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Radiative leptonic B_c decay in the relativistic independent quark model

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The radiative leptonic decay $B_c^- \to \mu^- \bar{\nu}_\mu \gamma$ is analyzed in its leading order in a relativistic independent quark model based on a confining potential in an equally mixed scalar-vector harmonic form. The branching ratio for this decay in the vanishing lepton mass limit is obtained as $\mathcal{B}r(B_c \to \mu \nu_\mu \gamma) = 6.83 \times 10^{-5}$, which includes the contributions of the internal bremsstrahlung and structure-dependent diagrams at the level of the quark constituents. The contributions of the bremsstrahlung and the structure-dependent diagrams, as well as their additive interference parts, are compared and found to be of the same order of magnitude. Finally, the predicted photon energy spectrum is observed here to be almost symmetrical about the peak value of the photon energy at $\tilde{E}_\gamma \simeq \frac{M_{B_c}}{4}$, which may be quite accessible experimentally at LHC in near future.

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I. INTRODUCTION

After the discovery of the B_c-meson by the CDF-Collaboration [1] in the Fermilab Tevatron experiment of 1.8 TeV $p\bar{p}$ collision with its mass and lifetime measured, respectively, as [2] $M_{B_c} = 6.40 \pm 0.39(\text{stat.}) \pm 0.13(\text{syst.})\text{GeV}$ and $\tau_{B_c} = 0.46^{+0.18}_{-0.16}(\text{stat.}) \pm 0.03(\text{syst.})$ ps, a great deal of interest has been generated in the study of its production and the decay modes. The prospects for the realization of higher statistics $B_c^\pm$ events in the foreseeable future on the LHC with luminosity $\mathcal{L} = 10^{34}\ \text{cm}^{-2}\,\text{s}^{-1}$ and $\sqrt{s} = 14$ TeV [3] has further stimulated such studies widely in the current literature. Since the B_c-meson consists of two heavy quarks ($b\bar{c}$ or $\bar{b}c$), its decay channels are very rich unlike those of other B_q-mesons ($b\bar{q}$ or $\bar{b}q$) with $q = (u, d, s)$. Being a double heavy quark-antiquark bound state like (η_c, η_b) and $(J/\psi, \Upsilon)$ etc., QCD-inspired perturbative models are expected to describe well its properties and decay modes. Again since B_c is composed of heavy quarks of different flavors, unlike the well-studied ($c\bar{c}$) and ($b\bar{b}$) bound systems, it is stable against the strong and electromagnetic decays. It can therefore decay only weakly for which it has comparatively a very long lifetime. Thus, due to its unique features as described above, the B_c-meson is expected to provide opportunities for testing various potential models and understanding weak decay mechanisms for heavy flavors [4].

The B_c-meson decay channels, ignoring the Cabibbo-suppressed and penguin-induced transitions, are thought to be mediated by three mechanisms such as: (i) b-quark decay ($b \to cW^-$) with $\bar{c}$ as spectator as in ($B_c \to J/\psi\pi, J/\psi l\nu_l$), (ii) $\bar{c}$-quark decay $\bar{c} \to \bar{s}W^-$ with b-quark as spectator as in ($B_c \to B_s\pi, B_s l\nu_l$), and (iii) the annihilation modes ($\bar{c}b \to W^-, W^-\gamma$) as in ($B_c \to l\nu_l, B_c \to l\nu_l\gamma$). The first two mechanisms provide dominant contributions to B_c total decay width, where as the third one corresponding to pure leptonic and radiative leptonic decay modes contribute a minor fraction only. Nevertheless their study and analysis is of particular interest from the theoretical and phenomenological point of view and we would be particularly concerned, in our present work, with the radiative leptonic decay mode of B_c-meson.

The radiative leptonic decay of B_q-mesons in general are of great interest both theoretically and experimentally. It is believed to provide a better alternative to probe various aspects of strong and weak interactions of heavy quark systems. An additional photon emission characterizing the decay $B_q \to l\nu_l\gamma$ makes it possible to overcome the well-known helicity suppression inevitable in pure leptonic decay $B_q \to l\nu_l$. As a result the branching ratio for the radiative leptonic decays finds a significant enhancement over that of the pure leptonic decay mode. Thus the study of radiative leptonic decay $B_q \to l\nu_l\gamma$ can serve as an independent and alternative way for extracting some fundamental parameters such as the Cabibbo-Kobayashi-Maskawa (CKM)-matrix elements ($\mathcal{V}_{bq}$) [5] and the leptonic decay constant f_{B_q} [6]. The B_c-annihilation mode $B_c \to l\nu_l\gamma$ in particular being governed by the CKM parameter $\mathcal{V}_{cb}$ is Cabibbo-enhanced in comparison to the corresponding B_u-decay mode. Therefore, from a phenomenological point of view, the B_c-decay mode is of particular interest since it may contribute significantly

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some background in the B_q-decays with the same final states [7].

In the tree-level descriptions of such decay modes $B_q \to l\nu_l\gamma$, generally two types of diagrams are considered which contribute to the decay amplitudes. They are (i) the structure-dependent (SD) diagrams where the photon line is attached to the charged quark lines and (ii) the internal bremsstrahlung (IB) diagram where the photon line is attached to the charged lepton line. It is usually believed that from helicity arguments the contribution of the bremsstrahlung part must be proportional to the lepton mass m_l for which it can be safely neglected, when $l = e$, μ [8]. In the study of $B_u \to l\nu_l\gamma$ by several models [9], it has been shown that the IB contributions to the decay amplitude are minimal compared to that of the dominant SD part. Following this assumption, we analyzed the $B_u \to \mu\nu_\mu\gamma$ process in the relativistic independent quark model (RIQM) [10] to find our predictions for the branching ratio along with the photon energy spectrum in reasonable agreement with other model predictions. However, such an assumption may not be quite justified in general terms and particularly so in the case of $B_c \to l\nu_l\gamma$, where both the structure-dependent and internal bremsstrahlung parts are found to give comparable contributions to the decay width. Moreover, the inclusion of the IB diagram is necessary to satisfy the requirement of gauge invariance for the decay amplitude in a more general way. There have been several theoretical studies of $B_c \to l\nu_l\gamma$ based on the models such as the QCD-inspired potential models [11,12], light cone QCD [8], effective field theoretic approach [13], and the light front model [14], etc. All of these model studies predict the branching ratio $\mathcal{B}r(B_c \to \mu\nu_\mu\gamma)$ in the range $(1-6)\times 10^{-5}$. We would like to test our RIQM approach used earlier in the study of $B_u \to \mu\nu_\mu\gamma$ [10] in analyzing the decay $B_c \to \mu\nu_\mu\gamma$, for which both the IB diagram and the SD diagrams would be taken into account at the level of the quark constituents. We would also like to compare the contributions of the bremsstrahlung and structure-dependent part along with their interference parts to the predicted branching ratio $\mathcal{B}r(B_c^- \to \mu^-\bar{\nu}_\mu\gamma)$. We would extend this analysis to reassess the dominance of the structure-dependent diagrams in case of $B_u \to \mu\nu_\mu\gamma$ for which we had ignored the IB diagram in our earlier calculation [10].

In the following section we briefly outline the model frame work to be adopted here for the present study which is based on a RIQM that has been found successful in providing satisfactory description of a wide ranging hadronic phenomena in the light as well as the heavy flavor sector [15,16]. Section III provides the calculation of the transition amplitude for the decay process $B_c^- \to \mu^-\bar{\nu}_\mu\gamma$ which includes the contributions of the structure-dependent and bremsstrahlung diagrams at the level of the quark constituents $(b\bar{c})$. In Sec. IV, we outline the required kinematics and obtain the decay width as well as the branching ratio. Section V provides the discussion of our prediction with those of other models. Finally, Sec. VI deals with a brief summary and conclusion.

