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Realistic lattice determination of $\alpha_s(M_Z)$ revisited

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We revisit the earlier determination of $\alpha_s(M_Z)$ via perturbative analyses of short-distance-sensitive lattice observables, incorporating new lattice data and performing a modified version of the original analysis. We focus on two high-intrinsic-scale observables, $\log(W_{11})$ and $\log(W_{12})$, and one lower-intrinsic-scale observable, $\log(W_{12}/u_0^6)$, finding improved consistency among the values extracted using the different observables and a final result, $\alpha_s(M_Z) = 0.1192 \pm 0.0011$, $\sim 2\sigma$ higher than the earlier result, in excellent agreement with recent nonlattice determinations and, in addition, in good agreement with the results of a similar, but not identical, reanalysis by the HPQCD Collaboration. A discussion of the relation between the two reanalyses is given, focusing on the complementary aspects of the two approaches.

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I. INTRODUCTION

The strong coupling α_s is usually characterized by giving the value $\alpha_s(M_Z)$ in the $\overline{\rm MS}$ scheme at the conventionally chosen $n_f = 5$ reference scale $\mu = M_Z$. A high precision determination of $\alpha_s(M_Z)$ based on the perturbative analysis of short-distance-sensitive lattice observables computed using the $a \sim 0.09$, 0.12, and 0.18 fm $n_f = 2 + 1$ MILC data was presented in Ref. [1]. The result, $\alpha_s(M_Z) = 0.1170(12)$, plays a dominant role in fixing the central value of the current PDG assessment [2], $\alpha_s(M_Z) = 0.1176(20)$.

Over the last year, a number of improved nonlattice determinations of $\alpha_s(M_Z)$ have appeared, in a variety of independent processes, over a wide range of scales [3–12]. The results, given in Table I (with all errors combined in quadrature), yield a weighted average, $\alpha_s(M_Z) = 0.1190(10)$, $\sim 2\sigma$ higher than the lattice determination. This difference, though not large, motivates revisiting the lattice analysis, especially in light of the existence of new high-scale ($a \sim 0.06$ fm) lattice data not available at the time of the earlier study. We perform such an extended reanalysis in this paper.

The rest of the paper is organized as follows. In Sec. II, we outline the original analysis, specify our own strategy for implementing the underlying approach, and clarify the difference between our implementation and that of the earlier study and recent HPQCD reanalysis. In Sec. III, we discuss the details of, and input to, our version of the analysis. Finally, in Sec. IV, we present and discuss our results.

II. THE LATTICE DETERMINATION OF $\alpha_s(M_Z)$

A. The original HPQCD/UKQCD analysis

In Ref. [1], $\alpha_s(M_Z)$ was extracted by studying perturbative expansions for a number of UV-sensitive lattice observables, O_k. The generic form of this expansion is

$$O_k = \sum_{N=1} \bar{c}_N^{(k)} \alpha_V(Q_k)^N \equiv D_k \alpha_V(Q_k) \sum_{M=0} c_M^{(k)} \alpha_V(Q_k)^M \quad (1)$$

where $Q_k = d_k/a$ are the Brodsky-Lepage-Mackenzie (BLM) scales [14] for the O_k, and $c_0^{(k)} \equiv 1$. The coefficients $\bar{c}_{1,2,3}^{(k)}$ (equivalently, D_k, $c_1^{(k)}$, and $c_2^{(k)}$) have been computed in 3-loop lattice perturbation theory [15], and, with the corresponding d_k, tabulated for a number of O_k in Refs. [1,15,16]. In Eq. (1), $\alpha_V(\mu)$ is a coupling with the same expansion to $O(\alpha_s^3)$ (with α_s the $\overline{\rm MS}$ coupling) as the

TABLE I. Recent nonlattice determinations of $\alpha_s(M_Z)$.

Source	$\alpha_s(M_Z)$
Global EW fit [3,4]	0.1191 ± 0.0027
H1 + ZEUS NLO inclusive jets [5]	0.1198 ± 0.0032
H1 high-Q^2 NLO jets [6]	0.1182 ± 0.0045
NNLO LEP event shapes [7]	0.1240 ± 0.0033
NNNLL ALEPH + OPAL thrust distributions [8]	0.1172 ± 0.0022
$\sigma[e^+e^- \to \text{hadrons}]$ (2–10.6 GeV) [9]	$0.1190^{+0.0090}_{-0.0110}$
$\frac{\Gamma[\Upsilon(1s)\to\gamma X]}{\Gamma[\Upsilon(1s)\to X]}$ [10]	$0.1190^{+0.0060}_{-0.0050}$
Hadronic τ decay [11–13]	0.1187 ± 0.0016

