]. From the same spectral function, one can also determine the $\Delta I = 3/2$ contribution of the $\Delta S = 1$ four-quark operators Q_7 and Q_8 to $\varepsilon'_K/\varepsilon_K$, in the chiral limit [23].

Using chiral perturbation theory (χPT) [24–26], the hadronic τ-decay data can also be related to order parameters of the spontaneous chiral symmetry breaking (SχSB) of QCD [27]. χPT is the effective field theory of QCD at very low energies; it describes the SχSB Nambu-Goldstone boson physics through an expansion in external momenta and quark masses. The coefficients of that expansion are related to order parameters of SχSB. At lowest order (LO), i.e. $\mathcal{O}(p^2)$, all low-energy observables are described in terms of the pion decay constant $f_\pi \simeq$ 92.4 MeV and the light quark condensate. At next-to-leading order (NLO), $\mathcal{O}(p^4)$, the SU(3) χPT Lagrangian contains 12 low-energy constants (LECs), $L_{i=1,\ldots,10}$ and $H_{1,2}$ [26]. At $\mathcal{O}(p^6)$, 94 (23) additional parameters $C_{i=1,\ldots,94}$ ($C^W_{i=1,\ldots,23}$) appear in the even (odd) intrinsic parity sector [28]. These LECs are not fixed by symmetry requirements alone and have to be determined phenomenologically or using nonperturbative techniques. The $\mathcal{O}(p^4)$ L_i couplings have been determined in the past to an acceptable accuracy; a recent compilation can be found in Ref. [29]. Much less well determined are the $\mathcal{O}(p^6)$ couplings C_i.

There has been a lot of recent activity to determine the chiral LECs from theory, using as much as possible QCD information [30–39]. This strong effort is motivated by the precision required in present phenomenological applications, which makes necessary to include corrections of $\mathcal{O}(p^6)$. The huge number of unknown couplings is the major source of theoretical uncertainty.

In this paper we present an accurate determination of the χPT couplings L_{10} and C_{87}, using the most recent experimental data on hadronic τ decays [1]. Previous work on L_{10} using τ-decay data can be found in Refs. [18,19,21,40].

Our analysis is the first one which includes the known two-loop χPT contributions and, therefore, provides also the $\mathcal{O}(p^6)$ coupling C_{87}.

II. THEORETICAL FRAMEWORK

The basic objects of the theoretical analysis are the two-point correlation functions of the vector and axial-vector quark currents, defined as follows:

$$\begin{aligned}\Pi^{\mu\nu}_{ij,\mathcal{J}}(q) &\equiv i\int d^4x e^{iqx}\langle 0|T(\mathcal{J}^{\mu}_{ij}(x)\mathcal{J}^{\nu}_{ij}(0)^{\dagger})|0\rangle \\ &= (-g^{\mu\nu}q^2 + q^{\mu}q^{\nu})\Pi^{(1)}_{ij,\mathcal{J}}(q^2) \\ &\quad + q^{\mu}q^{\nu}\Pi^{(0)}_{ij,\mathcal{J}}(q^2). \end{aligned} \tag{1}$$

Here, we just need the nonstrange correlators, i.e. $\mathcal{J}^{\mu}_{ij}(x)$ denotes the Cabibbo-allowed vector or axial-vector currents, $V^{\mu}_{ud}(x) = \bar{u}\gamma^{\mu}d$ and $A^{\mu}_{ud} = \bar{u}\gamma^{\mu}\gamma_5 d$. Moreover, our analysis will concentrate in the difference,

$$\begin{aligned}\Pi(s) &\equiv \Pi^{(0+1)}_{ud,V-A}(s) = \Pi^{(0+1)}_{ud,V}(s) - \Pi^{(0+1)}_{ud,A}(s) \\ &\equiv \frac{2f_\pi^2}{s-m_\pi^2} + \bar{\Pi}(s),\end{aligned} \tag{2}$$

where we have made explicit the contribution of the pion pole to the longitudinal axial-vector two-point function. We will work in the isospin limit $m_u = m_d$ where $\Pi^{(0)}_{ud,V}(q^2) = 0$.

The correlator $\bar{\Pi}(s)$ is analytic in the entire complex s plane, except for a cut on the positive real axis which starts at the threshold $s_{\rm th} = 4m_\pi^2$. Applying Cauchy's theorem to the circuit in Fig. 1, one gets the exact relation:

$$\begin{aligned}&\int_{s_{\rm th}}^{s_0} ds s^n \frac{1}{\pi}\,{\rm Im}\Pi(s) + \frac{1}{2\pi i}\oint_{|s|=s_0} ds s^n \Pi(s) \\ &\quad = 2f_\pi^2 m_\pi^{2n} + {\rm Res}[s^n\Pi(s), s=0].\end{aligned} \tag{3}$$

For non-negative values of the integer power n, the pion pole is the only singularity within the contour and one gets the so-called finite energy sum rules (FESR), widely used in the literature. When n takes negative values, the weight factor s^n introduces a pole at the origin which gives rise to the additional contribution in the right-hand side of the equation, given by the residue of $s^n\Pi(s)$ at $s = 0$.