II. MODEL FRAMEWORK

The model framework based on the relativistic independent quark model to analyze the decay process of the B_q-meson in the annihilation mode has already been described in detail in our previous work [10]. However, for the sake of completeness in the present context, we provide only a brief outline of the same. In this model the decaying meson B_c^- is represented by a suitably constructed definite momentum and spin state $|B_c(\vec{p})>_{S_B}$ in the form of a momentum wave packet reflecting the momentum and spin distribution of its constituent quark (b) and antiquark $(\bar{c})$ as

$$|B_c^-(\vec{p})\rangle_{S_B} = \hat{\Lambda}_{B_c}(\vec{p})\,|(\vec{p}_b,\lambda_b);(\vec{p}_c,\lambda_c)\rangle, \tag{1}$$

where

$$|(\vec{p}_b,\lambda_b);(\vec{p}_c,\lambda_c)\rangle = \hat{b}_b^\dagger(\vec{p}_b,\lambda_b)\hat{\tilde{b}}_c^\dagger(\vec{p}_c,\lambda_c)\,|0\rangle \tag{2}$$

is a Fock-space representation of an unbound quark "b" and antiquark "$\bar{c}$" in a color-singlet configuration with their respective momenta and spin as $(\vec{p}_b,\lambda_b)$ and $(\vec{p}_c,\lambda_c)$. Here $\hat{b}_q^\dagger(\vec{p}_q,\lambda_q)$ and $\hat{\tilde{b}}_q^\dagger(\vec{p}_q,\lambda_q)$ are, respectively, the quark and antiquark creation operators. $\hat{\Lambda}_{B_c}(\vec{p},S_B)$ stands here to represent an integral operator in the form

$$\hat{\Lambda}_{B_c}(\vec{p},S_B) \equiv \frac{\sqrt{3}}{\sqrt{N(\vec{p})}}\sum_{\lambda_b,\lambda_c}\zeta^{B_c}(\lambda_b,\lambda_c)\int d^3\vec{p}_b d^3\vec{p}_c\,\delta^{(3)} \times(\vec{p}_b+\vec{p}_c-\vec{p})\mathcal{G}_{B_c}(\vec{p}_b,\vec{p}_c) \tag{3}$$

where "$\sqrt{3}$" is an effective color factor and $\zeta^{B_c}(\lambda_b,\lambda_c)$ stands for the SU(6)-spin flavor coefficients for the meson $B_c^-(b\bar{c})$. The meson-state normalization factor $N(\vec{p})$ is realized from $\langle B_c(\vec{p})\,|\,B_c(\vec{p}\,')\rangle = \delta^{(3)}(\vec{p}-\vec{p}\,')$ in an integral form as

$$N(\vec{p}) = \int d^3\vec{p}_b\,|\,\mathcal{G}_{B_c}(\vec{p}_b,\vec{p}-\vec{p}_b)|^2. \tag{4}$$

Finally, $\mathcal{G}_{B_c}(\vec{p}_b,\vec{p}_c)$ represents the effective momentum distribution function for the constituent quark b and antiquark $\bar{c}$, which is taken in the form

$$\mathcal{G}_{B_c}(\vec{p}_b,\vec{p}_c) = [G_b(\vec{p}_b)\tilde{G}_c(\vec{p}_c)]^{1/2}, \tag{5}$$

where $G_b(\vec{p}_b)$ and $\tilde{G}_c(\vec{p}_c)$ are the momentum probability amplitude of the bound quark b with momentum $\vec{p}_b$ and antiquark $\bar{c}$ with momentum $\vec{p}_c$, respectively. The bound quark and antiquark $(b\bar{c})$ inside the meson core are in fact in a definite energy state with no definite momenta, which are represented by the bound quark orbitals corresponding to the ground state of the meson in the usual form

$$\phi_q^{(+)}(\vec{r},\lambda_q) = \frac{1}{\sqrt{4\pi}}\begin{pmatrix}\frac{ig_q(r)}{r}\\ \frac{\vec{\sigma}.\hat{r}f_q(r)}{r}\end{pmatrix}\chi(\lambda_q)$$
$$\phi_q^{(-)}(\vec{r},\lambda_q) = \frac{1}{\sqrt{4\pi}}\begin{pmatrix}\frac{i(\vec{\sigma}.\hat{r})f_q(r)}{r}\\ \frac{g_q(r)}{r}\end{pmatrix}\tilde{\chi}(\lambda_q), \tag{6}$$

where the spinors are

$$\chi(\uparrow) = \begin{pmatrix}1\\0\end{pmatrix} = -i\tilde{\chi}(\downarrow) \quad \text{and} \quad \chi(\downarrow) = \begin{pmatrix}0\\1\end{pmatrix} = i\tilde{\chi}(\uparrow).$$

In the relativistic independent quark model that we refer here, the meson is pictured as a color-singlet assembly of quark-antiquark ($b\bar{c}$) independently confined by an effective and average flavor-independent potential of the form [15,16] $U(r) = \frac{1}{2}(1+\gamma^0)(ar^2+V_0)$. The reduced radial parts as the upper and lower components of the quark orbitals in Eq. (6) can be realized in this potential model as

$$g_q(r) = \mathcal{N}_q\left(\frac{r}{r_{0q}}\right)\exp\left(-\frac{r^2}{2r_{0q}^2}\right)$$
$$f_q(r) = -\frac{\mathcal{N}_q}{\omega_q r_{0q}}\left(\frac{r}{r_{0q}}\right)^2\exp\left(-\frac{r^2}{2r_{0q}^2}\right), \tag{7}$$

where the quark binding energy E_q in the meson ground state is derivable from the bound-state condition $(E_q+m_q)(E_q-m_q-V_0)^2 = 9a$. With $E_q' = (E_q - \frac{V_0}{2})$, $m_q' = (m_q + \frac{V_0}{2})$, $\omega_q = (E_q + m_q)$, and $r_{0q} = (a\omega_q)^{-(1/4)}$, the normalization factor $\mathcal{N}_q$ appearing in Eq. (7) is obtained in the form

$$\mathcal{N}_q^2 = \frac{8\omega_q}{\sqrt{\pi}r_{0q}}\frac{1}{(3E_q'+m_q')}. \tag{8}$$

Then by taking suitable momentum space projections of the bound quark orbitals in Eqs. (6) and (7), the momentum probability amplitudes $G_b(\vec{p}_b)$ and $\tilde{G}_c(\vec{p}_c)$ can be obtained in the form

$$G_b(\vec{p}_b) = \frac{i\pi\mathcal{N}_b}{2\alpha_b\omega_b}\left[\frac{\epsilon_b(\vec{p}_b)}{E_b(\vec{p}_b)}\right]^{1/2}[E_b(\vec{p}_b)+E_b]\exp\left(-\frac{\vec{p}_b^{\,2}}{4\alpha_b}\right)$$
$$\tilde{G}_c(\vec{p}_c) = \frac{-i\pi\mathcal{N}_c}{2\alpha_c\omega_c}\left[\frac{\epsilon_c(\vec{p}_c)}{E_c(\vec{p}_c)}\right]^{1/2}[E_c(\vec{p}_c)+E_c]\exp\left(-\frac{\vec{p}_c^{\,2}}{4\alpha_c}\right). \tag{9}$$

Here $E_q(\vec{p}_q) = \sqrt{\vec{p}_q^{\,2}+m_q^2}$, $\epsilon_q(\vec{p}_q) = (E_q(\vec{p}_q)+m_q) \equiv \epsilon_q$, and $\alpha_q = \frac{1}{(2r_{0q}^2)}$. Now referring back to Eq. (1) we can consider that $\hat{\Lambda}_{B_c}(\vec{p},S_B)$ effectively transforms the free quark-antiquark Fock state $|(\vec{p}_b,\lambda_b);(\vec{p}_c,\lambda_c)\rangle$ to a baglike meson state $|B_c^-(\vec{p},S_B)\rangle$ of definite momentum $\vec{p}$ and spin $S_B=0$. The dynamics responsible for the bound-state character of the color-singlet assembly ($b\bar{c}$) inside the meson can thus be considered to have been factored out in terms of this operator $\hat{\Lambda}_{B_c}(\vec{p},S_B)$. We may point out that the bound-state picture of the meson that we have adopted here to construct a definite momentum state $|B_c^-(\vec{p},S_B)\rangle$ of the decaying meson, is not relativistically covariant in its strict sense. This is in fact true with almost all the potential models describing mesons as bound states of valence quarks and antiquarks interacting via some instantaneous potential. However, such models are often required to derive at the mesonic level decay amplitudes starting with the Feynmann amplitudes at the constituent quark level. Here one encounters a problem where the 4-momentum conservation at the meson level S-matrix does not follow automatically. This is due to the fact that although the 3-momentum conservation at the composite level of the meson has been ensured through $\delta^{(3)}(\vec{p}_b+\vec{p}_c-\vec{p})$ in the expression for the meson state $|B_c^-(\vec{p},S_B)\rangle$ in Eqs. (1)–(3), the energy conservation $E_{B_c} = E_b(\vec{p}_b)+E_c(\vec{p}_c)$ in such a definite momentum state cannot be specified so explicitly. This is indeed a pathological problem common to all such models attempting to explain the hadron decays in terms of constituent level dynamics in zeroth order. However, it is quite reassuring to note here that the effective momentum profile function $\mathcal{G}_{B_c}(\vec{p}_b,\vec{p}_c)$, defined through Eqs. (5) and (9) in our model, somehow ensures the energy conservation in an average sense satisfying $E_{B_c} = \langle B_c(\vec{p},S_B)|[E_b(\vec{p}_b)+E_c(\vec{p}_c)]|B_c(\vec{p},S_B)\rangle$. This has been shown in our earlier work [10] in the case of the meson state $|B_u(\vec{p},S_B)\rangle$. However, this point would be illustrated further in the present context for $|B_c(\vec{p},S_B)\rangle$ in the appropriate section hereafter.