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heavy quark potential coupling α_V^p, but differing from it, beginning at $O(\alpha_s^4)$, in a way that will be specified below. The expansion coefficients are known to $O(\alpha_s^4)$, and hence the β function of α_V, defined in our conventions by $\mu^2 da_V(\mu)/d\mu^2 = -\sum_{n=0}\beta_n^V a_V^{n+2}(\mu)$, with $a_V \equiv \alpha_V/\pi$, is determined to 4 loops by the known coefficients, $\beta_0, \cdots, \beta_3$, of the 4-loop $\overline{\rm MS}$ β function [17]. The coefficients $\bar{c}_1^{(k)}$, $\bar{c}_2^{(k)}$, and $\bar{c}_3^{(k)}$ tabulated in Refs. [1,15,16] are valid for expansions of the O_k in terms of any variable, α_T, sharing the same expansion as α_V out to $O(\alpha_s^3)$.

With only the known, third order terms in the expansions of the O_k, no value for the reference scale coupling, $\alpha_V(7.5\ \text{GeV}) \equiv \alpha_V^0$, was found to produce a simultaneous fit to the data at all three lattice spacings employed [1]. In consequence, terms out to tenth order in the expansion of Eq. (1) were incorporated, the unknown coefficients $\bar{c}_{4,\cdots,10}^{(k)}$ being fitted using input Bayesian prior constraints. The 4-loop version of β^V was used to run α_V^0 to the scales Q_k relevant to each of the given observables at each of the three lattice spacings. Linear extrapolation in the quark masses was employed, and possible residual mass-independent nonperturbative (NP) contributions estimated, and subtracted, using the known leading order (LO) gluon condensate contributions to the relevant Wilson loops [18].

The scales r_1/a and r_1, which determine the lattice spacing a in physical units, as well as the gluon condensate $\langle \alpha_s G^2/\pi \rangle$ required for the mass-independent NP subtraction, were determined as part of the independent fit performed for each of the O_k. This was accomplished using an augmented χ^2 function in which the squared deviations of the relevant parameters from their input central values were scaled by the squares of the input prior widths. For r_1/a and r_1 the central values and widths were provided by the measured values and their uncertainties. For $\langle \alpha_s G^2/\pi \rangle$, a central value 0 and uncertainty $\pm 0.010\ \text{GeV}^4$ ($\sim$ the conventional SVZ value $0.012\ \text{GeV}^4$ [19]) were employed [20]. While this procedure allows r_1/a and r_1 (which should be characteristic of the lattice under consideration) to take on values which vary slightly with the O_k being analyzed, one should bear in mind that the measured uncertainties, which set the range of these variations, are small compared to the variation of scales across the $a \sim 0.09$, 0.12, and 0.18 fm lattices employed in the analysis. The impact of any potential unphysical observable dependence of the physical scales on the fitted α_V^0 and $\bar{c}_n^{(k)}$ should thus be safely negligible. The situation with regard to the independent fitting of $\langle \alpha_s G^2/\pi \rangle$ for each O_k is potentially more complicated, and will be discussed further below.

The resulting best fit value for α_V^0, averaged over the various observables, was then matched to the $n_f = 3$ $\overline{\rm MS}$ coupling, and the corresponding $n_f = 5$ result, $\alpha_s(M_Z)$, obtained via standard running and matching at the flavor thresholds [21,22], yielding the result, $\alpha_s(M_Z) = 0.1170(12)$, already quoted above.

Regarding the conversion from α_V to α_s, one should bear in mind that, while the expansion for α_V in terms of α_s is, in principle, defined to all orders (see below for more on this point), the coefficients beyond $O(\alpha_s^4)$ involve the currently unknown $\overline{\rm MS}$ β function coefficients $\beta_4, \beta_5, \cdots$. The $n_f = 3$ conversion step is thus subject to a (hopefully small) higher order perturbative uncertainty. As will be explained in Sec. II C, with the definition of α_V employed in Ref. [1], the higher order perturbative uncertainties are, in fact, entirely isolated in the $V \to \overline{\rm MS}$ conversion step of the analysis.

