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Shadows, currents, and AdS fields

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Conformal totally symmetric arbitrary spin currents and shadow fields in flat space-time of dimension greater than or equal to four are studied. A gauge invariant formulation for such currents and shadow fields is developed. Gauge symmetries are realized by involving the Stueckelberg fields. A realization of global conformal boost symmetries is obtained. Gauge invariant differential constraints for currents and shadow fields are obtained. AdS/CFT correspondence for currents and shadow fields and the respective normalizable and non-normalizable solutions of massless totally symmetric arbitrary spin AdS fields are studied. The bulk fields are considered in a modified de Donder gauge that leads to decoupled equations of motion. We demonstrate that leftover on shell gauge symmetries of bulk fields correspond to gauge symmetries of boundary currents and shadow fields, while the modified de Donder gauge conditions for bulk fields correspond to differential constraints for boundary conformal currents and shadow fields. Breaking conformal symmetries, we find interrelations between the gauge invariant formulation of the currents and shadow fields, and the gauge invariant formulation of massive fields.

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I. INTRODUCTION

In view of the aesthetic features of conformal field theory, an interest in this theory has been periodically renewed (see [1] and references therein). Conjectured duality [2] of large N conformal $\mathcal{N}=4$ supersymmetric Yang-Mills (SYM) theory and type IIB superstring theory in $\mathrm{AdS}_5\times S^5$ has triggered intensive and in-depth study of various aspects of conformal fields. In space-time of dimension $d\geq 4$, the conformal fields studied in this paper can be separated into two groups: conformal currents and shadow fields. That is to say that a field having Lorentz algebra spin s and conformal dimension $\Delta=s+d-2$, is referred to as a conformal current with canonical dimension,[1] while a field having Lorentz algebra spin s and dual conformal dimension $\Delta=2-s$ is referred to as a shadow field.[2] In the framework of AdS/CFT correspondence, the conformal currents and shadow fields manifest themselves in two related ways at least. First, the conformal currents appear as boundary values of *normalizable* solutions of equations of motion for bulk fields of IIB supergravity in $\mathrm{AdS}_5\times S^5$ background, while the shadow fields appear as boundary values of *non-normalizable* solutions of equations of motion for bulk fields of IIB supergravity (see e.g. [9–13][3]). Second, the conformal currents, which are dual to string theory states, can be built in terms of fields of SYM theory. In view of these relations to IIB supergravity/superstring in $\mathrm{AdS}_5\times S^5$ and SYM theory, we think that various alternative formulations of conformal currents and shadow fields will be useful to understand string/gauge theory dualities better.

The purpose of this paper is to develop gauge invariant formulation for conformal currents and shadow fields. In this paper, we discuss bosonic arbitrary spin conformal currents and shadow fields in space-time of dimension $d\geq 4$. Our approach to the conformal currents and shadow fields can be summarized as follows.

(i) Starting with the field content of the standard formulation of currents (and shadow fields), we introduce additional field degrees of freedom (D.o.F), i.e., we extend the space of fields entering the standard conformal field theory. We note that these additional field D.o.F are similar to the ones used in the gauge invariant formulation of massive fields. Sometimes, such additional field D.o.F are referred to as Stueckelberg fields.

(ii) On the extended space of currents (and shadow fields), we introduce new differential constraints, gauge transformations, and conformal algebra transformations.

(iii) The new differential constraints are invariant under the gauge transformations and the conformal algebra transformations.

(iv) The gauge symmetries and the new differential constraints make it possible to match our approach and the standard one, i.e., by appropriate gauge fixing of

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[1]We note that conformal currents with $s=1$, $\Delta=d-1$ and $s=2$, $\Delta=d$, correspond to a conserved vector current and a conserved traceless rank-2 tensor field (energy-momentum tensor), respectively. Conserved conformal currents can be built from massless scalar, spinor, and spin-1 fields (see e.g. [3]). Discussion of higher-spin conformal conserved charges bilinear in $4d$ massless fields of arbitrary spins may be found in [4].

[2]It is the shadow fields that are used to discuss conformal invariant equations of motion and Lagrangian formulations (see e.g. [1,5–7]). Discussion of equations for mixed-symmetry conformal fields with discrete Δ may be found in [8].

[3]A discussion of shadow field dualities may be found in Refs. [14,15].

the Stueckelberg fields and by solving differential constraints, we obtain standard formulation of conformal currents and shadow fields.

We apply our approach to the study of AdS/CFT correspondence at the level of massless modes/currents, shadow fields matching. We shall demonstrate that normalizable modes of massless AdS fields are related to conformal currents, while non-normalizable modes of massless AdS fields are related to shadow fields. Such correspondence was studied for scalar fields in [9,12], and for massless arbitrary spin fields taken to be in light-cone gauge in Ref. [11]. In the latter reference, we have also developed light-cone formulation of conformal field theories (CFT). Light-cone formulation of CFT breaks boundary Lorentz symmetries and, therefore, is not commonly used. It is desirable, therefore, to develop AdS/CFT correspondence for arbitrary spin fields by maintaining boundary Lorentz symmetries.[4] Our approach to the study of AdS/CFT correspondence can be summarized as follows.

(i) We use a modified Lorentz gauge (for a spin-1 field) found in Ref. [11] and a modified de Donder gauge (for spin $s \geq 2$ fields) found in Ref. [16]. A remarkable property of these gauges is that they lead to the simple *decoupled* bulk equations of motion which can be solved in terms of the Bessel function, and this simplifies considerably the study of AdS/CFT correspondence.[5]

(ii) The number of boundary gauge conformal currents (or shadow fields) involved in our gauge invariant approach coincides with the number of bulk massless gauge AdS fields involved in the approach of Ref. [17]. Note however that, instead of using the approach presented in Ref. [17], we use the CFT adapted formulation of arbitrary spin AdS field theory developed in [16].

(iii) The number of gauge transformation parameters involved in our gauge invariant approach to currents (or shadow fields) coincides with the number of gauge transformation parameters of bulk massless gauge AdS fields involved in the standard approach of Ref. [17].

(iv) Our modified Lorentz gauge (for a spin-1 field) and modified de Donder gauge (for spin $s \geq 2$ fields) turn out to be related to the new differential constraints we obtained in the framework of our gauge invariant approach to conformal currents (and shadow fields).

(v) *Leftover on shell* gauge symmetries of massless bulk AdS fields are related to the gauge symmetries of boundary conformal currents (or shadow fields).

The rest of the paper is organized as follows. In Sec. II, we summarize the notation used in this paper and briefly review the standard approach to conformal currents and shadow fields. In Secs. III and IV, we start with the respective examples of spin-1 conformal currents and spin-1 shadow fields. We illustrate our gauge invariant approach to describing conformal currents and shadow fields.

Sections V and VI are devoted to spin-2 conformal currents and spin-2 shadow fields, respectively. We develop our gauge invariant approach and demonstrate how our spin-2 current is related with the standard energy-momentum tensor of CFT. We also discuss how our spin-2 shadow field is related to the one appearing in the standard approach to CFT.

In Secs. VII and VIII, we develop a gauge invariant approach to arbitrary spin-s conformal currents and shadow fields, respectively. Fixing Stueckelberg gauge symmetries and solving differential constraints for currents and shadow fields, we prove equivalence of our gauge invariant approach and the standard approach to CFT.

In Sec. IX, we discuss the two-point current-shadow field interaction vertex. Section X is devoted to the study of AdS/CFT correspondence for massless low spin, $s = 1$, 2, bulk AdS fields, and boundary low spin, $s = 1$, 2, currents and shadow fields. Section XI is devoted to the study of AdS/CFT correspondence for massless arbitrary spin bulk AdS fields and boundary arbitrary spin currents and shadow fields.

In Sec. XII, we discuss interrelations between our gauge invariant approach to currents (and shadow fields) and a gauge invariant (Stueckelberg) approach to massive fields in flat space. In due course, we discuss the de Donder-like gauge condition for arbitrary spin-s, $s \geq 2$, massive field in the framework of the Stueckelberg approach to massive fields. The de Donder-like gauge we find leads to a surprisingly simple gauge-fixed action for massive arbitrary spin fields.

We collect various technical details in four appendices. In Appendix A, we discuss restrictions imposed on the two-point current-shadow field interaction vertex by gauge symmetries and by dilatation symmetries. In Appendix B, we discuss restrictions imposed on conformal boost transformations by gauge symmetries. In Appendix C, we review the modified Lorentz condition for spin-1 massless AdS fields and the modified de Donder gauge for massless spin-2 AdS fields, while Appendix D is devoted to the modified de Donder gauge for fields propagating in conformal space. In Appendix E, we present some details of matching of the leftover gauge symmetries of bulk AdS fields and the gauge symmetries of boundary currents and shadow fields.

[4]One of popular gauges that respects boundary Lorentz symmetries is the radial gauge. However, in contrast to our approach, the radial gauge does not allow us to treat normalizable and non-normalizable solutions of bulk equations of motion on an equal footing.

[5]To our knowledge, our modified Lorenz gauge (for a spin-1 field) and modified de Donder gauge (for spin $s \geq 2$ fields) are unique, first-derivative gauges that lead to decoupled bulk equations of motion. Another gauge that also leads to decoupled bulk equations of motion is the light-cone gauge (see Ref. [11]). But, light-cone gauge breaks boundary Lorentz symmetries.

II. PRELIMINARIES

A. Notation

Our conventions are as follows. x^a denotes coordinates in d-dimensional flat space-time, while ∂_a denotes derivatives with respect to x^a, $\partial_a \equiv \partial/\partial x^a$. Vector indices of the Lorentz algebra $so(d-1,1)$ take the values $a, b, c, e = 0, 1, \ldots, d-1$. We use the mostly positive flat metric tensor η^{ab}. To simplify our expressions we drop η_{ab} in scalar products, i.e., we use $X^a Y^a \equiv \eta_{ab} X^a Y^b$.

We use a set of the creation operators α^a, α^z, and the respective set of annihilation operators $\bar\alpha^a$, $\bar\alpha^z$,

$$[\bar\alpha^a, \alpha^b] = \eta^{ab}, \qquad [\bar\alpha^z, \alpha^z] = 1, \tag{2.1}$$

$$\bar\alpha^a|0\rangle = 0, \qquad \bar\alpha^z|0\rangle = 0, \tag{2.2}$$

$$\alpha^{a\dagger} = \bar\alpha^a, \qquad \alpha^{z\dagger} = \bar\alpha^z. \tag{2.3}$$

These operators will often be referred to as oscillators in what follows.[6] The oscillators α^a, $\bar\alpha^a$ and α^z, $\bar\alpha^z$, transform in the respective vector and scalar representations of the $so(d-1,1)$ Lorentz algebra.

Throughout this paper we use operators constructed out of the derivatives and the oscillators,

$$\Box = \partial^a\partial^a, \qquad \alpha\partial = \alpha^a\partial^a, \qquad \bar\alpha\partial = \bar\alpha^a\partial^a, \tag{2.4}$$

$$\alpha^2 = \alpha^a\alpha^a, \qquad \bar\alpha^2 = \bar\alpha^a\bar\alpha^a, \tag{2.5}$$

$$N_\alpha \equiv \alpha^a\bar\alpha^a, \qquad N_z \equiv \alpha^z\bar\alpha^z. \tag{2.6}$$

B. Global conformal symmetries

In d-dimensional flat space-time, the conformal algebra $so(d,2)$ consists of translation generators P^a, dilatation generator D, conformal boost generators K^a, and generators of the $so(d-1,1)$ Lorentz algebra J^{ab}. We assume the following normalization for commutators of the conformal algebra:

$$[D, P^a] = -P^a, \qquad [P^a, J^{bc}] = \eta^{ab}P^c - \eta^{ac}P^b, \tag{2.7}$$

$$[D, K^a] = K^a, \qquad [K^a, J^{bc}] = \eta^{ab}K^c - \eta^{ac}K^b, \tag{2.8}$$

$$[P^a, K^b] = \eta^{ab}D - J^{ab}, \tag{2.9}$$

$$[J^{ab}, J^{ce}] = \eta^{bc}J^{ae} + 3\,\text{terms}. \tag{2.10}$$

Let $|\phi\rangle$ denote the conformal current (or shadow field) in flat space-time of dimension $d \geq 4$. Under conformal algebra transformations the $|\phi\rangle$ transforms as

$$\delta_{\hat G}|\phi\rangle = \hat G|\phi\rangle, \tag{2.11}$$

where a realization of the conformal algebra generators $\hat G$ in terms of differential operators takes the form

$$P^a = \partial^a, \tag{2.12}$$

$$J^{ab} = x^a\partial^b - x^b\partial^a + M^{ab}, \tag{2.13}$$

$$D = x\partial + \Delta, \tag{2.14}$$

$$K^a = K^a_{\Delta,M} + R^a, \tag{2.15}$$

and we use the notation

$$K^a_{\Delta,M} \equiv -\frac{1}{2}x^2\partial^a + x^a D + M^{ab}x^b, \tag{2.16}$$

$$x\partial \equiv x^a\partial^a, \qquad x^2 = x^a x^a. \tag{2.17}$$

In (2.13), (2.14), and (2.15), Δ is the operator of conformal dimension, M^{ab} is the spin operator of the Lorentz algebra,

$$[M^{ab}, M^{ce}] = \eta^{bc}M^{ae} + 3\,\text{terms}. \tag{2.18}$$

The spin operator of the Lorentz algebra is well known for arbitrary spin conformal currents and shadow fields. The spin operator of currents and shadow fields studied in this paper takes the form

$$M^{ab} \equiv \alpha^a\bar\alpha^b - \alpha^b\bar\alpha^a. \tag{2.19}$$

R^a is operator depending, in general, on derivatives with respect to space-time coordinates[7] and not depending on space-time coordinates x^a,

$$[P^a, R^b] = 0. \tag{2.20}$$

In standard formulation of conformal currents and shadow fields, the operator R^a is equal to zero, while in the gauge invariant approach that we develop in this paper, the operator R^a is nontrivial. This implies that, in the framework of gauge invariant approach, a complete description of the conformal currents and shadow fields requires, among other things, finding the operator R^a.

