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Calculation of f_π and m_π at finite chemical potential

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Based on the previous work in Y. Jiang, Y. M. Shi, H. T. Feng, W. M. Sun, and H. S. Zong, Phys. Rev. C **78**, 025214 (2008) on the quark-meson vertex and pion properties at finite quark chemical potential, we provide an analytical analysis of the weak decay constant of the pion ($f_\pi[\mu]$) and the pion mass ($m_\pi[\mu]$) at finite quark chemical potential using the model quark propagator proposed in R. Alkofer, W. Detmold, C. S. Fischer, and P. Maris, Phys. Rev. D **70**, 014014 (2004). It is found that when μ is below a threshold value μ_0 (which equals 0.350, 0.377, and 0.341 GeV, for the 2CC, 1R1CC, and 3R parametrizations of the model quark propagator, respectively), $f_\pi[\mu]$ and $m_\pi[\mu]$ are kept unchanged from their vacuum values. The value of μ_0 is intimately connected with the pole distribution of the model quark propagator and is found to coincide with the threshold value below which the quark-number density vanishes identically. Numerical calculations show that when μ becomes larger than μ_0, $f_\pi[\mu]$ exhibits a sharp decrease whereas $m_\pi[\mu]$ exhibits a sharp increase. A comparison is given between the results obtained in this paper and those obtained in previous literature.

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The in-medium modification of the properties of the pion is of fundamental interest in hadron physics. The pion is identified as a Goldstone boson arising from the spontaneous breakdown of chiral symmetry which is essential for describing low-energy hadronic phenomena. Since chiral symmetry is expected to be restored at high enough density, the change of pion properties in medium will provide crucial information on the restoration of chiral symmetry. Among these, the weak decay constant of the pion f_π and the pion mass m_π are the two most important quantities, since they are closely related to the spontaneous breakdown of chiral symmetry of quantum chromodynamics (QCD). Unfortunately, so far it has not been possible to obtain detailed information about modification of pion properties in medium directly from QCD. In this situation, different models have been used to study this sort of problem [1–10]. Just as was pointed out in Ref. [11], the pion has a dual role: it can be identified as a quark-antiquark bound state as well as a Goldstone boson arising from the spontaneous breakdown of chiral symmetry. From the point of view that the pion can be regarded as a quark-antiquark bound state, the full dynamical information of the pion is contained in the corresponding Bethe-Salpeter amplitude: $\Gamma_\pi(k, p)$ (k is the relative and p the total momentum of the quark-antiquark pair), which is the one-particle-irreducible, fully amputated quark-meson vertex. The Dyson-Schwinger equations (DSEs) of QCD provide a nonperturbative, continuum framework for analyzing such quark-meson vertices directly [11–15]. The aim of this paper is to study the change of f_π and m_π with quark chemical potential μ in the framework of this nonperturbative QCD model.

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The DSEs of QCD have been used extensively at zero temperature and zero quark chemical potential to extract hadronic observables [12–15]. However, this is very difficult at finite quark chemical potential due to the fact that the number of independent Lorentz structures of the quark-meson vertex at finite μ is much larger than that of the corresponding one at $\mu = 0$. In Ref. [16], using the method of studying the dressed quark propagator at finite μ given in Ref. [17], the authors have given a new approach for tackling this problem. Based on the rainbow-ladder approximation of the DSEs and the assumption of analyticity of the quark-meson vertex in the neighborhood of $\mu = 0$ and neglecting the μ dependence of the dressed gluon propagator, the authors show that the general quark-meson vertex at finite μ can be obtained from the corresponding one at $\mu = 0$ by a shift of variable: $\Gamma[\mu](k, p) = \Gamma(\tilde{k}, p)$, where $\tilde{k} = (\vec{k}, k_4 + i\mu)$. From this result the authors of Ref. [16] numerically calculated $f_\pi[\mu]$ and $m_\pi[\mu]$ for $\mu < 300$ MeV. It is found that $f_\pi[\mu]$ increases slowly (with an increase of less than about 0.01%) and $m_\pi[\mu]$ falls slowly (with a decrease of less than about 0.06%) with increasing μ. Numerically the change of $f_\pi[\mu]$ and $m_\pi[\mu]$ is so small that one can think $f_\pi[\mu]$ and $m_\pi[\mu]$ do not change with μ for $\mu < 300$ MeV within numerical errors. One of our motivations for this work is to explore the mathematical reason behind this. Based on the work in [16], in this paper we provide an analytic analysis of $f_\pi[\mu]$ and $m_\pi[\mu]$. It is found that when μ is below a critical value μ_0, $f_\pi[\mu]$ and $m_\pi[\mu]$ are kept unchanged from their vacuum values. Moreover, numerical calculations show that when μ becomes larger than μ_0, $f_\pi[\mu]$ exhibits a sharp decrease whereas $m_\pi[\mu]$ exhibits a sharp increase.

