PHYSICAL REVIEW D **78,** 116007 (2008)

# Discrete and continuous symmetries in multi-Higgs-doublet models

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(Received 30 September 2008; published 8 December 2008)

We consider the Higgs sector of multi-Higgs-doublet models in the presence of simple symmetries relating the various fields. We construct basis-invariant observables which may in principle be used to detect these symmetries for any number of doublets. A categorization of the symmetries into classes is required, which we perform in detail for the case of two and three Higgs doublets.

DOI: 10.1103/PhysRevD.78.116007 PACS numbers: 11.30.Er, 11.30.Ly, 12.60.Fr, 14.80.Cp

## I. INTRODUCTION

Many features of the standard model of electroweak interactions have been accurately tested. Still, the Higgs sector remains largely unknown. Indeed, even after one scalar particle is directly detected, there may well be further scalars awaiting discovery (as required, for example, by supersymmetry). It is easy to construct an $N$-Higgs-doublet model (NHDM), but the number of parameters in the Higgs potential grows very rapidly with $N$. A generic two-Higgs-doublet model (THDM) has 14 real parameters in the Higgs potential, while the generic three-Higgs-doublet model (3HDM) already has 54 real parameters.

The number of parameters will be reduced if the theory has discrete (or continuous) symmetries relating the various Higgs fields, which we denote by Higgs Family symmetries or HF symmetries. Besides parameter reduction, such symmetries may also be desirable features of a theory in order to preclude flavor changing neutral currents or to explain relations among different observables. This article presents some features of HF symmetries in the NHDM.

Two deceptively simple questions about HF- symmetries arise. First, classifying the symmetries by their impact on the Higgs potential, one would like to know how many distinct classes of symmetries may be implemented. Surprisingly, this turns out to be a rather nontrivial question. Second, a simple basis change among the various Higgs fields alters the Lagrangian but, obviously, not its physical consequences. Signals of HF symmetries should be invariant under these transformations.

The need to seek basis-invariant observables in models with many Higgs was pointed out by Lavoura and Silva [1], and by Botella and Silva [2], stressing applications to $CP$ violation. References [2,3] indicate how to construct basis-invariant quantities in a systematic fashion for any model, including multi-Higgs-doublet models. Work on basis invariance in the THDM was much expanded upon by Davidson and Haber [4], by Gunion and Haber [5,6], and by Haber and O'Neil [7]. Basis invariance in the THDM was also considered in Refs. [8–11]. In particular, Davidson and Haber [4] develop several strategies to construct basis-invariant descriptions of HF symmetries, in the context of the THDM [12]. One of our aims is to extend their work into multi-Higgs systems.

The paper is organized as follows. In Sec. II we introduce our notation and show that a simple HF symmetry may always be reduced to a standard diagonal form through a basis transformation. Then, we turn to the problem of classifying the HF symmetries according to their action on the Higgs potential. We cover the THDM in Sec. III, and we discuss the 3HDM in Sec. IV. In Sec. V we define a set of basis-invariant observables applicable to any NHDM, which may in principle be used in order to identify the presence of HF symmetries. We present our conclusions in Sec. VI. Appendix A includes the implications that the different classes of symmetries of the 3HDM have on the quadratic and quartic coupling coefficients. This provides the fingerprint database against which the basis-invariant observables of Sec. V should be compared.

## II. THE SCALAR SECTOR OF A GENERIC $N$-HIGGS-DOUBLET MODEL

### A. The scalar potential and basis transformations

In this article we follow the notation of Refs. [2–4]. Let us consider a $SU(2) \otimes U(1)$ gauge theory with $N$-Higgs doublets $\Phi_i$, with the same hypercharge 1/2, and with vacuum expectation values

$$\langle \Phi_i \rangle = \begin{pmatrix} 0 \\ v_i/\sqrt{2} \end{pmatrix}. \qquad (1)$$

The index $i$ runs from 1 to $N$, and we use the standard definition for the electric charge, whereby the upper components of the $SU(2)$ doublets are charged and the lower components neutral.

The scalar potential may be written as

$$V_H = Y_{ij}(\Phi_i^\dagger \Phi_j) + Z_{ij,kl}(\Phi_i^\dagger \Phi_j)(\Phi_k^\dagger \Phi_l), \qquad (2)$$

where Hermiticity implies

 

$$Y_{ij} = Y_{ji}^*, \qquad Z_{ij,kl} \equiv Z_{kl,ij} = Z_{ji,lk}^*. \tag{3}$$

The number of independent parameters of this potential are shown in Table I.

The stationarity conditions are

$$[Y_{ij} + 2Z_{ij,kl}v_k^* v_l]v_j = 0 \qquad (\text{for } i = 1, \cdots, N). \tag{4}$$

Multiplying by $v_i^*$ leads to

$$Y_{ij}(v_i^* v_j) = -2Z_{ij,kl}(v_i^* v_j)(v_k^* v_l). \tag{5}$$

We may rewrite the potential in terms of new fields $\Phi_i'$, obtained from the original ones by a simple basis transformation

$$\Phi_i \rightarrow \Phi_i' = U_{ij}\Phi_j, \tag{6}$$

where $U$ is an $N \times N$ unitary matrix. Under this unitary basis transformation, the gauge-kinetic terms remain the same but the coefficients $Y_{ij}$ and $Z_{ij,kl}$ are transformed as

$$Y_{ij} \rightarrow Y_{ij}' = U_{ik}Y_{kl}U_{jl}^*, \tag{7}$$

$$Z_{ij,kl} \rightarrow Z_{ij,kl}' = U_{im}U_{ko}Z_{mn,op}U_{jn}^*U_{lp}^*, \tag{8}$$

and the vacuum expectation values are transformed as

$$v_i \rightarrow v_i' = U_{ij}v_j. \tag{9}$$

Thus, the basis transformations $U$ may be utilized in order to absorb some of the parameters in $Y$ and/or $Z$, meaning that not all parameters in Table I have physical significance.