Thus we can represent the decaying meson state in the form of a momentum wave packet of the constituent quark-antiquark ($b\bar{c}$), where the bound-state character has been embedded in the momentum profile function $\mathcal{G}_{B_c}(\vec{p}_b,\vec{p}_c)$ appearing in the integral operator $\hat{\Lambda}_{B_c}(\vec{p},S_B)$ of Eq. (3). Then any residual internal dynamics causing the ultimate annihilation decay of the meson B_c^-, can be studied at the level of the otherwise free quark b and antiquark $\bar{c}$ using the relevant Feynmann diagrams. The total contributions of these diagrams taken in momentum space is to be finally operated upon by the baglike integral operator $\hat{\Lambda}_{B_c}(\vec{p},S_B)$ so as to obtain the effective transition amplitude for the decay process $B_c \to \mu\nu_\mu\gamma$ as

$$S_{fi}^{B_c} = \hat{\Lambda}_{B_c}(\vec{p},S_B)S_{fi}^{b\bar{c}} \tag{10}$$

when $S_{fi}^{b\bar{c}}$ is the S-matrix element at the constituent level describing the annihilation process $b\bar{c} \to W^-\gamma \equiv \mu^-\bar{\nu}_\mu\gamma$ and $S_{fi}^{B_c}$ is the effective meson level S-matrix element describing the transition $B_c^- \to \mu^-\bar{\nu}_\mu\gamma$. Following this prescription of the model framework, we obtain the transition amplitude for the decay process $B_c^- \to \mu^- \bar{\nu}_\mu \gamma$ in the following section.

III. TRANSITION AMPLITUDE FOR $B_c^-(p) \to \mu^-(p_1, r)\bar{\nu}_\mu(p_2, s)\gamma(k, \lambda)$

The contribution to the transition amplitude for the process $B_c^-(p, S_{B_c} = 0) \to \mu^-(p_1, r)\bar{\nu}_\mu(p_2, s)\gamma(k, \lambda)$ is usually calculated from four possible tree-level diagrams shown in Figs. 1(a)–1(d). However, the contribution of the diagram in Fig. 1(d), where the photon line is attached to the internal gauge boson W^- is quite negligible as it is suppressed by a factor of $M_{B_c}^2/M_W^2$. Therefore, for the present analysis we take into account the contribution of the other three possible diagrams corresponding to the structure-dependent (SD) parts [Figs. 1(a) and 1(b)] and the internal bremsstrahlung (IB) part [Fig. 1(c)].

Following the same approach as in Ref. [10], we can find the contributions of the two SD diagrams [Figs. 1(a) and 1(b)] to express the S-matrix element at the level of the unbound quark and antiquark $(b\bar{c})$ in the form

$$S_{fi}^{b\bar{c}}(\mathrm{SD}) = \left[-ie\frac{G_F}{\sqrt{2}}\mathcal{V}_{cb}L^\mu \epsilon^{\star\nu}(k, \lambda)\Gamma_{\mu\nu}(p_b, p_c, k)\right]\frac{(2\pi)^4\delta^{(4)}(p_b + p_c - p_1 - p_2 - k)}{\sqrt{(2\pi)^3 2E_b(\vec{p}_b)(2\pi)^3 2E_c(\vec{p}_c)(2\pi)^3 2E_1(2\pi)^3 2E_2(2\pi)^3 2E_\gamma}}, \quad (11)$$

where G_F is the Fermi-coupling constant, $\mathcal{V}_{cb}$ is the CKM-matrix element, $q = (p_1 + p_2) = p - k$ is the 4-momentum transfer, $\epsilon(k, \lambda)$ is the outgoing photon polarization vector. With $\Gamma^\mu = \gamma^\mu(1 - \gamma^5)$, we represent here the weak leptonic current as

$$L^\mu = \bar{U}_\mu(\vec{p}_1, r)\Gamma^\mu V_{\nu_\mu}(\vec{p}_2, s) \quad (12)$$

and the hadronic tensor part $\Gamma_{\mu\nu}(p_b, p_c, k)$ as

$$\Gamma_{\mu\nu}(p_b, p_c, k) = \bar{V}_c(\vec{p}_c, \lambda_c)\Big[Q_c\gamma_\nu\frac{(\not{k} - \not{p}_c) + m_c}{(k - p_c)^2 - m_c^2}\Gamma_\mu + Q_b\Gamma_\mu\frac{(\not{p}_b - \not{k}) + m_b}{(p_b - k)^2 - m_b^2}\gamma_\nu\Big]U_b(\vec{p}_b, \lambda_b). \quad (13)$$

Here eQ_c and eQ_b are the charm and beauty quark electric charges. Similarly we can obtain the contribution of the IB diagram in Fig. 1(c) as

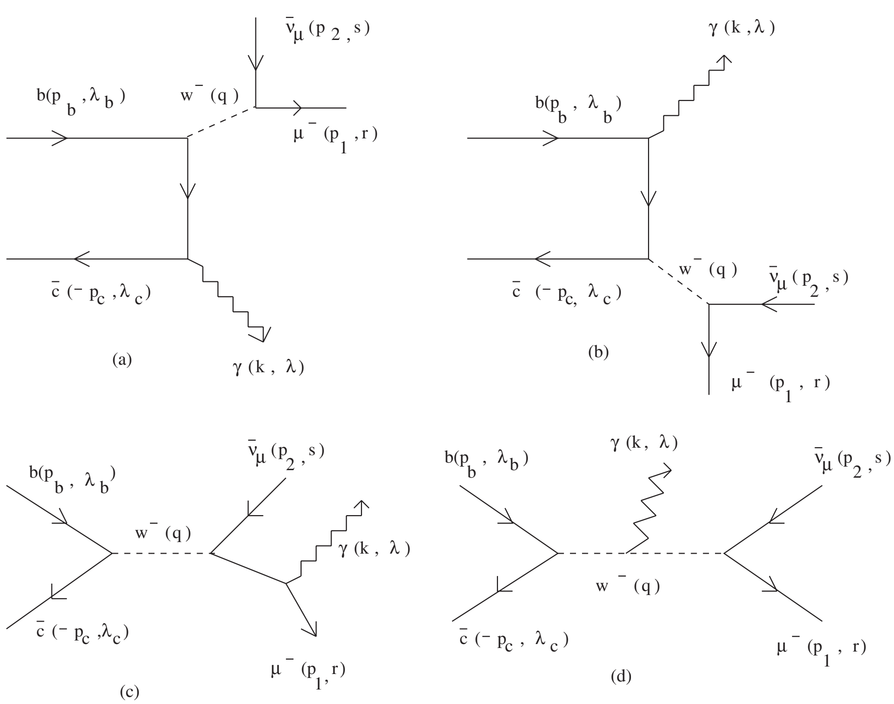

FIG. 1. Feynmann diagrams for the decay $B_c \to \gamma l \bar{\nu}_l$.

$$S_{fi}^{b\bar{c}}(\mathrm{IB}) = \left[-ie\frac{G_F}{\sqrt{2}}\mathcal{V}_{cb}\epsilon^{\star\nu}(k,\lambda)L_{\nu\mu}h^{\mu}\right]\frac{(2\pi)^4\delta^{(4)}(p_b+p_c-p_1-p_2-k)}{\sqrt{(2\pi)^3 2E_b(\vec{p}_b)(2\pi)^3 2E_c(\vec{p}_c)(2\pi)^3 2E_1(2\pi)^3 2E_2(2\pi)^3 2E_\gamma}}, \tag{14}$$

where in the vanishing lepton mass limit ($m_\mu \simeq 0$)

$$L_{\nu\mu} = \left[\bar{U}_\mu(p_1,r)\gamma_\nu\frac{\not{p}_1+\not{k}}{(p_1+k)^2}\Gamma_\mu V_{\nu_\mu}(p_2,s)\right] \tag{15}$$

$$h^\mu = \bar{V}_c(p_c,\lambda_c)\Gamma^\mu U_b(p_b,\lambda_b). \tag{16}$$

We have taken the free particle spinors $U(\vec{p},\lambda)$ and $V(\vec{p},\lambda)$ as

$$\begin{aligned} U(\vec{p},\lambda) &= \sqrt{\epsilon(\vec{p})}\begin{pmatrix}\chi(\lambda)\\ \frac{\vec{\sigma}.\vec{p}}{\epsilon(\vec{p})}\chi(\lambda)\end{pmatrix}\\ V(\vec{p},\lambda) &= \sqrt{\epsilon(\vec{p})}\begin{pmatrix}\frac{\vec{\sigma}.\vec{p}}{\epsilon(\vec{p})}\tilde{\chi}(\lambda)\\ \tilde{\chi}(\lambda)\end{pmatrix}, \end{aligned} \tag{17}$$

where

$$\chi(\uparrow) = -\tilde{\chi}(\downarrow) = \begin{pmatrix}1\\0\end{pmatrix}, \qquad \chi(\downarrow) = \tilde{\chi}(\uparrow) = \begin{pmatrix}0\\1\end{pmatrix}.$$