According to Ref. [16], the pion decay constant at finite μ can be expressed as the following:

$$\delta^{ij} f_\pi[\mu] p_\nu = \int_q \mathrm{tr}\left[\frac{\tau^i}{2}\gamma_5\gamma_\nu S[\mu](q_+) \times \Gamma_\pi^j[\mu](q;p) S[\mu](q_-)\right] = \int_q \mathrm{tr}\left[\frac{\tau^i}{2}\gamma_5\gamma_\nu S(\tilde{q}_+)\Gamma_\pi^j(\tilde{q};p)S(\tilde{q}_-)\right], \quad (1)$$

where $S[\mu](q)$ is the full dressed quark propagator at finite μ and $S(q)$ is the full dressed quark propagator at zero μ, $\tilde{q}_\pm = \tilde{q} \pm p/2$, $\tilde{q} = (\vec{q}, q_4 + i\mu)$, $\frac{\tau^i}{2}$ are the flavor $SU(2)$ generators, and $\int_q \equiv \int d^4q/(2\pi)^4$. In the present paper we will not write the renormalization constants explicitly because one would find that in the final result the renormalization constants cancel each other. In fact, Eq. (1) is the expression of f_π which is independent of the renormalization point and the regularization mass scale [11]. Before proceeding we shall give some remarks concerning the range of validity of Eq. (1). In the calculation of $f_\pi[\mu]$ in this work, we shall make use of the following two relations:

$$S[\mu](q_\pm) = S[\tilde{q}_\pm], \qquad \Gamma_\pi^j[\mu](q;p) = \Gamma_\pi^j(\tilde{q};p).$$

In deriving these two relations we have assumed that μ lies within the circle of convergence of Taylor expansion of $S[\mu](q_\pm)$ [or $\Gamma_\pi^j[\mu](q;p)$] around $\mu = 0$ (for details, see Refs. [16,17]). But mathematically the relation $S[\mu]\times(q_\pm) = S[\tilde{q}_\pm]$ [or $\Gamma_\pi^j[\mu](q;p) = \Gamma_\pi^j(\tilde{q};p)$] in fact holds in the whole domain of analyticity of $S[\mu](q_\pm)$ [or $\Gamma_\pi^j[\mu](q;p)$] in the complex μ plane, not only within the circle of convergence of μ expansion. This is a result of a well-known theorem in complex analysis [18]: Suppose each of two functions $f(z)$ and $g(z)$ is analytic in a common domain D. If $f(z)$ and $g(z)$ coincide in some subportion $D' \subset D$, then $f(z) = g(z)$ everywhere in D. So, under the assumptions and approximations we have taken (adopting the rainbow approximation of the DSEs, assuming the analyticity of the dressed quark propagator and the quark-meson vertex in the neighborhood of $\mu = 0$, and neglecting the μ dependence of the dressed gluon propagator), Eq. (1) is valid in the whole range of μ. However, physically, when μ is large enough, it is not a good approximation to neglect the μ dependence of the dressed gluon propagator. Therefore when one uses Eq. (1) to calculate $f_\pi[\mu]$ at very large μ, one should be cautious.

The integral of the right-hand side of Eq. (1) can be rewritten as

$$\int_q \equiv \int \frac{d^4q}{(2\pi)^4} \equiv \int \frac{d^4\tilde{q}}{(2\pi)^4}. \quad (2)$$

Contracting both sides of Eq. (1) with p_ν and using Eq. (2), we obtain the following:

$$\delta^{ij} f_\pi[\mu] = \frac{1}{p^2}\int_{-\infty}^{+\infty}\frac{d^3\vec{q}}{(2\pi)^3} \times \int_{-\infty+i\mu}^{+\infty+i\mu}\frac{dq_4}{(2\pi)}\mathrm{tr}\left[\frac{\tau^i}{2}\gamma_5 \not{p} S(q_+)\Gamma_\pi^j(q;p)S(q_-)\right] = \frac{1}{p^2}\int_{-\infty}^{+\infty}\frac{d^3\vec{q}}{(2\pi)^3} \times \int_{C_1}\frac{dq_4}{(2\pi)}\mathrm{tr}\left[\frac{\tau^i}{2}\gamma_5 \not{p} S(q_+)\Gamma_\pi^j(q;p)S(q_-)\right], \quad (3)$$

where the integration path C_1 is depicted in Fig. 1.

Let us use $z_n = \chi_n + i\omega_n$ ($\omega_n > 0$), $n = 1, 2 \cdots$ to denote the poles of the function

$$F^{ij}(q_4) \equiv \frac{1}{p^2}\mathrm{tr}\left[\frac{\tau^i}{2}\gamma_5 \not{p} S(q_+)\Gamma_\pi^j(q;p)S(q_-)\right] \quad (4)$$

located in the upper half complex q_4 plane. According to Cauchy's theorem we obtain the following from Eq. (3):

$$\begin{aligned}\delta^{ij} f_\pi[\mu] &= \frac{1}{p^2}\int_{-\infty}^{+\infty}\frac{d^3\vec{q}}{(2\pi)^3}\int_{C_1}\frac{dq_4}{(2\pi)}\mathrm{tr}\left[\frac{\tau^i}{2}\gamma_5 \not{p} S(q_+)\Gamma_\pi^j(q;p)S(q_-)\right] \\ &= \frac{1}{p^2}\int_{-\infty}^{+\infty}\frac{d^3\vec{q}}{(2\pi)^3}\int_{C_0}\frac{dq_4}{(2\pi)}\mathrm{tr}\left[\frac{\tau^i}{2}\gamma_5 \not{p} S(q_+)\Gamma_\pi^j(q;p)S(q_-)\right] - i\int_{-\infty}^{+\infty}\frac{d^3\vec{q}}{(2\pi)^3}\sum_n \theta(\mu - \omega_n)\mathrm{Res}\{F^{ij}(z); z_n\} \\ &= \delta^{ij} f_\pi - i\int_{-\infty}^{+\infty}\frac{d^3\vec{q}}{(2\pi)^3}\sum_n \theta(\mu - \omega_n)\mathrm{Res}\{F^{ij}(z); z_n\}. \end{aligned} \quad (5)$$

From Eq. (5) it is easily seen that when $\mu < \min\{\omega_n\}$, the function $F^{ij}(q_4)$ has no pole in the region Ω (the region enclosed by C_1 and C_0, see Fig. 1), and therefore $f_\pi[\mu] = f_\pi$, which means that for small enough μ the pion decay constant should be independent of μ. Of course, when $\mu > \min\{\omega_n\}$ the pion decay constant can have an explicit μ dependence.