C. Standard approach to conformal currents and shadow fields

We begin with a brief review of the standard approach to conformal currents and shadow fields. To keep our presentation as simple as possible we restrict our attention to the case of arbitrary spin *totally symmetric* conformal currents and shadow fields which have the appropriate canonical conformal dimensions given below. In this section, we

[6] We use oscillator formulation [18–20] to handle the many indices appearing for tensor fields. It can also be reformulated as an algebra acting on the symmetric-spinor bundle on the manifold M [21].

[7] For the conformal currents and shadow fields studied in this paper, the operator R^a does not depend on derivatives. Dependence on derivatives of R^a appears e.g. in the ordinary-derivative approach to conformal fields [22].

recall the main facts of conformal field theory about these currents and shadow fields.

Conformal current with the canonical conformal dimension. Consider the totally symmetric rank-s tensor field $T^{a_1\ldots a_s}$ of the Lorentz algebra $so(d-1,1)$. The field is referred to as spin-s *conformal current with canonical dimension* if $T^{a_1\ldots a_s}$ satisfies the constraints

$$T^{aaa_3\ldots a_s} = 0, \qquad \partial^a T^{aa_2\ldots a_s} = 0 \tag{2.21}$$

and has the conformal dimension[8]

$$\Delta = s + d - 2, \tag{2.22}$$

which is referred to as *the canonical conformal dimension of spin-s conformal current*. Taking into account that the operator R^a of the conformal current $T^{a_1\ldots a_s}$ is equal to zero, using the well-known spin operator M^{ab} of the totally symmetric traceless current $T^{a_1\ldots a_s}$ and Δ in (2.22), one can make sure that constraints (2.21) are invariant under conformal algebra transformations (2.11).

Shadow field with the canonical conformal dimension. Consider the totally symmetric rank-s tensor field $\Phi^{a_1\ldots a_s}$ of the Lorentz algebra $so(d-1,1)$. The field $\Phi^{a_1\ldots a_s}$ is referred to as shadow field if it meets the following requirements:

(i) The field $\Phi^{a_1\ldots a_s}$ is traceless,

$$\Phi^{aaa_3\ldots a_s} = 0. \tag{2.23}$$

(ii) The field $\Phi^{a_1\ldots a_s}$ transforms under the conformal algebra symmetries so that the following two-point current-shadow field interaction vertex

$$\mathcal{L} = \frac{1}{s!}\Phi^{a_1\ldots a_s}T^{a_1\ldots a_s} \tag{2.24}$$

is invariant (up to total derivative) under conformal algebra transformations.

We now note that:

(i) Taking into account conformal dimension of current (2.22) and requiring vertex $\mathcal{L}$ (2.24) to be invariant under the dilatation transformation, we obtain conformal dimension of the spin-s shadow field,

$$\Delta = 2 - s, \tag{2.25}$$

which is referred to as *the canonical conformal dimension of spin-s shadow field*. Taking into account that the operator R^a of the conformal current $T^{a_1\ldots a_s}$ is equal to zero and requiring vertex $\mathcal{L}$ (2.24) to be invariant under the conformal boost transformations we find that the operator R^a of the shadow field $\Phi^{a_1\ldots a_s}$ is also equal to zero.

(ii) The divergence-free constraint (2.21) and requirement for the vertex $\mathcal{L}$ to be invariant imply that the shadow field is defined by a module of gauge transformation

$$\delta\Phi^{a_1\ldots a_1} = \Pi^{tr}\partial^{(a_1}\xi^{a_2\ldots a_s)}, \tag{2.26}$$

where $\xi^{a_1\ldots a_{s-1}}$ is the traceless parameter of gauge transformation and the projector Π^{tr} is inserted to respect the traceless constraint (2.23).

III. GAUGE INVARIANT FORMULATION OF SPIN-1 CONFORMAL CURRENT

To discuss the gauge invariant formulation of the spin-1 conformal current in flat space of dimension $d \geq 4$ we use one vector field $\phi_{\rm cur}^a$ and one scalar field $\phi_{\rm cur}$:

$$\phi_{\rm cur}^a, \qquad \phi_{\rm cur}. \tag{3.1}$$

The fields $\phi_{\rm cur}^a$ and $\phi_{\rm cur}$ transform in the respective vector and scalar irreps of the Lorentz algebra $so(d-1,1)$. We note that fields $\phi_{\rm cur}^a$ and $\phi_{\rm cur}$ (3.1) have the conformal dimensions

$$\Delta_{\phi_{\rm cur}^a} = d - 1, \qquad \Delta_{\phi_{\rm cur}} = d - 2. \tag{3.2}$$

We now introduce the following differential constraint[9]:

$$\partial^a\phi_{\rm cur}^a + \Box\phi_{\rm cur} = 0. \tag{3.3}$$

It is easy to see that this constraint is invariant under gauge transformations

$$\delta\phi_{\rm cur}^a = \partial^a\xi_{\rm cur}, \tag{3.4}$$

$$\delta\phi_{\rm cur} = -\xi_{\rm cur}, \tag{3.5}$$

where $\xi_{\rm cur}$ is a gauge transformation parameter.

In order to obtain a realization of conformal algebra symmetries we use the oscillators. We collect fields (3.1) into a ket-vector $|\phi_{\rm cur}\rangle$ defined by

$$|\phi_{\rm cur}\rangle = (\phi_{\rm cur}^a\alpha^a + \phi_{\rm cur}\alpha^z)|0\rangle. \tag{3.6}$$

A realization of the spin operator M^{ab} on $|\phi_{\rm cur}\rangle$ is given in (2.19), while a realization of the operator Δ,

$$\Delta = d - 1 - N_z, \tag{3.7}$$

can be read from (3.2). We then find that a realization of the operator R^a on $|\phi_{\rm cur}\rangle$ takes the form

$$R^a = (2-d)\alpha^a\bar{\alpha}^z. \tag{3.8}$$

[8]The fact that expression in the right-hand side of (2.22) is the lowest energy value of the totally symmetric spin-s massless fields propagating in AdS$_{d+1}$ space was demonstrated in Ref. [23]. Generalization of relation (2.22) to mixed symmetric fields in AdS may be found in Ref. [24].

[9]Constraint (3.3) can simply be obtained by adapting the standard procedure of introducing the Stueckelberg field for a massive spin-1 field. Namely, representing the standard conserved spin-1 current as $T_{\rm cur}^a = \phi_{\rm cur}^a + \partial^a\phi_{\rm cur}$ and using conservation law $\partial^a T_{\rm cur}^a = 0$, we obtain (3.3). For spin $s > 2$ fields, such procedure involves complicated higher-derivative expressions and turns out to be inconvenient for developing a gauge invariant approach to both massive and conformal theories.

Using this, we make sure that constraint (3.3) is invariant under transformations of the conformal algebra (2.11). In terms of the fields $\phi_{\rm cur}^a$, $\phi_{\rm cur}$, action of operator R^a (3.8) can be represented as

$$R^a\phi_{\rm cur}^b = (2-d)\eta^{ab}\phi_{\rm cur}, \tag{3.9}$$

$$R^a\phi_{\rm cur} = 0. \tag{3.10}$$

From (3.5), we see that the scalar field $\phi_{\rm cur}$ transforms as the Stueckelberg field, i.e., this field can be gauged away via Stueckelberg gauge fixing, $\phi_{\rm cur}=0$. If we gauge away the scalar field, then the remaining vector field $\phi_{\rm cur}^a$ becomes, according to constraint (3.3), divergence-free. In other words, our constraint (3.3) taken to be in the gauge $\phi_{\rm cur}=0$ leads to the well-known divergence-free constraint of the standard approach.[10]

We note that our approach can be related with the standard one without gauge fixing. Consider vector field

$$T_{\rm cur}^a = \phi_{\rm cur}^a + \partial^a\phi_{\rm cur}. \tag{3.11}$$

It is easy to see that

(i) $T_{\rm cur}^a$ is invariant under gauge transformations (3.4) and (3.5).

(ii) Denoting the left hand side of (3.3) by $C_{\rm cur}$ we get

$$\partial^a T_{\rm cur}^a = C_{\rm cur}, \tag{3.12}$$

i.e., constraint $C_{\rm cur}=0$ (3.3) amounts to

$$\partial^a T_{\rm cur}^a = 0. \tag{3.13}$$

In our approach, the gauge invariant vector field $T_{\rm cur}^a$ (3.11) is counterpart of the conserved current in the standard formulation of CFT.

IV. GAUGE INVARIANT FORMULATION OF SPIN-1 SHADOW FIELD

To discuss the gauge invariant formulation of the spin-1 shadow field in space of dimension $d\geq 4$ we use one vector field $\phi_{\rm sh}^a$ and one scalar field $\phi_{\rm sh}$:

$$\phi_{\rm sh}^a, \qquad \phi_{\rm sh}. \tag{4.1}$$

The fields $\phi_{\rm sh}^a$ and $\phi_{\rm sh}$ transform in the respective vector and scalar representations of the Lorentz algebra $so(d-1,1)$. We note that these fields have the conformal dimensions

$$\Delta_{\phi_{\rm sh}^a} = 1, \qquad \Delta_{\phi_{\rm sh}} = 2. \tag{4.2}$$

We now introduce the following differential constraint:

$$\partial^a\phi_{\rm sh}^a + \phi_{\rm sh} = 0. \tag{4.3}$$

[10] As in the standard approach to CFT, our currents can be considered either as fundamental field degrees of freedom or as composite operators. At the group theoretical level that we study in this paper, this distinction is immaterial.

It is easy to see that this constraint is invariant under gauge transformations

$$\delta\phi_{\rm sh}^a = \partial^a\xi_{\rm sh}, \tag{4.4}$$

$$\delta\phi_{\rm sh} = -\Box\xi_{\rm sh}, \tag{4.5}$$

where $\xi_{\rm sh}$ is a gauge transformation parameter.

As before, to obtain a realization of conformal algebra symmetries we use the oscillators and introduce a ket-vector $|\phi_{\rm sh}\rangle$ defined by

$$|\phi_{\rm sh}\rangle = (\phi_{\rm sh}^a\alpha^a + \phi_{\rm sh}\alpha^z)|0\rangle. \tag{4.6}$$

A realization of the spin operator M^{ab} on $|\phi_{\rm sh}\rangle$ is given in (2.19), while a realization of the operator Δ,

$$\Delta = 1 + N_z, \tag{4.7}$$

can be read from (4.2). We then find that a realization of the operator R^a on $|\phi_{\rm sh}\rangle$ takes the form

$$R^a = (d-2)\alpha^z\bar\alpha^a. \tag{4.8}$$

Using this, we check that constraint (4.3) is invariant under transformations of the conformal algebra (2.11). In terms of the fields $\phi_{\rm sh}^a$, $\phi_{\rm sh}$, action of operator R^a (4.8) can be represented as

$$R^a\phi_{\rm sh}^b = 0, \tag{4.9}$$

$$R^a\phi_{\rm sh} = (d-2)\phi_{\rm sh}^a. \tag{4.10}$$

Gauge transformation of the scalar field $\phi_{\rm sh}$ (4.5) involves d'Alembertian operator $\Box$, i.e., this transformation is not realized as the standard Stueckelberg (Goldstone) gauge symmetry. Therefore the scalar field appearing in the gauge invariant formulation of the spin-1 shadow field cannot be referred to as a Stueckelberg field. We note that our field $\phi_{\rm sh}^a$ can be identified with the shadow field Φ^a of the standard approach to CFT.

As in the case of the conformal current, we can introduce the gauge invariant field $T_{\rm sh}^a$,

$$T_{\rm sh}^a = \Box\phi_{\rm sh}^a + \partial^a\phi_{\rm sh}. \tag{4.11}$$

One can check that

(i) $T_{\rm sh}^a$ is invariant under gauge transformations (4.4) and (4.5).

(ii) The differential constraint for gauge fields (4.3) leads to a divergence-free constraint for the field $T_{\rm sh}^a$,

$$\partial^a T_{\rm sh}^a = 0. \tag{4.12}$$

However, constraint (4.3) is not equivalent to (4.12). Namely, if we denote the left-hand side of (4.3) by $C_{\rm sh}$, then we get

$$\partial^a T_{\rm sh}^a = \Box C_{\rm sh}. \tag{4.13}$$

We see that constraint $C_{\rm sh}=0$ (4.3) leads to constraint

(4.12), while constraint (4.12) does not imply the constraint $C_{\rm sh} = 0$, in general.