### B. Higgs Family symmetries

Let us assume that the scalar potential in Eq. (2) has some explicit internal symmetry. That is, we assume that the coefficients of $V_H$ stay *exactly the same* under a transformation

$$\Phi_i \rightarrow \Phi_i^S = S_{ij}\Phi_j. \tag{10}$$

$S$ is a unitary matrix, so that the gauge-kinetic couplings are also left invariant by this HF symmetry. As a result of this symmetry

$$Y_{ij} = Y_{ij}^S = S_{ik}Y_{kl}S_{jl}^*, \tag{11}$$

$$Z_{ij,kl} = Z_{ij,kl}^S = S_{im}S_{ko}Z_{mn,op}S_{jn}^*S_{lp}^*. \tag{12}$$

TABLE I. Number of parameters in the $Y$ and $Z$ coefficients of the Higgs potential.

| | Parameters | Magnitudes | Phases |
|---|---|---|---|
| $Y$ | $N^2$ | $\frac{N(N+1)}{2}$ | $\frac{N(N-1)}{2}$ |
| $Z$ | $\frac{N^2(N^2+1)}{2}$ | $\frac{N^2(N^2+3)}{4}$ | $\frac{N^2(N^2-1)}{4}$ |
| $Y$ and $Z$ | $\frac{N^2(N^2+3)}{2}$ | $\frac{N^4+5N^2+2N}{4}$ | $\frac{N^4+N^2-2N}{4}$ |

Notice that this is *not* the situation considered in Eqs. (6)–(8). There, the coefficients of the Lagrangian *do change*. What we said there was that, although the coefficients do change, the quantities which are physically measurable cannot. What we consider in Eqs. (10)–(12) is different. Here we consider the possibility that $V_H$ has some HF symmetry $S$ which leaves the coefficients unchanged.

We now turn to the complicated interplay between HF symmetries and basis transformations. Let us imagine that, when written in the basis of fields $\Phi_i$, $V_H$ has a symmetry $S$. Then we perform a basis transformation from the basis $\Phi_i$ to the basis $\Phi_i'$, as given by Eq. (6). Clearly, when written in the new basis, $V_H$ does *not* remain invariant under $S$. Rather, it will be invariant under

$$S' = USU^\dagger. \tag{13}$$

As we change the basis, the form of the potential changes in a way which may obscure the presence of an HF symmetry. Equation (13) means that many HF symmetries which might look distinct on the surface, will actually imply exactly the same physical predictions. Any two symmetries $S$ and $S'$ related by Eq. (13), for some basis transformation $U$, will make the same predictions. Now $S$, $S'$, and $U$ are all matrices of the $U(N)$ group, within which Eq. (13) constitutes a conjugacy relation. Thus, Eq. (13) means that HF symmetries associated with matrices $S$ and $S'$ in the same conjugacy class of $U(N)$ correspond to the same model. This result is easy to generalize because an overall phase transformation on $U$ or $S$ has no impact on the potential $V_H$. This can be seen directly from Eqs. (7), (8), (11), and (12), and is due to the fact that the Higgs potential $V_H$ in Eq. (2) only depends on the Higgs fields through bilinear combinations. So, symmetries $S$ and $S'$ belonging to conjugacy classes related by a global phase transformation lead to the same physics.

### C. A special basis

One can show that a $N \times N$ complex matrix $S$ belongs to $U(N)$ if and only if there exists a unitary matrix $U$ such that $S'$ in Eq. (13) is diagonal, with all entries of magnitude 1 [13]. This means that, by a suitable basis transformation, any symmetry $S$ may be brought to the form

$$\begin{pmatrix} e^{i\theta_1} & & & \\ & e^{i\theta_2} & & \\ & & \ddots & \\ & & & e^{i\theta_N} \end{pmatrix}, \tag{14}$$

where $0 \leq \theta_i < 2\pi$ ($i = 1 \ldots N$). The conjugacy classes can thus be classified by matrices of the type in Eq. (14).

In the basis where the symmetry is represented by Eq. (14), the coefficients must obey

$$Y_{ij} = e^{i(\theta_i - \theta_j)}Y_{ij}, \tag{15}$$

$$Z_{ij,kl} = e^{i(\theta_i - \theta_j)} e^{i(\theta_k - \theta_l)} Z_{ij,kl}, \tag{16}$$

where there is no sum over repeated indexes. This result is obtained by substituting the special form of $S$ in Eq. (14) onto Eqs. (11) and (12). The result in Eq. (15) applies not only to the matrix $Y$ but to any matrix whose two indexes transform as $U^\dagger$ and $U$, as shown for $Y$ in Eq. (7). In particular, in this special basis, the matrices

$$Z^{(1)}_{ij} = \sum_k Z_{ik,kj}, \tag{17}$$

$$Z^{(2)}_{ij} = \sum_k Z_{ij,kk}, \tag{18}$$

introduced by Davidson and Haber [4] must also obey a relation like Eq. (15). (Although sum over repeated indexes is assumed unless explicitly stated, we have shown it here explicitly for clarity.)

For symmetries corresponding to $\theta_i \neq \theta_j$ for all $i \neq j$, Eq. (15) implies that the matrix $Y$ is diagonal. If the Higgs potential only had quadratic terms, this information would be useless, since any Hermitian matrix can be diagonalized by a suitable unitary basis change. Said otherwise, without quartic terms in the potential, imposing HF symmetries (or not) would make no difference. Thus, any sign of HF symmetries must necessarily involve the quartic terms. For the special case of the THDM, this can be seen explicitly in Eqs. (39–50) of Ref. [4].

But given two matrices (for example, $A = Y$ and $B = Z^{(1)}$), Eq. (15) already gives crucial information in the case where $\theta_i \neq \theta_j$ for all $i \neq j$. Indeed it states that, in the special basis, $A$ and $B$ are simultaneously diagonal. As a result, their commutator vanishes; $[A, B] = 0$. Now, the commutator is a matrix and the null matrix is always mapped onto the null matrix, regardless of which basis transformation one chooses. Thus, we conclude that symmetries which are represented in the special basis by $\theta_i \neq \theta_j$ (for all $i \neq j$) will lead to the basis-invariant result $[A, B] = 0$. This can be used to define basis-invariant fingerprints of HF symmetries, explaining Eqs. (39–41) of Ref. [4]

One could now ask whether all possible impositions due to HF symmetries can be cast in the form $[A, B] = 0$ for suitably chosen matrices $A$ and $B$. The answer is negative, as Davidson and Haber found when trying to disentangle the Peccei-Quinn [14] symmetry from the usual $Z_2$ symmetry [15][c.f. their Eq. (46)].

### D. Further simplifications due to global phase invariance

Because the Lagrangian is invariant under global phase transformations, there are infinitely many conjugacy classes which imply the same physical predictions. Indeed, classes represented by the diagonal elements

$$e^{i\theta_1} \begin{pmatrix} 1 & & & \\ & e^{i\theta_2} & & \\ & & \ddots & \\ & & & e^{i\theta_N} \end{pmatrix} \tag{19}$$

for fixed values of $\theta_j$ ($j = 2 \ldots N$) lead to the same physical predictions, regardless of the value of $\theta_1$. This means that we can concentrate on symmetries of the type shown in Eq. (19), without the prefactor $e^{i\theta_1}$. Thenceforth, we shall classify each class of symmetries by their diagonal representative:

$$S = \begin{pmatrix} 1 & & & \\ & e^{i\theta_2} & & \\ & & \ddots & \\ & & & e^{i\theta_N} \end{pmatrix}. \tag{20}$$

Alternatively, we could use the $e^{i\theta_1}$ phase freedom in order to restrict our attention to symmetries in $SU(N)$.