In the chiral limit ($m_u = m_d = 0$) the correlator $\Pi(s)$ vanishes identically to all orders in perturbation theory. For

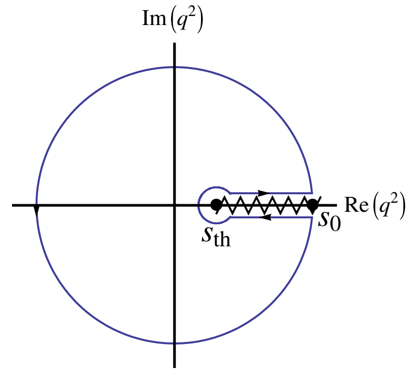

FIG. 1 (color online). Analytic structure of $\bar{\Pi}(s)$.

large enough Euclidean values of $s = -Q^2$ its operator product expansion (OPE), $\Pi(Q^2) = \sum_k C_{2k}^{V-A}/Q^{2k}$, contains only power-suppressed contributions from dimension $d = 2k$ operators, starting at $d = 6$. The nonzero up and down quark masses induce tiny corrections with dimensions two and four, which are negligible at high values of Q^2. Therefore, with $n \geq 0$ and s_0 large enough so that the OPE can be applied in the entire circle $s = s_0$, the integral over the spectral function from $s_{\rm th}$ to s_0 is equal to the pion pole term $2f_\pi^2 m_\pi^{2n}$ plus the OPE contribution $(-1)^n C^{V-A}_{2(n+1)}$ generated by the integration along the circle. For $n = 0$ and $n = 1$, $C^{V-A}_{2(n+1)}$ is zero in the chiral limit and one gets the celebrated first and second WSRs [20], respectively.

For negative values of $n \equiv -m < 0$, the OPE does not give any contribution to the integration along the circle $s = s_0$. One gets then

$$\begin{aligned}\int_{s_{\rm th}}^{s_0}\frac{ds}{s^m}\frac{1}{\pi}\,{\rm Im}\Pi(s) &= \frac{2f_\pi^2}{m_\pi^{2m}} + \frac{1}{(m-1)!}\Pi^{(m-1)}(0) \\ &= \frac{1}{(m-1)!}\bar{\Pi}^{(m-1)}(0),\end{aligned} \tag{4}$$

where $\bar{\Pi}^{(m-1)}(0)$ denotes the $(m-1)$th derivative of $\bar{\Pi}(s)$ at $s = 0$. The interest of this relation stems from the fact that at low values of s the correlator can be rigorously calculated within χPT. At present $\Pi(s)$ is known to $\mathcal{O}(p^6)$ [41], in terms of the LECs that we want to determine. The choices $m = 1$ and $m = 2$ allow us then to relate the spectral function measured in τ decays with the theoretical expressions of $\bar{\Pi}(0)$ and $\bar{\Pi}'(0)$, which can be derived from the results obtained in Ref. [41]:

$$\begin{aligned}L_{10}^{\rm eff} &\equiv -\frac{1}{8}\bar{\Pi}(0) = L_{10}^r(\mu) + \frac{1}{128\pi^2}\left[1 - \log\left(\frac{\mu^2}{m_\pi^2}\right) + \frac{1}{3}\log\left(\frac{m_K^2}{m_\pi^2}\right)\right] \\ &\quad + 4m_\pi^2(C_{61}^r - C_{12}^r - C_{80}^r)(\mu) + 4(2m_K^2 + m_\pi^2)(C_{62}^r - C_{13}^r - C_{81}^r)(\mu) - 2(2\mu_\pi + \mu_K)(L_9^r + 2L_{10}^r)(\mu) \\ &\quad + G_{2L}(\mu, s=0) + \mathcal{O}(p^8),\end{aligned} \tag{5}$$

$$C_{87}^{\mathrm{eff}} \equiv \frac{1}{16}\bar{\Pi}'(0) = C_{87}^r(\mu) + \frac{1}{7680\pi^2}\left(\frac{1}{m_K^2} + \frac{2}{m_\pi^2}\right) - \frac{1}{64\pi^2 f_\pi^2}\left[1 - \log\left(\frac{\mu^2}{m_\pi^2}\right) + \frac{1}{3}\log\left(\frac{m_K^2}{m_\pi^2}\right)\right]L_9^r(\mu) - \frac{1}{2}G'_{2L}(\mu, s=0) + \mathcal{O}(p^8), \tag{6}$$

where $\mu_i = m_i^2 \log(m_i/\mu)/(16\pi^2 f_\pi^2)$.