Then the meson level S-matrix elements corresponding to the structure-dependent and bremsstrahlung parts can be obtained effectively by operating the integral operator $\hat{\Lambda}_{B_c}(\vec{p},S_B)$ respectively on $S_{fi}^{b\bar{c}}(SD)$ and $S_{fi}^{b\bar{c}}(IB)$. We must point out here that, in the present model, the energy conservation at the composite meson level is expected to be satisfied in an average sense through the momentum distribution function $\mathcal{G}_{B_c}(\vec{p}_b,\vec{p}_c)$. Therefore, we may assume $E_{B_c} = E_b(\vec{p}_b) + E_c(\vec{p}_c)$, which together with the 3-momentum conservation $\vec{p} = \vec{p}_b + \vec{p}_c$ ensured by $\delta^3(\vec{p}_b+\vec{p}_c-\vec{p})$ appearing in $\hat{\Lambda}_{B_c}(\vec{p},S_B)$, can enable us to write $p = (p_b + p_c)$. With this assumption we pull out $\delta^{(4)}(p_b+p_c-q-k)$ appearing in $S_{fi}^{b\bar{c}}(\mathrm{SD})$ and $S_{fi}^{b\bar{c}}(\mathrm{IB})$ from within the integral operator in the form $\delta^{(4)}(p-q-k)$ which ensures the desired 4-momentum conservation in the decay process. Then we can write

$$S_{fi}^{B_c}(\mathrm{SD}) = (2\pi)^4\delta^{(4)}(p-q-k)\times[-i\mathcal{M}_{fi}(\mathrm{SD})]\prod_f\left(\frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2E_f}}\right) \tag{18}$$

and

$$S_{fi}^{B_c}(\mathrm{IB}) = (2\pi)^4\delta^{(4)}(p-q-k)[-i\mathcal{M}_{fi}(\mathrm{IB})]\times\prod_f\left(\frac{1}{(2\pi)^{3/2}}\frac{1}{\sqrt{2E_f}}\right). \tag{19}$$

We may further point out that unlike the final-state normalization factors the initial meson-state normalization factor $\frac{1}{\sqrt{(2\pi)^3 2E_{B_c}}}$ is not explicitly available in the kinematic expressions. We therefore incorporate this factor by adequately compensating the same in the numerator. This compensating factor $\sqrt{2E_{B_c}}$ is then pushed inside the integral as $[2E_b(\vec{p}_b)+2E_c(\vec{p}_c)]^{1/2}$ under the assumption of the energy conservation mentioned earlier. Thus realizing the meson level S-matrix elements for the decay process in the desired form, the corresponding invariant transition amplitudes can be extracted as

$$\mathcal{M}_{fi}(\mathrm{SD}) = -\frac{G_F}{\sqrt{2}}e\mathcal{V}_{cb}\epsilon^{\star\nu}(k,\lambda)L^\mu\Pi_{\nu\mu} \tag{20}$$

$$\mathcal{M}_{fi}(\mathrm{IB}) = \frac{G_F}{\sqrt{2}}e\mathcal{V}_{cb}\epsilon^{\star\nu}(k,\lambda)\mathcal{H}^\mu L_{\nu\mu} \tag{21}$$

where, with

$$\begin{aligned} X_{\nu\mu}(p_b,p_c,k) = \sum_{\lambda_b,\lambda_c}&\zeta^{B_c}(\lambda_b,\lambda_c)\bar{V}_c(\vec{p}_c,\lambda_c)\\ &\times\left[-2\gamma_\nu\frac{(\not{k}-\not{p}_c)+m_c}{(k-p_c)^2-m_c^2}\Gamma_\mu\right.\\ &\left.+\Gamma_\mu\frac{(\not{p}_b-\not{k})+m_b}{(p_b-k)^2-m_b^2}\gamma_\nu\right]U_b(\vec{p}_b,\lambda_b) \end{aligned} \tag{22}$$

and

$$\tilde{\mathcal{G}}_{B_c}(\vec{p}_b,\vec{p}-\vec{p}_b) = \left[\frac{E_b(\vec{p}_b)+E_c(\vec{p}_c)}{2E_b(\vec{p}_b)E_c(\vec{p}_c)}\right]^{1/2}\mathcal{G}_{B_c}(\vec{p}_b,\vec{p}-\vec{p}_b) \tag{23}$$

the hadronic contribution of the SD part is obtained in the form

$$\Pi_{\nu\mu} = \frac{1}{\sqrt{(2\pi)^3 3N(\vec{p})}}\int d^3\vec{p}_b\tilde{\mathcal{G}}_{B_c}(\vec{p}_b,\vec{p}-\vec{p}_b)\times X_{\nu\mu}(p_b,p_c,k). \tag{24}$$

Similarly the leptonic tensor $L_{\nu\mu}$ and hadronic term $\mathcal{H}^\mu$ in Eq. (21) corresponding to the (IB) part are obtained in the form

$$L_{\nu\mu} = \bar{U}_\mu(p_1,r)\gamma_\nu\frac{\not{p}_1+\not{k}}{(p_1+k)^2}\Gamma_\mu V_{\nu_\mu}(p_2,s) \tag{25}$$

and

$$\mathcal{H}^\mu = \frac{\sqrt{3}}{\sqrt{(2\pi)^3 N(\vec{p})}}\int d^3\vec{p}_b\tilde{\mathcal{G}}_{B_c}(\vec{p}_b,\vec{p}-\vec{p}_b)\times X^\mu(\vec{p}_b,\vec{p}-\vec{p}_b) \tag{26}$$

when

$$X^{\mu}(\vec{p}_b, \vec{p} - \vec{p}_b) = \sum_{\lambda_b, \lambda_c} \zeta^{B_c}(\lambda_b, \lambda_c) \times [\bar{V}_c(\vec{p}_c, \lambda_c)\Gamma^{\mu} U_b(\vec{p}_b, \lambda_b)]. \quad (27)$$

In fact $\mathcal{H}^{\mu}$ here can be identified as the hadronic matrix element $\langle 0|\bar{c}\Gamma^{\mu} b|B_c(\vec{p})\rangle$ which can be expressed in terms of the B_c-meson decay constant f_{B_c} as

$$\mathcal{H}^{\mu} = \langle 0|\bar{c}\Gamma^{\mu} b|B_c(\vec{p})\rangle = f_{B_c}(p^2)p^{\mu}.$$

Then using the result $L_{\nu\mu}p^{\mu} = L^{\mu} g_{\mu\nu}$ in the limit $m_{\mu} \to 0$, Eq. (21) can be modified to the form as

$$\mathcal{M}_{fi}(\mathrm{IB}) = \frac{G_F}{\sqrt{2}} \mathcal{V}_{cb} \epsilon^{\star\nu}(k, \lambda)[e f_{B_c} L^{\mu} g_{\mu\nu}]. \quad (28)$$

Now combining Eqs. (20) and (28) we obtain the total invariant transition amplitude for the decay process $B_c^-(p) \to \mu^-(p_1, r)\bar{\nu}_{\mu}(p_2, s)\gamma(k, \lambda)$ as

$$\mathcal{M}_{fi}(B_c^- \to \mu^- \bar{\nu}_{\mu}\gamma) = -\frac{eG_F}{\sqrt{2}} \mathcal{V}_{cb} \epsilon^{\star\nu}(k, \lambda) M_{\nu}, \quad (29)$$

where

$$M_{\nu} = L^{\mu}[\Pi_{\nu\mu} - f_{B_c} g_{\mu\nu}]. \quad (30)$$

However, one must ensure here the gauge invariance of the total transition amplitude by imposing conditions on which $k^{\nu} M_{\nu} = 0$. For this we may expand the hadronic tensor $\Pi_{\nu\mu}(p, k)$ in terms of independent Lorentz-invariant structure as

$$\Pi_{\nu\mu} = \alpha p_{\nu} p_{\mu} + \beta k_{\nu} k_{\mu} + \zeta p_{\nu} k_{\mu} + \delta k_{\nu} p_{\mu} + \xi g_{\nu\mu} + i\eta \varepsilon_{\nu\mu\rho\sigma} p^{\rho} k^{\sigma}, \quad (31)$$

where (α, β, ζ, δ, ξ, and η) are the form factors. Since $(\epsilon . k) = 0$; the form factors β and δ would be inconsequential and hence can be omitted. Again the term proportional to the form factor η, which is in fact the vector part of $\Pi_{\nu\mu}$, independently satisfies the gauge-invariance condition. Hence the gauge-invariance condition $k^{\nu} M_{\nu} = 0$ effectively boils down to

$$k^{\nu} L^{\mu}[\alpha p_{\nu} p_{\mu} + \zeta p_{\nu} k_{\mu} + (\xi - f_{B_c}) g_{\mu\nu}] = 0. \quad (32)$$

But since in the zero mass limit of the leptons one can find $L^{\mu} p_{\mu} \equiv L^{\mu} k_{\mu}$, Eq. (32) becomes

$$L^{\mu} k_{\mu}[(\alpha + \zeta)(p.k) + (\xi - f_{B_c})] = 0. \quad (33)$$

This implies the condition

$$(\alpha + \zeta)(p.k) + (\xi - f_{B_c}) = 0. \quad (34)$$

Then the gauge-invariant structure of the total transition amplitude consisting of both the structure-dependent and the bremsstrahlung parts can be written in the form