A first question now arises: do classes corresponding to different values of $\theta_j$ ($j = 2 \ldots N$) necessarily imply different physical predictions? The answer is negative. A second question arises: can one classify the different types of HF symmetries according to their impact on the Higgs potential? The answer is affirmative, but that must be done separately for each value of $N$. We review the case of $N = 2$ in Sec. III, and we turn to the more difficult case of $N = 3$ in Sec. IV.

## III. HF SYMMETRIES IN THE THDM

### A. Simple symmetries

In the previous sections we learned the following. Two symmetries in the same conjugacy class yield the same physics. Thus, we can go into a special basis and consider a diagonal matrix with complex entries of unit magnitude. In fact, there are infinitely many such diagonal matrices which yield the same physics, because global phase transformations have no impact on the Lagrangian. Therefore, we can concentrate on symmetries of the type

$$S = \begin{pmatrix} 1 & \\ & e^{i\alpha} \end{pmatrix}. \tag{21}$$

Substituting into Eq. (15), we obtain

$$\begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} = \begin{pmatrix} Y_{11} & e^{-i\alpha} Y_{12} \\ e^{i\alpha} Y_{21} & Y_{22} \end{pmatrix}. \tag{22}$$

We conclude that the $Y$ matrix elements come affected by the following phase factors:

$$\begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}. \tag{23}$$

This table is a shorthand notation to keep track of the exponents which appear in Eq. (22). If $S$ is indeed a symmetry of the potential, then these phase factors must equal 0 (mod $2\pi$) for any nonzero value of the corresponding entry in the $Y_{ij}$ matrix. If $\alpha = 0$, we have the uninteresting identity transformation. We shall ignore this possibility henceforth. If $0 < \alpha < 2\pi$, then any matrix in the problem (built from the coefficients in the scalar potential) must be diagonal. This leads to conditions of the type $[A, B] = 0$ discussed above. Notice that this condition does not distinguish $\alpha = \pi$, corresponding to the $Z_2$ symmetry

$$S_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{24}$$

from the transformation with $\alpha = \pi/3$, etc. Accordingly, Davidson and Haber [4] were unable to find a condition to distinguish $Z_2$ from Peccei-Quinn based exclusively on matrix conditions of the type $[A, B] = 0$.

The $Z_2$ symmetry will only be distinguished from the symmetries with other values of $\alpha$ by the quartic terms. Substituting Eq. (21) into Eq. (16), we conclude that the $Z$ matrix elements come affected by the following phase factors:

$$\begin{bmatrix} \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix} & \begin{bmatrix} -\alpha & -2\alpha \\ 0 & -\alpha \end{bmatrix} \\ \begin{bmatrix} \alpha & 0 \\ 2\alpha & \alpha \end{bmatrix} & \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix} \end{bmatrix}. \tag{25}$$

This is represented as a table of tables. The uppermost-leftmost table corresponds to the phases affecting $Z_{11,kl}$. The next table along the same line corresponds to the phases affecting $Z_{12,kl}$, and so on. If $S$ is indeed a symmetry of the potential, then these phase factors must equal 0 (mod $2\pi$) for any nonzero value in the corresponding entry of the $Z_{ij,kl}$ tensor. Unlike what happened for the quadratic terms (and, in general, for any matrix built out of quadratic and/or quartic terms) we see that there is a distinction between two cases, according to whether $2\alpha = 2\pi$ or $2\alpha \neq 0, 2\pi$. If $\alpha = \pi$, then the terms $Z_{12,12}$ and $Z_{21,21}$ (which are related to a parameter denoted by $\lambda_5$ in usual presentations of the THDM) may be different from zero. In contrast, $\lambda_5$ must vanish for symmetries with $\alpha \neq 0, \pi$.

So, the quartic terms do distinguish $S_1$ in Eq. (24) from

$$S_2 = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}_{(\alpha \neq 0, \pi)}. \tag{26}$$

But they do not distinguish among the symmetries

$$S_{2/3} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i2\pi/3} \end{pmatrix}, \qquad S_{2/5} = \begin{pmatrix} 1 & 0 \\ 0 & e^{i2\pi/5} \end{pmatrix}. \tag{27}$$

These symmetries are actually quite curious. Suppose we impose the symmetry $S_{2/3}$ on the Lagrangian. Clearly, applying the symmetry again must also leave the Lagrangian invariant. As a result, the Lagrangian is invariant under $S_{2/3}$, $S_{2/3}^2$, and $S_{2/3}^3 = 1$, which form a closed group. The Lagrangian is always invariant under a group; if it is invariant under symmetries $S_a$ and $S_b$, it is obviously also invariant under $S_a S_b$. And if we choose $\alpha/\pi$ irrational, the group $S_2, S_2^2, S_2^3, \ldots$ will even have an infinite number of elements.

But there is a further important point. We have just shown that if the potential is invariant with respect to a symmetry $S_2$ for some value of $\alpha \neq 0$, $\pi$, then it will necessarily be invariant with respect to a symmetry $S_2$ with any other value of $\alpha$. That is, we have imposed a discrete symmetry but the resulting potential is invariant with respect to a continuous symmetry—the Peccei-Quinn symmetry [14]. This is an important point because continuous symmetries, if broken, imply the presence of massless Goldstone bosons. Suppose we build an NHDM with an innocent-looking discrete symmetry. It may happen that imposing this symmetry has the same effect on the potential as a global symmetry and, thus, the possibility exists for undesired massless scalars. We have just seen one such example. We impose the discrete symmetry $S_{2/3}$ (the corresponding group of symmetries, to be precise) only to find that the resulting potential is invariant under the continuous Peccei-Quinn symmetry.

Notice that the analysis of the quartic terms is sufficient to isolate all cases of interest. Indeed, the uppermost-leftmost $2 \times 2$ block of Eq. (25) coincides with Eq. (23). A similar situation occurs for any other value of $N$.

In conclusion, as far as simple HF symmetries are concerned, we have only three possibilities. Either we have the most general Lagrangian, or we have the $Z_2$ symmetry, or we have the Peccei-Quinn symmetry. This exhausts all simple symmetries. We are not considering here $CP$-type symmetries, and we comment briefly on multiple symmetries in Sec. .