To a first approximation the effective parameters L_{10}^{eff} and C_{87}^{eff} correspond to the LECs $L_{10}^r(\mu)$ and $C_{87}^r(\mu)$, respectively. At $\mathcal{O}(p^4)$, the only relevant correction is given by the logarithmic terms in the first line of (5), which cancel the χPT renormalization scale dependence of $L_{10}^r(\mu)$; these contributions are suppressed by one power of $1/N_C$ with respect to $L_{10}^r(\mu)$, where N_C is the number of quark colors. The rest of the lines in (5) contain the $\mathcal{O}(p^6)$ corrections: the tree-level contributions from the $\mathcal{O}(p^6)$ χPT Lagrangian are given in the second line, the term proportional to $(L_9^r + 2L_{10}^r)(\mu)$ in the same line is the one-loop contribution of the $\mathcal{O}(p^4)$ χPT Lagrangian, and the function $G_{2L}(\mu, s=0)$ in the last line, which does not depend on any LEC, contains the proper two-loop contributions.

In Eq. (6) the tree-level contribution is given by $C_{87}^r(\mu)$, whereas the term proportional to $L_9^r(\mu)$ is a one-loop correction, which is suppressed by one power of $1/N_C$, and the two-loop contributions are contained in $G'_{2L}(\mu, s) \equiv \frac{d}{ds}G_{2L}(\mu, s)$. The derivative operation, when acting over the one-loop contribution to $\Pi(s)$, generates the terms proportional to inverse powers of the pion and kaon masses in the second line. For simplicity, we omit the explicit analytic forms of $G_{2L}(\mu)$ and $G'_{2L}(\mu)$, which are very lengthy and not too enlightening; these two functions contain a $1/N_C^2$ suppression factor with respect to $L_{10}^r(\mu)$ and $C_{87}^r(\mu)$.

III. DETERMINATION OF EFFECTIVE COUPLINGS

We will use the 2005 ALEPH data on semileptonic τ decays [1], which provides the most recent and precise measurement of the $V-A$ spectral function. The effective chiral couplings can be directly extracted from the following integrals over the hadronic spectrum:

$$-8L_{10}^{\mathrm{eff}} \equiv \bar{\Pi}(0) = \frac{1}{\pi}\int_{s_{\mathrm{th}}}^{s_0} \frac{ds}{s}\,\mathrm{Im}\Pi(s), \tag{7}$$

$$16C_{87}^{\mathrm{eff}} \equiv \bar{\Pi}'(0) = \frac{1}{\pi}\int_{s_{\mathrm{th}}}^{s_0} \frac{ds}{s^2}\,\mathrm{Im}\Pi(s). \tag{8}$$

These relations are exactly satisfied at $s_0 \to \infty$. At finite values of s_0, they assume that the OPE approximates well the correlator $\Pi(s)$ over the entire complex circle [42] $|s| = s_0$. The OPE is expected to be a valid approximation for high-enough values of s_0 and away from the real axis. While the kinematics of τ decay restrict the upper limit of integration to the range $s_0 \le m_\tau^2$, the main source of theoretical uncertainty in the contour integration originates in the region close to the point $s = s_0$ in the real axis. Studying the sensitivity to s_0 of the integrals (7) and (8), one can test validity of the OPE and assess the size of the associated systematic errors.

In Fig. 2, we plot the value of L_{10}^{eff} obtained from Eq. (7) for different values of s_0. The band between the continuous lines shows the corresponding experimental uncertainties (at one sigma). As expected, the result is far from a horizontal line at low values of s_0, where the applicability of the OPE is suspect. The oscillatory behavior stabilizes quite fast reaching a rather stable and flat result at values of s_0 between 2 and 3 GeV^2. The weight factor $1/s$ decreases the impact of the high-energy region, minimizing the size of quark-hadron duality violations around s_0. This integral appears then to be much better behaved than the corresponding FESRs with s^n $(n \ge 0)$ weights.

There are several possible strategies to estimate the central value for L_{10}^{eff} and the unavoidable theoretical uncertainties. One is to give the predictions fixing s_0 at the so-called "duality points," where the first and second WSRs happen to be satisfied. Owing to the oscillatory behavior of the WSRs results, this happens at two different values of s_0. At the highest "duality point," which is obviously the more reliable, we obtain $L_{10}^{\mathrm{eff}} = -(6.45 \pm 0.09) \times 10^{-3}$, where the quoted error only includes the experimental uncertainty. Being very conservative, one could also take into account the first "duality point"; performing a weighted average of both results, we get $L_{10}^{\mathrm{eff}} = -(6.50 \pm 0.13) \times 10^{-3}$, where the uncertainty covers the values obtained at the two "duality points."