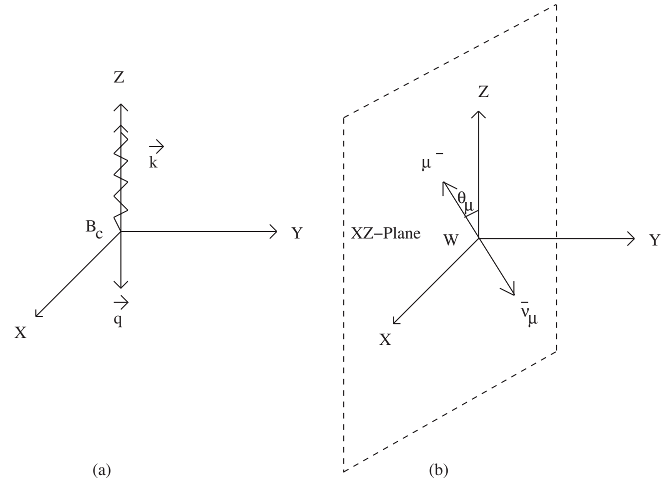

FIG. 2. Coordinate system for the radiative leptonic decay of a heavy meson: (a) Meson (B_c) rest frame and (b) $\mu\bar{\nu}_{\mu}$–CM frame.

$$\mathcal{M}_{fi}(B_c \to \mu\nu_{\mu}\gamma) = -\frac{eG_F}{\sqrt{2}} \mathcal{V}_{cb} \epsilon^{\star\nu}(k, \lambda) \times L^{\mu}[F_A(q^2)\{p_{\nu} k_{\mu} - (p.k) g_{\nu\mu}\} + iF_V(q^2)\varepsilon_{\nu\mu\rho\sigma} p^{\rho} k^{\sigma}], \quad (35)$$

where $F_V(q^2) = \eta$ and $F_A(q^2) = (\alpha + \zeta)$ are the form factors related to the vector and axial vector weak current contributions, respectively, which can be evaluated from Eqs. (24) and (31) imposing the gauge-invariance condition (34).

It is convenient to evaluate $F_V(q^2)$ and $F_A(q^2)$ in the hadronic part of the transition amplitude using the kinematic relation in the decaying meson rest frame [Fig. 2(a)] where the meson momentum $p \equiv (M_{B_c}, 0, 0, 0)$, the photon momentum $k \equiv (\tilde{E}_{\gamma}, 0, 0, \tilde{E}_{\gamma})$, and the photon energy $\tilde{E}_{\gamma} = \frac{M_{B_c}(1-y)}{2}$. Here we introduce a dimensionless variable $y = \frac{q^2}{M_{B_c}^2}$, which in the vanishing lepton mass limit has the range, $0 \leq y \leq 1$. Then, from Eq. (31) with gauge-invariant condition (34), we can realize in B_c-rest frame

$$F_A(q^2) = \frac{1}{M_{B_c}\tilde{E}_{\gamma}}(\tilde{\Pi}_{11} + f_{B_c}) \qquad F_V(q^2) = -i\frac{\tilde{\Pi}_{21}}{M_{B_c}\tilde{E}_{\gamma}}. \quad (36)$$

Finally, $\tilde{\Pi}_{11}(q^2)$ and $\tilde{\Pi}_{21}(q^2)$ can be obtained in the B_c-rest frame from the model expression for the hadronic tensor $\tilde{\Pi}_{\nu\mu}(q^2)$ in Eq. (24).

IV. THE DECAY WIDTH

The invariant transition amplitude in Eq. (35) can be rewritten as

$$\mathcal{M}_{fi} = -\frac{eG_F}{\sqrt{2}}\mathcal{V}_{cb}L^{\mu}H_{\mu} \tag{37}$$

with

$$H_{\mu} = \epsilon^{\star\nu}(k,\lambda)[F_A(q^2)\{k_{\mu}p_{\nu} - (p.k)g_{\mu\nu}\} + iF_V(q^2)\varepsilon_{\nu\mu\rho\sigma}p^{\rho}k^{\sigma}]. \tag{38}$$

Then the partial decay rate for $B_c^- \to \mu^-\bar{\nu}_{\mu}\gamma$ is written in its generic form as

$$d\Gamma = \frac{1}{2E_{B_c}}\sum_{r,s,\lambda}|\mathcal{M}_{fi}|^2 d\Omega_3, \tag{39}$$

where the three body phase-space factor $d\Omega_3$ is expressed as

$$d\Omega_3 = (2\pi)^4\delta^{(4)}(p-q-k)\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2} \times \frac{d^3\vec{k}}{(2\pi)^3 2k_0} \tag{40}$$

and

$$\sum_{r,s,\lambda}|\mathcal{M}_{fi}|^2 = \frac{e^2G_F^2}{2}|\mathcal{V}_{cb}|^2\Big(\sum_{r,s}L^{\mu}L^{\sigma\dagger}\Big)\Big(\sum_{\lambda}H_{\mu}H_{\sigma}^{\dagger}\Big). \tag{41}$$

We write $L^{\mu\sigma} = \sum_{r,s}(L^{\mu}L^{\sigma\dagger})$ representing the summations over the lepton spin indices r and s. Similarly we write $H_{\mu\sigma} = \sum_{\lambda}(H_{\mu}H_{\sigma}^{\dagger})$ representing the summation over the photon polarization index λ

Now for the sake of convenience we adopt the $\mu\bar{\nu}_{\mu}$-–center-of-mass (CM) frame [Fig. 2(b)] to account for the leptonic contribution, whereas the hadronic part is accounted for in the B_c-rest frame [Fig. 2(a)]. In the $\mu\bar{\nu}_{\mu}$–CM frame we have the respective energies of the outgoing particles μ, $\bar{\nu}_{\mu}$, and γ as

$$E_1 = \frac{1}{2}M_{B_c}\sqrt{y} = E_2, \qquad E_{\gamma} = \frac{M_{B_c}(1-y)}{2\sqrt{y}}. \tag{42}$$

In this reference frame it is easy to check that the timelike part of $L^{\mu\sigma}$, which is obtained as

$$L^{\mu\sigma} = 8[g^{\mu\rho}g^{\sigma\delta} + g^{\sigma\rho}g^{\mu\delta} - g^{\rho\delta}g^{\mu\sigma} + i\epsilon^{\mu\rho\sigma\delta}]p_{1\rho}p_{2\delta}, \tag{43}$$

vanishes. Thus with the nonvanishing spacelike terms L^{ij}, the product $(L^{\mu\sigma}H_{\mu\sigma})$ in Eq. (41) reduces to the form $(L^{ij}H_{ij})$. Then integrating the L^{ij} part over the lepton phase space in the $\mu\bar{\nu}_{\mu}$–CM frame, we obtain

$$\iint\frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2}L^{ij}\delta^{(4)}(p-p_1-p_2-k) = \frac{4\pi}{3}q^2\delta^{ij}. \tag{44}$$

As a result the photon polarization sum is further reduced to $\sum_{\lambda}H_iH_i^{\dagger}$. Using now the completeness relation of the photon polarization in transverse gauge, i.e., $\sum_{\lambda}\epsilon_{\alpha}^*(k,\lambda)\epsilon_{\beta}(k,\lambda) = (\delta_{\alpha\beta} - \hat{k}_{\alpha}\hat{k}_{\beta})$, we can obtain in the $(\mu\bar{\nu}_{\mu})$–CM frame

$$\sum_{\lambda}H_iH_i^{\dagger} = 2(p.k)^2[|F_A(q^2)|^2 + |F_V(q^2)|^2]. \tag{45}$$

Then the partial decay rate can be obtained in an invariant form

$$d\Gamma = \frac{G_F^2|\mathcal{V}_{cb}|^2e^2}{24\pi^4(2E_{B_c})}q^2(p.k)^2[|F_A(q^2)|^2 + |F_V(q^2)|^2]\frac{d^3\vec{k}}{2k_0}. \tag{46}$$

Now it is convenient to simplify the differential decay rate by integrating over the photon phase-space solid angle in the B_c-rest frame, such that

$$\frac{d\Gamma}{d\tilde{E}_{\gamma}} = \frac{G_F^2}{24\pi^3}|\mathcal{V}_{cb}|^2[|F_A(q^2)|^2 + |F_V(q^2)|^2] \times M_{B_c}^2(M_{B_c} - 2\tilde{E}_{\gamma})\tilde{E}_{\gamma}^3. \tag{47}$$