### B. Multiple symmetries

In this article we concentrate on what we call simple symmetries. By this we mean the following: we choose some symmetry $S$ and we impose *only that symmetry* on the Higgs potential; the Higgs potential is the most general consistent with invariance under

$$\Phi_1 \rightarrow S_{11}\Phi_1 + S_{12}\Phi_2, \qquad \Phi_2 \rightarrow S_{21}\Phi_1 + S_{22}\Phi_2. \tag{28}$$

In our notation, a multiple symmetry arises if and only if the Higgs potential is invariant under Eq. (28), and also under

$$\Phi_1 \rightarrow T_{11}\Phi_1 + T_{12}\Phi_2, \qquad \Phi_2 \rightarrow T_{21}\Phi_1 + T_{22}\Phi_2, \tag{29}$$

for some symmetry $T$ which forces new and independent constraints on the Higgs potential.

We recall two points. First, under a basis change, a symmetry $S$ will look different. For example, if we impose the symmetry $\Phi_1 \rightarrow \Phi_1$, $\Phi_2 \rightarrow -\Phi_2$ on some basis, then that symmetry will turn into $\Phi_1' \leftrightarrow \Phi_2'$ if we change into the basis $\Phi_1' = (\Phi_1 + \Phi_2)/\sqrt{2}$, $\Phi_2' = (\Phi_1 - \Phi_2)/\sqrt{2}$. Indeed,

$$D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{30}$$

is the corresponding Eq. (13)

Second, it may be that imposing a symmetry $S$ alone will yield a potential with a larger symmetry. We are not referring only to the obvious possibility that the potential becomes automatically invariant to the group of all powers of $S$ (as exemplified above in connections with $S_{2/3}$). It may be that the potential becomes automatically invariant under other symmetries, such as continuous symmetries, as indeed happens in the THDM.

All the possibilities discussed thus far fall under what we call simple symmetries. But we may have more complicated situations. We may impose simultaneously two symmetries. For example, we may ask that the potential be invariant *both* under $S_1$ in Eq. (24) and $D$ in Eq. (30), *in the same basis*. Notice that now it is irrelevant that $S_1$ and $D$ are in the same conjugacy class when considered individually. We are imposing both in the same basis. Bringing one to diagonal form will make the other off diagonal, and vice-versa. In this case, the potential becomes automatically invariant under the group of four symmetries $S_1$, $D$, $S_1D$, and 1. This case is considered by Davidson and Haber [4] after their Eq. (37). Other multiple symmetries of the THDM were studied by Ivanov [11]. The general analysis of such cases in the NHDM is much more difficult and it is not considered in this article. Indeed, what we dubbed simple symmetries will prove surprisingly demanding, even for $N = 3$.

## IV. HF SYMMETRIES IN THE 3HDM

The analogues of Eqs. (21), (23), and (25), for $N = 3$ are

$$S = \begin{pmatrix} 1 & & \\ & e^{i\alpha} & \\ & & e^{i\beta} \end{pmatrix}, \tag{31}$$

$$\begin{bmatrix} 0 & -\alpha & -\beta \\ \alpha & 0 & \alpha-\beta \\ \beta & \beta-\alpha & 0 \end{bmatrix}, \tag{32}$$

and

$$\begin{bmatrix} \begin{bmatrix} 0 & -\alpha & -\beta \\ \alpha & 0 & \alpha-\beta \\ \beta & \beta-\alpha & 0 \end{bmatrix} & \begin{bmatrix} -\alpha & -2\alpha & -\alpha-\beta \\ 0 & -\alpha & -\beta \\ \beta-\alpha & \beta-2\alpha & -\alpha \end{bmatrix} & \begin{bmatrix} -\beta & -\alpha-\beta & -2\beta \\ \alpha-\beta & -\beta & \alpha-2\beta \\ 0 & -\alpha & -\beta \end{bmatrix} \\ \begin{bmatrix} \alpha & 0 & \alpha-\beta \\ 2\alpha & \alpha & 2\alpha-\beta \\ \alpha+\beta & \beta & \alpha \end{bmatrix} & \begin{bmatrix} 0 & -\alpha & -\beta \\ \alpha & 0 & \alpha-\beta \\ \beta & \beta-\alpha & 0 \end{bmatrix} & \begin{bmatrix} \alpha-\beta & -\beta & \alpha-2\beta \\ 2\alpha-\beta & \alpha-\beta & 2\alpha-2\beta \\ \alpha & 0 & \alpha-\beta \end{bmatrix} \\ \begin{bmatrix} \beta & \beta-\alpha & 0 \\ \alpha+\beta & \beta & \alpha \\ 2\beta & 2\beta-\alpha & \beta \end{bmatrix} & \begin{bmatrix} \beta-\alpha & \beta-2\alpha & -\alpha \\ \beta & \beta-\alpha & 0 \\ 2\beta-\alpha & 2\beta-2\alpha & \beta-\alpha \end{bmatrix} & \begin{bmatrix} 0 & -\alpha & -\beta \\ \alpha & 0 & \alpha-\beta \\ \beta & \beta-\alpha & 0 \end{bmatrix} \end{bmatrix}, \tag{33}$$

respectively.

Let us first look at the impact of the symmetries on the quadratic terms in Eq. (32). One interesting situation is $\alpha = 0$, $\beta \neq 0$. In this situation $\Phi_1$ and $\Phi_2$ have the same transformation, while $\Phi_3$ transforms differently. One could think of $\alpha \neq 0$, $\beta = 0$ as a different situation, but it is not. It is the same as the previous situation, with the interchange of fields 2 and 3. Since such a field permutation corresponds to a basis change (achievable through some unitary matrix $U$), the two situations correspond to exactly the same symmetry viewed in different basis, and lead to the same physics. Now we consider $\alpha \neq 0$, $\beta \neq 0$. The (2, 3) and (3, 2) entries in Eq. (32) show that we must distinguish $\alpha = \beta$ from $\alpha \neq \beta$. But $\alpha = \beta$ corresponds to the symmetry

$$\begin{pmatrix} 1 & & \\ & e^{i\beta} & \\ & & e^{i\beta} \end{pmatrix} = e^{i\beta}\begin{pmatrix} e^{-i\beta} & & \\ & 1 & \\ & & 1 \end{pmatrix}, \tag{34}$$

which, aside from the irrelevant overall phase, is just a $1 \leftrightarrow 3$ permutation of the situation already considered. In conclusion, the quadratic terms in the Higgs potential distinguish among the following two types of symmetries:

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & e^{i\beta} \end{pmatrix}_{(\beta\neq 0)}, \qquad \begin{pmatrix} 1 & & \\ & e^{i\alpha} & \\ & & e^{i\beta} \end{pmatrix}_{(\alpha\neq 0|\beta\neq\alpha,0)}. \tag{35}$$