V. GAUGE INVARIANT FORMULATION OF SPIN-2 CONFORMAL CURRENT

To discuss the gauge invariant formulation of the spin-2 conformal current in flat space of dimension $d \geq 4$ we use one rank-2 tensor field $\phi_{\rm cur}^{ab}$, one vector field $\phi_{\rm cur}^a$, and one scalar field $\phi_{\rm cur}$:

$$\phi_{\rm cur}^{ab}, \qquad \phi_{\rm cur}^a, \qquad \phi_{\rm cur}. \tag{5.1}$$

The fields $\phi_{\rm cur}^{ab}$, $\phi_{\rm cur}^a$, and $\phi_{\rm cur}$ transform in the respective rank-2 tensor, vector, and scalar representations of the Lorentz algebra $so(d-1,1)$. Note that the field $\phi_{\rm cur}^{ab}$ is not traceless. We note that fields (5.1) have the conformal dimensions

$$\Delta_{\phi_{\rm cur}^{ab}} = d, \qquad \Delta_{\phi_{\rm cur}^a} = d-1, \qquad \Delta_{\phi_{\rm cur}} = d-2. \tag{5.2}$$

We now introduce the following differential constraints:

$$\partial^b \phi_{\rm cur}^{ab} - \frac{1}{2}\partial^a \phi_{\rm cur}^{bb} + \Box \phi_{\rm cur}^a = 0, \tag{5.3}$$

$$\partial^a \phi_{\rm cur}^a + \frac{1}{2}\phi_{\rm cur}^{aa} + u\Box\phi_{\rm cur} = 0, \tag{5.4}$$

$$u \equiv \sqrt{2}\Big(\frac{d-1}{d-2}\Big)^{1/2}. \tag{5.5}$$

One can make sure that these constraints are invariant under gauge transformations

$$\delta\phi_{\rm cur}^{ab} = \partial^a \xi_{\rm cur}^b + \partial^b \xi_{\rm cur}^a + \frac{2}{d-2}\eta^{ab}\Box\xi_{\rm cur}, \tag{5.6}$$

$$\delta\phi_{\rm cur}^a = \partial^a \xi_{\rm cur} - \xi_{\rm cur}^a, \tag{5.7}$$

$$\delta\phi_{\rm cur} = -u\xi_{\rm cur}, \tag{5.8}$$

where $\xi_{\rm cur}^a$, $\xi_{\rm cur}$ are gauge transformation parameters.

In order to obtain a realization of conformal algebra symmetries in an easy-to-use form we use oscillators (2.1) and collect fields (5.1) into a ket-vector $|\phi_{\rm cur}\rangle$ defined by

$$|\phi_{\rm cur}\rangle = \Big(\frac{1}{2}\phi_{\rm cur}^{ab}\alpha^a\alpha^b + \phi_{\rm cur}^a\alpha^a\alpha^z + \frac{1}{\sqrt{2}}\phi_{\rm cur}\alpha^z\alpha^z\Big)|0\rangle. \tag{5.9}$$

A realization of the spin operator M^{ab} on $|\phi_{\rm cur}\rangle$ is given in (2.19), while a realization of the operator Δ,

$$\Delta = d - N_z, \tag{5.10}$$

can be read from (5.2). We then find that a realization of the operator R^a on $|\phi_{\rm cur}\rangle$ takes the form

$$R^a = \bar{r}\Big(\tilde{C}^a + \frac{2}{d(d-2)}\alpha^2\bar{C}_\perp^a\Big), \tag{5.11}$$

$$\tilde{C}^a \equiv \alpha^a - \frac{1}{d-2}\alpha^2\bar{\alpha}^a, \tag{5.12}$$

$$\bar{C}_\perp^a \equiv \bar{\alpha}^a - \frac{1}{2}\alpha^a\bar{\alpha}^2, \tag{5.13}$$

$$\bar{r} \equiv -\sqrt{(d-N_z)(d-2N_z)}\,\bar{\alpha}^z. \tag{5.14}$$

Using this, we check that constraints (5.3) and (5.4) are invariant under transformations of the conformal algebra (2.11).

From (5.7) and (5.8), we see that the vector and scalar fields $\phi_{\rm cur}^a$, $\phi_{\rm cur}$ transform as Stueckelberg fields, i.e., these fields can be gauged away via Stueckelberg gauge fixing, $\phi_{\rm cur}^a = 0$, $\phi_{\rm cur} = 0$. If we gauge away these fields, then the remaining rank-2 tensor field $\phi_{\rm cur}^{ab}$ becomes, according to constraints (5.3) and (5.4), divergence-free and traceless. In other words, our constraints taken to be in the gauge $\phi_{\rm cur}^a = 0$, $\phi_{\rm cur} = 0$ lead to the well-known divergence-free and tracelessness constraints of the standard approach.

Our approach can be related with the standard one without gauge fixing, i.e., by maintaining gauge symmetries. We construct the following tensor field:

$$T_{\rm cur}^{ab} = \phi_{\rm cur}^{ab} + \partial^a\phi_{\rm cur}^b + \partial^b\phi_{\rm cur}^a + \frac{2}{u}\partial^a\partial^b\phi_{\rm cur} + \frac{2}{(d-2)u}\eta^{ab}\Box\phi_{\rm cur}. \tag{5.15}$$

One can make sure that

(i) $T_{\rm cur}^{ab}$ is invariant under gauge transformations (5.6), (5.7), and (5.8).

(ii) Denoting the respective left-hand sides of (5.3) and (5.4) by $C_{\rm cur}^a$ and $C_{\rm cur}$ we get

$$\partial^b T_{\rm cur}^{ab} - \frac{1}{2}\partial^a T_{\rm cur}^{bb} = C_{\rm cur}^a, \qquad T_{\rm cur}^{aa} = 2C_{\rm cur}, \tag{5.16}$$

i.e., the constraints $C_{\rm cur}^a = 0$, $C_{\rm cur} = 0$ amount to

$$\partial^b T_{\rm cur}^{ab} = 0, \qquad T_{\rm cur}^{aa} = 0. \tag{5.17}$$

In our approach, the gauge invariant tensor field $T_{\rm cur}^{ab}$ (5.15) is counterpart of the energy-momentum tensor appearing in standard formulation of CFT.

VI. GAUGE INVARIANT FORMULATION OF SPIN-2 SHADOW FIELD

To discuss the gauge invariant formulation of the spin-2 shadow field in flat space of dimension $d \geq 4$ we use one rank-2 tensor field $\phi_{\rm sh}^{ab}$, one vector field $\phi_{\rm sh}^a$, and one scalar field $\phi_{\rm sh}$:

$$\phi_{\rm sh}^{ab}, \qquad \phi_{\rm sh}^a, \qquad \phi_{\rm sh}. \tag{6.1}$$

The fields $\phi_{\rm sh}^{ab}$, $\phi_{\rm sh}^a$, and $\phi_{\rm sh}$ transform in the respective rank-2 tensor, vector, and scalar representations of the Lorentz algebra $so(d-1,1)$. We note that these fields have the conformal dimensions

$$\Delta_{\phi_{\rm sh}^{ab}} = 0, \qquad \Delta_{\phi_{\rm sh}^{a}} = 1, \qquad \Delta_{\phi_{\rm sh}} = 2. \tag{6.2}$$

We now introduce the following differential constraints:

$$\partial^b \phi_{\rm sh}^{ab} - \frac{1}{2}\partial^a \phi_{\rm sh}^{bb} + \phi_{\rm sh}^a = 0, \tag{6.3}$$

$$\partial^a \phi_{\rm sh}^{a} + \frac{1}{2}\Box \phi_{\rm sh}^{aa} + u\phi_{\rm sh} = 0, \tag{6.4}$$

where u is given in (5.5). One can make sure that these constraints are invariant under gauge transformations

$$\delta\phi_{\rm sh}^{ab} = \partial^a \xi_{\rm sh}^b + \partial^b \xi_{\rm sh}^a + \frac{2}{d-2}\eta^{ab}\xi_{\rm sh}, \tag{6.5}$$

$$\delta\phi_{\rm sh}^{a} = \partial^a \xi_{\rm sh} - \Box \xi_{\rm sh}^a, \tag{6.6}$$

$$\delta\phi_{\rm sh} = -u\Box\xi_{\rm sh}, \tag{6.7}$$

where $\xi_{\rm sh}^a$, $\xi_{\rm sh}$ are gauge transformation parameters.

In order to obtain a realization of conformal algebra symmetries, we use the oscillators and introduce a ket-vector $|\phi_{\rm sh}\rangle$ defined by

$$|\phi_{\rm sh}\rangle = \Big(\frac{1}{2}\phi_{\rm sh}^{ab}\alpha^a\alpha^b + \phi_{\rm sh}^{a}\alpha^a\alpha^z + \frac{1}{\sqrt{2}}\phi_{\rm sh}\alpha^z\alpha^z\Big)|0\rangle. \tag{6.8}$$

A realization of the spin operator M^{ab} on $|\phi_{\rm sh}\rangle$ is given in (2.19), while a realization of the operator Δ,

$$\Delta = N_z, \tag{6.9}$$

can be read from (6.2). We then find that a realization of the operator R^a on $|\phi_{\rm sh}\rangle$ takes the form

$$R^a = r\Big(\bar\alpha^a - \frac{1}{d}\alpha^a\bar\alpha^2\Big), \tag{6.10}$$

$$r \equiv \alpha^z\sqrt{(d-N_z)(d-2N_z)}. \tag{6.11}$$

Using this, we check that constraints (6.3) and (6.4) are invariant under transformations of the conformal algebra (2.11).

Gauge transformations of the scalar field $\phi_{\rm sh}$ (6.7) and the vector field $\phi_{\rm sh}^a$ (6.6) involve d'Alembertian operator $\Box$. Therefore these transformations are not realized as the standard Stueckelberg gauge symmetries, i.e., the scalar and vector fields cannot be referred to as Stueckelberg fields. In contrast with the gauge invariant approach to the spin-2 current, the scalar and the vector fields appearing in the gauge invariant approach to spin-2 shadow field are not Stueckelberg fields and they cannot be gauged away via Stueckelberg gauge fixing. All that we can do is to express these fields in terms of the rank-2 tensor field $\phi_{\rm sh}^{ab}$ by using constraints (6.3) and (6.4). On the other hand, from (6.5), we see that the trace of the rank-2 tensor field $\phi_{\rm sh}^{ab}$ transforms as a Stueckelberg field, i.e., $\phi_{\rm sh}^{aa}$ can be gauged away via Stueckelberg gauge fixing, $\phi_{\rm sh}^{aa} = 0$. Imposing the gauge $\phi_{\rm sh}^{aa} = 0$, we obtain traceless field $\phi_{\rm sh}^{ab}$ which can be identified with the shadow field Φ^{ab} of the standard approach to CFT.

As in the case of the conformal current, we can introduce the gauge invariant field $T_{\rm sh}^{ab}$,

$$T_{\rm sh}^{ab} = \Box^2\phi_{\rm sh}^{ab} + \Box(\partial^a\phi_{\rm sh}^b + \partial^b\phi_{\rm sh}^a) + \frac{2}{u}\partial^a\partial^b\phi_{\rm sh} + \frac{2}{(d-2)u}\eta^{ab}\Box\phi_{\rm sh}. \tag{6.12}$$

One can check that

(i) $T_{\rm sh}^{ab}$ is invariant under gauge transformations (6.5)–(6.7).

(ii) Differential constraints for gauge fields (6.3) and (6.4) lead to divergence-free and tracelessness constraints for the field $T_{\rm sh}^{ab}$,

$$\partial^b T_{\rm sh}^{ab} = 0, \qquad T_{\rm sh}^{aa} = 0. \tag{6.13}$$

However, constraints (6.13) are not equivalent to (6.3) and (6.4). Namely, if we denote the respective left-hand sides of (6.3) and (6.4) by $C_{\rm sh}^a$ and $C_{\rm sh}$, then we obtain

$$\partial^b T_{\rm sh}^{ab} - \frac{1}{2}\partial^a T_{\rm sh}^{bb} = \Box^2 C_{\rm sh}^a, \qquad T_{\rm sh}^{aa} = 2\Box C_{\rm sh}. \tag{6.14}$$

From (6.14), we see that the constraints $C_{\rm sh}^a = 0$, $C_{\rm sh} = 0$ lead to constraints (6.13), while constraints (6.13) do not imply the constraints $C_{\rm sh}^a = 0$, $C_{\rm sh} = 0$, in general.

VII. GAUGE INVARIANT FORMULATION OF ARBITRARY SPIN CONFORMAL CURRENT

Field content. To discuss the gauge invariant formulation of an arbitrary spin-s conformal current in flat space of dimension $d \geq 4$ we use the following fields:

$$\phi_{{\rm cur},s'}^{a_1\ldots a_{s'}}, \qquad s' = 0,1,\ldots,s; \tag{7.1}$$

where the subscript s' denotes that the field $\phi_{{\rm cur},s'}^{a_1\ldots a_{s'}}$ is the rank-s' tensor field of the Lorentz algebra $so(d-1,1)$.

We note that

(i) In (7.1), the fields $\phi_{{\rm cur},0}$ and $\phi_{{\rm cur},1}^a$ are the respective scalar and vector fields of the Lorentz algebra, while the fields $\phi_{{\rm cur},s'}^{a_1\ldots a_{s'}}$, $s' > 1$, are the rank-s' totally symmetric tensor fields of the Lorentz algebra $so(d-1,1)$.