Finally, we integrate Eq. (47) over the kinematically allowed range of photon energy $0 \le \tilde{E}_{\gamma} \le M_{B_c}/2$ and use Eq. (36) to obtain the decay width $\Gamma(B_c \to \mu\nu_{\mu}\gamma)$ in the vanishing lepton mass limit ($m_{\mu} \simeq 0$) as

$$\Gamma(B_c \to \mu\nu_{\mu}\gamma) = \frac{\alpha G_F^2|\mathcal{V}_{cb}|^2}{6\pi^2}\int_0^{M_{B_c}/2} d\tilde{E}_{\gamma}\tilde{E}_{\gamma}(M_{B_c} - 2\tilde{E}_{\gamma}) \times [|\tilde{\Pi}_{11} + f_{B_c}|^2 + |\tilde{\Pi}_{21}|^2]. \tag{48}$$

Hence the study of the $B_c \to \mu\bar{\nu}_{\mu}\gamma$ process essentially reduces to the evaluation of the hadronic tensor components $\tilde{\Pi}_{11}(q^2)$ and $\tilde{\Pi}_{21}(q^2)$ as well as the decay constant f_{B_c} within the model framework. Equation (24) providing the model expression for the hadronic tensor components $\Pi_{\nu\mu}$ can be reduced in the B_c-rest frame to the form

$$\tilde{\Pi}_{\nu\mu}(q^2) = \frac{1}{\sqrt{6\pi N(0)}}\int_{-1}^{+1}d(\cos\theta) \times \int_0^{|p_b|_{\max}} d|\vec{p}_b||\vec{p}_b|^2\tilde{\mathcal{G}}_{B_c}(\vec{p}_b, -\vec{p}_b) \times \tilde{X}_{\mu\nu}(q^2, |\vec{p}_b|, \cos\theta). \tag{49}$$

Using free quark-antiquark spinors $U_b(\vec{p}_b, \lambda_b)$ and $V_c(\vec{p}_c, \lambda_c)$ and the necessary spin algebra, we can extract the relevant components of $\tilde{X}_{\nu\mu}$ in the B_c-rest frame from Eq. (22) for calculating $\tilde{\Pi}_{11}(q^2)$ and $\tilde{\Pi}_{21}(q^2)$. We obtain

$$\begin{aligned}\tilde{X}_{11}(q^2, \vec{p}_b) &= \sqrt{\frac{2}{\epsilon_c\epsilon_b}}\Big[\frac{2\tilde{Z}_{11}^{(c)}}{D_c} + \frac{\tilde{Z}_{11}^{(b)}}{D_b}\Big]\\ \tilde{X}_{21}(q^2, \vec{p}_b) &= i\sqrt{\frac{2}{\epsilon_c\epsilon_b}}\Big[\frac{2\tilde{Z}_{21}^{(c)}}{D_c} - \frac{\tilde{Z}_{21}^{(b)}}{D_b}\Big],\end{aligned} \tag{50}$$

where,

$$D_c = [2(M_{B_c} - \tilde{E}_\gamma)\{M_{B_c} - E_b(\vec{p}_b) + |\vec{p}_b|\cos\theta\} - (M_{B_c}^2 - m_b^2 + m_c^2 + 2M_{B_c}|\vec{p}_b|\cos\theta)], \tag{51}$$

$$\tilde{Z}_{11}^{(c)} = [(M_{B_c} - \tilde{E}_\gamma - E_b)(\epsilon_c\epsilon_b - |\vec{p}_b|^2) - m_c(\epsilon_c\epsilon_b + |\vec{p}_b|^2) - \tilde{E}_\gamma(\epsilon_b - \epsilon_c)|\vec{p}_b|\cos\theta - (\epsilon_b - \epsilon_c)|\vec{p}_b|^2\cos^2\theta], \tag{52}$$

$$\tilde{Z}_{21}^{(c)} = [\tilde{E}_\gamma(\epsilon_c\epsilon_b - |\vec{p}_b|^2) + \{(E_b - M_{B_c} + \tilde{E}_\gamma)(\epsilon_b - \epsilon_c) - m_c(\epsilon_b + \epsilon_c) + (\epsilon_c\epsilon_b - |\vec{p}_b|^2)\}|\vec{p}_b|\cos\theta]. \tag{53}$$

Similar expressions for D_b, $\tilde{Z}_{11}^{(b)}$, and $\tilde{Z}_{21}^{(b)}$ are found to be related to D_c, $\tilde{Z}_{11}^{(c)}$, and $\tilde{Z}_{21}^{(c)}$ respectively through the interchange of $m_c \leftrightarrow m_b$, $\epsilon_c \leftrightarrow \epsilon_b$, $E_c(-\vec{p}_b) \leftrightarrow E_b(\vec{p}_b)$, and $\vec{p}_b \leftrightarrow (-\vec{p}_b)$. Next we obtain the decay constant f_{B_c} from Eqs. (26) and (27) in the B_c-meson rest frame in the form

$$f_{B_c} = \frac{2\sqrt{6}}{M_{B_c}\sqrt{(2\pi)N(0)}}\int_0^{|p_b|_{\max}} d|\vec{p}_b||\vec{p}_b|^2\tilde{\mathcal{G}}_{B_c}(\vec{p}_b, -\vec{p}_b) \times \sqrt{\epsilon_c\epsilon_b}\left[1 - \frac{|\vec{p}_b|^2}{\epsilon_c\epsilon_b}\right]. \tag{54}$$

Thus with the model expressions for $\tilde{\Pi}_{11}$ and $\tilde{\Pi}_{21}$ obtainable from Eq. (49) using Eqs. (50)–(53) and that of the decay constant f_{B_c} in Eq. (54), we can evaluate numerically the decay width $\Gamma(B_c \to \mu\nu_\mu\gamma)$ from Eq. (48). Here $\tilde{\Pi}_{11}(q^2)$ and $\tilde{\Pi}_{21}(q^2)$ decide the contribution of the SD diagrams, while the f_{B_c}-term provides the bremsstrahlung contribution.

V. NUMERICAL RESULTS AND DISCUSSION

For numerical calculation, we take the quark masses (m_c, m_b), the potential parameters (a, V_0) and the resulting quark binding energies (E_c, E_b) as in Refs. [10,15,16]

$$\begin{aligned}(a, V_0) &\equiv (0.017\,166\ \text{GeV}^3, -0.1375\ \text{GeV})\\ (m_c, m_b) &\equiv (1.492\,76\ \text{GeV}, 4.776\,59\ \text{GeV})\\ (E_c, E_b) &\equiv (1.579\,51\ \text{GeV}, 4.766\,33\ \text{GeV}).\end{aligned} \tag{55}$$

The CKM parameter $\mathcal{V}_{cb}$, B_c-meson life time τ_{B_c}, and the decaying meson mass M_{B_c} are taken from the Particle Data Group [17] as

$$\mathcal{V}_{cb} = 0.0416, \qquad \tau_{B_c} = 0.46\ \text{ps}, \qquad M_{B_c} = 6.286\ \text{GeV}. \tag{56}$$

At the outset we must point out that the energy conservation constraint $E_b(\vec{p}_b) + E_c(-\vec{p}_b) = M_{B_c}$ assumed in our model framework corresponding to the B_c-rest frame, may present spurious kinematic singularities. This can be dealt with in the same manner as in [10] by assigning a running mass $m_b(|\vec{p}_b|)$ instead of the model value m_b which satisfies the energy conservation constraint in the form

$$m_b^2(|\vec{p}_b|) = M_{B_c}^2 + m_c^2 - 2M_{B_c}\sqrt{|\vec{p}_b|^2 + m_c^2}. \tag{57}$$

As a result, there would be an upper bound on the quark momentum $|\vec{p}_b| < \frac{M_{B_c}^2 - m_c^2}{2M_{B_c}} = |\vec{p}_b|_{\max}$ in order to retain $m_b^2(|\vec{p}_b|)$ positive definite.

Now it may be quite interesting to find the radial quark momentum distribution amplitude $|\vec{p}_b|\mathcal{G}_{B_c}(\vec{p}_b, -\vec{p}_b)$ in the B_c-meson rest frame and its behavior in the allowed range of momenta $0 < |\vec{p}_b| < |\vec{p}_b|_{\max}$. The radial quark momentum distribution for b-quark within the meson B_c (as well as B_u) has been depicted in Fig. 3. From the expectation value $\langle B_c(0)|\vec{p}_b^2|B_c(0)\rangle = \langle\vec{p}_b^2\rangle$, we find the rms-value of the b-quark momentum $\sqrt{\langle\vec{p}_b{}^2\rangle} = 0.66$ GeV, which is much less than $|\vec{p}_b|_{\max}$. Similarly we find $\langle E_b(\vec{p}_b)\rangle =$ 4.657 GeV and $\langle E_c(-\vec{p}_b)\rangle = 1.629$ GeV, which are very close to the respective model solutions for the quark binding energies $E_b = 4.766\,33$ GeV and $E_c = 1.579\,51$ GeV, respectively. We also obtain $\langle E_b(\vec{p}_b) + E_c(-\vec{p}_b)\rangle =$ 6.286 GeV which is equal to M_{B_c}. Thus we see from these results that our ansatz for the energy conservation is somehow justified since it is satisfied in an average sense.