As happened for $N = 2$, looking at the quartic terms will further open up these classes of symmetries. Said otherwise, two symmetries may have the same impact on the quadratic terms but different impact on the quartic terms. The zeros in Eq. (33) correspond to the entries of the $Z$ tensor which are real. In order to see how the symmetries affect the quartic terms we start by collecting all distinct combinations in Eq. (33): $\alpha$, $2\alpha$, $\beta$, $2\beta$, $\alpha + \beta$, $\alpha - \beta$, $2\alpha - \beta$, $\alpha - 2\beta$, and $2\alpha - 2\beta$. Each may be equal to 0 (mod $2\pi$) or not. We study all possible combinations, making sure that each new class of symmetries found does not correspond to a mere basis transformation of a class considered previously. Proceeding in this fashion, we find that (as far as simple HF symmetries are concerned, and aside from the most general Lagrangian) there are seven types of symmetries having distinct impacts on the Higgs potential:

$$\begin{aligned} S_1 &= \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \\ S_2 &= \begin{pmatrix} 1 & & \\ & 1 & \\ & & e^{i\alpha} \end{pmatrix}_{(\alpha\neq 0,\pi)}, \\ S_3 &= \begin{pmatrix} 1 & & \\ & e^{2i\pi/3} & \\ & & e^{-2i\pi/3} \end{pmatrix}, \\ S_4 &= \begin{pmatrix} 1 & & \\ & i & \\ & & -i \end{pmatrix}, \\ S_5 &= \begin{pmatrix} 1 & & \\ & e^{i\alpha} & \\ & & e^{-i\alpha} \end{pmatrix}_{(\alpha\neq 0,\pi/2,2\pi/3,\pi)}, \\ S_6 &= \begin{pmatrix} 1 & & \\ & -1 & \\ & & e^{i\alpha} \end{pmatrix}_{(\alpha\neq 0,\pi)}, \\ S_7 &= \begin{pmatrix} 1 & & \\ & e^{i\alpha} & \\ & & e^{i\beta} \end{pmatrix}_{(\alpha\neq 0,\pi|\beta\neq\pm\alpha,0,\pi)}. \end{aligned} \tag{36}$$

Comparing with Eq. (35), we see that $S_1$ and $S_2$ have the same impact on the quadratic terms. This is different from the impact of $S_3$–$S_7$ on the quadratic terms. The impact of the symmetries $S_1$–$S_7$ on the coefficients of the Higgs potential is presented in the Appendix.

Recall that the potential is always invariant under a group of symmetries. For example, imposing $S_3$, the potential is automatically invariant under $S_3$, $S_3^2$, and $S_3^3 = 1$. The symmetries $S_2$, $S_5$, and $S_6$ are special in this respect. Imagine that we impose $S_2$ from some value of $\alpha \neq 0, \pi$. Then, the potential is automatically invariant with respect to $S_2$ with any other value for $\alpha$. That is, we wish to impose a discrete symmetry, but the resulting potential turns out to be invariant under a $U(1)$ continuous symmetry. The case for $S_7$ is even worse. Imposing $S_7$ for some chosen numerical values for $\alpha(\neq 0, \pi)$ and $\beta(\neq \pm\alpha, 0, \pi)$ will automatically generate a potential invariant under all symmetries $S_7$ for any values of $\alpha$ and $\beta$.

## V. BASIS-INVARIANT DESCRIPTIONS OF THE SYMMETRY CLASSES IN THE NHDM

Let us look back at Eq. (35). We recall that the analysis of the quadratic terms applies equally well to any matrix (even if built out of the quartic terms, as $Z^{(1)}$ and $Z^{(2)}$). Therefore, a (basis-invariant) commutator type condition will distinguish one symmetry of the first type in Eq. (35) ($S_1 - S_2$, where $[A, B] \neq 0$) from one of the second type ($S_3 - S_7$, where $[A, B] = 0$). But it will not distinguish among two symmetries of the same type in Eq. (35). For example, it will not distinguish $S_3$ from $S_7$, even though they have a different impact on the quartic terms. As a result, commutator conditions, which were so central to Davidson and Haber's study of the THDM [4], have a very limited use for $N \geq 3$.

Basis-invariant fingerprints of the HF symmetries may be found by combining eigenvectors of the matrix of quadratic couplings $Y$ with the quartic couplings $Z$. In the context of the THDM this seemed a curiosity and was left by Davidson and Haber [4] to the end of their Appendix B; Eqs. (B17–B22). Inspired by this remark, we developed a technique which, when suitably extended and interpreted, will become central to our definition of the basis-invariant observables identifying HF symmetries.

Consider a general 3HDM. We define the three eigenvectors of the matrix $Y$ by $\hat{y}^1$, $\hat{y}^2$, and $\hat{y}^3$. For simplicity, we assume that the matrix $Y$ does not have degenerate eigenvalues. The components of the $\hat{y}^\alpha$ eigenvector are denoted by $\hat{y}_i^\alpha$. Under a basis change $U$, the components of the eigenvectors change as

$$\hat{y}_i^{\,\alpha} \longrightarrow U_{ij}\hat{y}_j^\alpha. \tag{37}$$

Combining this with Eq. (8) we see that the quantities

$$I^{\alpha\beta,\gamma\delta} \equiv Z_{ij,kl}(\hat{y}_i^\alpha)^*(\hat{y}_j^\beta)(\hat{y}_k^\gamma)^*(\hat{y}_l^\delta) \tag{38}$$

are basis invariant for any values of $\alpha$, $\beta$, $\gamma$, and $\delta$ between 1 and 3. Therefore, we can evaluate them in any basis. In particular, in a basis where $Y$ is diagonal, we may choose the eigenvectors as

$$\hat{y}^{\,1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \hat{y}^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \hat{y}^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{39}$$

That is, $\hat{y}_i^\alpha = \delta_{\alpha i}$, where $\delta_{\alpha i}$ is the Kronecker symbol. In the basis of Eq. (39), the quantities in Eq. (38) become

$$I^{\alpha\beta,\gamma\delta} \equiv Z_{ij,kl}(\hat{y}_i^\alpha)^*(\hat{y}_j^\beta)(\hat{y}_k^\gamma)^*(\hat{y}_l^\delta) = Z_{\alpha\beta,\gamma\delta} \quad \text{(mod permutations)}. \tag{40}$$

This means that the quantities $I^{\alpha\beta,\gamma\delta}$ (permutations aside) equal the quartic couplings $Z_{\alpha\beta,\gamma\delta}$ calculated in the basis where $Y$ is diagonal. As a result, $I^{\alpha\beta,\gamma\delta}$ has the same symmetries of the $Z$ couplings and, according to Table I, only $N^2(N^2 + 1)/2$ of these are independent.