(ii) The tensor fields $\phi_{{\rm cur},s'}^{a_1\ldots a_{s'}}$ with $s' \geq 4$ satisfy the double-tracelessness constraint

$$\phi_{{\rm cur},s'}^{aabba_5\ldots a_{s'}} = 0, \qquad s' = 4,5,\ldots,s. \tag{7.2}$$

(iii) The fields $\phi_{\mathrm{cur},s'}^{a_1\ldots a_{s'}}$ have the following conformal dimensions:

$$\Delta(\phi_{\mathrm{cur},s'}^{a_1\ldots a_{s'}}) = s' + d - 2. \tag{7.3}$$

In order to obtain the gauge invariant description in an easy-to-use form, we use the oscillators (2.1) and introduce a ket-vector $|\phi_{\mathrm{cur}}\rangle$ defined by

$$|\phi_{\mathrm{cur}}\rangle \equiv \sum_{s'=0}^{s} \alpha_z^{s-s'}|\phi_{\mathrm{cur},s'}\rangle, \tag{7.4}$$

$$|\phi_{\mathrm{cur},s'}\rangle \equiv \frac{\alpha^{a_1}\ldots\alpha^{a_{s'}}}{s'!\sqrt{(s-s')!}}\phi_{\mathrm{cur},s'}^{a_1\ldots a_{s'}}|0\rangle. \tag{7.5}$$

From (7.4) and (7.5), we see that the ket-vector $|\phi_{\mathrm{cur}}\rangle$ is a degree-s homogeneous polynomial in the oscillators α^a, α^z, while the ket-vector $|\phi_{s'}\rangle$ is a degree-s' homogeneous polynomial in the oscillators α^a, i.e., these ket-vectors satisfy the relations

$$(N_\alpha + N_z - s)|\phi_{\mathrm{cur}}\rangle = 0, \tag{7.6}$$

$$(N_\alpha - s')|\phi_{\mathrm{cur},s'}\rangle = 0. \tag{7.7}$$

In terms of the ket-vector $|\phi\rangle$, double-tracelessness constraint (7.2) takes the form[11]

$$(\bar\alpha^2)^2|\phi_{\mathrm{cur}}\rangle = 0. \tag{7.8}$$

Differential constraint. We find the following differential constraint for the conformal current:

$$\bar C_{\mathrm{cur}}|\phi_{\mathrm{cur}}\rangle = 0, \tag{7.9}$$

$$\bar C_{\mathrm{cur}} = \bar C_\perp + c_1\bar\alpha^2 + c_2\Box\Pi^{[1,2]}, \tag{7.10}$$

$$\bar C_\perp = \bar\alpha\partial - \frac{1}{2}\alpha\partial\bar\alpha^2, \tag{7.11}$$

$$\Pi^{[1,2]} = 1 - \alpha^2\frac{1}{2(2N_\alpha + d)}\bar\alpha^2, \tag{7.12}$$

$$c_1 = \frac{1}{2}\alpha^z\tilde e_1, \qquad c_2 = \tilde e_1\bar\alpha^z, \tag{7.13}$$

$$\tilde e_1 = \left(\frac{2s + d - 4 - N_z}{2s + d - 4 - 2N_z}\right)^{1/2}. \tag{7.14}$$

One can make sure that constraint (7.9) is invariant under gauge transformation and conformal algebra transformations which we discuss below. Details of the derivation of constraint (7.9) may be found in Appendix A.

Gauge symmetries. We now discuss the gauge symmetries of the conformal current. To this end, we introduce the following gauge transformation parameters:

$$\xi_{\mathrm{cur},s'}^{a_1\ldots a_{s'}}, \qquad s' = 0, 1, \ldots, s-1. \tag{7.15}$$

We note that

(i) In (7.15), the gauge transformation parameters $\xi_{\mathrm{cur},0}$ and $\xi_{\mathrm{cur},1}^a$ are the respective scalar and vector fields of the Lorentz algebra, while the gauge transformation parameters $\xi_{\mathrm{cur},s'}^{a_1\ldots a_{s'}}$, $s' > 1$, are the rank-s' totally symmetric tensor fields of the Lorentz algebra $so(d-1,1)$.

(ii) The gauge transformation parameters $\xi_{\mathrm{cur},s'}^{a_1\ldots a_{s'}}$ with $s' \geq 2$ satisfy the tracelessness constraint

$$\xi_{\mathrm{cur},s'}^{aaa_3\ldots a_{s'}} = 0, \qquad s' = 2, 3, \ldots, s-1. \tag{7.16}$$

(iii) The gauge transformation parameters $\xi_{\mathrm{curs}'}^{a_1\ldots a_{s'}}$ have the conformal dimensions

$$\Delta(\xi_{\mathrm{cur},s'}^{a_1\ldots a_{s'}}) = s' + d - 3. \tag{7.17}$$

Now, as usual, we collect the gauge transformation parameters in ket-vector $|\xi_{\mathrm{cur}}\rangle$ defined by

$$|\xi_{\mathrm{cur}}\rangle \equiv \sum_{s'=0}^{s-1} \alpha_z^{s-1-s'}|\xi_{\mathrm{cur},s'}\rangle, \tag{7.18}$$

$$|\xi_{\mathrm{cur},s'}\rangle \equiv \frac{\alpha^{a_1}\ldots\alpha^{a_{s'}}}{s'!\sqrt{(s-1-s')!}}\xi_{\mathrm{cur},s'}^{a_1\ldots a_{s'}}|0\rangle. \tag{7.19}$$

The ket-vectors $|\xi_{\mathrm{cur}}\rangle$, $|\xi_{\mathrm{cur},s'}\rangle$ satisfy the algebraic constraints

$$(N_\alpha + N_z - s + 1)|\xi_{\mathrm{cur}}\rangle = 0, \tag{7.20}$$

$$(N_\alpha - s')|\xi_{\mathrm{cur},s'}\rangle = 0, \tag{7.21}$$

which tell us that $|\xi_{\mathrm{cur}}\rangle$ is a degree-$(s-1)$ homogeneous polynomial in the oscillators α^a, α^z, while the ket-vector $|\xi_{\mathrm{cur},s'}\rangle$ is a degree-s' homogeneous polynomial in the oscillators α^a.

In terms of the ket-vector $|\xi_{\mathrm{cur}}\rangle$, the tracelessness constraint (7.16) takes the form

$$\bar\alpha^2|\xi_{\mathrm{cur}}\rangle = 0. \tag{7.22}$$

Gauge transformation can be written entirely in terms of $|\phi_{\mathrm{cur}}\rangle$ and $|\xi_{\mathrm{cur}}\rangle$. That is to say that gauge transformation takes the form

$$\delta|\phi_{\mathrm{cur}}\rangle = (\alpha\partial + b_1 + b_2\alpha^2\Box)|\xi_{\mathrm{cur}}\rangle, \tag{7.23}$$

[11]In this paper, we adapt the formulation in terms of the double traceless gauge fields [17]. An adaptation of the approach in Ref. [17] to massive fields may be found in [25,26]. A discussion of various formulations in terms of unconstrained gauge fields may be found in [27–32]. For a recent review, see [33]. A discussion of other formulations which seem to be most suitable for the theory of interacting fields may be found e.g. in [34,35].

$$b_1 = -\alpha^z \tilde{e}_1, \qquad (7.24)$$

$$b_2 = \frac{1}{2s+d-6-2N_z}\tilde{e}_1\bar{\alpha}^z, \qquad (7.25)$$

where $\tilde{e}_1$ is given in (7.14). We note that constraint (7.9) is invariant under gauge transformation (7.23). Details of the derivation of gauge transformation (7.23) may be found in Appendix A.

Realization of conformal algebra symmetries. To complete the gauge invariant formulation of the spin-s conformal current we provide a realization of the conformal algebra symmetries on space of the ket-vector $|\phi_{\rm cur}\rangle$. All that is required is to fix the operators M^{ab}, Δ, and R^a and then insert these operators into (2.12)–(2.15). A realization of the spin operator M^{ab} on ket-vector $|\phi_{\rm cur}\rangle$ (7.4) is given in (2.19), while a realization of the operator Δ,

$$\Delta = s+d-2-N_z, \qquad (7.26)$$

can be read from (7.3). In the gauge invariant formulation, finding the operator R^a provides the real difficulty. Representation of the operator R^a we find is given by

$$R^a = \bar{r}\Big(\tilde{C}^a + \alpha^2\frac{2}{(2N_\alpha+d-2)(2N_\alpha+d)}\bar{C}^a_\perp\Big), \qquad (7.27)$$

$$\tilde{C}^a \equiv \alpha^a - \alpha^2\frac{1}{2N_\alpha+d-2}\bar{\alpha}^a, \qquad (7.28)$$

$$\bar{C}^a_\perp \equiv \bar{\alpha}^a - \frac{1}{2}\alpha^a\bar{\alpha}^2, \qquad (7.29)$$

$$\bar{r} \equiv -((2s+d-4-N_z)(2s+d-4-2N_z))^{1/2}\bar{\alpha}^z. \qquad (7.30)$$

Details of the derivation of operator R^a (7.27) may be found in Appendix B.

Equivalence of the gauge invariant and standard approaches. We begin with comment on the structure of gauge transformation (7.23). Making use of simplified notation for conformal currents, gauge transformation parameters, derivatives, and flat metric tensor

$$\phi_{{\rm cur},s'} \sim \phi_{{\rm cur},s'}^{a_1\ldots a_{s'}}, \qquad \xi_{{\rm cur},s'} \sim \xi_{{\rm cur},s'}^{a_1\ldots a_{s'}}, \qquad \partial \sim \partial^a, \qquad \eta \sim \eta^{ab}, \qquad (7.31)$$

gauge transformation (7.23) can schematically be represented as

$$\delta\phi_{{\rm cur},s'} \sim \partial\xi_{{\rm cur},s'-1} + \xi_{{\rm cur},s'} + \eta\Box\xi_{{\rm cur},s'-2}, \quad s' = 2,3,\ldots,s, \qquad (7.32)$$

$$\delta\phi_{{\rm cur},1} \sim \partial\xi_{{\rm cur},0} + \xi_{{\rm cur},1}, \qquad (7.33)$$

$$\delta\phi_{{\rm cur},0} \sim \xi_{{\rm cur},0}, \qquad (7.34)$$

where we assume $\xi_{{\rm cur},s} \equiv 0$. From (7.32)–(7.34), we see that all gauge transformations are realized as Stueckelberg (Goldstone) gauge transformations. We now find currents that are realized as Stueckelberg fields. To this end, we note that the currents $\phi_{{\rm cur},s'}$ with $s' \geq 2$ can be decomposed into traceless tensor fields as

$$\phi_{{\rm cur},s'} = \phi^{\rm T}_{{\rm cur},s'} \oplus \phi^{\rm TT}_{{\rm cur}s'-2}, \qquad s' = 2,3,\ldots,s, \qquad (7.35)$$

where $\phi^{\rm T}_{{\rm cur},s'}$ and $\phi^{\rm TT}_{{\rm cur}s'-2}$ stand for the respective rank-s' and rank-$(s'-2)$ traceless tensors of the Lorentz algebra $so(d-1,1)$. From (7.32)–(7.34), we see that we can impose the gauge conditions

$$\phi_{{\rm cur},0} = 0, \quad \phi_{{\rm cur},1} = 0, \quad \phi^{\rm T}_{{\rm cur},s'} = 0, \quad s' = 2,3,\ldots,s-1. \qquad (7.36)$$

Currents given in (7.36) are Stueckelberg fields in our approach.

We now discuss restrictions imposed by differential constraint (7.9). To this end, we note that our gauge conditions (7.36) can be written in terms of the ket-vectors $|\phi_{{\rm cur},s'}\rangle$ as[12]

$$\Pi^{[1,2]}|\phi_{{\rm cur},s'}\rangle = 0, \qquad s' = 0,1,\ldots,s-1, \qquad (7.37)$$

which, in turn, can be represented as

$$|\phi_{{\rm cur},s'}\rangle = \alpha^2\frac{1}{2(2N_\alpha+d)}\bar{\alpha}^2|\phi_{{\rm cur},s'}\rangle, \quad s' = 0,1,\ldots,s-1. \qquad (7.38)$$

Making use of gauge conditions (7.37) in differential constraint (7.9) leads to

$$\Big(\bar{\alpha}\partial - \frac{1}{2}\alpha\partial\bar{\alpha}^2\Big)|\phi_{{\rm cur},s'}\rangle + \frac{1}{2}\tilde{e}_{1,s-s'-1}\bar{\alpha}^2|\phi_{{\rm cur},s'+1}\rangle = 0, \quad s' = 0,1,\ldots,s, \qquad (7.39)$$

where $\tilde{e}_{1,n} \equiv \tilde{e}_1|_{N_z=n}$. Using (7.38), equations (7.39) can be represented as

$$\frac{2N_\alpha+d-4}{2N_\alpha+d-2}\Big(\alpha\partial - \alpha^2\frac{1}{2N_\alpha+d}\bar{\alpha}\partial\Big)\bar{\alpha}^2|\phi_{{\rm cur},s'}\rangle + \frac{1}{2}\tilde{e}_{1,s-s'-1}\bar{\alpha}^2|\phi_{{\rm cur},s'+1}\rangle = 0, \qquad (7.40)$$

when $s' = 0,1,2,\ldots,s-1$, while for $s' = s$, (7.39) amounts to

$$\Big(\bar{\alpha}\partial - \frac{1}{2}\alpha\partial\bar{\alpha}^2\Big)|\phi_{{\rm cur},s}\rangle = 0. \qquad (7.41)$$

Taking into account (7.40) and gauge conditions $|\phi_{{\rm cur},0}\rangle = 0$, $|\phi_{{\rm cur},1}\rangle = 0$ we obtain

$$\bar{\alpha}^2|\phi_{{\rm cur},s'}\rangle = 0, \qquad s' = 0,1,\ldots,s. \qquad (7.42)$$

[12]In terms of the ket-vector $|\phi_{\rm cur}\rangle$, gauge conditions (7.36) can simply be represented as $\bar{\alpha}^z\Pi^{[1,2]}|\phi_{\rm cur}\rangle = 0$.

Relations (7.38) and (7.42) imply

$$|\phi_{\mathrm{cur},s'}\rangle = 0, \qquad s' = 0, 1, \ldots, s-1. \tag{7.43}$$

Thus, we are left with the one spin-s traceless current $|\phi_{\mathrm{cur},s}\rangle$ which turns out to be divergence-free because of (7.41),

$$\bar{\alpha}\partial|\phi_{\mathrm{cur},s}\rangle = 0. \tag{7.44}$$

This implies that our gauge invariant approach is equivalent to the standard one.