In the chiral limit, expanding the trace term of the right-hand side of Eq. (4) to $\mathcal{O}(p^2)$ near $p = 0$ [12], we have the following:

$$F^{ij}(q_4) = \frac{1}{p^2}\mathrm{tr}\left\{\frac{\tau^i}{2}\gamma_5 \not{p}\left[S + \frac{1}{2}p\cdot\partial S\right][\Gamma_\pi^j(q,0) + \mathcal{O}(p)\gamma_5]\left[S - \frac{1}{2}p\cdot\partial S\right]\right\}, \quad (6)$$

where we have adopted the approximation [12]

$$\Gamma_\pi^j(q,p) = \Gamma_\pi^j(q,0) + \mathcal{O}(p)\gamma_5. \quad (7)$$

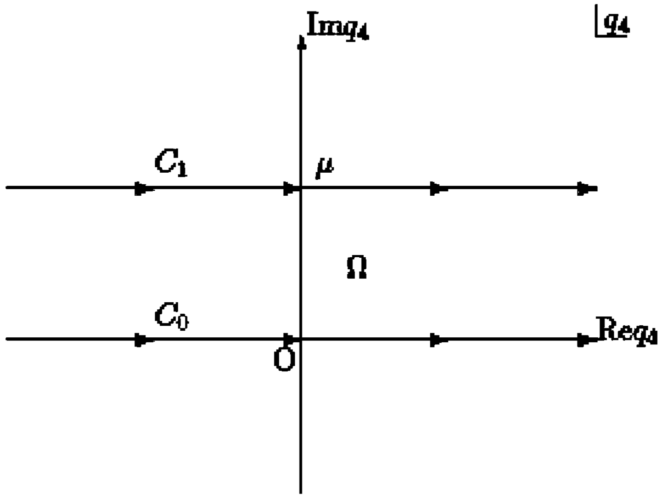

FIG. 1. The integration path in the complex q_4 plane.

With this approximation $\Gamma_\pi^j(q,0)$ can be expressed as [12,19]

$$\Gamma_\pi^j(q,0) = \tau^j\gamma_5 \cdot \frac{iB(q^2)}{f_\pi}, \tag{8}$$

where $B(q^2)$ is the scalar part of $S^{-1}(q)$. Noticing that $\mathrm{tr}[\gamma_5 \not{p} S\gamma_5 S] = 0$, we obtain the following:

$$F^{ij}(q_4) = \frac{1}{p^2}\,\mathrm{tr}\left\{\frac{\tau^i}{4}\gamma_5\not{p}[p\cdot\partial S\Gamma_\pi^j(q,0)S - S\Gamma_\pi^j(q,0)p\cdot\partial S]\right\} + \mathcal{O}(p). \tag{9}$$

Substituting Eq. (8) into Eq. (9) and using $\mathrm{tr}(\tau^i\tau^j) = 2\delta^{ij}$, we obtain

$$F^{ij}(q_4) \simeq \frac{1}{2p^2}\delta^{ij}\frac{iB(q^2)}{f_\pi}\,\mathrm{tr}\{\gamma_5\not{p}[p\cdot\partial S\gamma_5 S - S\gamma_5 p\cdot\partial S]\}. \tag{10}$$

The dressed quark propagator at $\mu = 0$ can be written as

$$S(q) = \frac{1}{i\not{q}A(q^2) + B(q^2)} = -i\not{q}\sigma_v(q^2) + \sigma_s(q^2), \tag{11}$$

which contains two independent Lorentz structures. The introduction of a nonzero μ to Euclidean QCD breaks the original $O(4)$ symmetry of the theory to $O(3)$ symmetry. In this case, the most general form of the dressed quark propagator reads (due to the presence of the medium)

$$S^{-1}[\mu](p) = i\vec{\gamma}\cdot\vec{p}\mathcal{A}(p^2, u\cdot p) - \mu\gamma_4\mathcal{C}(p^2, u\cdot p) + \mathcal{B}(p^2, u\cdot p) - \mu\gamma_4\vec{\gamma}\cdot\vec{p}\mathcal{D}(p^2, u\cdot p),$$

where u_μ denotes the relative velocity of the medium which in the rest frame of the medium can be written as $u_\mu = (\vec{0}, 1)$. Therefore, at finite μ the dressed quark propagator contains four independent Lorentz structures. However, based on the rainbow approximation of DSEs and the assumption of the analyticity of the dressed quark propagator at $\mu = 0$ and neglecting the μ dependence of the dressed gluon propagator, one can show [17] that the dressed quark propagator at finite μ is obtained from the $\mu = 0$ propagator by a simple shift of variable $q_4 \to q_4 + i\mu$: $S[\mu](q) = S(\tilde{q}) = -i\tilde{\not{q}}\sigma_v(\tilde{q}^2) + \sigma_s(\tilde{q}^2)$ and thus only two Lorentz structures may be used here. From Eqs. (10) and (11), we obtain

$$F^{ij}(q_4) \simeq \frac{1}{2}\delta^{ij}\frac{1}{f_\pi}\frac{8\sigma_s}{\sigma_v^2 q^2 + \sigma_s^2}\left[\sigma_s\sigma_v + \frac{2(p\cdot q)^2}{p^2}\times(\sigma_s\sigma_v' - \sigma_s'\sigma_v)\right] \tag{12}$$

$$= \delta^{ij}\frac{1}{f_\pi}F(q_4), \tag{13}$$

where $'$ means d/dq^2 and

$$F(q_4) \equiv \frac{4\sigma_s}{\sigma_v^2 q^2 + \sigma_s^2}\left[\sigma_s\sigma_v + 2\frac{(p\cdot q)^2}{p^2}\times(\sigma_s\sigma_v' - \sigma_s'\sigma_v)\right]. \tag{14}$$