B. An alternate implementation of the HPQCD/UKQCD approach

The higher order perturbative uncertainty encountered in matching α_V to α_s can be removed entirely by working with any expansion parameter, α_T, whose expansion in α_s is fully specified. We take α_T to be defined by the third-order-truncated form of the relation between $\alpha_V^p(\mu^2)$ and $\alpha_s(\mu^2)$ [23] which, for $n_f = 3$, yields

$$\alpha_T(\mu^2) = \alpha_s(\mu^2)[1 + 0.5570\alpha_s(\mu^2) + 1.702\alpha_s^2(\mu^2)]. \tag{2}$$

The β function for α_T, β^T, is then determined to 4 loops by the known values of $\beta_0, \cdots, \beta_3$. With all coefficients on the right-hand side (RHS) positive, α_T runs much faster than α_s, a fact reflected in the significantly larger values of the nonuniversal β function coefficients, $\beta_2^T = 33.969$ and $\beta_3^T = -324.393$. This makes running α_T using the 4-loop-truncated β^T function typically unreliable at the BLM scales corresponding to the coarsest ($a \sim 0.18$ fm) lattices considered here. Since, however, the 4-loop-truncated $\overline{\rm MS}$ running of α_s remains reliable down to these scales, and the relation Eq. (2) is, by definition, exact, the running of α_T may be performed by converting from α_T to α_s at the initial scale, running α_s to the final scale, and then converting back to α_T. This procedure will be especially reliable for O_k like $\log(W_{11})$ and $\log(W_{12})$ with lowest BLM scales >3 GeV.

Though the conversion from the fitted reference scale α_T value to the equivalent $\overline{\rm MS}$ coupling α_s can be accomplished without perturbative uncertainties, higher order perturbative uncertainties do remain in the analysis. To see where, define $\alpha_0 \equiv \alpha_T(Q_0)$, with $Q_0 = Q_k^{\rm max} = d_k/a_{\rm min}$ the maximum of the BLM scales (corresponding to the finest of the lattice spacings, $a_{\rm min}$) for the observable in question. Expanding the couplings at those BLM scales corresponding to coarser lattices, but the same observable, in the standard manner as a power series in α_0, $\alpha_T(Q_k) = \sum_{N=1} p_N(t_k)\alpha_0^N$ [where $t_k = \log(Q_k^2/Q_0^2)$, and the $p_N(t)$ are polynomials in t], one finds, on substitution into Eq. (1),

$$\frac{O_k}{D_k} = \cdots + \alpha_0^4(c_3^{(k)} + \cdots) + \alpha_0^5(c_4^{(k)} - 2.87c_3^{(k)}t_k + \cdots) + \alpha_0^6(c_5^{(k)} - 0.0033\beta_4^T t_k - 3.58c_4^{(k)}t_k + [5.13t_k^2 - 1.62t_k]c_3^{(k)} + \cdots) + \alpha_0^7(c_6^{(k)} - 0.0010\beta_5^T t_k + [0.0094t_k^2 - 0.0065c_1^{(k)}t_k]\beta_4^T - 4.30c_5^{(k)}t_k + [7.69t_k^2 - 2.03t_k]c_4^{(k)} + [-7.35t_k^3 + 6.39t_k^2 - 4.38t_k]c_3^{(k)} + \cdots) + \cdots, \quad (3)$$

where the known numerical values of $\beta_0^T, \cdots, \beta_3^T$ have been employed, and we display only terms involving one or more of the unknown quantities $\beta_4^T, \beta_5^T, \cdots, c_3^{(k)}, c_4^{(k)}, \cdots$.

Running the $\overline{\text{MS}}$ coupling numerically using the 4-loop-truncated β function is equivalent to keeping terms involving $\beta_0, \cdots, \beta_3$ to all orders, and setting $\beta_4 = \beta_5 = \cdots = 0$. The neglect of $\beta_4, \beta_5 \cdots$ means that $\beta_4^T, \beta_5^T, \cdots$ do not take on their correct physical values either, leading to an alteration of the true t_k dependence, beginning at $O(\alpha_0^6)$. Since it is the scale dependence of O_k which is used to fit the unknown coefficients $c_{3,4,\ldots}^{(k)}$, as well as α_0, we see immediately that the 4-loop truncation necessarily forces compensating changes in at least the coefficients $c_{4,5,\ldots}^{(k)}$. A shift in $c_4^{(k)}$, however, also alters the $O(\alpha_0^5)$ coefficient, which will, in general, necessitate an approximate compensating shift in $c_3^{(k)}$ as well, and, in consequence, a further compensating shift in α_0. From Eq. (3), the size of such effects, associated with the truncation of the running and unavoidable at some level, can be minimized by taking Q_0 as large as possible (achieved by working with the observable with the highest intrinsic BLM scale) and keeping t_k from becoming too large (achieved by restricting one's attention, if possible, to a subset of finer lattices) [24].