Before we proceed, we must point out a subtlety concerning Eqs. (38)–(40). Suppose that we have a $Y$ matrix whose eigenvalues are not degenerate, and that we find its three eigenvectors. Now we have a problem in attributing to them the labels 1, 2, and 3. There are six possibilities, which differ by permutations. What one author chooses as

$I^{1,1,1,2}$ may be what another author chooses as $I^{2,2,2,1}$. Once this choice is made, then the quantity is basis invariant. But the choices of different authors may differ by permutations connected with their specific choices for the ordering of the eigenvectors. Equation (40) identifies one possibility; the other five possibilities differ by permutations in the choice of basis eigenvectors in Eq. (39). Thus, permutations must be considered when using the Appendix, as explained below.

We now show how the quantities in Eq. (38) can be used to identify the various discrete symmetries. Let us assume that the Higgs potential is invariant under some symmetry $S_3$–$S_7$; for example $S_3$. The potential may be originally written in a basis where this symmetry is not diagonal. But that is irrelevant; we may always consider what happens in a basis where the symmetry has the form in Eq. (36). In this basis the $Z$ coefficients have the structure (of zero and nonzero entries) presented in the Appendix for the $S_3$ symmetry. Also in this basis, the $Y$ matrix is diagonal. Therefore, its eigenvectors are given by Eq. (39) or some permutation thereof. We conclude from Eq. (40) that the observables in Eq. (38) must fall into the pattern shown in the Appendix which corresponds to $S_3$ (or some permutation thereof).

The discussion in the previous paragraph invoked a special basis, only to show that the presence of a symmetry $S_3 - S_7$ will force the observables of Eq. (38) to fall onto the corresponding pattern shown in the Appendix (or some permutation thereof). But the invariant need not be calculated in this basis. Because it is basis invariant, it can be calculated in any basis whatsoever; the result must be the same. So, the algorithm to identify the presence of a symmetry is straightforward:

(i) We start with the potential in some original basis. The potential has the symmetry $S_i$ in that basis (in general $S_i$ will not have the simple diagonal form when written in that original basis).

(ii) We find the eigenvectors of $Y$ (which, in general, will also not be diagonal in the original basis).

(iii) We combine the $Y$ eigenvectors with $Z$ to calculate the basis-invariant $I^{\alpha\beta,\gamma\delta}$ observables in Eq. (38).

(iv) We check whether the resulting pattern matches the patterns in the Appendix (or some permutation thereof).

This procedure identifies which symmetry we have, even when the potential is written in an original basis where $S_i$ has a very obscure form.

We have postponed the proof that this procedure also works for $S_i$ ($i = 1, 2$) because the basis where this symmetry is diagonal does not guarantee that $Y$ is diagonal. Indeed, in a basis where $S_1$ (say) is diagonal, $Y$ is block diagonal, c.f. the Appendix. In order to make $Y$ diagonal and guarantee that its eigenvectors can be cast in the form of Eq. (39) (aside from permutations), we need a further diagonalization of the uppermost-leftmost $2 \times 2$ block of $Y$. But because the two first eigenvalues of $S_1$ and $S_2$ are degenerate, a unitary $2 \times 2$ rotation on the uppermost-leftmost block has no effect on the form of the symmetry. This shows that a basis may be found where $S_1 - S_2$ have the form in Eq. (36) and $Y$ is diagonal, completing our proof.

Clearly, Eqs. (38) and (40) hold for any value of $N$. As a result, we have succeeded in defining basis-invariant quantities which can *in principle* be utilized in order to identify any HF symmetry in NHDM, for any value of $N$. But in order to perform this identification *in practice* we need to have a set of textures to compare with, as we have done in the Appendix for $N = 3$. The problem of categorizing the different classes of HF symmetries which may affect the Higgs potential (and, thus, the corresponding textures) becomes demanding as $N$ increases. For example, in a cursory analysis of $N = 4$ we have identified at least 15 distinct classes of symmetries.

One final remark concerns the possibility that the matrix $Y$ has degenerate eigenvalues. In this case we must define new $I^{\alpha,\beta,\gamma,\delta}$ parameters invoking the eigenvectors of $Z^{(1)}$ (or $Z^{(2)}$) rather than the eigenvectors of $Y$. Some regions of parameter space may require special care. These types of questions were already present in the various methods proposed for the THDM [4].

An apt analogy to our procedure is the following. We wish to identify a symmetry. The $I^{\alpha\beta,\gamma\delta}$ in Eq. (38) provide us with a (basis-independent) fingerprint of the symmetry. But we can only use this information in order to identify the symmetry, if we have a database with all the distinct fingerprints which may show up, one for each symmetry class. This is what we provide explicitly in the Appendix for $N = 3$. Anyone interested may construct a similar database for $N \geq 4$.

## VI. CONCLUSIONS

We have constructed a set of basis-invariant quantities which may in principle be used to identify the presence of HF symmetries in an NHDM, regardless of the value of $N$. HF symmetries can be classified according to their impact on the Higgs potential. Surprisingly, this classification is already involved for $N = 3$. We have discussed the cases of $N = 2$ and $N = 3$ in detail showing how to combine the $I^{\alpha\beta,\gamma\delta}$ observables with the classification scheme in order to identify any HF symmetry, regardless of the basis in which the Higgs potential may be originally written in. Our basis invariants $I^{\alpha\beta,\gamma\delta}$ may be applied to any other value of $N$, by constructing the database of the classes of symmetries possible for that value of $N$.

This classification is also important because sometimes one imposes a discrete symmetry only to find that the potential becomes automatically invariant under a much larger class of symmetries. These may even be continuous, implying the danger that Goldstone bosons might appear. We provide explicit examples of this problem. This may even be relevant for studies of the fermion sector. For example, Grimus *et al.* discuss very general symmetry realizations of texture zeros in the fermion sector with the help of scalar fields onto which certain discrete symmetries are imposed [16]. When using such techniques, one must inspect also the Higgs potential in some detail, including the symmetry breaking, lest there be undesired massless scalars.

## ACKNOWLEDGMENTS

We are grateful to Owen Brison for useful discussions. The work of P. M. F. is supported in part by the Portuguese *Fundação para a Ciência e a Tecnologia* (FCT) under Contract No. PTDC/FIS/70156/2006. The work of J. P. S. is supported in part by FCT under Contract No. CFTP-Plurianual (U777).