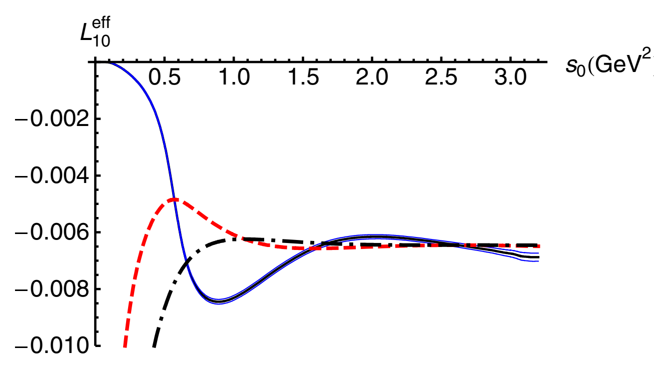

FIG. 2 (color online). Determinations of L_{10}^{eff} at different values of s_0. The continuous lines show the results obtained from Eq. (7). The modified expressions in Eqs. (9) and (10) give rise to the dashed and dot-dashed lines, respectively. For clarity, we do not include their corresponding error bands.

Assuming that the integral (7) oscillates around his asymptotic value with decreasing oscillations, one can get another estimate performing an average between the maxima and minima of the successive oscillations. This procedure gives a value $L_{10}^{\mathrm{eff}} = -(6.5 \pm 0.2) \times 10^{-3}$, that is perfectly compatible with the previous results based on the "duality points." Our last method of estimating the quark-hadron duality violation uses appropriate oscillating functions defined in [43] which mimic the real quark-hadron oscillations above the data. These functions are defined such that they match the data at approximately 3 GeV^2, go to zero with decreasing oscillations, and satisfy the first and second WSRs. We find in this way $L_{10}^{\mathrm{eff}} = -(6.50 \pm 0.12) \times 10^{-3}$, where the error spans the range generated by the different functions used. This result agrees well with our previous estimates.

We can take advantage of the WSRs to construct modified sum rules with weight factors proportional to $(1 - s/s_0)$, in order to suppress numerically the role of the suspect region around $s \sim s_0$ [8]:

$$-8L_{10}^{\mathrm{eff}} = \frac{1}{\pi}\int_{s_{\mathrm{th}}}^{s_0}\frac{ds}{s}\left(1-\frac{s}{s_0}\right)\mathrm{Im}\Pi(s) + \Delta_1(s_0), \qquad (9)$$

$$= \frac{1}{\pi}\int_{s_{\mathrm{th}}}^{s_0}\frac{ds}{s}\left(1-\frac{s}{s_0}\right)^2\mathrm{Im}\Pi(s) + 2\Delta_1(s_0) - \Delta_2(s_0). \qquad (10)$$

The factors $\Delta_1(s_0) = (2f_\pi^2 + C_2^{V-A})/s_0$ and $\Delta_2(s_0) = (2f_\pi^2 m_\pi^2 - C_4^{V-A})/s_0^2$ are small corrections dominated by the f_π^2 term, since $C_{2,4}^{V-A}$ vanish in the chiral limit. The sum rule (10) has been previously used in Refs. [21,40].

The dashed and dot-dashed lines in Fig. 2 show the results obtained from Eqs. (9) and (10), respectively. As already found in Refs. [21,40], the modified weight factors minimize the theoretical uncertainties in a very sizable way, giving rise to very stable results over a quite wide range of s_0 values. One gets then $L_{10}^{\mathrm{eff}} = -(6.51 \pm 0.06) \times 10^{-3}$ using Eq. (9), and $L_{10}^{\mathrm{eff}} = -(6.45 \pm 0.06) \times 10^{-3}$ from Eq. (10).

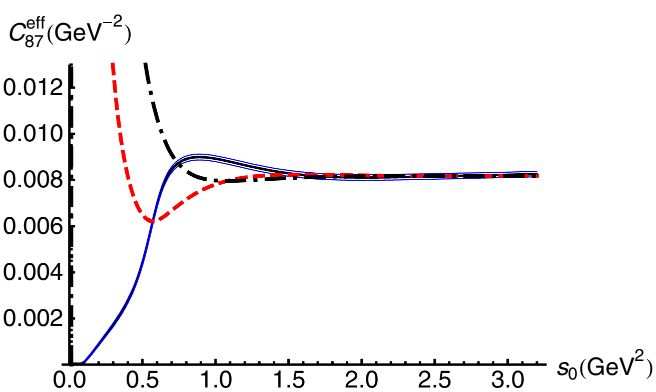

FIG. 3 (color online). Determinations of C_{87}^{eff} at different values of s_0. The continuous lines show the results obtained from Eq. (8). The modified expressions in Eqs. (12) and (13) give rise to the dashed and dot-dashed lines, respectively. For clarity, we do not include their corresponding error bands.