C. More on the relation between the two implementations

For $n_f = 3$, in our notation, the relation between α_V^p and α_s, to $O(\alpha_s^3)$, is [23]

$$\alpha_V^p(q^2) = \alpha_s(\mu^2)[1 + \kappa_1(\mu^2/q^2)\alpha_s(\mu^2) + \kappa_2(\mu^2/q^2)\alpha_s(\mu^2)] \quad (4)$$

where $\kappa_2(x) = [a_2 + 16\beta_0^2\log^2(x) + (16\beta_1 + 8\beta_0 a_1) \times \log(x)]/16\pi^2$, with $a_2 = \frac{695}{6} + 36\pi^2 - \frac{9}{4}\pi^4 + 14\zeta(3)$, and $\kappa_1(x) = [7 + 4\beta_0\log(x)]/4\pi$. Our expansion parameter $\alpha_T(q^2)$ is defined to be equal to the RHS of Eq. (4) with $\mu^2 = q^2$, leading to the numerical result given in Eq. (2). The conversion from α_T to α_s can be performed exactly but the absence in $\beta_{4,5,\ldots}^T$ of terms $\propto \beta_{4,5,\ldots}$ induces a perturbative uncertainty in the values of our fitted parameters, one which can, however, be reduced by working with high-scale observables and fine lattices. It is also possible to test for its presence by expanding the fits to include coarser lattices, where the effects of the omitted contributions will be larger.

The construction of the expansion parameter α_V is somewhat more complicated, but turns out to be equivalent to the following [25]. One first takes the RHS of Eq. (4), with $\mu^2 = e^{-5/3}q^2$, to define an intermediate coupling, $\alpha_V'(q^2)$. The corresponding β function, β', is then determined to 4 loops by $\beta_0, \cdots, \beta_3$. The higher order coefficients, $\beta_{4,5,\ldots}'$, however, depend on the presently unknown $\beta_{4,5,\ldots}$, and hence are themselves unknown. The final HPQCD coupling, α_V, is obtained from α_V' by adding terms of $O(\alpha_s^5)$ and higher with coefficients chosen to make $\beta_4^V = \beta_5^V = \cdots = 0$. Since $\beta_{4,5,\ldots}$ are not known, the values of the coefficients needed to implement these constraints are also not known. The coupling is nonetheless, in principle, well defined, with higher order coefficients computable as soon as the corresponding higher order β_k become available. Since the 4-loop-truncated β^V function is, by definition, exact, the distortions of the fit parameters induced, in general, by the 4-loop truncation of the running are absent for the α_V coupling. The price to be paid for this advantage is the unknown perturbative uncertainty in the relation between α_V and α_s, which affects the conversion and running to $\alpha_s(M_Z)$. With this definition, α_V differs from α_T beginning at $O(\alpha_s^4)$.

The other difference between the two reanalyses lies in the treatment of r_1/a, r_1, and $\langle\alpha_s G^2/\pi\rangle$. In Ref. [1], these are allowed to vary independently, though within the range of the input prior constraints, for each O_k, whereas in our analysis, they are treated as fixed external input, and have the same central values for all O_k. As noted above, the difference in the treatment of r_1/a and r_1 is expected to have a negligible impact. The impact of the differing treatments of $\langle\alpha_s G^2/\pi\rangle$ should be similarly negligible for observables with intrinsic scales high enough that the associated correction is small.

The two different implementations of the original HPQCD/UKQCD approach will thus, when restricted to high-scale observables, correspond to isolating residual higher order perturbative uncertainties in different sectors of the analysis. If these uncertainties are, as desired, small in both cases, the two analyses should be in good agreement. Such agreement (which is, in fact, observed, provided comparison is made to the very recent HPQCD update) serves to increase confidence in the results of both analyses.

III. DETAILS OF OUR REANALYSIS

In our analysis, we have calculated the desired Wilson loops using the publicly available $a \sim 0.09, 0.12, 0.15$, and 0.18 fm MILC $n_f = 2 + 1$ ensembles and incorporated information on W_{11} and W_{12} for the three $a \sim 0.06$ fm

USQCD ensembles provided to us by Doug Toussaint of the collaboration.