## APPENDIX: COUPLING STRUCTURES FOR THE DIFFERENT CLASSES OF SYMMETRIES

In this appendix we discuss the impact that the seven classes of symmetries identified in the 3HDM have on the coupling constants in the scalar potential. We show the result in the special basis in which the symmetry has one of the diagonal forms in Eq. (36). In Sec. V we show how to turn this information into a basis-invariant fingerprint for the discrete symmetries.

The quadratic couplings distinguish three cases: i) the most general potential, where all entries of $Y_{ij}$ may be nonzero; ii) the potential with one of the symmetries $S_1 - S_2$, where the matrix $Y_{ij}$ is block diagonal and the uppermost-leftmost $2 \times 2$ block is left unconstrained; and iii) the potential with one of the symmetries $S_3 - S_7$, where the matrix $Y_{ij}$ is diagonal. The first case corresponds to three real and three complex parameters (for a sum of nine real variables); the second case corresponds to three real and one complex parameters (five real variables); and the third case corresponds to three real parameters.

To see the impact on the quartic potential, we organize the $Z_{ij,kl}$ tensor into a matrix of matrices. The uppermost-leftmost matrix corresponds to the phases affecting $Z_{11,kl}$. The next matrix along the same line corresponds to the phases affecting $Z_{12,kl}$, and so on. We use the following notation for the various entries

$$\begin{bmatrix} \begin{bmatrix} r_1 & c_1 & c_2 \\ c_1^* & r_4 & c_6 \\ c_2^* & c_6^* & r_5 \end{bmatrix} & \begin{bmatrix} c_1 & c_3 & c_4 \\ r_7 & c_7 & c_8 \\ c_9^* & c_{12} & c_{13} \end{bmatrix} & \begin{bmatrix} c_2 & c_4 & c_5 \\ c_9 & c_{10} & c_{11} \\ r_8 & c_{14} & c_{15} \end{bmatrix} \\ \begin{bmatrix} c_1^* & r_7 & c_9 \\ c_3^* & c_7^* & c_{12}^* \\ c_4^* & c_8^* & c_{13}^* \end{bmatrix} & \begin{bmatrix} r_4 & c_7 & c_{10} \\ c_7^* & r_2 & c_{16} \\ c_{10}^* & c_{16}^* & r_6 \end{bmatrix} & \begin{bmatrix} c_6 & c_8 & c_{11} \\ c_{12}^* & c_{16} & c_{17} \\ c_{14}^* & r_9 & c_{18} \end{bmatrix} \\ \begin{bmatrix} c_2^* & c_9^* & r_8 \\ c_4^* & c_{10}^* & c_{14}^* \\ c_5 & c_{11}^* & c_{15}^* \end{bmatrix} & \begin{bmatrix} c_6^* & c_{12} & c_{14} \\ c_8^* & c_{16}^* & r_9 \\ c_{11}^* & c_{17}^* & c_{18}^* \end{bmatrix} & \begin{bmatrix} r_5 & c_{13} & c_{15} \\ c_{13}^* & r_6 & c_{18} \\ c_{15}^* & c_{18}^* & r_3 \end{bmatrix} \end{bmatrix}, \qquad \text{(A1)}$$

where $r_i$ $(i = 1 \ldots 9)$ are real and $c_i$ $(i = 1 \ldots 18)$ are complex. In the basis of Eq. (36), the symmetries $S_1 - S_7$ set different combinations of $c_i$ to zero but leave the real coefficients $r_i$ unconstrained.

The most general 3HDM has nine real and 18 complex quartic couplings for a total of 45 real variables. Combining with the quadratic parameters, we have 12 real and 21 complex parameters, for a total of 54 real variables (33 magnitudes and 21 phases). However, not all the variables have physical significance due to the possibility of changing basis through any $3 \times 3$ unitary matrix (which can be parametrized with three magnitudes, five relative phases, and one global phase). Thus, the most general 3HDM has 30 magnitudes and 16 phases with physical significance.

If the potential obeys the symmetry $S_1$, in the basis of Eq. (36) the quartic couplings have the following structure,

$$\begin{bmatrix} \begin{bmatrix} r_1 & c_1 & 0 \\ c_1^* & r_4 & 0 \\ 0 & 0 & r_5 \end{bmatrix} & \begin{bmatrix} c_1 & c_3 & 0 \\ r_7 & c_7 & 0 \\ 0 & 0 & c_{13} \end{bmatrix} & \begin{bmatrix} 0 & 0 & c_5 \\ 0 & 0 & c_{11} \\ r_8 & c_{14} & 0 \end{bmatrix} \\ \begin{bmatrix} c_1^* & r_7 & 0 \\ c_3^* & c_7^* & 0 \\ 0 & 0 & c_{13}^* \end{bmatrix} & \begin{bmatrix} r_4 & c_7 & 0 \\ c_7^* & r_2 & 0 \\ 0 & 0 & r_6 \end{bmatrix} & \begin{bmatrix} 0 & 0 & c_{11} \\ 0 & 0 & c_{17} \\ c_{14}^* & r_9 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & r_8 \\ 0 & 0 & c_{14}^* \\ c_5 & c_{11}^* & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & c_{14} \\ 0 & 0 & r_9 \\ c_{11}^* & c_{17}^* & 0 \end{bmatrix} & \begin{bmatrix} r_5 & c_{13} & 0 \\ c_{13}^* & r_6 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \end{bmatrix}. \qquad \text{(A2)}$$

This corresponds to nine (12) real and eight (nine) complex parameters in the quartic couplings (in the scalar potential).

If the potential obeys the symmetry $S_2$, in the basis of Eq. (36) the quartic couplings have the following structure,

$$\begin{bmatrix} \begin{bmatrix} r_1 & c_1 & 0 \\ c_1^* & r_4 & 0 \\ 0 & 0 & r_5 \end{bmatrix} & \begin{bmatrix} c_1 & c_3 & 0 \\ r_7 & c_7 & 0 \\ 0 & 0 & c_{13} \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_8 & c_{14} & 0 \end{bmatrix} \\ \begin{bmatrix} c_1^* & r_7 & 0 \\ c_3^* & c_7^* & 0 \\ 0 & 0 & c_{13}^* \end{bmatrix} & \begin{bmatrix} r_4 & c_7 & 0 \\ c_7^* & r_2 & 0 \\ 0 & 0 & r_6 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_{14}^* & r_9 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & r_8 \\ 0 & 0 & c_{14}^* \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & c_{14} \\ 0 & 0 & r_9 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} r_5 & c_{13} & 0 \\ c_{13}^* & r_6 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \end{bmatrix}. \qquad \text{(A3)}$$

This corresponds to nine (12) real and five (six) complex parameters in the quartic couplings (in the scalar potential).