Then Eq. (5) can be written as

$$f_\pi[\mu] \simeq f_\pi - \frac{i}{f_\pi}\int_{-\infty}^{+\infty}\frac{d^3\vec{q}}{(2\pi)^3}\sum_n \theta(\mu - \omega_n)\mathrm{Res}\{F(z); z_n\}. \tag{15}$$

To determine the pole distribution of function $F(q_4)$, we should first specify the form of the dressed quark propagator. Here, as in Refs. [16,20] we adopt the following propagator proposed in Ref. [21]:

$$S(q) = \sum_{j=1}^{n_P}\left(\frac{r_j}{i\not{q} + a_j + ib_j} + \frac{r_j}{i\not{q} + a_j - ib_j}\right). \tag{16}$$

The propagator of this form has n_P pairs of complex conjugate poles located at $a_j \pm ib_j$. When some b_j is set to zero, the pair of complex conjugate poles degenerates to a real pole. The restrictions of the parameters r_j, a_j, and b_j in the chiral limit are [21]

$$\sum_{j=1}^{n_P} r_j = \frac{1}{2}, \tag{17}$$

$$\sum_{j=1}^{n_P} r_j a_j = 0. \tag{18}$$

If we are not in the chiral limit, the right-hand side of Eq. (18) should be replaced by the current quark mass. The value of these parameters is shown in Table I, where 2CC, 1R1CC, and 3R stand for three meromorphic forms of the quark propagator, respectively: two pairs of complex conjugate poles, one real pole and one pair of complex conjugate poles, three real poles.

TABLE I. The parameters used in the calculation of $F(q_4)$ and f_π. These parameters are taken directly from Ref. [21].

Parametrization	r_1	a_1 (GeV)	b_1 (GeV)	r_2	a_2 (GeV)	b_2 (GeV)	r_3	a_3 (GeV)
2CC	0.360	0.351	0.08	0.140	−0.899	0.436	⋯	⋯
1R1CC	0.354	0.377	⋯	0.146	−0.91	0.45	⋯	⋯
3R	0.365	0.341	⋯	1.2	−1.31	⋯	-1.06	-1.40

Without losing generality we assume $p_\nu = (\vec{0}, p)$ (i.e. the pion is at rest) and write

$$\frac{(p\cdot q)^2}{p^2} = \frac{q_4^2 p^2}{p^2} = q_4^2. \tag{19}$$

Now let us calculate $F(q_4)$. With the quark propagator given in Eq. (16) we can obtain

$$F(q_4) = \frac{\Xi(q_4^2)}{\prod_j [q^2 + (a_j + ib_j)^2]^2 [q^2 + (a_j - ib_j)^2]^2 \prod_k (q^2 + \eta_k^2)}, \tag{20}$$

where Ξ is a polynomial of q_4^2 [for the detailed calculation of $F(q_4)$, Ξ, and η_k, see the Appendix]. The values of η_k are shown in Table II (η_k are ordered from small to large according to their real part).

Here it should be noticed that when some $b_j = 0$ (the quark propagator has a real pole), some η_k must exactly equal the corresponding $|a_j|$ (see the Appendix). For 1R1CC case, $b_1 = 0$ and $\eta_1 = |a_1|$. For 3R case, all $b_j = 0$ and $\eta_1 = |a_1|$, $\eta_3 = |a_2|$, $\eta_4 = |a_3|$. For 2CC case, because $b_1 = 0.08$ GeV is very close to zero, the value of η_1 is very close to a_1.

Because $q^2 = q_4^2 + \vec{q}^2$, according to Eq. (20) the poles of $F(q_4)$: $z_n = \chi_n + i\omega_n$ are decided by the following equation:

$$(\chi_n + i\omega_n)^2 + \vec{q}^2 + (\xi_{nR} + i\xi_{nI})^2 = 0, \tag{21}$$

where ξ_{nR} and ξ_{nI} are the real and imaginary part of η_k or $a_j \pm ib_j$. One can easily find

$$\omega_n = \sqrt{\frac{(\vec{q}^2 + \xi_{nR}^2 - \xi_{nI}^2) + \sqrt{(\vec{q}^2 + \xi_{nR}^2 - \xi_{nI}^2)^2 + 4\xi_{nR}^2\xi_{nI}^2}}{2}}, \tag{22}$$

$$\chi_n = -\frac{\xi_{nR}\xi_{nI}}{\omega_n}. \tag{23}$$

From Eq. (22) we find that for $\mu < |\xi_{nR}|$ the corresponding ω_n is always larger than μ, irrespective of $\vec{q}$. For $\mu > |\xi_{nR}|$, $\omega_n < \mu$ when $\vec{q}^2 < \mu^2 - (\xi_{nR}^2\xi_{nI}^2/\mu^2) - \xi_{nR}^2 + \xi_{nI}^2$, and $\omega_n > \mu$ when $\vec{q}^2 > \mu^2 - (\xi_{nR}^2\xi_{nI}^2/\mu^2) - \xi_{nR}^2 + \xi_{nI}^2$. Therefore Eq. (15) can be written as

$$f_\pi[\mu] = f_\pi - \frac{i}{2\pi^2 f_\pi}\sum_n \theta(\mu - |\xi_{nR}|) \times \int_0^{\Lambda_n(\mu)} d|\vec{q}|\vec{q}^2\, \mathrm{Res}\{F(z); z_n\}, \tag{24}$$

where

$$\Lambda_n(\mu) = \sqrt{\mu^2 - (\xi_{nR}^2\xi_{nI}^2/\mu^2) - \xi_{nR}^2 + \xi_{nI}^2}. \tag{25}$$

From Eq. (24) and the values of a_j, b_j, and η_k in Tables I and II we find that when μ is below some threshold value μ_0, the pion decay constant at finite chemical potential $f_\pi[\mu]$ is kept unchanged from its vacuum value. The threshold value μ_0, which equals the minimum of the real part of $a_j \pm ib_j$ and η_k, is shown in Table III.