We follow the basic strategy of the earlier analysis, using the same 3-loop perturbative input, but with the following differences in implementation. First, we employ the expansion parameter α_T throughout. All running of α_T is carried out using exact 4-loop-truncated running of the intermediate variable α_s, whose relation to α_T is given by Eq. (2). Second, to minimize the effect of our incomplete knowledge of the running of α_T beyond 4-loop order, the impact of which will be larger for coarser lattices, we perform ''central'' 3-fold versions of our fits using the three finest lattices, with $a \sim 0.12$, 0.09, and 0.06 fm. Expanded 5-fold fits then serve as a way of studying the impact of the truncated running, as well as of the truncation of the perturbative expansion for the O_k. Since we do not currently have access to the actual $a \sim 0.06$ fm configurations, we are restricted to analyzing the three observables indicated above. One of these, $\log(W_{12}/u_0^6)$, has a significantly lower BLM scale, and hence is particularly useful for studying the impact of these truncations. As in Ref. [1], we extrapolate linearly in the quark masses [26], and estimate (and subtract) residual mass-independent NP effects using the known form of the leading order gluon condensate contributions to the relevant Wilson loops.

Regarding the mass extrapolation, the sets of configurations for different mass combinations am_ℓ/am_s corresponding to approximately the same lattice spacing a in fact have slightly different measured r_1/a. Since the O_k we study are themselves scale dependent, full consistency requires converting the results corresponding to the different am_ℓ/am_s to a common scale before extrapolation. This could be done with high accuracy if the parameters appearing in the perturbative expansion of the O_k were already known. Since, however, some of these parameters are to be determined as part of the fit, the extrapolation and fitting procedure must be iterated. With sensible starting points, convergence is achieved in a few iterations. The dominant uncertainty in the converged iterated extrapolated values is that associated with the uncertainties in r_1/a. There is also a 100%-correlated global scale uncertainty associated with that on r_1. We employ $r_1 = 0.318(7)$ fm, as given in the MILC Lattice 2007 pseudoscalar project update [27].

The mass-independent NP subtractions are estimated using the LO $D = 4$ gluon condensate contribution, $\delta_g W_{mn}$, to the $m \times n$ Wilson loop, W_{mn} [18],

$$\delta_g W_{mn} = \frac{-\pi^2}{36} m^2 n^2 a^4 \langle \alpha_s G^2/\pi \rangle \tag{5}$$

and the central value, $\langle \alpha_s G^2/\pi \rangle = (0.009 \pm 0.007)$ GeV^4, of the updated charmonium sum rule analysis [28]. Since the error here is already close to 100%, we take the difference between results obtained with and without the related subtraction as a measure of the associated uncertainty. This should be sufficiently conservative if the correction is small. If not, the measured O_k values may contain additional non-negligible mass-independent contributions, of dimension $D > 4$, which we do not know how to estimate and subtract. O_k for which this occurs will thus provide a less reliable determination of α_s.

Fortunately, for the observables we consider, the gluon condensate correction is, as desired, small. For $O_k = \log(W_{11})$, the corrections required for the 3-fold (5-fold) fit do not exceed $\sim$0.1% ($\sim$0.5%). The corrections remain small [less than $\sim$0.4% ($\sim$1.8%)] for $O_k = \log(W_{12})$. The effect is somewhat larger for $\log(W_{12}/u_0^6)$, as a consequence of cancellations encountered in combining the uncorrected $\log(W_{11})$ and $\log(W_{12})$ values, but still reaches only $\sim$1.3% ($\sim$5.6%) for the 3-fold (5-fold) fit [29].

In line with what was seen in Ref. [1], we find that the known terms in the perturbative expansions of the O_k are insufficient to provide a description of the observed scale dependence, even when only the three finest lattices are considered. When $c_3^{(k)}$ is added to the fit, however, we find very good fits, with $\chi^2/\text{dof} < 1$ (very significantly so for the 3-fold fits). With current errors, it is thus not possible to sensibly fit additional coefficients $c_{m>3}^{(k)}$. This raises concerns about possible truncation uncertainties. Comparison of the results of the 3-fold and 5-fold fits provides one handle on such an uncertainty since the relative weight of higher order to lower order terms grows with decreasing scale. If neglected higher order terms are, in fact, *not* negligible, the growth with decreasing scale of the resulting fractional error should show up as an instability in the values of the parameters extracted using the different fits. We see no signs for such an instability within the errors of our fits, but nonetheless include the difference of central values obtained from the 3-fold and 5-fold fits as a component of our error estimate.