Now using Eq. (49), we evaluate the relevant hadronic tensor components $\tilde{\Pi}_{11}(q^2)$ and $\tilde{\Pi}_{21}(q^2)$ representing the contributions of the SD diagrams in Figs. 1(a) and 1(b) separately at any particular q^2-value. We find the relative magnitudes of the contribution due to these two diagrams as

$$\frac{\tilde{\Pi}_{11}^b}{\tilde{\Pi}_{11}^c} = 0.20 \qquad \frac{\tilde{\Pi}_{21}^b}{\tilde{\Pi}_{21}^c} = 0.18, \tag{58}$$

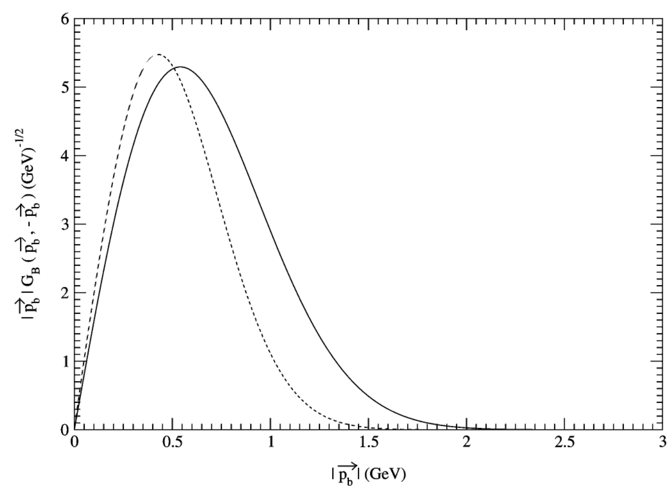

FIG. 3. Radial quark momentum distribution amplitude $|\vec{p}_b|\mathcal{G}_B(\vec{p}_b, -\vec{p}_b)$ for B_u-decay (dashed line) and for B_c-decay (solid line).

which indicates that the contribution of the structure-dependent process where the photon couples to the lighter quark $\bar{c}$ is dominant. This is expected since the electromagnetic coupling of the photon to the quark corresponds to the magnetic transition for which it is inversely proportional to the quark mass. This trend has been shown in our earlier work [10] rather more predominantly, as expected, in the decay process, $B_u \to \mu\nu_\mu\gamma$, as the mass of the antiquark "$\bar{u}$" is much more smaller than the "b"-quark mass.

Numerical values of $\tilde{\Pi}_{11}(q^2)$, $\tilde{\Pi}_{21}(q^2)$, and f_{B_c}, obtained from Eqs. (49)–(54) can enable one to find the photon energy dependence of the invariant form factors $F_A(q^2)$ and $F_V(q^2)$ through Eq. (36). It is noticed that at low enough photon energy $\tilde{E}_\gamma = 41$ MeV $< \Lambda_{\rm QCD}$, uncontrollable divergences occur, which are usually accounted for by appropriate radiative corrections. But in the tree-level discussion such as this, where we have not taken into account any possible radiative corrections, we prefer instead a lower cutoff for the photon energy at $|\tilde{E}_\gamma|_{\rm min} =$ 41 MeV. Thus $F_A(q^2)$ and $F_V(q^2)$ are considered here reliable only at the photon energy range 41 MeV $< \tilde{E}_\gamma \leq \frac{M_{B_c}}{2}$, which has been depicted in Fig. 4. The asymptotic behavior of the form factors obtained here is consistent with the prediction of Ref. [14] showing approximate equality $F_A(q^2) \simeq F_V(q^2)$ at large photon energy corresponding to $q^2 \to 0$.

Now using Eq. (48), we obtain the partial decay width for $B_c^- \to \mu^-\bar{\nu}_\mu\gamma$ as

$$\Gamma(B_c \to \mu\nu_\mu\gamma) = 9.77 \times 10^{-17}\left(\frac{\mathcal{V}_{cb}}{4.16 \times 10^{-2}}\right)^2 \text{ GeV}, \tag{59}$$

which leads to the branching ratio as

$$\mathcal{B}r(B_c \to \mu\nu_\mu\gamma) = 6.83 \times 10^{-5}. \tag{60}$$

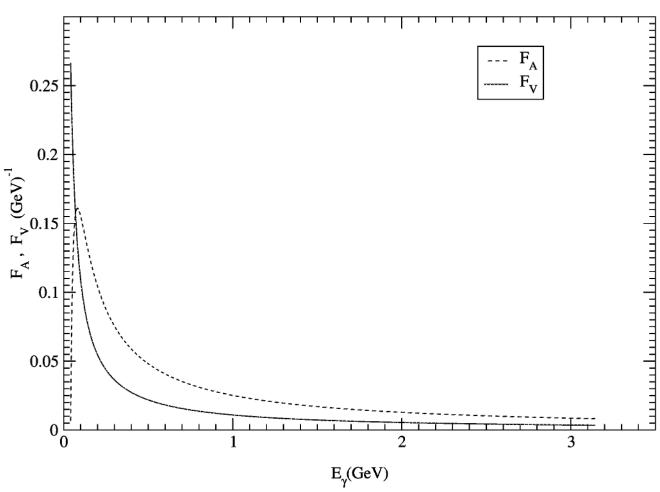

FIG. 4. Form factor $F_A(q^2)$ and $F_V(q^2)$ versus $\tilde{E}_\gamma$.

TABLE I. Results for $\mathcal{B}r(B_c \to \mu\nu_\mu\gamma)$ by different formalisms.

Formalism	Reference	$\mathcal{B}r(B_c \to \mu\nu_\mu\gamma)$	r_μ
RIQM	This paper	6.83×10^{-5}	0.025
NRQCD	[13]	6.0×10^{-5}	...
NRQM	[11]	6.0×10^{-5}	0.06–0.1
Rel. QM	[12]	3.0×10^{-5}	0.03
LFQM	[14]	2.3×10^{-5}	...
LCSR	[8]	2.0×10^{-5}	0.1

Here we have noticed that by varying the photon energy cutoff over a small range of values around $|\tilde{E}_\gamma|_{\rm min} =$ 41 MeV chosen earlier in the evaluation of the form factors, the branching ratio remains unaltered, which indicates that there is practically no uncertainty related to the choice of the lower cutoff $|\tilde{E}_\gamma|_{\rm min} = 41$ MeV. We now compare our prediction on $\mathcal{B}r(B_c \to \mu\nu_\mu\gamma)$ with those of other model calculations as shown in Table I. We find that our result, although quite comparable with those of Refs. [11,13], is certainly higher than other model predictions [8,12,14]. This is because the interference of the bremsstrahlung part with the structure-dependent part is found to be large and constructive.

In fact, the contribution of the interference term is found to be significant, being of the same order of magnitude as that due to the pure SD part and pure IB part. This result (Table II) is quite contrary to the observations of Ref. [14], where the SD contribution was dominant with the bremsstrahlung and interference term contributing one to two orders less, respectively. If we extend this analysis to $B_u \to \mu\nu_\mu\gamma$, which was studied earlier [10] with the SD diagrams only, the contribution of the bremsstrahlung part is found to be lesser by 1 order of magnitude compared to that of the pure SD part. This brings in a marginal change in the predicted branching ratio $\mathcal{B}r(B_u \to \mu\nu_\mu\gamma)$ from its earlier value 1.70×10^{-6} [10] to 2.17×10^{-6} as shown in Table II. Therefore, we find that neglecting the bremsstrahlung diagram in the case of $B_u \to \mu\nu_\mu\gamma$ decay may be a reasonable approximation. But such an assumption is not justified in the analysis of $B_c \to \mu\nu_\mu\gamma$ where the bremsstrahlung contribution is quite significant.

The decay width of the purely leptonic decay mode $B_c \to \mu\nu_\mu$ had been estimated earlier in the present model [18] as

$$\Gamma(B_c \to \mu\nu_\mu) = 3.41 \times 10^{-17} \text{ GeV}$$

with

$$\mathcal{B}r(B_c \to \mu\nu_\mu) = 2.38 \times 10^{-5}. \tag{61}$$

Then the radiative leptonic decay $B_c \to \mu\nu_\mu\gamma$ corresponds to an enhancement factor $R_\mu^{(B_c)}$ as

TABLE II. Branching ratios for B_c- and B_u-meson radiative leptonic decay modes:

Decay mode	$\mathcal{B}r$(SD)	$\mathcal{B}r$(IB)	$\mathcal{B}r$(IN)	$\mathcal{B}r$(Total)
$\mathcal{B}r(B_c \to \mu \bar{\nu}_\mu \gamma)$	3.08×10^{-5}	0.97×10^{-5}	2.78×10^{-5}	6.83×10^{-5}
$\mathcal{B}r(B_u \to \mu \bar{\nu}_\mu \gamma)$	1.70×10^{-6}	0.07×10^{-6}	0.40×10^{-6}	2.17×10^{-6}