Here we note that in Ref. [22] it is found that when μ is below the same threshold value μ_0, the quark-number density vanishes identically. Namely, $\mu = \mu_0$ is a singularity which separates two regions with different quark-number densities. In fact, in Ref. [23], based on a universal argument, it is pointed out that the existence of some singularity at the point $\mu = \mu_0$ and $T = 0$ is a robust and model-independent prediction. Below $\mu = \mu_0$, the QCD system at finite μ remains in the vacuum (ground state) of QCD at $\mu = 0$, so the properties of the Goldstone boson excited from this vacuum does not change with μ. Thus the result that $f_\pi[\mu]$ is kept unchanged from its vacuum value is just to be expected. Here it should also be noticed that in our method the value of μ_0 is intimately connected with the pole distribution of the quark propagator.

In Ref. [20], with the same quark propagator the authors find that the quark condensate at finite chemical potential is kept unchanged from its vacuum value when $\mu < \mu_0$. From the Gell-Mann-Oakes-Renner relation $f_\pi^2[\mu]m_\pi^2[\mu] = 2m\langle\bar{q}q\rangle_0[\mu] + \mathcal{O}(m^2)$ [11,16] (where $m_\pi[\mu]$ is the pion mass at finite μ, m is the current quark mass and $\langle\bar{q}q\rangle_0[\mu]$ is the quark condensate in the chiral limit at finite μ) one would also conclude that $m_\pi[\mu]$ is kept unchanged from its value at $\mu = 0$ when $\mu < \mu_0$. In Ref. [16], the authors did not made an analytical analysis of $f_\pi[\mu]$ and $m_\pi[\mu]$ by the method of pole analysis, but instead made a direct numerical calculation. There exist numerical errors in this calculation. Within numerical errors $f_\pi[\mu]$ and $m_\pi[\mu]$ do not change with μ for $\mu <$ 300 MeV. The analytical analysis made in this paper explains the numerical results obtained in [16]. This result is quite different from the result in previous literature. For

TABLE II. The calculated values of η_k.

Parametrization	η_1 (GeV)	η_2 (GeV)	η_3 (GeV)	η_4 (GeV)	η_5 (GeV)
2CC	0.350	$0.723 - 0.351i$	$0.723 + 0.351i$	...	...
1R1CC	0.377	$0.723 - 0.328i$	$0.723 + 0.328i$	...	...
3R	0.341	0.617	1.31	1.40	1.849

TABLE III. The calculated values of μ_0.

Parametrization	μ_0 (GeV)
2CC	0.350
1R1CC	0.377
3R	0.341

example, in a recent work [7], the authors also investigated f_π and m_π at finite density within the framework of the nonlocal quark model from the instanton vacuum. Their results show that in the range $0 \leq \mu \leq 320$ MeV, f_π falls slowly whereas m_π increases slowly. This behavior of f_π and m_π is qualitatively different from that found in this paper.

In [16], we only calculated $f_\pi[\mu]$ and $m_\pi[\mu]$ for $\mu <$ 300 MeV, and in the present work we try to go beyond $\mu \sim 300$ MeV. For $\mu > \mu_0$ one can calculate $f_\pi[\mu]$ and $m_\pi[\mu]$ numerically based on Eq. (24) and the Gell-Mann-Oakes-Renner relation. As was pointed out earlier in this paper, one should be cautious when using our approach to calculate $f_\pi[\mu]$ at large μ. However, let us now assume that the calculation based on Eq. (24) in the vicinity of $\mu = \mu_0$ is still reasonable and see what results it will give. The results for the behaviors of $f_\pi[\mu]$ and $m_\pi[\mu]$ for $\mu > \mu_0$ are shown in Figs. 2 and 3. One sees that $f_\pi[\mu]$ exhibits a sharp decrease whereas $m_\pi[\mu]$ exhibits a sharp increase near μ_0 for all three cases.

Finally, we should emphasize that in obtaining our results about $f_\pi[\mu]$, $m_\pi[\mu]$, and $\langle\bar{q}q\rangle_0[\mu]$ in this paper, we have made these approximations and assumptions: (1) we adopt the rainbow-ladder approximation of the DSEs; (2) we assume the quark propagator and quark-meson vertex are analytic in the neighborhood of $\mu = 0$; (3) we have neglected the μ dependence of the dressed gluon propagator (for a discussion about these approximations and assumptions, see Ref. [16]). For further study one should consider improvements on these approximations.