VIII. GAUGE INVARIANT FORMULATION OF ARBITRARY SPIN SHADOW FIELD

Field content. To discuss the gauge invariant formulation of the arbitrary spin-s shadow field in flat space of dimension $d \geq 4$ we use the following fields:

$$\phi_{\mathrm{sh},s'}^{a_1\ldots a_{s'}}, \qquad s' = 0, 1, \ldots, s; \tag{8.1}$$

where the subscript s' denotes that the field $\phi_{\mathrm{sh},s'}^{a_1\ldots a_{s'}}$ is a rank-s' tensor field of the Lorentz algebra $so(d-1,1)$.

We note that

(i) In (8.1), the fields $\phi_{\mathrm{sh},0}$ and $\phi_{\mathrm{sh},1}^a$ are the respective scalar and vector fields of the Lorentz algebra, while the fields $\phi_{\mathrm{sh},s'}^{a_1\ldots a_{s'}}$, $s' > 1$, are rank-s' totally symmetric tensor fields of the Lorentz algebra $so(d-1,1)$.

(ii) The tensor fields $\phi_{\mathrm{sh},s'}^{a_1\ldots a_{s'}}$ with $s' \geq 4$ satisfy the double-tracelessness constraint

$$\phi_{\mathrm{sh},s'}^{aabba_5\ldots a_{s'}} = 0, \qquad s' = 4, 5, \ldots, s. \tag{8.2}$$

(iii) The fields $\phi_{\mathrm{sh},s'}^{a_1\ldots a_{s'}}$ have the following conformal dimensions:

$$\Delta(\phi_{\mathrm{sh},s'}^{a_1\ldots a_{s'}}) = 2 - s'. \tag{8.3}$$

In order to obtain the gauge invariant description in an easy-to-use form we use the oscillators and introduce a ket-vector $|\phi_{\mathrm{sh}}\rangle$ defined by

$$|\phi_{\mathrm{sh}}\rangle \equiv \sum_{s'=0}^{s} \alpha_z^{s-s'}|\phi_{\mathrm{sh},s'}\rangle, \tag{8.4}$$

$$|\phi_{\mathrm{sh},s'}\rangle \equiv \frac{\alpha^{a_1}\ldots\alpha^{a_{s'}}}{s'!\sqrt{(s-s')!}}\phi_{\mathrm{sh},s'}^{a_1\ldots a_{s'}}|0\rangle. \tag{8.5}$$

From (8.4) and (8.5), we see that the ket-vectors $|\phi_{\mathrm{sh}}\rangle$, $|\phi_{\mathrm{sh},s'}\rangle$ satisfy the algebraic constraints

$$(N_\alpha + N_z - s)|\phi_{\mathrm{sh}}\rangle = 0, \tag{8.6}$$

$$(N_\alpha - s')|\phi_{\mathrm{sh},s'}\rangle = 0. \tag{8.7}$$

These constraints tell us that $|\phi_{\mathrm{sh}}\rangle$ is a degree-s homogeneous polynomial in the oscillators α^a, α^z, while the ket-vector $|\phi_{\mathrm{sh},s'}\rangle$ is a degree-s' homogeneous polynomial in the oscillators α^a. In terms of the ket-vector $|\phi_{\mathrm{sh}}\rangle$, double-tracelessness constraint (8.2) takes the form

$$(\bar{\alpha}^2)^2|\phi_{\mathrm{sh}}\rangle = 0. \tag{8.8}$$

Differential constraint. We find the following differential constraint for the shadow field:

$$\bar{C}_{\mathrm{sh}}|\phi_{\mathrm{sh}}\rangle = 0, \tag{8.9}$$

$$\bar{C}_{\mathrm{sh}} = \bar{C}_\perp + c_1\bar{\alpha}^2\Box + c_2\Pi^{[1,2]}, \tag{8.10}$$

$$\bar{C}_\perp = \bar{\alpha}\partial - \frac{1}{2}\alpha\partial\bar{\alpha}^2, \tag{8.11}$$

$$\Pi^{[1,2]} = 1 - \alpha^2\frac{1}{2(2N_\alpha + d)}\bar{\alpha}^2, \tag{8.12}$$

$$c_1 = \frac{1}{2}\alpha^z\tilde{e}_1, \qquad c_2 = \tilde{e}_1\bar{\alpha}^z, \tag{8.13}$$

$$\tilde{e}_1 = \left(\frac{2s + d - 4 - N_z}{2s + d - 4 - 2N_z}\right)^{1/2}. \tag{8.14}$$

One can make sure that constraint (8.9) is invariant under gauge transformation and conformal algebra transformations which we discuss below. Details of the derivation of constraint (8.9) may be found in Appendix A.

Gauge symmetries of shadow field. We now discuss gauge symmetries of the shadow field. To this end, we introduce the following gauge transformation parameters:

$$\xi_{\mathrm{sh},s'}^{a_1\ldots a_{s'}}, \qquad s' = 0, 1, \ldots, s-1. \tag{8.15}$$

We note that

(i) In (8.15), the gauge transformation parameters $\xi_{\mathrm{sh},0}$ and $\xi_{\mathrm{sh},1}^a$ are the respective scalar and vector fields of the Lorentz algebra, while the gauge transformation parameters $\xi_{\mathrm{sh},s'}^{a_1\ldots a_{s'}}$, $s' > 1$, are rank-s' totally symmetric tensor fields of the Lorentz algebra $so(d-1,1)$.

(ii) The gauge transformation parameters $\xi_{\mathrm{sh},s'}^{a_1\ldots a_{s'}}$ with $s' \geq 2$ satisfy the tracelessness constraint

$$\xi_{\mathrm{sh},s'}^{aaa_3\ldots a_{s'}} = 0, \qquad s' = 2, 3, \ldots, s-1. \tag{8.16}$$

(iii) The gauge transformation parameters $\xi_{\mathrm{sh},s'}^{a_1\ldots a_{s'}}$ have the conformal dimensions

$$\Delta(\xi_{\mathrm{sh},s'}^{a_1\ldots a_{s'}}) = 1 - s'. \tag{8.17}$$

Now, as usual, we collect gauge transformation parameters in ket-vector $|\xi_{\mathrm{sh}}\rangle$ defined by

$$|\xi_{\rm sh}\rangle \equiv \sum_{s'=0}^{s-1} \alpha_z^{s-1-s'} |\xi_{{\rm sh},s'}\rangle, \qquad (8.18)$$

$$|\xi_{{\rm sh},s'}\rangle \equiv \frac{\alpha^{a_1}\dots\alpha^{a_{s'}}}{s'!\sqrt{(s-1-s')!}}\xi_{{\rm sh},s'}^{a_1\dots a_{s'}}|0\rangle. \qquad (8.19)$$

The ket-vectors $|\xi_{\rm sh}\rangle$, $|\xi_{{\rm sh},s'}\rangle$ satisfy the algebraic constraints

$$(N_\alpha + N_z - s + 1)|\xi_{\rm sh}\rangle = 0, \qquad (8.20)$$

$$(N_\alpha - s')|\xi_{{\rm sh},s'}\rangle = 0, \qquad (8.21)$$

which tell us that $|\xi_{\rm sh}\rangle$ is a degree-$(s-1)$ homogeneous polynomial in the oscillators α^a, α^z, while the ket-vector $|\xi_{{\rm sh},s'}\rangle$ is a degree-s' homogeneous polynomial in the oscillators α^a. In terms of the ket-vector $|\xi_{\rm sh}\rangle$, tracelessness constraint (8.16) takes the form

$$\bar\alpha^2|\xi_{\rm sh}\rangle = 0. \qquad (8.22)$$

Gauge transformation can entirely be written in terms of $|\phi_{\rm sh}\rangle$ and $|\xi_{\rm sh}\rangle$. This is to say that gauge transformation takes the form

$$\delta|\phi_{\rm sh}\rangle = (\alpha\partial + b_1\Box + b_2\alpha^2)|\xi_{\rm sh}\rangle, \qquad (8.23)$$

$$b_1 = -\alpha^z\tilde e_1, \qquad (8.24)$$

$$b_2 = \frac{1}{2s+d-6-2N_z}\tilde e_1\bar\alpha^z, \qquad (8.25)$$

where $\tilde e_1$ is given in (8.14). We note that constraint (8.9) is invariant under gauge transformation (8.23). Details of the derivation of gauge transformation (8.23) may be found in Appendix A.

Realization of conformal algebra symmetries. To complete gauge invariant formulation of a spin-s shadow field we should provide a realization of the conformal algebra symmetries on the space of the ket-vector $|\phi_{\rm sh}\rangle$, i.e., we should find operators M^{ab}, Δ, and R^a to insert them into (2.12)–(2.15). A realization of the spin operator M^{ab} on ket-vector $|\phi_{\rm sh}\rangle$ (8.4) is given in (2.19), while a realization of the operator Δ,

$$\Delta = 2 - s + N_z, \qquad (8.26)$$

can be read from (8.3). Representation of the operator R^a we find is given by

$$R^a = r\Big(\bar\alpha^a - \alpha^a\frac{1}{2N_\alpha + d}\bar\alpha^2\Big), \qquad (8.27)$$

$$r \equiv \alpha^z((2s+d-4-N_z)(2s+d-4-2N_z))^{1/2}. \qquad (8.28)$$

Details of the derivation of operator R^a (8.27) may be found in Appendix B.

Equivalence of the gauge invariant and standard approaches. We begin with comments on the structure of gauge transformation (8.23) and identification of Stueckelberg shadow fields in the gauge invariant approach. Making use of simplified notation for shadow fields, gauge transformation parameters, derivatives, and flat metric tensor

$$\phi_{{\rm sh},s'} \sim \phi_{{\rm sh},s'}^{a_1\dots a_{s'}}, \qquad \xi_{{\rm sh},s'} \sim \xi_{{\rm sh},s'}^{a_1\dots a_{s'}}, \qquad \partial\sim\partial^a, \qquad \eta\sim\eta^{ab}, \qquad (8.29)$$

gauge transformation (8.23) can schematically be represented as

$$\delta\phi_{{\rm sh},s'} \sim \partial\xi_{{\rm sh},s'-1} + \Box\xi_{{\rm sh},s'} + \eta\xi_{{\rm sh},s'-2}, \quad s' = 2,3,\dots,s, \qquad (8.30)$$

$$\delta\phi_{{\rm sh},1} \sim \partial\xi_{sh,0} + \Box\xi_{{\rm sh},1}, \qquad (8.31)$$

$$\delta\phi_{{\rm sh},0} \sim \Box\xi_{{\rm sh},0}, \qquad (8.32)$$

where we assume $\xi_{{\rm sh},s} \equiv 0$. We now find shadow fields that are realized as Stueckelberg fields. To this end, we note that the fields $\phi_{{\rm sh},s'}$ with $s' \ge 2$ can be decomposed into traceless tensor fields as

$$\phi_{{\rm sh},s'} = \phi_{{\rm sh},s'}^{\rm T} \oplus \phi_{{\rm sh},s'-2}^{\rm TT}, \qquad s' = 2,3,\dots,s, \qquad (8.33)$$

where $\phi_{{\rm sh},s'}^{\rm T}$ and $\phi_{{\rm sh},s'-2}^{\rm TT}$ stand for the respective rank-s' and rank-$(s'-2)$ traceless tensors of the Lorentz algebra $so(d-1,1)$. From (8.30), (8.31), and (8.32), we see that, in contrast to conformal currents, the following shadow fields

$$\phi_{{\rm sh},0}, \qquad \phi_{{\rm sh},1}, \qquad \phi_{{\rm sh},s'}^{\rm T}, \qquad s' = 2,3,\dots,s, \qquad (8.34)$$

cannot be gauged away via Stueckelberg gauge fixing. From (8.30), it is easy to see that by using gauge symmetries related to the gauge transformation parameters

$$\xi_{{\rm sh},s'}, \qquad s' = 0,1,\dots,s-2, \qquad (8.35)$$

we can impose the following gauge conditions

$$\phi_{{\rm sh},s'-2}^{\rm TT} = 0, \qquad s' = 2,3,\dots,s, \qquad (8.36)$$

and we note that fields in (8.36) are the Stueckelberg fields in the framework of the gauge invariant approach.

We now discuss restrictions imposed by differential constraint (8.9). To this end, we note that our gauge conditions (8.36) can be written in terms of the ket-vectors $|\phi_{{\rm sh},s'}\rangle$ as

$$\bar\alpha^2|\phi_{{\rm sh},s'}\rangle = 0, \qquad s' = 2,3,\dots,s. \qquad (8.37)$$

Making use of gauge conditions (8.37) in (8.9) leads to

$$\bar\alpha\partial|\phi_{{\rm sh},s'}\rangle + (s-s'+1)\tilde e_{1,s-s'}|\phi_{{\rm sh},s'-1}\rangle = 0, \quad s' = 0,1,\dots,s. \qquad (8.38)$$

Relations (8.38) imply that the fields $|\phi_{{\rm sh},s'}\rangle$, $s' = 0,1,\dots,s-1$, can be expressed in terms of the one field $|\phi_{{\rm sh},s}\rangle$ subject to the tracelessness constraint [see (8.37) when $s' = s$]. Thus, we are left with the one spin-s traceless shadow field $|\phi_{{\rm sh},s}\rangle$ and one surviving gauge symmetry generated by the gauge transformation parameter $\xi_{\mathrm{sh},s-1}$. This implies that our gauge invariant approach is equivalent to the standard one.

IX. TWO-POINT CURRENT-SHADOW FIELD INTERACTION VERTEX

We now discuss the two-point current-shadow field interaction vertex. In the gauge invariant approach, the interaction vertex is determined by requiring the vertex to be invariant under gauge transformations of both currents and shadow fields. Also, the interaction vertex should be invariant under conformal algebra transformations.