Taking into account all the previous discussion, we quote as our final result:

$$L_{10}^{\mathrm{eff}} = -(6.48 \pm 0.06) \times 10^{-3}. \qquad (11)$$

We have made a completely analogous analysis to determine the effective coupling C_{87}^{eff}. The results are shown in Fig. 3. The continuous lines, obtained from Eq. (8), are much more stable than the corresponding results for L_{10}^{eff}, owing to the $1/s^2$ factor in the integrand. The discontinuous and dotted lines correspond to the results obtained from the modified sum rules:

$$16C_{87}^{\mathrm{eff}} = \frac{1}{\pi}\int_{s_{\mathrm{th}}}^{s_0}\frac{ds}{s^2}\left(1-\frac{s^2}{s_0^2}\right)\mathrm{Im}\Pi(s) + \frac{\Delta_1}{s_0}, \qquad (12)$$

$$= \frac{1}{\pi}\int_{s_{\mathrm{th}}}^{s_0}\frac{ds}{s^2}\left(1-\frac{s}{s_0}\right)^2\left(1+2\frac{s}{s_0}\right)\mathrm{Im}\Pi(s) + \frac{3\Delta_1 - 2\Delta_2}{s_0}. \qquad (13)$$

The agreement among the different estimates is quite remarkable. We quote as our final conservative result,

$$C_{87}^{\mathrm{eff}} = (8.18 \pm 0.14) \times 10^{-3}\ \mathrm{GeV}^{-2}. \qquad (14)$$

IV. DETERMINATION OF L_{10}^r AND C_{87}^r

The χPT coupling $L_{10}^r(\mu)$ can be obtained from L_{10}^{eff}, using the relation (5). At $\mathcal{O}(p^4)$ the determination is straightforward, since one only needs to subtract from L_{10}^{eff} the term $[1 - \log(\mu^2/m_\pi^2) + \frac{1}{3}\log(m_K^2/m_\pi^2)]/(128\pi^2)$. Taking $\mu = M_\rho$ as the reference value for the χPT renormalization scale, one gets

$$L_{10}^r(M_\rho) = -(5.22 \pm 0.06) \times 10^{-3}. \qquad (15)$$

At order p^6, the numerical relation is more subtle because it gets small corrections from other LECs. It is useful to classify the $\mathcal{O}(p^6)$ contributions through their ordering within the $1/N_C$ expansion. The tree-level term $4m_\pi^2(C_{61}^r - C_{12}^r - C_{80}^r)(M_\rho)$, which is the only $\mathcal{O}(p^6)$ correction in the large-N_C limit, is numerically small because it appears suppressed by a factor m_π^2. The three relevant couplings have been determined phenomenologically with a moderate accuracy: $C_{61}^r(M_\rho) = (1.24 \pm 0.44) \times 10^{-3}\ \mathrm{GeV}^{-2}$ [34] [from $\Pi_{ud,V}^{(0+1)}(0) - \Pi_{us,V}^{(0+1)}(0)$], $C_{12}^r(M_\rho) = (0.4 \pm 6.3) \times 10^{-5}\ \mathrm{GeV}^{-2}$ [44] (from the $K\pi$ scalar form factor) and $C_{80}^r(M_\rho) = (2.1 \pm 0.5) \times 10^{-3}\ \mathrm{GeV}^{-2}$ [45] (from a_1/K_1 mass and width differences). These determinations agree reasonably well with published meson-exchange estimates [35,41] and lead to a total contribution $4m_\pi^2(C_{61}^r - C_{12}^r - C_{80}^r)(M_\rho) = -(6.7 \pm 5.2) \times 10^{-5}$. The scale dependence of this combination of $\mathcal{O}(p^6)$ couplings [28] between $\mu = 0.6$ GeV and $\mu = 1.1$ GeV is within its quoted uncertainty.