To summarize, based on the previous work in Ref. [16] on the quark-meson vertex and pion properties at finite quark chemical potential, we provide an analytical analysis of the weak decay constant of the pion ($f_\pi[\mu]$) and the pion mass ($m_\pi[\mu]$) at finite quark chemical potential using the model quark propagator proposed in Ref. [21]. It is found that when μ is below a threshold value μ_0 (which equals 0.350 GeV, 0.377 GeV, and 0.341 GeV, for the 2CC, 1R1CC, and 3R parametrizations of the model quark propagator, respectively), $f_\pi[\mu]$ and $m_\pi[\mu]$ are kept unchanged from their vacuum values. The value of μ_0 is intimately connected with the pole distribution of the model quark propagator and is found to coincide with the threshold value below which the quark-number density vanishes identically. Numerical calculations show that when μ becomes larger than μ_0, $f_\pi[\mu]$ exhibits a sharp decrease whereas $m_\pi[\mu]$ exhibits a sharp increase. These results are quite different from those obtained in previous literature. For example, our results are qualitatively different from those reported in Ref. [7], which uses the nonlocal chiral quark model from the instanton vacuum to investigate f_π and m_π at finite density.

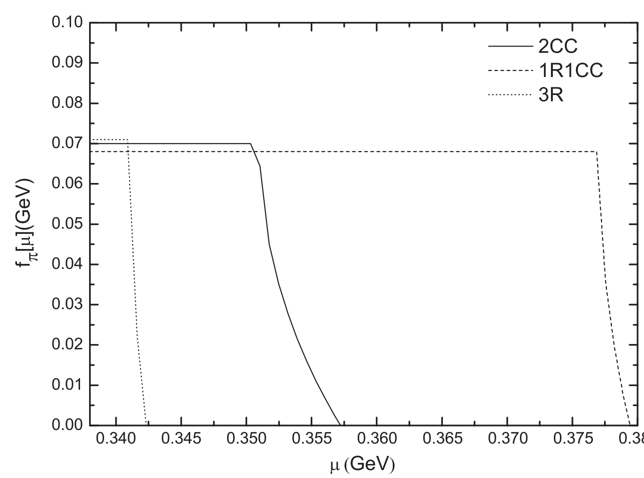

FIG. 2. The μ dependence of f_π near μ_0.

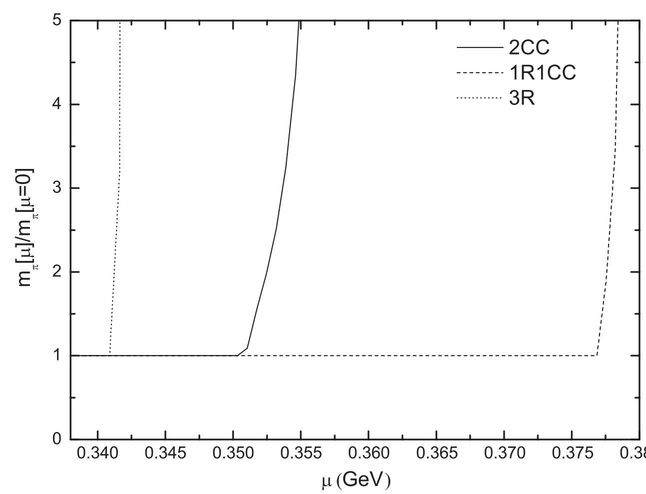

FIG. 3. The μ dependence of m_π near μ_0.

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APPENDIX A: THE ANALYSIS OF THE POLES

1. General analysis

With the quark propagator given by Eq. (16) one can find the following:

$$\sigma_v = \sum_j \left[\frac{r_j}{q^2 + (a_j + ib_j)^2} + \frac{r_j}{q^2 + (a_j - ib_j)^2}\right] = \frac{f_v}{f_0}, \tag{A1}$$

$$\sigma_s = \sum_j \left[\frac{r_j(a_j + ib_j)}{q^2 + (a_j + ib_j)^2} + \frac{r_j(a_j - ib_j)}{q^2 + (a_j - ib_j)^2}\right] = \frac{f_s}{f_0}, \tag{A2}$$

with

$$f_v = \sum_j 2r_j(q^2 + a_j^2 - b_j^2)\prod_{k\neq j}[q^2 + (a_k + ib_k)^2] \times [q^2 + (a_k - ib_k)^2], \tag{A3}$$

$$f_s = \sum_j 2r_j a_j(q^2 + a_j^2 + b_j^2)\prod_{k\neq j}[q^2 + (a_k + ib_k)^2] \times [q^2 + (a_k - ib_k)^2], \tag{A4}$$

$$f_0 = \prod_j [q^2 + (a_j + ib_j)^2][q^2 + (a_j - ib_j)^2]. \tag{A5}$$

Then one has

$$F(q_4) = \frac{4\sigma_s}{\sigma_v^2 q^2 + \sigma_s^2}[\sigma_s\sigma_v + 2q_4^2(\sigma_s\sigma_v' - \sigma_s'\sigma_v)] = \frac{1}{f_0}\frac{\Xi}{f_v^2 q^2 + f_s^2}, \tag{A6}$$

where

$$\Xi = 4f_s[f_s f_v + 2(q^2 - \vec{q}^2)(f_s f_v' - f_s' f_v)]. \tag{A7}$$

For convenience let us use $x^2 = q^2$ with x a complex number. Then the denominator of the right-hand side of Eq. (A6) can be decomposed as

$$f_v^2 x^2 + f_s^2 = (f_v x + if_s)(f_v x - if_s). \tag{A8}$$

f_v and f_s can be expressed as

$$f_v = \sum_j r_j\left[\frac{f_0}{x^2 + (a_j + ib_j)^2} + \frac{f_0}{x^2 + (a_j - ib_j)^2}\right], \tag{A9}$$

$$f_s = \sum_j r_j\left[\frac{f_0(a_j + ib_j)}{x^2 + (a_j + ib_j)^2} + \frac{f_0(a_j - ib_j)}{x^2 + (a_j - ib_j)^2}\right], \tag{A10}$$

so one has the following:

$$\begin{aligned}(f_v x + if_s)(f_v x - if_s) &= \left\{\sum_j r_j f_0\left[\frac{x + i(a_j + ib_j)}{x^2 + (a_j + ib_j)^2} + \frac{x + i(a_j - ib_j)}{x^2 + (a_j - ib_j)^2}\right]\right\} \\ &\quad\times\left\{\sum_j r_j f_0\left[\frac{x - i(a_j + ib_j)}{x^2 + (a_j + ib_j)^2} + \frac{x - i(a_j - ib_j)}{x^2 + (a_j - ib_j)^2}\right]\right\} \\ &= \left\{\sum_j r_j f_0\left[\frac{1}{x - i(a_j + ib_j)} + \frac{1}{x - i(a_j - ib_j)}\right]\right\}\left\{\sum_j r_j f_0\left[\frac{1}{x + i(a_j + ib_j)} + \frac{1}{x + i(a_j - ib_j)}\right]\right\}.\end{aligned} \tag{A11}$$

f_0 can be expressed as

$$f_0 = \prod_{k_1}[x + i(a_{k_1} + ib_{k_1})][x + i(a_{k_1} - ib_{k_1})]\prod_{k_2}[x - i(a_{k_1} + ib_{k_1})][x - i(a_{k_1} - ib_{k_1})]. \tag{A12}$$

Therefore one obtains

$$\begin{aligned}\sum_j r_j f_0\left[\frac{1}{x - i(a_j + ib_j)} + \frac{1}{x - i(a_j - ib_j)}\right] &= \sum_j \{r_j[x - i(a_j - ib_j) + x - i(a_j + ib_j)]\prod_{k_1}[x + i(a_{k_1} + ib_{k_1})] \\ &\quad\times[x + i(a_{k_1} - ib_{k_1})]\prod_{k_2\neq j}[x - i(a_{k_2} + ib_{k_2})][x - i(a_{k_2} - ib_{k_2})]\} \\ &= \prod_{k_1}[x + i(a_{k_1} + ib_{k_1})][x + i(a_{k_1} - ib_{k_1})]\sum_j 2r_j(x - ia_j) \\ &\quad\times\prod_{k\neq j}[x - i(a_k + ib_k)][x - i(a_k - ib_k)]\end{aligned} \tag{A13}$$

and

$$\sum_j r_j f_0\left[\frac{1}{x+i(a_j+ib_j)}+\frac{1}{x+i(a_j-ib_j)}\right]=\sum_j\{r_j[x+i(a_j-ib_j)+x+i(a_j+ib_j)]\prod_{k_1\neq j}[x+i(a_{k_1}+ib_{k_1})]\times[x+i(a_{k_1}-ib_{k_1})]\prod_{k_2}[x-i(a_{k_2}+ib_{k_2})][x-i(a_{k_2}-ib_{k_2})]\}$$
$$=\prod_{k_2}[x-i(a_{k_2}+ib_{k_2})][x-i(a_{k_2}-ib_{k_2})]\sum_j 2r_j(x+ia_j)\times\prod_{k\neq j}[x+i(a_k+ib_k)][x+i(a_k-ib_k)]. \quad \text{(A14)}$$

With Eq. (A12) one can find the following:

$$f_v^2x^2+f_s^2=(f_vx+if_s)(f_vx-if_s)=f_0\left\{\sum_j 2r_j(x-ia_j)\prod_{k\neq j}[x-i(a_k+ib_k)]\times[x-i(a_k-ib_k)]\right\}\left\{\sum_j 2r_j(x+ia_j)\times\prod_{k\neq j}[x+i(a_k+ib_k)][x+i(a_k-ib_k)]\right\}. \quad \text{(A15)}$$

Hence, in order to determine the poles of $F(q_4)$, one should solve the following three equations:

$$f_0=\prod_j[x^2+(a_j+ib_j)^2][x^2+(a_j-ib_j)^2]=0, \quad \text{(A16)}$$

$$\sum_j 2r_j(x-ia_j)\prod_{k\neq j}[x-i(a_k+ib_k)][x-i(a_k-ib_k)]=0, \quad \text{(A17)}$$

$$\sum_j 2r_j(x+ia_j)\prod_{k\neq j}[x+i(a_k+ib_k)][x+i(a_k-ib_k)]=0. \quad \text{(A18)}$$

Here it should be noted that if some $b_j=0$, then $x=ia_j$ (or $x=-ia_j$) must be the solution of Eq. (A17) [or Eq. (A18)]. One should also be aware that after finding the roots of the above equations one should substitute them into Ξ to ensure that $\Xi(x)\neq 0$ (we will see it in the discussion of 1R1CC and 3R case below). For general n_P Eq. (A17) [or Eq. (A18)] is an equation of degree $2n_P-1$ in x, and it is almost impossible to give the analytic form of the solution for general r_j, a_j, and b_j when $n_P\geq 2$.