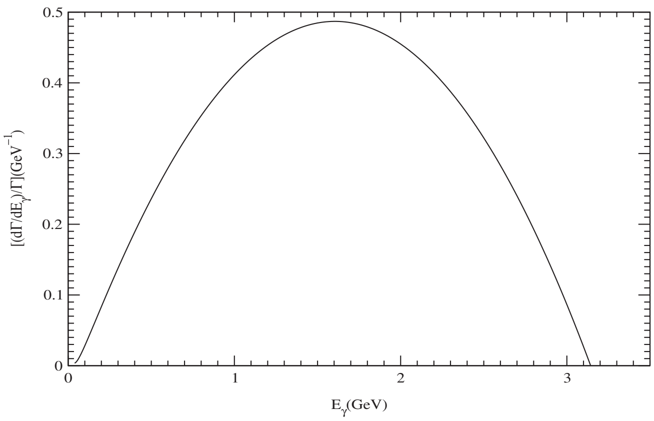

FIG. 5. Predicted photon energy spectrum.

$$R_\mu^{(B_c)} = \frac{\mathcal{B}r(B_c \to \mu \nu_\mu \gamma)}{\mathcal{B}r(B_c \to \mu \nu_\mu)} = 2.9. \qquad (62)$$

Thus we find that the decay width for the radiative leptonic B_c-meson decay is of the same order of magnitude as that for the pure leptonic mode. On the other hand, it is 1 order of magnitude higher in the context of B_u-meson decays as is evident from the calculated value of the corresponding enhancement factor $R_\mu^{(B_u)} = \frac{\mathcal{B}r(B_u \to \mu \nu_\mu \gamma)}{\mathcal{B}r(B_u \to \mu \nu_\mu)} = 11.5$. We can also estimate here the relative fraction of the same final state $(\mu \nu_\mu \gamma)$ coming from different sources such as B_u and B_c in a decay process by the ratio r_μ, which is obtained here as

$$r_\mu = \frac{N_{B_u}}{N_{B_c}} = \frac{\mathcal{B}r(B_u \to \mu \bar{\nu}_\mu \gamma)}{\mathcal{B}r(B_c \to \mu \bar{\nu}_\mu \gamma)} = 0.025. \qquad (63)$$

This indicates that a large number of $\mu \nu_\mu \gamma$ final states should be expected from B_c-decays compared to what is expected from B_u-decays.

Finally, using Eqs. (47) and (48) we evaluate $(\frac{1}{\Gamma} \frac{d\Gamma}{d\tilde{E}_\gamma})$ for different photon energies in the allowed range $|\tilde{E}_\gamma|_{\min} \leq \tilde{E}_\gamma \leq M_{B_c}/2$ to obtain the photon energy spectrum as shown in Fig. 5. We find that the shape of the photon energy spectrum is almost symmetrical about the peak at $\tilde{E}_\gamma \simeq \frac{M_{B_c}}{4}$ which is in good agreement with other model observations [11,13].

VI. SUMMARY AND CONCLUSION

We have calculated the decay width and branching ratio for the radiative leptonic decay $B_c \to \mu \nu_\mu \gamma$ in the relativistic independent quark model taking into account both the inner bremsstrahlung and structure-dependent diagrams. The contribution of the IB diagram is found to be of the same order of magnitude as that of the SD diagrams. Our predicted branching ratio for this decay $\mathcal{B}r(B_c \to \mu \nu_\mu \gamma) = 6.83 \times 10^{-5}$ is in agreement with some other model predictions [11,13]. The photon energy spectrum for this decay is also found to be symmetric about the peak value around $\tilde{E}_\gamma \simeq \frac{M_{B_c}}{4}$ as in other model predictions, which should render it quite accessible to the experimental observation at the LHC in the near future.

Thus we conclude that the present model (RIQM), which has already successfully been tested earlier in explaining wide ranging hadronic phenomena as cited in Refs. [15,16] and the references therein, predicts the radiative leptonic B_c decay in agreement with other model predictions. Since we do not use any free parameter in the present study based on the RIQM model, we do not so much claim any fine-tuning for quantitative precision in our predictions. Neither do we emphasize any improvement whatsoever due to the inclusion of relativistic effects which in any case are expected to be not so significant in the case of the B_c meson consisting of two heavy flavored quark-antiquark $(b\bar{c})$.

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[1] C. H. Chang and Y. Q. Chen, Phys. Rev. D **46**, 3845 (1992); Phys. Lett. B **284**, 127 (1992); S. S. Gershtein, A. K. Likhoded, and S. R. Slabospitsky, Int. J. Mod. Phys. A **6**, 2309 (1991).

[2] F. Abe *et al.* (CDF Collaboration), Phys. Rev. Lett. **81**, 2432 (1998); Phys. Rev. D **58**, 112004 (1998).

[3] K. Cheung, Phys. Rev. Lett. **71**, 3413 (1993); C. H. Chang and Y. Q. Chen, Phys. Rev. D **48**, 4086 (1993).

[4] G. T. Bodwin, E. Braaten, and G. P. Lepage, Phys. Rev. D **51**, 1125 (1995); **55**, 5853(E) (1997).

[5] N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963); M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).

[6] E. Braaten and S. Fleming, Phys. Rev. D **52**, 181 (1995); V. V. Kislev and A. V. Tkabladge, Yad. Fiz. **505**, 1714 (1989) [Sov. J. Nucl. Phys. **50**, 1063 (1989)].

[7] L. M. Michelangelo and S. R. Slabospitskili, Phys. Lett. B **410**, 299 (1997).

[8] T. M. Aliev and M. Savci, Phys. Lett. B **434**, 358 (1998); J. Phys. G **25**, 1205 (1999).

[9] G. Burdman, T. Goldman, and D. Wyler, Phys. Rev. D **51**, 111 (1995); A. Khodjamirian, G. Stoll, and D. Wyler, Phys. Lett. B **358**, 129 (1995); E. Lunghi, D. Pirjol, and D. Wyler, Nucl. Phys. **B649**, 349 (2003); S. Descotes-Genon and C. T. Sachrajda, Nucl. Phys. **B650**, 356 (2003); P. Colangelo, F. De Fazio, and G. Nardulli, Phys. Lett. B **372**, 331 (1996); **386**, 328 (1996); D. Atwood, G. Eilam, and A. Soni, Mod. Phys. Lett. A **11**, 1061 (1996); P. Ball and E. Kou, J. High Energy Phys. 04 (2003) 029; G. Eilam, I. Halperin, and R. R. Mendel, Phys. Lett. B **361**, 137 (1995); C. Q. Geng, C. C. Lih, and Wei-Min Zhang, Phys. Rev. D **57**, 5697 (1998); **59**, 114002 (1999); G. P. Korchemsky, D. Pirjol, and T.-M. Yan, Phys. Rev. D **61**, 114510 (2000); G. A. Chelkov, M. I. Gostkin, and Z. K. Silagadze, Phys. Rev. D **64**, 097503 (2001); Chien-Wen Hwang, Eur. Phys. J. C **46**, 379 (2006).

[10] N. Barik, Sk. Naimuddin, P. C. Dash, and Susmita Kar, Phys. Rev. D **77**, 014038 (2008).

[11] C. H. Chang, J. P. Cheng, and C. D. Lu, Phys. Lett. B **425**, 166 (1998); C. H. Chang, C. D. Lu, G. L. Wang, and H. S. Zong, Phys. Rev. D **60**, 114013 (1999).

[12] P. Colangelo and F. De Fazio, Mod. Phys. Lett. A **14**, 2303 (1999).

[13] G. Chiladze, A. F. Falk, and A. A. Petrov, Phys. Rev. D **60**, 034011 (1999).

[14] C. C. Lih, C. Q. Geng, and W-M. Zhang, Phys. Rev. D **59**, 114002 (1999).

[15] N. Barik, B. K. Dash, and P. C. Dash, Pramana **29**, 543 (1987); N. Barik, Sk. Naimuddin, S. Kar, and P. C. Dash, Phys. Rev. D **59**, 037301 (1998).

[16] N. Barik, P. C. Dash, and A. R. Panda, Phys. Rev. D **46**, 3856 (1992); N. Barik and P. C. Dash, *ibid.* **49**, 299 (1994); N. Barik and P. C. Dash, Mod. Phys. Lett. A **10**, 103 (1995); N. Barik, S. Kar, and P. C. Dash, Phys. Rev. D **57**, 405 (1998); N. Barik, Sk. Naimuddin, S. Kar, and P. C. Dash, Phys. Rev. D **63**, 014024 (2000); N. Barik, P. C. Dash, and A. R. Panda, Phys. Rev. D **47**, 1001 (1993); N. Barik and P. C. Dash, Phys. Rev. D **47**, 2788 (1993); **53**, 1366 (1996); N. Barik, S. K. Tripathy, S. Kar, and P. C. Dash, *ibid.* **56**, 4238 (1997); N. Barik, S. Kar, and P. C. Dash, Phys. Rev. D **63**, 114002 (2001).

[17] W-M Yao *et al.* (Particle Data Group), J. Phys. G **33**, 1 (2006).

[18] N. Barik and P. C. Dash, Phys. Rev. D **47**, 2788 (1993).