If the potential obeys the symmetry $S_3$, in the basis of Eq. (36) the quartic couplings have the following structure,

$$\begin{bmatrix} \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_4 & 0 \\ 0 & 0 & r_5 \end{bmatrix} & \begin{bmatrix} 0 & 0 & c_4 \\ r_7 & 0 & 0 \\ 0 & c_{12} & 0 \end{bmatrix} & \begin{bmatrix} 0 & c_4 & 0 \\ 0 & 0 & c_{11} \\ r_8 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & r_7 & 0 \\ 0 & 0 & c_{12}^* \\ c_4^* & 0 & 0 \end{bmatrix} & \begin{bmatrix} r_4 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_6 \end{bmatrix} & \begin{bmatrix} 0 & 0 & c_{11} \\ c_{12}^* & 0 & 0 \\ 0 & r_9 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & r_8 \\ c_4^* & 0 & 0 \\ 0 & c_{11}^* & 0 \end{bmatrix} & \begin{bmatrix} 0 & c_{12} & 0 \\ 0 & 0 & r_9 \\ c_{11}^* & 0 & 0 \end{bmatrix} & \begin{bmatrix} r_5 & 0 & 0 \\ 0 & r_6 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \end{bmatrix}. \tag{A4}$$

This corresponds to nine (12) real and three (three) complex parameters in the quartic couplings (in the scalar potential).

If the potential obeys the symmetry $S_4$, in the basis of Eq. (36) the quartic couplings have the following structure,

$$\begin{bmatrix} \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_4 & 0 \\ 0 & 0 & r_5 \end{bmatrix} & \begin{bmatrix} 0 & 0 & c_4 \\ r_7 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & c_4 & 0 \\ 0 & 0 & 0 \\ r_8 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & r_7 & 0 \\ 0 & 0 & 0 \\ c_4^* & 0 & 0 \end{bmatrix} & \begin{bmatrix} r_4 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_6 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c_{17} \\ 0 & r_9 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & r_8 \\ c_4^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & r_9 \\ 0 & c_{17}^* & 0 \end{bmatrix} & \begin{bmatrix} r_5 & 0 & 0 \\ 0 & r_6 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \end{bmatrix}. \tag{A5}$$

This corresponds to nine (12) real and two (two) complex parameters in the quartic couplings (in the scalar potential).

If the potential obeys the symmetry $S_5$, in the basis of Eq. (36) the quartic couplings have the following structure,

$$\begin{bmatrix} \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_4 & 0 \\ 0 & 0 & r_5 \end{bmatrix} & \begin{bmatrix} 0 & 0 & c_4 \\ r_7 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & c_4 & 0 \\ 0 & 0 & 0 \\ r_8 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & r_7 & 0 \\ 0 & 0 & 0 \\ c_4^* & 0 & 0 \end{bmatrix} & \begin{bmatrix} r_4 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_6 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & r_9 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & r_8 \\ c_4^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & r_9 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} r_5 & 0 & 0 \\ 0 & r_6 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \end{bmatrix}. \tag{A6}$$

This corresponds to nine (12) real and one (one) complex parameter(s) in the quartic couplings (in the scalar potential). The single complex parameter appears in four entries.

If the potential obeys the symmetry $S_6$, in the basis of Eq. (36) the quartic couplings have the following structure,

$$\begin{bmatrix} \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_4 & 0 \\ 0 & 0 & r_5 \end{bmatrix} & \begin{bmatrix} 0 & c_3 & 0 \\ r_7 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_8 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & r_7 & 0 \\ c_3^* & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} r_4 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_6 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & r_9 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & r_8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & r_9 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} r_5 & 0 & 0 \\ 0 & r_6 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \end{bmatrix}. \tag{A7}$$

This corresponds to nine (12) real and one (one) complex parameter(s) in the quartic couplings (in the scalar potential). Unlike what happened for $S_5$, here the single complex parameter appears in two entries.

If the potential obeys the symmetry $S_7$, in the basis of Eq. (36) the quartic couplings have the following structure,

$$\begin{bmatrix} \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_4 & 0 \\ 0 & 0 & r_5 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ r_7 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ r_8 & 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & r_7 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} r_4 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_6 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & r_9 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & r_8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & r_9 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} r_5 & 0 & 0 \\ 0 & r_6 & 0 \\ 0 & 0 & r_3 \end{bmatrix} \end{bmatrix}. \tag{A8}$$

This corresponds to nine (12) real parameters in the quartic couplings (in the scalar potential). All complex parameters in the scalar potential vanish.

[1] L. Lavoura and J. P. Silva, Phys. Rev. D **50**, 4619 (1994).
[2] F. J. Botella and J. P. Silva, Phys. Rev. D **51**, 3870 (1995).
[3] G. C. Branco, L. Lavoura, and J. P. Silva, *CP Violation* (Oxford University Press, Oxford, 1999).
[4] S. Davidson and H. E. Haber, Phys. Rev. D **72**, 035004 (2005); **72**, 099902(E) (2005).
[5] J. F. Gunion and H. E. Haber, Phys. Rev. D **72**, 095002 (2005).
[6] J. F. Gunion, CPNSH, CERN, Switzerland, 2004 (unpublished).
[7] H. E. Haber and D. O'Neil, Phys. Rev. D **74**, 015018 (2006).

[8] G. C. Branco, M. N. Rebelo, and J. I. Silva-Marcos, Phys. Lett. B **614**, 187 (2005).

[9] I. F. Ginzburg and M. Krawczyk Phys. Rev. D **72**, 115013 (2005).

[10] C. C. Nishi, Phys. Rev. D **74**, 036003 (2006); **76**, 055013 (2007); **77**, 055009 (2008).

[11] For a discussion of all types of symmetries (simple, multiple, and *CP*) in the THDM, from the point of view of field bilinears, see I. P. Ivanov, Phys. Rev. D **77**, 015017 (2008). We prefer to discuss the symmetries of the fields themselves, which are most useful for model building.

[12] L. Lavoura, Phys. Rev. D **74**, 015018 (2006), had previously analyzed discrete symmetries in the THDM, but more from the perspective of its possible experimental signals and not aiming at the construction of Lagrangian inspired basis-invariant signals of discrete symmetries.

[13] We are very grateful to Owen Brison for many discussions regarding this question.

[14] R. D. Peccei and H. R. Quinn, Phys. Rev. Lett. **38**, 1440 (1977).

[15] S. L. Glashow and S. Weinberg, Phys. Rev. D **15**, 1958 (1977); E. A. Paschos, Phys. Rev. D **15**, 1966 (1977).

[16] W. Grimus *et al.*, Eur. Phys. J. C **36**, 227 (2004).