Spin-1. We begin with spin-1 fields. Let us consider the following vertex:

$$\mathcal{L} = \phi_{\mathrm{cur}}^a \phi_{\mathrm{sh}}^a + \phi_{\mathrm{cur}} \phi_{\mathrm{sh}}. \tag{9.1}$$

Under gauge transformations of the current (3.4) and (3.5), the variation of vertex (9.1) takes the form (up to total derivative)

$$\delta_{\xi_{\mathrm{cur}}} \mathcal{L} = -\xi_{\mathrm{cur}}(\partial^a \phi_{\mathrm{sh}}^a + \phi_{\mathrm{sh}}). \tag{9.2}$$

From this expression, we see that the vertex $\mathcal{L}$ is invariant under gauge transformations of the current provided the shadow field satisfies differential constraint (4.3). We then find that under gauge transformations of the shadow field (4.4) and (4.5) the variation of the vertex $\mathcal{L}$ takes the form (up to total derivative)

$$\delta_{\xi_{\mathrm{sh}}} \mathcal{L} = -\xi_{\mathrm{sh}}(\partial^a \phi_{\mathrm{cur}}^a + \Box \phi_{\mathrm{cur}}), \tag{9.3}$$

i.e., the vertex $\mathcal{L}$ is invariant under gauge transformations of the shadow field provided the current satisfies differential constraint (3.3).

Making use of the representation for generators of the conformal algebra obtained in Secs. III and IV, we check that vertex $\mathcal{L}$ (9.1) is also invariant under the conformal algebra transformations.

Spin-2. We proceed with spin-2 fields. One can make sure that the following vertex

$$\mathcal{L} = \frac{1}{2} \phi_{\mathrm{cur}}^{ab} \phi_{\mathrm{sh}}^{ab} - \frac{1}{4} \phi_{\mathrm{cur}}^{aa} \phi_{\mathrm{sh}}^{bb} + \phi_{\mathrm{cur}}^a \phi_{\mathrm{sh}}^a + \phi_{\mathrm{cur}} \phi_{\mathrm{sh}} \tag{9.4}$$

is invariant under gauge transformations of the spin-2 shadow field (6.5)–(6.7) provided the spin-2 current satisfies differential constraints (5.3) and (5.4). Vertex (9.4) is also invariant under gauge transformations of the spin-2 current (5.6), (5.7), and (5.8) provided the spin-2 shadow field satisfies differential constraints (6.3) and (6.4). Using the representation for generators of the conformal algebra obtained in Secs. V and VI we check that vertex $\mathcal{L}$ (9.4) is invariant under the conformal algebra transformations.

Arbitrary spin current and shadow field. For the case of arbitrary spin current and shadow field, the gauge invariant vertex takes the form

$$\mathcal{L} = \langle \phi_{\mathrm{cur}} | \Big(1 - \frac{1}{4} \alpha^2 \bar{\alpha}^2 \Big) | \phi_{\mathrm{sh}} \rangle. \tag{9.5}$$

This vertex is invariant under gauge transformation of the shadow field (8.23) provided the current satisfies differential constraint (7.9). Vertex (9.5) is also invariant under gauge transformation of the current (7.23) provided the shadow field satisfies differential constraint (8.9). Using the representation for generators of the conformal algebra obtained in the Secs. VII and VIII, we check that $\mathcal{L}$ (9.5) is also invariant under the conformal algebra transformations. Details of the derivation of the vertex $\mathcal{L}$ may be found in Appendix A.

X. ADS/CFT CORRESPONDENCE

We now apply our results to the study of AdS/CFT correspondence for bulk massless fields and boundary conformal currents and shadow fields. We demonstrate that normalizable solutions of bulk equations of motion are related to conformal currents, while non-normalizable solutions of bulk equations of motion are related to shadow fields. As is well known, investigation of AdS/CFT correspondence for massless fields requires analysis of some subtleties related to the fact that global transformations of bulk massless fields are defined up to local gauge transformations. In our approach, these complications are easily controllable because of the following reasons:

(i) We use the modified Lorentz gauge for a spin-1 field and the modified de Donder gauge for spin $s \geq 2$ fields. These gauges lead to the *decoupled* bulk equations of motion for arbitrary spin AdS fields, and this considerably simplifies the study of AdS/CFT correspondence. We note that the most convenient way to dial with the modified gauges is to use the Poincaré parametrization of AdS_{d+1} space,

$$ds^2 = \frac{1}{z^2}(dx^a dx^a + dz dz). \tag{10.1}$$

(ii) The modified gauges are invariant under the leftover on shell gauge symmetries of bulk AdS fields. Note however that, in our approach, we have gauge symmetries not only at the AdS side, but also at the boundary CFT.[13] It turns out that these gauge symmetries are also related via AdS/CFT correspondence. Namely, the leftover on shell gauge symmetries of bulk AdS fields are related with the gauge symmetries of currents and shadow fields we obtained in the framework of our gauge invariant approach to CFT in Secs. III, IV, V, VI, VII, and VIII.

(iii) In AdS space and at the boundary, we have the same number of gauge fields and the same number of

[13]Note that in the standard approach to CFT only the shadow fields are transformed under gauge transformations, while in our gauge invariant approach both the currents and shadow fields are transformed under gauge transformations. Thus, our approach allows us to study the currents and shadow fields on an equal footing.

gauge transformation parameters. Also, our AdS fields, currents, and shadow fields satisfy the same algebraic constraints.

A. AdS/CFT correspondence for spin-1 fields

As a warm-up let us consider the spin-1 Maxwell field. In AdS_{d+1} space, the massless spin-1 field is described by fields $\phi^a(x,z)$ and $\phi(x,z)$ which are the respective vector and scalar fields of the $so(d-1,1)$ algebra. In the modified Lorentz gauge,[14] found in Ref. [11],

$$\partial^a\phi^a + \Big(\partial_z - \frac{d-3}{2z}\Big)\phi = 0, \tag{10.2}$$

we obtain the decoupled equations of motion (for details, see Appendix C),

$$\Big(\Box + \partial_z^2 - \frac{1}{z^2}\Big(\nu_1^2 - \frac{1}{4}\Big)\Big)\phi^a = 0, \tag{10.3}$$

$$\Big(\Box + \partial_z^2 - \frac{1}{z^2}\Big(\nu_0^2 - \frac{1}{4}\Big)\Big)\phi = 0, \tag{10.4}$$

$$\nu_1 = \frac{d-2}{2}, \qquad \nu_0 = \frac{d-4}{2}. \tag{10.5}$$

Gauge condition (10.2) and equations of motion (10.3), (10.4) are invariant under the leftover on shell gauge transformations

$$\delta\phi^a = \partial^a\xi, \tag{10.6}$$

$$\delta\phi = \Big(\partial_z + \frac{d-3}{2z}\Big)\xi, \tag{10.7}$$

where the gauge transformation parameter ξ satisfies the equation of motion

$$\Big(\Box + \partial_z^2 - \frac{1}{z^2}\Big(\nu_1^2 - \frac{1}{4}\Big)\Big)\xi = 0. \tag{10.8}$$

It is easy to see that the normalizable solution of Eqs. (10.3) and (10.4) takes the form

$$\phi^a_{\rm norm}(x,z) = U_{\nu_1}\phi^a_{\rm cur}(x), \tag{10.9}$$

$$\phi_{\rm norm}(x,z) = U_{\nu_0}(-\phi_{\rm cur}(x)), \tag{10.10}$$

while the non-normalizable solution is given by[15]

$$\phi^a_{\text{non-norm}}(x,z) = U_{-\nu_1}\phi^a_{\rm sh}(x), \tag{10.11}$$

$$\phi_{\text{non-norm}}(x,z) = U_{-\nu_0}\phi_{\rm sh}(x), \tag{10.12}$$

where we introduce operator U_ν defined by

$$U_\nu \equiv \sqrt{qz}J_\nu(qz)q^{-\nu-(1/2)}, \qquad q^2 \equiv \Box, \tag{10.13}$$

and J_ν stands for the Bessel function. Taking into account the well-known properties of the Bessel function, we find that the asymptotic behavior of the normalizable solution is given by

$$\phi^a_{\rm norm}(x,z) \stackrel{z\to 0}{\longrightarrow} z^{\nu_1+(1/2)}\phi^a_{\rm cur}(x), \tag{10.14}$$

$$\phi_{\rm norm}(x,z) \stackrel{z\to 0}{\longrightarrow} z^{\nu_0+(1/2)}\phi_{\rm cur}(x), \tag{10.15}$$

while the asymptotic behavior of the non-normalizable solution takes the form

$$\phi^a_{\text{non-norm}}(x,z) \stackrel{z\to 0}{\longrightarrow} z^{-\nu_1+(1/2)}\phi^a_{\rm sh}(x), \tag{10.16}$$

$$\phi_{\text{non-norm}}(x,z) \stackrel{z\to 0}{\longrightarrow} z^{-\nu_0+(1/2)}\phi_{\rm sh}(x). \tag{10.17}$$

In (10.14)–(10.17), we drop overall factors that do not depend on z and $\Box$. From (10.14)–(10.17), we see that $\phi^a_{\rm cur}$, $\phi_{\rm cur}$ are indeed boundary values of the normalizable solution, while $\phi^a_{\rm sh}$, $\phi_{\rm sh}$ are boundary values of the non-normalizable solution.

In the right-hand side of (10.9)–(10.12), we use the respective notation $\phi^a_{\rm cur}$, $\phi_{\rm cur}$ and $\phi^a_{\rm sh}$, $\phi_{\rm sh}$ since we are going to demonstrate that these boundary values are indeed the conformal currents and shadow fields entering our gauge invariant formulation in the Secs. III and IV. Namely, one can prove the following statements:

(i) *Leftover on shell* gauge transformations (10.6) and (10.7) of the normalizable solution (10.9) and (10.10) lead to gauge transformations (3.4) and (3.5) of the conformal currents $\phi^a_{\rm cur}$, $\phi_{\rm cur}$, while *leftover on-shell* gauge transformations (10.6) and (10.7) of the non-normalizable solution (10.11) and (10.12) lead to gauge transformations (4.4) and (4.5) of the shadow fields $\phi^a_{\rm sh}$, $\phi_{\rm sh}$.

(ii) For the normalizable solution (10.9) and (10.10), the modified Lorentz gauge condition (10.2) leads to the differential constraint (3.3) of the conformal currents $\phi^a_{\rm cur}$, $\phi_{\rm cur}$, while, for the non-normalizable solution (10.11) and (10.12), the modified Lorentz gauge condition (10.2) leads to the differential constraint (4.3) of the shadow fields $\phi^a_{\rm sh}$, $\phi_{\rm sh}$.

(iii) Global $so(d,2)$ symmetries of the normalizable (non-normalizable) massless spin-1 modes in AdS_{d+1} become global $so(d,2)$ conformal symmetries of the conformal spin-1 current (shadow field).[16]

[14]A discussion of AdS/CFT correspondence for the spin-1 Maxwell field by using radial gauge may be found in [36].

[15]To keep the discussion from becoming unwieldy here and below, we restrict our attention to odd d. In this case, solutions given in (10.9)–(10.12) are independent.

[16]In this section, to avoid repetition, we do not demonstrate matching of the global $so(d,2)$ symmetries. Matching of the global $so(d,2)$ symmetries for arbitrary spin fields is studied in the Sec. XI.

These statements can easily be proved by using the following relations for the operator U_ν:

$$\Big(\partial_z + \frac{\nu - \frac{1}{2}}{z}\Big)U_\nu = U_{\nu-1}, \qquad (10.18)$$

$$\Big(\partial_z - \frac{\nu + \frac{1}{2}}{z}\Big)U_\nu = U_{\nu+1}(-\Box), \qquad (10.19)$$

$$\Big(\partial_z + \frac{\nu - \frac{1}{2}}{z}\Big)U_{-\nu} = U_{-\nu+1}(-\Box), \qquad (10.20)$$

$$\Big(\partial_z - \frac{\nu + \frac{1}{2}}{z}\Big)U_{-\nu} = U_{-\nu-1}, \qquad (10.21)$$

which, in turn, can be obtained by using the following well-known identities for the Bessel function:

$$\Big(\partial_z + \frac{\nu}{z}\Big)J_\nu(z) = J_{\nu-1}(z), \qquad \Big(\partial_z - \frac{\nu}{z}\Big)J_\nu(z) = -J_{\nu+1}(z). \qquad (10.22)$$

As an illustration, we demonstrate how a constraint for the conformal current (3.3) can be obtained from the modified Lorentz gauge condition (10.2). To this end, adapting relation (10.19) for $\nu = \nu_0$ (10.5), we obtain

$$\Big(\partial_z - \frac{d-3}{2z}\Big)U_{\nu_0} = U_{\nu_1}(-\Box). \qquad (10.23)$$

Plugging normalizable solutions $\phi_{\rm norm}^a$ (10.9), $\phi_{\rm norm}$ (10.10) in the modified Lorentz gauge condition (10.2) and using (10.23) we obtain the relation

$$\partial^a\phi_{\rm norm}^a + \Big(\partial_z - \frac{d-3}{2z}\Big)\phi_{\rm norm} = U_{\nu_1}(\partial^a\phi_{\rm cur}^a + \Box\phi_{\rm cur}), \qquad (10.24)$$

i.e., our modified Lorentz gauge condition (10.2) indeed leads to a differential constraint for the conformal current (3.3).