At NLO in $1/N_C$ we need to consider the tree-level contribution proportional to the combination of LECs $(C_{62}^r - C_{13}^r - C_{81}^r)(M_\rho)$. We are not aware of any published estimate of these $1/N_C$ suppressed couplings, beyond the trivial statement that they do not get any tree-level contribution from resonance exchange [35]. We will adopt the conservative range $|C_{62}^r - C_{13}^r - C_{81}^r|(M_\rho) \leq |C_{61}^r - C_{12}^r - C_{80}^r|(M_\rho)/3$, which gives a contribution $4(2m_K^2 + m_\pi^2)(C_{62}^r - C_{13}^r - C_{81}^r)(M_\rho) = (0.0 \pm 5.8) \times 10^{-4}$. The scale dependence between $\mu = 0.6$ GeV and $\mu = 1.1$ GeV of this combination of $\mathcal{O}(p^6)$ couplings [28] is within its quoted uncertainty. The uncertainty on this term will dominate our final error on the $L_{10}^r(M_\rho)$ determination. At the same NLO in $1/N_C$, there is also a one-loop correction proportional to $L_9^r(M_\rho)$; using the $\mathcal{O}(p^6)$ determination $L_9^r(M_\rho) = (5.93 \pm 0.43) \times 10^{-3}$ [46], this contribution can be estimated to be $2(2\mu_\pi + \mu_K)L_9^r(M_\rho) = -(1.56 \pm 0.11) \times 10^{-3}$. Finally, the $1/N_C^2$ suppressed two-loop function which collects the nonanalytic contributions takes the value $G_{2L}(M_\rho) = -0.524 \times 10^{-3}$, 1 order of magnitude smaller than $L_{10}^{\rm eff}$, but still 8 times larger than the uncertainty quoted for $L_{10}^{\rm eff}$ in (11). Taking all these contributions into account, we finally get the wanted $\mathcal{O}(p^6)$ result:

$$\begin{aligned} L_{10}^r(M_\rho) &= -(4.06 \pm 0.04_{L_{10}^{\rm eff}} \pm 0.39_{\rm LECs}) \times 10^{-3} \\ &= -(4.06 \pm 0.39) \times 10^{-3}, \end{aligned} \tag{16}$$

where the uncertainty has been split into its two main components. The final error is completely dominated by our ignorance on the $1/N_C$ suppressed LECs of $\mathcal{O}(p^6)$.

The determination of C_{87}^r from $C_{87}^{\rm eff}$ does not involve any unknown LEC. The relation (6) contains a one-loop correction of size $-(3.15 \pm 0.13) \times 10^{-3}$, which only depends on $L_9^r(M_\rho)$ and the pion and kaon masses, and small nonanalytic two-loop contributions collected in the term $G'_{2L}(M_\rho) = -0.277 \times 10^{-3}$ GeV^{-2}. In spite of its $1/N_C$ suppression, the one-loop correction is very sizable, decreasing the final value of the $\mathcal{O}(p^6)$ LEC:

$$C_{87}^r(M_\rho) = (4.89 \pm 0.19) \times 10^{-3} \text{ GeV}^{-2}. \tag{17}$$

V. SU(2) χPT

Up to now, we have discussed the LECs of the usual SU(3) χPT. It turns useful to consider also the effective low-energy theory with only two flavors of light quarks. In some cases, this allows one to perform high-accuracy phenomenological determinations of the corresponding LECs at NLO. Moreover, recent lattice calculations with two dynamical quarks are already able to obtain the SU(2) LECs with sufficient accuracy and this is an important check for them.

In SU(2) χPT, there are ten LECs, $l_{i=1,\ldots,7}$ and $h_{1,2,3}$, at $\mathcal{O}(p^4)$ (NLO) [25]. Using the $\mathcal{O}(p^6)$ relation between $l_5^r(\mu)$ and $L_{10}^r(\mu)$, recently obtained in Ref. [47], and the definition of the invariant couplings $\bar{l}_i$ adopted in [25], we get

$$\begin{aligned} \bar{l}_5 = &-192\pi^2 L_{10}^{\rm eff} + 1 + \log\left(\frac{m_K}{\hat{m}_K}\right) + 768\pi^2 m_\pi^2 (C_{61}^r + C_{62}^r - C_{12}^r - C_{13}^r - C_{80}^r - C_{81}^r)(\mu) \\ &+ 1536\pi^2(m_K^2 - \hat{m}_K^2)(C_{62}^r - C_{13}^r - C_{81}^r)(\mu) - 384\pi^2(2\mu_\pi + \mu_K - \hat{\mu}_K)(L_9^r + 2L_{10}^r)(\mu) \\ &- x_K\left[-\frac{67}{48} + \frac{21}{16}\rho_1 + \frac{5}{8}\log\left(\frac{4}{3}\right) - \frac{17}{4}\log\left(\frac{\mu^2}{\hat{m}_K^2}\right) + \frac{3}{4}\log^2\left(\frac{\mu^2}{\hat{m}_K^2}\right)\right] + 192\pi^2 G_{2L}(\mu) + \mathcal{O}(p^8), \end{aligned} \tag{18}$$

where $\hat{m}_K^2 = m_K^2 - m_\pi^2/2$ is the kaon mass squared in the limit $m_u = m_d = 0$, $x_K = \hat{m}_K^2/(16\pi^2 f_\pi^2)$, $\hat{\mu}_K = \hat{m}_K^2 \log(\hat{m}_K/\mu)/(16\pi^2 f_\pi^2)$, and $\rho_1 \simeq 1.416\,02$.