IV. RESULTS

Central inputs for our fits are the measured lattice observables (whose errors are tiny on the scale of the other uncertainties), the computed D_k, $c_1^{(k)}$ and $c_2^{(k)}$ [1,15], r_1/a, r_1 and $\langle \alpha_s G^2/\pi \rangle$, and the choice of the 3-fold fitting procedure. In addition to the uncertainties generated by the errors on r_1/a, r_1, and $\langle \alpha_s G^2/\pi \rangle$ are those due to uncertainties in numerical evaluations of the D_k, $c_1^{(k)}$, and $c_2^{(k)}$.

We construct an ''overall scale uncertainty error'' by adding *linearly* the fit uncertainties generated by those on r_1 and r_1/a. This combined error is added in quadrature to (1) uncertainties produced by varying the $c_2^{(k)}$ (and, if relevant, $c_1^{(k)}$) within their errors, (2) the difference between results obtained with and without the gluon condensate correction, and (3) the difference between the results of the 3-fold and 5-fold fits. Because of the iterative nature of the fit procedure, the mass extrapolation uncertainty is

TABLE II. Central fit results for $\alpha_s(M_Z)$ and the $c_3^{(k)}$.

O_k	$\alpha_s(M_Z)$	$c_3^{(k)}$
$\log(W_{11})$	0.1192 ± 0.0011	-3.8 ± 0.6
$\log(W_{12})$	0.1193 ± 0.0011	-4.0 ± 0.9
$\log(W_{12}/u_0^6)$	0.1193 ± 0.0011	-1.7 ± 0.8

incorporated into what we have here identified as the overall scale uncertainty.

We run our $n_f = 3$ results to M_Z using the self-consistent combination of 4-loop running and 3-loop matching at the flavor thresholds, taking the flavor thresholds to lie at $rm_c(m_c)$ and $rm_b(m_b)$, with $m_c(m_c) = 1.286(13)$ GeV and $m_b(m_b) = 4.164(25)$ GeV [30], and r allowed to vary between 1 and 3. These uncertainties in the matching thresholds, together with standard estimates for the impact of the truncated running and matching, produce an evolution contribution to the uncertainty on $\alpha_s(M_Z)$ of ± 0.0003 [3].

Our central fit results for $\alpha_s(M_Z)$ and the $c_3^{(k)}$ are given in Table II. For comparison, the results for $\alpha_s(M_Z)$ obtained in Ref. [1] were 0.1171(12), 0.1170(12), and 0.1162(12), for $\log(W_{11})$, $\log(W_{12})$, and $\log(W_{12}/u_0^6)$, respectively. Our $\alpha_s(M_Z)$ are significantly larger, and in closer mutual agreement. The recent HPQCD update [16] also finds significantly larger values. (We will return to a more detailed comparison of the two updates below.) The very good agreement between the $\alpha_s(M_Z)$ values obtained in our fits using both low- and high-scale observables suggests

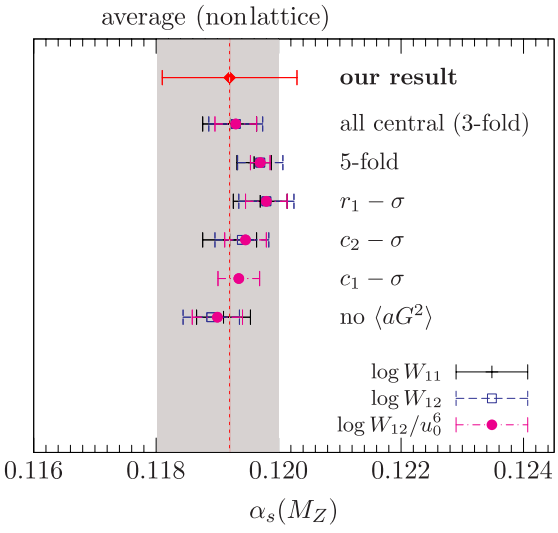

FIG. 1 (color online). Contributions to the errors on $\alpha_s(M_Z)$. Shown are the $\alpha_s(M_Z)$ obtained using (i) the 3-fold fit strategy, with all central input, (ii) the alternate 5-fold fit strategy, with all central input, and (iii) the 3-fold fit strategy, with, one at a time, each input shifted from its central value by 1σ, retaining central values for the remaining input parameters. The error bars shown are those associated with the uncertainties in r_1/a.