As second illustration, we demonstrate how gauge transformations of the conformal current (3.4) and (3.5) can be obtained from leftover on shell gauge transformations of the massless AdS field (10.6) and (10.7). To this end, we note that the respective normalizable and non-normalizable solutions of equation for the gauge transformation parameter (10.8) take the form

$$\xi_{\rm norm}(x, z) = U_{\nu_1}\xi_{\rm cur}(x), \qquad (10.25)$$

$$\xi_{\rm non\text{-}norm}(x, z) = U_{-\nu_1}\xi_{\rm sh}(x). \qquad (10.26)$$

Plugging (10.9) and (10.25) in (10.6), we see that (10.6) indeed leads to (3.4). To match the remaining gauge transformations (3.5) and (10.7), we adapt relation (10.18) with $\nu = \nu_1$ to obtain

$$\Big(\partial_z + \frac{d-3}{2z}\Big)U_{\nu_1} = U_{\nu_0}. \qquad (10.27)$$

Plugging (10.25) in (10.7) and using (10.27), we obtain

$$\delta\phi_{\rm norm} = U_{\nu_0}\xi_{\rm cur}. \qquad (10.28)$$

Taking into account (10.10), we see that the gauge transformations (3.5) and (10.7) match.

In similar way, one can match: (i) the leftover on shell gauge transformations of the non-normalizable massless AdS modes and the gauge transformations of the shadow field; (ii) the modified Lorentz gauge condition for the non-normalizable solution and the differential constraint for the shadow field.

Gauge invariant fields $T_{\rm cur}^a$, $T_{\rm sh}^a$ given in (3.11) and (4.11) can also be obtained via AdS/CFT correspondence. We consider a field strength W^a constructed out of the massless fields ϕ^a, ϕ,

$$W^a = \Big(\partial_z + \frac{d-3}{2z}\Big)\phi^a - \partial^a\phi, \qquad (10.29)$$

and note that

(i) W^a is invariant under gauge transformations (10.6) and (10.7).

(ii) Plugging the normalizable and non-normalizable solutions (10.9)–(10.12) in (10.29) and using (3.11) and (4.11), we obtain the respective relations

$$W_{\rm norm}^a = U_{\nu_0}T_{\rm cur}^a, \qquad (10.30)$$

$$W_{\rm non\text{-}norm}^a = U_{-\nu_0}(-T_{\rm sh}^a), \qquad (10.31)$$

i.e., for the normalizable solution, bulk field W^a (10.29) corresponds to the boundary gauge invariant field $T_{\rm cur}^a$ (3.11), while, for the non-normalizable solution, bulk field W^a (10.29) corresponds to the boundary gauge invariant field $T_{\rm sh}^a$ (4.11).

(iii) Denoting the left-hand side of (10.2) by $C_{\rm mod}$, we get

$$\partial^a W^a = \Big(\partial_z + \frac{d-3}{2z}\Big)C_{\rm mod}. \qquad (10.32)$$

We then check that plugging the normalizable solution in (10.32) and using (10.30) gives (3.12), while plugging the non-normalizable solution in (10.32) and using (10.31) gives (4.13).

B. AdS/CFT correspondence for spin-2 fields

We now proceed with the discussion of AdS/CFT correspondence for a bulk massless spin-2 AdS field and a boundary spin-2 conformal current and shadow field. To this end, we use the modified de Donder gauge condition for the massless spin-2 AdS field [16].[17] In Ref. [16], we

[17] A discussion of AdS/CFT correspondence for a massless spin-2 field taken to be in radial gauge may be found in [37,38].

found that the suitable modification of the standard de Donder gauge condition leads to the decoupled equations of motion for the massless spin-2 AdS field. We begin therefore with a presentation of our results from Ref. [16]. Some useful details may be found in the Appendix C.

In AdS_{d+1} space, a massless spin-2 field is described by fields $\phi^{ab}(x,z)$, $\phi^a(x,z)$, $\phi(x,z)$. The field ϕ^{ab} is the rank-2 tensor field of the $so(d-1,1)$ algebra, while ϕ^a and ϕ are the respective vector and scalar fields of the $so(d-1,1)$ algebra. The gauge condition, which we refer to as the modified de Donder gauge condition, is defined to be

$$\partial^b\phi^{ab} - \frac{1}{2}\partial^a\phi^{bb} + \left(\partial_z - \frac{d-1}{2z}\right)\phi^a = 0, \qquad (10.33)$$

$$\partial^a\phi^a - \frac{1}{2}\left(\partial_z + \frac{d-1}{2z}\right)\phi^{aa} + u\left(\partial_z - \frac{d-3}{2z}\right)\phi = 0, \qquad (10.34)$$

where u is given in (5.5). A remarkable property of this gauge condition is that it leads to the decoupled equations of motion for the fields ϕ^{ab}, ϕ^a, ϕ,

$$\left(\Box + \partial_z^2 - \frac{1}{z^2}\left(\nu_2^2 - \frac{1}{4}\right)\right)\phi^{ab} = 0, \qquad (10.35)$$

$$\left(\Box + \partial_z^2 - \frac{1}{z^2}\left(\nu_1^2 - \frac{1}{4}\right)\right)\phi^{a} = 0, \qquad (10.36)$$

$$\left(\Box + \partial_z^2 - \frac{1}{z^2}\left(\nu_0^2 - \frac{1}{4}\right)\right)\phi = 0, \qquad (10.37)$$

$$\nu_2 = \frac{d}{2}, \qquad \nu_1 = \frac{d-2}{2}, \qquad \nu_0 = \frac{d-4}{2}. \qquad (10.38)$$

These equations and the gauge condition (10.33) and (10.34) are invariant under the leftover on shell gauge transformations,

$$\delta\phi^{ab} = \partial^a\xi^b + \partial^b\xi^a + \frac{2}{d-2}\left(\partial_z - \frac{d-1}{2z}\right)\eta^{ab}\xi, \qquad (10.39)$$

$$\delta\phi^a = \partial^a\xi + \left(\partial_z + \frac{d-1}{2z}\right)\xi^a, \qquad (10.40)$$

$$\delta\phi = u\left(\partial_z + \frac{d-3}{2z}\right)\xi, \qquad (10.41)$$

where the gauge transformation parameters ξ^a and ξ satisfy the respective equations of motion,

$$\left(\Box + \partial_z^2 - \frac{1}{z^2}\left(\nu_2^2 - \frac{1}{4}\right)\right)\xi^{a} = 0, \qquad (10.42)$$

$$\left(\Box + \partial_z^2 - \frac{1}{z^2}\left(\nu_1^2 - \frac{1}{4}\right)\right)\xi = 0. \qquad (10.43)$$

Thus, we see that our modified de Donder gauge leads to the decoupled equations of motion for both the gauge fields and gauge transformation parameters. This streamlines the investigation of AdS/CFT correspondence.

First of all, we note that the normalizable solution of equations of motion (10.35), (10.36), and (10.37) is given by

$$\phi_{\rm norm}^{ab}(x,z) = U_{\nu_2}\phi_{\rm cur}^{ab}(x), \qquad (10.44)$$

$$\phi_{\rm norm}^{a}(x,z) = U_{\nu_1}(-\phi_{\rm cur}^{a}(x)), \qquad (10.45)$$

$$\phi_{\rm norm}(x,z) = U_{\nu_0}\phi_{\rm cur}(x), \qquad (10.46)$$

while the non-normalizable solution takes the form

$$\phi_{\text{non-norm}}^{ab}(x,z) = U_{-\nu_2}\phi_{\rm sh}^{ab}(x), \qquad (10.47)$$

$$\phi_{\text{non-norm}}^{a}(x,z) = U_{-\nu_1}\phi_{\rm sh}^{a}(x), \qquad (10.48)$$

$$\phi_{\text{non-norm}}(x,z) = U_{-\nu_0}\phi_{\rm sh}(x), \qquad (10.49)$$

where the operator U_ν is defined in (10.13). From these relations, we find the asymptotic behavior of the normalizable solution

$$\phi_{\rm norm}^{ab}(x,z) \stackrel{z\to 0}{\longrightarrow} z^{\nu_2+(1/2)}\phi_{\rm cur}^{ab}(x), \qquad (10.50)$$

$$\phi_{\rm norm}^{a}(x,z) \stackrel{z\to 0}{\longrightarrow} z^{\nu_1+(1/2)}\phi_{\rm cur}^{a}(x), \qquad (10.51)$$

$$\phi_{\rm norm}(x,z) \stackrel{z\to 0}{\longrightarrow} z^{\nu_0+(1/2)}\phi_{\rm cur}(x), \qquad (10.52)$$

while the asymptotic behavior of the non-normalizable solution takes the form

$$\phi_{\text{non-norm}}^{ab}(x,z) \stackrel{z\to 0}{\longrightarrow} z^{-\nu_2+(1/2)}\phi_{\rm sh}^{ab}(x), \qquad (10.53)$$

$$\phi_{\text{non-norm}}^{a}(x,z) \stackrel{z\to 0}{\longrightarrow} z^{-\nu_1+(1/2)}\phi_{\rm sh}^{a}(x), \qquad (10.54)$$

$$\phi_{\text{non-norm}}(x,z) \stackrel{z\to 0}{\longrightarrow} z^{-\nu_0+(1/2)}\phi_{\rm sh}(x). \qquad (10.55)$$

From (10.50)–(10.55), we see that the fields $\phi_{\rm cur}^{ab}$, $\phi_{\rm cur}^{a}$, $\phi_{\rm cur}$ are indeed boundary values of the normalizable solution, while $\phi_{\rm sh}^{ab}$, $\phi_{\rm sh}^{a}$, $\phi_{\rm sh}$ are boundary values of the non-normalizable solution.

In the right-hand side of (10.44)–(10.49), we use the respective notation $\phi_{\rm cur}^{ab}$, $\phi_{\rm cur}^{a}$, $\phi_{\rm cur}$ and $\phi_{\rm sh}^{ab}$, $\phi_{\rm sh}^{a}$, $\phi_{\rm sh}$ because these boundary values turn out to be the spin-2 conformal currents and shadow fields entering our gauge invariant formulation in Secs. V and VI. Namely, one can prove the following statements:

(i) *Leftover on shell* gauge transformations (10.39)–(10.41) of the normalizable solution (10.44)–(10.46) lead to the gauge transformations (5.6), (5.7), and (5.8) of the conformal currents $\phi_{\rm cur}^{ab}$, $\phi_{\rm cur}^{a}$, $\phi_{\rm cur}$, while *leftover on shell* gauge transformations (10.39)–(10.41) of the non-normalizable solution (10.47)–(10.49) lead to the gauge transformations (6.5)–(6.7) of the shadow fields $\phi_{\rm sh}^{ab}$, $\phi_{\rm sh}^{a}$, $\phi_{\rm sh}$.

(ii) For the normalizable solution (10.44)–(10.46), the modified de Donder gauge condition (10.33) and (10.34) leads to the differential constraints (5.3) and (5.4) of the conformal currents $\phi_{\rm cur}^{ab}$, $\phi_{\rm cur}^{a}$, $\phi_{\rm cur}$ while, for the non-normalizable solution (10.47)–(10.49), modified the de Donder gauge condition (10.33) and (10.34) leads to the differential constraints (6.3) and (6.4) of the shadow fields $\phi_{\rm sh}^{ab}$, $\phi_{\rm sh}^{a}$, $\phi_{\rm sh}$.

(iii) Global $so(d,2)$ bulk symmetries of the normalizable (non-normalizable) massless spin-2 modes in AdS_{d+1} become global $so(d,2)$ boundary conformal symmetries of the spin-2 current (shadow field).

These statements can easily be proved in the same way as in the case of the massless spin-1 field. To do that one needs to use the relations for the operator U_ν given in (10.18)–(10.21). Also, one needs to take into account the following normalizable solution of equations of motion for the gauge transformation parameters in (10.42) and (10.43):

$$\xi_{\rm norm}^a(x,z) = U_{\nu_2}\xi_{\rm cur}^a(x), \qquad (10.56)$$

$$\xi_{\rm norm}(x,z) = U_{\nu_1}(-\xi_{\rm cur}(x)), \qquad (10.57)$$

and the appropriate non-normalizable solution given by

$$\xi_{\rm non\text{-}norm}^a(x,z) = U_{\nu_2}\xi_{\rm sh}^a(x), \qquad (10.58)$$

$$\xi_{\rm non\text{-}norm}(x,z) = U_{\nu_1}\xi_{\rm sh}(x). \qquad (10.59)$$

We note that the gauge invariant fields $T_{\rm cur}^{ab}$, $T_{\rm sh}^{ab}$ given in (5.15) and (6.12) can also be obtained via AdS/CFT correspondence. Thus, we consider a field strength W^{ab} constructed out of the massless fields ϕ^{ab}, ϕ^a, ϕ,

$$\begin{aligned} W^{ab} &= \Big(\partial_z + \frac{d-3}{2z}\Big)\Big(\partial_z + \frac{d-1}{2z}\Big)\phi^{ab} \\ &\quad - \Big(\partial_z + \frac{d-3}{2z}\Big)(\partial^a\phi^b + \partial^b\phi^a) + \frac{2}{u}\partial^a\partial^b\phi \\ &\quad + \frac{2}{(d-2)u}\eta^{ab}\Box\phi, \end{aligned} \qquad (10.60)$$

where u is given in (5.5). We note that

(i) W^{ab} is invariant under the on shell gauge transformations (10.39)–(10.41).

(ii) Plugging normalizable and non-normalizable solutions (10.44)–(10.49) in (10.60) and using (5.15) and (6.12), we obtain the respective relations

$$W_{\rm norm}^{ab} = U_{\nu_0}T_{\rm cur}^{ab}, \qquad (10.61)$$

$$W_{\rm non\text{-}norm}^{ab} = U_{-\nu_0}(-T_{\rm sh}^{ab}), \qquad (10.62)$$

i.e., we see that, for the normalizable solution, the bulk tensor field W^{ab} (10.60) corresponds to the boundary gauge invariant field $T_{\rm cur}^{ab}$ (5.15), while, for the non-normalizable solution, the bulk tensor field W^{ab} (10.60) corresponds to the boundary gauge invariant field $T_{\rm sh}^{ab}$ (6.12).