2. Detailed calculation of the poles

For 2CC case one can find the following:

$$f_v=q^6+d_{v1}q^4+d_{v2}q^2+d_{v3}, \quad \text{(A19)}$$

$$f_s=d_{s1}q^4+d_{s2}q^2+d_{s3}, \quad \text{(A20)}$$

$$f_0=[q^4+2(a_1^2-b_1^2)q^2+(a_1^2+b_1^2)^2]\times[q^4+2(a_2^2-b_2^2)q^2+(a_2^2+b_2^2)^2], \quad \text{(A21)}$$

where d_{v1}, d_{v2}, d_{v3}, d_{s1}, d_{s2}, d_{s3} are coefficients decided by r_j, a_j, b_j. With parameters shown in Table I the solutions of Eqs. (A17) and (A18) are found to be $\eta_1=0.350$ GeV, $\eta_{2,3}=(0.723\pm 0.351i)$ GeV. Of course, one can directly verify

$$f_v^2q^2+f_s^2=f_0(q^2+\eta_1^2)(q^2+\eta_2^2)(q^2+\eta_3^2). \quad \text{(A22)}$$

So the poles of $F(q_4)$ for 2CC parameters (in the upper half complex q_4 plane) are

$$z_1=i\sqrt{\vec{q}^2+\eta_1^2} \quad \text{(simple pole)}, \quad \text{(A23)}$$

$$z_2=\chi_2+i\omega_2 \quad \text{(simple pole)}, \quad \text{(A24)}$$

$$z_3=\chi_3+i\omega_3 \quad \text{(simple pole)}, \quad \text{(A25)}$$

$$z_4=\chi_4+i\omega_4 \quad \text{(double pole)}, \quad \text{(A26)}$$

$$z_5=\chi_5+i\omega_5 \quad \text{(double pole)}, \quad \text{(A27)}$$

$$z_6=\chi_6+i\omega_6 \quad \text{(double pole)}, \quad \text{(A28)}$$

$$z_7=\chi_7+i\omega_7 \quad \text{(double pole)}, \quad \text{(A29)}$$

with

$$\omega_2=\omega_3=\sqrt{\frac{\vec{q}^2+(\mathrm{Re}\eta_2)^2-(\mathrm{Im}\eta_2)^2+\sqrt{[\vec{q}^2+(\mathrm{Re}\eta_2)^2-(\mathrm{Im}\eta_2)^2]^2+4(\mathrm{Re}\eta_2)^2(\mathrm{Im}\eta_2)^2}}{2}}, \quad \text{(A30)}$$

$$\chi_2=-\chi_3=-\frac{(\mathrm{Re}\eta_2)(\mathrm{Im}\eta_2)}{\omega_2}, \quad \text{(A31)}$$

$$\omega_4 = \omega_5 = \sqrt{\frac{\vec{q}^2 + a_1^2 - b_1^2 + \sqrt{(\vec{q}^2 + a_1^2 - b_1^2)^2 + 4a_1^2 b_1^2}}{2}}, \tag{A32}$$

$$\chi_4 = -\chi_5 = -\frac{a_1 b_1}{\omega_4}, \tag{A33}$$

$$\omega_6 = \omega_7 = \sqrt{\frac{\vec{q}^2 + a_2^2 - b_2^2 + \sqrt{(\vec{q}^2 + a_2^2 - b_2^2)^2 + 4a_2^2 b_2^2}}{2}}, \tag{A34}$$

$$\chi_6 = -\chi_7 = -\frac{a_2 b_2}{\omega_6}. \tag{A35}$$

For 1R1CC (and 3R) case, the analysis is similar except a little modification for correctly analyzing the degree of the poles. Because $b_1 = 0$ for 1R1CC case (for 3R case, all b_j equal zero), the functions f_v and f_s both have a factor of $q^2 + a_1^2$ [see Eqs. (A3) and (A4)] which would be canceled by the same factor in f_0. Therefore for 1R1CC case one should adopt the following modified expressions:

$$\begin{aligned} f_{v1} &= \frac{f_v}{q^2 + a_1^2} \\ &= 2r_1[q^2 + (a_2 + ib_2)^2][q^2 + (a_2 - ib_2)^2] \\ &\quad + r_2(q^2 + a_1^2)[q^2 + (a_2 - ib_2)^2] \\ &\quad + r_2(q^2 + a_1^2)[q^2 + (a_2 + ib_2)^2], \end{aligned} \tag{A36}$$

$$\begin{aligned} f_{s1} &= \frac{f_s}{q^2 + a_1^2} \\ &= 2r_1 a_1[q^2 + (a_2 + ib_2)^2][q^2 + (a_2 - ib_2)^2] \\ &\quad + r_2(a_2 + ib_2)(q^2 + a_1^2)[q^2 + (a_2 - ib_2)^2] \\ &\quad + r_2(a_2 - ib_2)(q^2 + a_1^2)[q^2 + (a_2 + ib_2)^2], \end{aligned} \tag{A37}$$

$$\begin{aligned} f_1 &= \frac{f_0}{q^2 + a_1^2} \\ &= (q^2 + a_1^2)[q^4 + 2(a_2^2 - b_2^2)q^2 + (a_2^2 + b_2^2)^2]. \end{aligned} \tag{A38}$$

According to the decomposition in Eq. (A15) one has

$$f_v^2 q^2 + f_s^2 = f_0(q^2 + \eta_1^2)(q^2 + \eta_2^2)(q^2 + \eta_3^2) \tag{A39}$$

with $\eta_{1,2,3}$ being obtained by solving Eqs. (A17) and (A18). Then

$$F(q_4) = \frac{4f_s[f_s f_v + 2(q^2 - \vec{q}^2)(f_s f_v' - f_v f_s')]}{f_0^2(q^2 + \eta_1^2)(q^2 + \eta_2^2)(q^2 + \eta_3^2)} \tag{A40}$$

$$= \frac{4f_{s1}[f_{s1} f_{v1} + 2(q^2 - \vec{q}^2)(f_{s1} f_{v1}' - f_{v1} f_{s1}')]}{f_1^2(q^2 + \eta_1^2)(q^2 + \eta_2^2)(q^2 + \eta_3^2)} \times (q^2 + a_1^2). \tag{A41}$$

Because η_1 equal a_1 exactly, one would find that $i\sqrt{\vec{q}^2 + a_1^2}$ is a double pole. The analysis for 3R case is similar.

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