The first three terms in the right-hand side of Eq. (18) are the $\mathcal{O}(p^4)$ contributions; the determination of $\bar{l}_5$ at this order is then straightforward. The full $\mathcal{O}(p^6)$ result, with the different tree-level, one-loop, and two-loop corrections, is given by the rest of the terms. Following the same procedure as in the SU(3) case, we get the results

$$\bar{l}_5 = \begin{cases} 13.30 \pm 0.11, & \mathcal{O}(p^4), \\ 12.24 \pm 0.21, & \mathcal{O}(p^6). \end{cases} \tag{19}$$

VI. SUMMARY

Using the most recent hadronic τ-decay data [1] on the $V - A$ spectral function, and general properties of QCD such as analyticity, the OPE, and χPT, we have determined very accurately the chiral LECs $L_{10}^r(M_\rho)$ and $C_{87}^r(M_\rho)$. Performing an $\mathcal{O}(p^4)$ analysis, we obtain

$$L_{10}^r(M_\rho) = -(5.22 \pm 0.06) \times 10^{-3}, \tag{20}$$

while a more elaborate study, including the $\mathcal{O}(p^6)$ χPT corrections provides the values

$$\begin{aligned} L_{10}^r(M_\rho) &= -(4.06 \pm 0.04_{L_{10}^{\rm eff}} \pm 0.39_{\rm LECs}) \times 10^{-3} \\ &= -(4.06 \pm 0.39) \times 10^{-3}, \end{aligned} \tag{21}$$

and

$$C_{87}^r(M_\rho) = (4.89 \pm 0.19) \times 10^{-3} \text{ GeV}^{-2}. \tag{22}$$

Our error estimate includes a careful analysis of the theoretical uncertainties associated with the use of the OPE in the dangerous region close to the physical cut. Moreover, in (21) we have explicitly separated the error into its two main components, showing that our present ignorance on the $1/N_C$ suppressed LECs dominates the final uncertainty of the $L_{10}^r(M_\rho)$ determination at $\mathcal{O}(p^6)$.

Several determinations of L_{10} have been performed before [18,19,40], using the older 1998 ALEPH data [2,3]. In Ref. [18] the result $L_{10}^r(M_\rho) = -(5.13 \pm 0.19) \times 10^{-3}$ was obtained to $\mathcal{O}(p^4)$, through a simultaneous fit of this parameter and the OPE corrections of dimensions six and eight to several spectral moments of the hadronic distribution. This determination is in good agreement with our $\mathcal{O}(p^4)$ result (20). Our quoted uncertainty has a smaller experimental contribution and includes a better assessment of the theoretical uncertainties. The value $L_{10}^{\rm eff} = (-5.8 \pm 0.2) \times 10^{-3}$ (3.2σ smaller than ours) was extracted from τ data in Ref. [19] using the first "duality point" of the WSRs. The difference comes from underestimated theoretical uncertainties in this reference, as can be easily seen by choosing instead the second duality point or varying slightly the value of the first duality point. In fact the same reference [19] [see Eq. (10) therein] presents also a different estimate of $L_{10}^{\rm eff}$ that is in very good agreement with our result. In Ref. [40] both $L_{10}^{\rm eff}$ and $C_{87}^{\rm eff}$ were determined, in good agreement with our findings which use the most recent 2005 data. An updated value of $L_{10}^{\rm eff}$, using the 2005 data, has also been given in Ref. [21].

Our determinations of $L_{10}^r(\mu)$ and $C_{87}^r(\mu)$ at $\mu = M_\rho$ agree within errors with the large-N_C estimates based on lowest-meson dominance [31,36,41,48]:

$$L_{10} = -\frac{F_V^2}{4M_V^2} + \frac{F_A^2}{4M_A^2} \approx -\frac{3f_\pi^2}{8M_V^2} \approx -5.4 \times 10^{-3}, \quad (23)$$

$$C_{87} = \frac{F_V^2}{8M_V^4} - \frac{F_A^2}{8M_A^4} \approx \frac{7f_\pi^2}{32M_V^4} \approx 5.3 \times 10^{-3}\ \text{GeV}^{-2}. \quad (24)$$