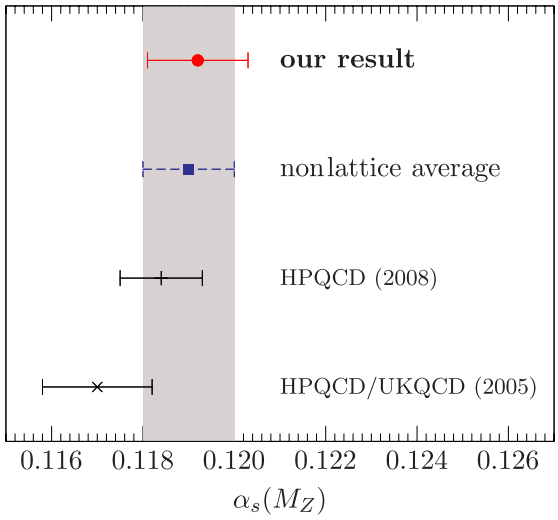

FIG. 2 (color online). Comparison of the results for $\alpha_s(M_Z)$ from our fits, the fits of Ref. [1], and the updated fits of Ref. [16] with the average of recent nonlattice determinations.

that the effects of the truncated running, present at some level in all such fits, are small in the cases we have studied.

One-sided versions of the various components of the total errors on $\alpha_s(M_Z)$ are displayed in Fig. 1. The difference of the 3-fold and 5-fold determinations is ~ 0.0004, significantly smaller than the ~ 0.0009 overall scale uncertainty. The results thus show no evidence for any instability associated with opening up the fit to lower scales.

While the total error on $\alpha_s(M_Z)$ is the same for all three O_k considered, the general arguments above lead us to believe that the most reliable determination is that obtained using the highest-scale observable, $\log(W_{11})$, and highest-scale (3-fold fit) analysis window. Our final assessment,

$$\alpha_s(M_Z) = 0.1192 \pm 0.0011, \tag{6}$$

is in excellent agreement with the nonlattice average and the result, 0.1184 ± 0.0009, of the independent HPQCD analysis. The various results are shown for comparison in Fig. 2. A more detailed discussion of the relation between our reanalysis and that of HPQCD may be found in the Appendix.

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the original lattice analysis and the recent HPQCD update, and providing information on ongoing work.

APPENDIX: MORE ON THE RELATION TO THE HPQCD REANALYSIS

After the completion of the work reported in this paper, the HPQCD Collaboration posted an update of their earlier 2005 analysis [16]. This update works with a subset of 11 of the available MILC ensembles, spanning the $a \sim 0.18$, 0.15, 0.12, 0.09, and 0.06 fm lattices and a range of am_ℓ/am_s. The fits follow the strategy of the earlier analysis [1], employing the expansion parameter α_V, and fitting the unknown $\bar{c}_n^{(k)}$ using priors. Linear mass extrapolation has been employed, and mass-independent NP $D = 4$ contributions estimated and subtracted using the LO formula for $\delta_g W_{mn}$. The fitting of r_1/a, r_1, and $\langle \alpha_s G^2/\pi \rangle$, observable by observable, using central input and prior widths, is also as in the earlier analysis, with the exception that the central value and width for $\langle \alpha_s G^2/\pi \rangle$ are now 0 and ± 0.012 GeV^4, respectively.

The HPQCD implementation differs from ours in the choice of expansion parameter, and in the implementation of the input information on r_1/a, r_1, and $\langle \frac{\alpha_s}{\pi} G^2 \rangle$. For the reasons discussed above, we expect the impact on $\alpha_s(M_Z)$ of the observable-by-observable fitting of r_1/a, r_1, and $\langle \alpha_s G^2/\pi \rangle$ in the HPQCD approach to be small for O_k having small gluon condensate corrections. Since the different choices of expansion parameter correspond to different ways of isolating residual higher order perturbative uncertainties, one expects the results of the two analyses to be in good agreement so long as (i) one is working with O_k having small mass-independent NP corrections, (ii) the same input values are used for both, and (iii) residual NP and higher order perturbative uncertainties are indeed small. The situation is likely to be more complicated for O_k with sizable estimated $D = 4$ gluon condensate corrections.

The results of the HPQCD fit for the three O_k we consider are $\alpha_s(M_Z) = 0.1186(9)$, 0.1186(9), and 0.1183 (8) for $\log(W_{11})$, $\log(W_{12})$, and $\log(W_{12}/u_0^6)$, respectively [16]. All are in good agreement within errors with the corresponding results from our analysis. This agreement is further improved if one takes into account the small difference in input r_1 values. Were we to switch from $r_1 = 0.318$ fm to the central value of the HPQCD determination, 0.321(5) fm, all three of our $\alpha_s(M_Z)$ results would decrease by 0.0002. Note also that use of the central charmonium sum rule input for $\langle \alpha_s G^2/\pi \rangle$ in our calculation raises the output $\alpha_s(M_Z)$ obtained from $\log(W_{11})$, $\log(W_{12})$, and $\log(W_{12}/u_0^6)$ by 0.0001, 0.0004, and 0.0005, respectively. Our fitted values would thus be in even closer agreement with those of the HPQCD update were we to impose the HPQCD central zero value of $\langle \alpha_s G^2/\pi \rangle$ in our fits. Since the fitted $\langle \alpha_s G^2/\pi \rangle$ values obtained by analyzing the various O_k are not quoted in Ref. [16], it is not possible to quantify further the role of this effect. The agreement for the three observables under discussion is, in any case, good, within expectations, independent of this question.