Equation (22) is also in good agreement with the result of Ref. [38] for C_{87} based on Padé approximants. These predictions, however, are unable to fix the scale dependence which is of higher order in $1/N_C$. More recently, the resonance chiral theory Lagrangian [36,49] has been used to analyze the correlator $\Pi(s)$ at NLO order in the $1/N_C$ expansion [39]. Matching the effective field theory description with the short-distance QCD behavior, the two LECs are determined, keeping full control of their μ dependence. The theoretically predicted values $L_{10}^r(M_\rho) = -(4.4 \pm 0.9) \times 10^{-3}$ and $C_{87}^r(M_\rho) = (3.6 \pm 1.3) \times 10^{-3}$ GeV^{-2} [39] are in perfect agreement with our determinations, although less precise. A recent lattice estimate [50] finds $L_{10}^r(M_\rho) = -(5.2 \pm 0.5) \times 10^{-3}$ at $\mathcal{O}(p^4)$, which is also in good agreement with our $\mathcal{O}(p^4)$ result in (20).

A recent reanalysis of the decay $\pi^+ \to e^+ \nu \gamma$ [45], using new experimental data, has provided quite accurate values for the combination of $\mathcal{O}(p^4)$ LECs $L_9 + L_{10}$. To $\mathcal{O}(p^4)$ one finds $L_9^r(M_\rho) + L_{10}^r(M_\rho) = (1.32 \pm 0.14) \times 10^{-3}$, while the $\mathcal{O}(p^6)$ result $L_9^r(M_\rho) + L_{10}^r(M_\rho) = (1.44 \pm 0.08) \times 10^{-3}$ is slightly more precise [45]. Combining these numbers with our results for $L_{10}^r(M_\rho)$, one obtains

$$L_9^r(M_\rho) = \begin{cases} (6.54 \pm 0.15) \times 10^{-3}, & \mathcal{O}(p^4), \\ (5.50 \pm 0.40) \times 10^{-3}, & \mathcal{O}(p^6), \end{cases} \quad (25)$$

in perfect agreement with the $\mathcal{O}(p^4)$ result $L_9^r(M_\rho) = (6.9 \pm 0.7) \times 10^{-3}$ of Ref. [29] and the $\mathcal{O}(p^6)$ result $L_9^r(M_\rho) = (5.93 \pm 0.43) \times 10^{-3}$ of Ref. [46]. This last comparison represents an indirect check (in fact the only possible one for the moment) of our $\mathcal{O}(p^6)$ result for L_{10}.

We have also determined the corresponding LEC of L_{10} in the SU(2) effective theory, both at LO and NLO:

$$\bar{l}_5 = \begin{cases} 13.30 \pm 0.11, & \mathcal{O}(p^4), \\ 12.24 \pm 0.21, & \mathcal{O}(p^6). \end{cases} \quad (26)$$

From a phenomenological analysis of the radiative decay $\pi \to l\nu\gamma$ within SU(2) χPT, the authors of Ref. [51] obtained $\bar{l}_6 - \bar{l}_5 = 2.57 \pm 0.35$ at $\mathcal{O}(p^4)$, and $\bar{l}_6 - \bar{l}_5 = 2.98 \pm 0.33$ at $\mathcal{O}(p^6)$. Using these results and our determinations for $\bar{l}_5$ in (26), one gets

$$\bar{l}_6 = \begin{cases} 15.87 \pm 0.37, & \mathcal{O}(p^4), \\ 15.22 \pm 0.39, & \mathcal{O}(p^6). \end{cases} \quad (27)$$

At $\mathcal{O}(p^4)$ the comparison of these estimates of SU(2) LECs with previous results is straightforward, since they are proportional to the corresponding SU(3) couplings, that we have already discussed. Our determination of $\bar{l}_5$ is the first one obtained at $\mathcal{O}(p^6)$, whereas for $\bar{l}_6$ Ref. [52] finds $\bar{l}_6 = 16.0 \pm 0.5 \pm 0.7$, where the last error is purely theoretical, in good agreement with ours, although less precise.

ACKNOWLEDGMENTS

We would like to thank Hans Bijnens and Pere Talavera for providing information on the two-loop functions in Ref. [41]. We are also grateful to Jorge Portolés for his help and useful comments. The work of M. G.-A. is supported by an FPU Grant (MICINN, Spain). This work has been supported in part by the EU RTN network FLAVIAnet [Contract No. MRTN-CT-2006-035482], by MICINN, Spain [Grants No. FPA2007-60323 (M. G.-A., A. P.), No. FPA2006-05294 (J. P.), and Consolider-Ingenio 2010 Programme No. CSD2007-00042-CPAN-] and by Junta de Andalucía (J. P.) [Grants No. P05-FQM 101, No. P05-FQM 467, and No. P07-FQM 03048].

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