We now turn to a more detailed discussion of the issue of the subtraction of the mass-independent NP contributions. If the estimated LO, $D = 4$ gluon condensate subtraction represents only a small fraction of the measured O_k at the scales under consideration, analogous mass-independent NP contributions with $D > 4$ should be even smaller, and hence safely negligible. If, however, the estimated $D = 4$ correction is sizable, analogous $D > 4$ corrections can no longer be expected to be small. These necessarily scale differently with lattice spacing than do the $D = 0$ perturbative and $D = 4$ NP contributions and hence, if not included when fitting the data, are likely to force shifts in both α_s and $\langle \alpha_s G^2/\pi \rangle$ if present at a non-negligible level.

We deal with this potential problem by focusing on O_k for which the impact of the estimated $D = 4$ gluon condensate subtraction is small compared to the variation of the O_k in question over the lattice scales employed in the fit. In the initial version of the HPQCD reanalysis, mass-independent NP subtractions were estimated using only the $D = 4$ gluon condensate form, even for observables where the estimated correction is sizable. In the more recent update, additional terms, scaling as would mass-independent contributions of $D > 4$, are added to the fit function for each observable, and the accompanying coefficients extracted as part of the augmented Bayesian fit. The impact of including the $D > 4$ terms is, as expected, small for those observables having small values of the estimated $D = 4$ subtraction. For observables with larger $D = 4$ subtractions, the fit errors are increased (by factors of ~ 2 for those observables having the largest $D = 4$ corrections) and some shifts in $\alpha_s(M_Z)$ of order 1/2 to 1 times the smaller preliminary errors are observed. The shifts serve to reduce the spread of $\alpha_s(M_Z)$ values compared to that seen in the original version of the reanalysis. The values of $\langle \alpha_s G^2/\pi \rangle$ obtained from the independent fits to the different observables are not quoted in Ref. [16], but a useful test of the self-consistency of the approach would be to verify that the inclusion of the $D > 4$ contributions has brought these values into good agreement with one another.

It is worth noting that the observables, $\log(W_{23}/u_0^{10})$, $\log(W_{14}/W_{23})$, and $\log(W_{11}W_{23}/W_{12}W_{13})$, which produce the three smallest results for $\alpha_s(M_Z)$, have estimated $D = 4$ corrections significantly larger than those for any of the other observables. The magnitudes of the corrections in these cases represent $\sim$50%–100% of the variation with scale of the uncorrected O_k between the lightest mass $a \sim 0.06$ and $a \sim 0.12$ fm ensembles. (This variation with scale provides a suitable measure for use in assessing the importance of NP corrections since it is the variation with scale which provides the input needed to fix the fit parameters, and, as explained in Ref. [16], the $a \sim 0.06$, 0.09, and

0.12 fm ensembles which dominate the HPQCD reanalysis.) These observables are thus, for the purposes of the analysis, rather nonperturbative. Were one to exclude observables with larger NP contributions from the HPQCD average, on the grounds that the related subtractions introduce additional theoretical systematic uncertainties, the HPQCD result would be brought into even closer agreement with ours, though the resulting shift would, in fact, be small (at the $\sim\frac{1}{4}\sigma$ level).

We stress that, independent of these questions, our results agree well within errors with those of the HPQCD update. This agreement is further improved by a shift to common input. We argue that the nonzero central value for $\langle\alpha_s G^2/\pi\rangle$ obtained from the updated charmonium sum rule analysis represents our best present knowledge of this quantity, and hence also the best choice as input for evaluating the small mass-independent NP subtractions needed for extracting $\alpha_s(M_Z)$. In addition, for the reasons just discussed, we believe that the most reliable determinations of $\alpha_s(M_Z)$ are those based on those observables for which the $D=4$ correction is as small as possible. Such an assessment produces the results already noted above, which are in extremely good agreement with what is known from other sources.

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