(iii) Denoting the respective left-hand sides of (10.33) and (10.34) by $C_{\rm mod}^a$ and $C_{\rm mod}$, we get

$$\partial^b W^{ab} - \frac{1}{2}\partial^a W^{bb} = \Big(\partial_z + \frac{d-3}{2z}\Big)\Big(\partial_z + \frac{d-1}{2z}\Big)C_{\rm mod}^a, \qquad (10.63)$$

$$W^{aa} = -2\Big(\partial_z + \frac{d-3}{2z}\Big)C_{\rm mod}. \qquad (10.64)$$

We then check that plugging the normalizable solution in (10.63) and (10.64) and using (10.61) gives (5.16), while plugging the non-normalizable solution in (10.63) and (10.64) and using (10.62) gives (6.14).

XI. ADS/CFT CORRESPONDENCE FOR ARBITRARY SPIN FIELDS

We proceed with the discussion of AdS/CFT correspondence for the bulk massless arbitrary spin-s AdS field and the boundary spin-s conformal current and shadow field. To discuss the correspondence, we use the *modified* de Donder gauge condition for the bulk massless arbitrary spin field.[18] In Ref. [16], we found that some modification of the standard de Donder gauge condition[19] leads to the *decoupled* equations of motion for the arbitrary spin AdS field.[20] We begin therefore with a presentation of our results from Ref. [16]. In AdS_{d+1} space, the massless spin-s field is described by the following scalar, vector, and totally symmetric tensor fields of the Lorentz algebra $so(d-1,1)$:

$$\phi_{s'}^{a_1\ldots a_{s'}}, \qquad s' = 0,1,\ldots,s. \qquad (11.1)$$

The fields $\phi_{s'}^{a_1\ldots a_{s'}}$ with $s' > 3$ are double-traceless,

$$\phi_{s'}^{aabba_5\ldots a_{s'}} = 0, \qquad s' = 4,5,\ldots,s. \qquad (11.2)$$

In order to obtain the gauge invariant description in an easy-to-use form, we use the oscillators and introduce a ket-vector $|\phi\rangle$ defined by

$$|\phi\rangle \equiv \sum_{s'=0}^{s}\alpha_z^{s-s'}|\phi_{s'}\rangle, \qquad (11.3)$$

[18] In light-cone gauge, AdS/CFT correspondence for arbitrary spin massless fields was studied in Ref. [11]. In radial gauge, AdS/CFT correspondence for arbitrary spin massless fields was considered in Ref. [39].

[19] Recent interesting applications of the *standard* de Donder gauge to the various problems of higher-spin fields may be found in Refs. [40,41].

[20] We believe that our modified de Donder gauge will also be useful for a better understanding of various aspects of AdS/QCD correspondence which are discussed e.g. in [42–45].

$$|\phi_{s'}\rangle \equiv \frac{\alpha^{a_1}\ldots\alpha^{a_{s'}}}{s'!\sqrt{(s-s')!}}\phi_{s'}^{a_1\ldots a_{s'}}|0\rangle. \qquad (11.4)$$

From (11.3) and (11.4), we see that the ket-vector $|\phi\rangle$ is a degree-s homogeneous polynomial in the oscillators α^a, α^z, while the ket-vector $|\phi_{s'}\rangle$ is a degree-s' homogeneous polynomial in the oscillators α^a, i.e., these ket-vectors satisfy the relations

$$(N_\alpha + N_z - s)|\phi\rangle = 0, \qquad (11.5)$$

$$(N_\alpha - s')|\phi_{s'}\rangle = 0. \qquad (11.6)$$

In terms of the ket-vector $|\phi\rangle$, the double-tracelessness constraint (11.2) takes the form

$$(\bar\alpha^2)^2|\phi\rangle = 0. \qquad (11.7)$$

The gauge condition, which we refer to as the modified de Donder gauge condition, is defined as

$$\bar C_{\rm mod}|\phi\rangle = 0, \qquad (11.8)$$

$$\bar C_{\rm mod} \equiv \bar\alpha\partial - \frac{1}{2}\alpha\partial\bar\alpha^2 + \frac{1}{2}e_1\bar\alpha^2 - \bar e_1\Pi^{[1,2]}, \qquad (11.9)$$

$$\Pi^{[1,2]} \equiv 1 - \alpha^2\frac{1}{2(2N_\alpha + d)}\bar\alpha^2, \qquad (11.10)$$

$$e_1 = e_{1,1}\Big(\partial_z + \frac{2s+d-5-2N_z}{2z}\Big), \qquad (11.11)$$

$$\bar e_1 = \Big(\partial_z - \frac{2s+d-5-2N_z}{2z}\Big)\bar e_{1,1}, \qquad (11.12)$$

$$e_{1,1} = -\alpha^z\tilde e_1 \qquad \bar e_{1,1} = -\tilde e_1\bar\alpha^z, \qquad (11.13)$$

$$\tilde e_1 = \Big(\frac{2s+d-4-N_z}{2s+d-4-2N_z}\Big)^{1/2}. \qquad (11.14)$$

In this gauge, we obtain the decoupled equations of motion for the massless arbitrary spin-s AdS field $|\phi\rangle$,

$$\Big(\Box + \partial_z^2 - \frac{1}{z^2}\Big(\nu^2 - \frac{1}{4}\Big)\Big)|\phi\rangle = 0, \qquad (11.15)$$

$$\nu \equiv s + \frac{d-4}{2} - N_z. \qquad (11.16)$$

The gauge condition (11.8) and Eqs. (11.15) are invariant under the leftover on shell gauge transformation

$$\delta|\phi\rangle = \Big(\alpha\partial - e_1 - \frac{\alpha^2}{2s+d-6-2N_z}\bar e_1\Big)|\xi\rangle, \quad (11.17)$$

where e_1, $\bar e_1$ are given in (11.11) and (11.12) and the gauge transformation ket-vector $|\xi\rangle$ satisfies the equations of motion

$$\Big(\Box + \partial_z^2 - \frac{1}{z^2}\Big(\nu^2 - \frac{1}{4}\Big)\Big)|\xi\rangle = 0, \qquad (11.18)$$

with ν given (11.16). In terms of $so(d-1,1)$ algebra tensor fields, the ket-vector $|\xi\rangle$ is represented as

$$|\xi\rangle \equiv \sum_{s'=0}^{s-1}\alpha_z^{s-1-s'}|\xi_{s'}\rangle, \qquad (11.19)$$

$$|\xi_{s'}\rangle \equiv \frac{\alpha^{a_1}\ldots\alpha^{a_{s'}}}{s'!\sqrt{(s-1-s')!}}\xi_{s'}^{a_1\ldots a_{s'}}|0\rangle, \qquad (11.20)$$

and satisfies the standard tracelessness constraint

$$\bar\alpha^2|\xi\rangle = 0. \qquad (11.21)$$

We note that the gauge invariant description of the conformal currents (or shadow fields) given in the Secs. VII and VIII and the description of AdS fields given in this section turn out to be very convenient for studying AdS/CFT correspondence because of the following reasons:

(i) The number of gauge fields involved in the gauge invariant description of the spin-s conformal current (or shadow field) in d-dimensional space is equal to the number of gauge fields involved in the gauge invariant description of the massless spin-s field in AdS_{d+1} [see (7.1), (8.1), and (11.1)]. Note also that the conformal current, shadow field, and AdS field satisfy the same double-tracelessness constraint [see (7.2), (8.2), and (11.2)].

(ii) The number of gauge transformation parameters involved in the gauge invariant description of the spin-s conformal current (or shadow field) in d-dimensional space is equal to the number of gauge transformation parameters involved in the gauge invariant description of the massless spin-s field in AdS_{d+1} [see (7.15), (8.15), (11.9), and (11.20)]. Also, all these gauge transformation parameters satisfy the same tracelessness constraint [see (7.22), (8.22), and (11.21)].

(iii) In the Poincaré parametrization of AdS_{d+1} space, the d-dimensional Poincaré symmetries of AdS_{d+1} field theory are manifest. In the conformal current/shadow field theory, the d-dimensional Poincaré symmetries are also manifest, i.e. manifest Poincaré symmetries of AdS field theory and CFT match.

We now discuss solutions of equations of motion in (11.15). It is easy to see that the respective normalizable and non-normalizable solutions of Eqs. (11.15) take the form

$$|\phi_{\rm norm}(x,z)\rangle = U_\nu(-)^{N_z}|\phi_{\rm cur}(x)\rangle, \qquad (11.22)$$

$$|\phi_{\text{non-norm}}(x,z)\rangle = U_{-\nu}|\phi_{\rm sh}(x)\rangle, \qquad (11.23)$$

where the operator U_ν is defined in (10.13). From these relations, we find the asymptotic behavior of our solutions

$$|\phi_{\rm norm}(x,z)\rangle \stackrel{z\to 0}{\longrightarrow} z^{\nu+(1/2)}|\phi_{\rm cur}(x)\rangle, \qquad (11.24)$$

$$|\phi_{\text{non-norm}}(x,z)\rangle \stackrel{z\to 0}{\longrightarrow} z^{-\nu+(1/2)}|\phi_{\rm sh}(x)\rangle. \qquad (11.25)$$

Now we are ready to formulate our statements:

(i) *Leftover on shell* gauge transformation (11.17) of the normalizable solution (11.22) leads to the gauge transformation (7.23) of the current $|\phi_{\rm cur}\rangle$, while *leftover on shell* gauge transformation (11.17) of the non-normalizable solution (11.23) leads to the gauge transformation (8.23) of the shadow field $|\phi_{\rm sh}\rangle$.

(ii) For the normalizable solution (11.22), the modified de Donder gauge condition (11.8) leads to the differential constraint (7.9) of the current $|\phi_{\rm cur}\rangle$, while, for the non-normalizable solution (11.23), the modified de Donder gauge condition (11.8) leads to the differential constraint (8.9) of the shadow field $|\phi_{\rm sh}\rangle$.[21]

(iii) Global $so(d,2)$ bulk symmetries of the normalizable (non-normalizable) massless spin-s modes in AdS$_{d+1}$ become global $so(d,2)$ boundary conformal symmetries of the spin-s current (shadow field).

We note that all these statements can straightforwardly be proved by using the following relations for the operator U_ν:

$$\mathbf{e}_1 U_\nu = U_\nu \alpha^z, \qquad (11.26)$$

$$\bar{\mathbf{e}}_1 U_\nu = U_\nu(-\Box\bar{\alpha}^z), \qquad (11.27)$$

$$\mathbf{e}_1 U_{-\nu} = U_{-\nu}(-\Box\alpha^z), \qquad (11.28)$$

$$\bar{\mathbf{e}}_1 U_{-\nu} = U_{-\nu}\bar{\alpha}^z, \qquad (11.29)$$

$$\mathbf{e}_1(zU_{\nu+1}) = zU_{\nu+1}\alpha^z, \qquad (11.30)$$

$$\bar{\mathbf{e}}_1(zU_{\nu+1}) = 2U_\nu\bar{\alpha}^z - z\Box U_{\nu+1}\bar{\alpha}^z, \qquad (11.31)$$

$$\mathbf{e}_1(zU_{-\nu+1}) = 2U_{-\nu}\alpha^z - z\Box U_{-\nu+1}\alpha^z, \qquad (11.32)$$

$$\bar{\mathbf{e}}_1(zU_{-\nu+1}) = zU_{-\nu+1}\bar{\alpha}^z, \qquad (11.33)$$

where ν is given in (11.16) and we use the notation

$$\mathbf{e}_1 \equiv \alpha^z\Big(\partial_z + \frac{\nu-\frac{1}{2}}{z}\Big), \qquad (11.34)$$

$$\bar{\mathbf{e}}_1 \equiv \Big(\partial_z - \frac{\nu-\frac{1}{2}}{z}\Big)\bar{\alpha}^z. \qquad (11.35)$$

Also, one needs to take into account the following normalizable and non-normalizable solutions of equations of motion for the gauge transformation parameters in (11.18),

$$|\xi_{\rm norm}(x,z)\rangle = U_\nu(-)^{N_z}|\xi_{\rm cur}(x)\rangle, \qquad (11.36)$$

$$|\xi_{\text{non-norm}}(x,z)\rangle = U_{-\nu}|\xi_{\rm sh}(x)\rangle. \qquad (11.37)$$

As an illustration, we demonstrate how the gauge transformation of the shadow field can be obtained from the leftover on shell gauge transformation of the massless non-normalizable AdS modes. To this end, we note that, one the one hand, gauge transformation of $|\phi_{\text{non-norm}}\rangle$ takes the form [see (11.23)]

$$\delta|\phi_{\text{non-norm}}(x,z)\rangle = U_{-\nu}\delta|\phi_{\rm sh}(x)\rangle. \qquad (11.38)$$

On the other hand, plugging (11.37) in (11.17) and using (11.28) and (11.29), we obtain the relations

$$\delta|\phi_{\text{non-norm}}(x,z)\rangle = \Big(\alpha\partial - e_1 - \frac{\alpha^2}{2s+d-6-2N_z}\bar{e}_1\Big)U_{-\nu}|\xi_{\rm sh}\rangle = U_{-\nu}(\alpha\partial + b_1\Box + b_2\alpha^2)|\xi_{\rm sh}\rangle, \qquad (11.39)$$

where the b_1, b_2 operators entering gauge transformation (8.23) of the shadow field are given in (8.24) and (8.25). Comparing (11.38) and (11.39), we see that the leftover on shell gauge transformation (11.17) of the massless non-normalizable AdS modes (11.23) indeed leads to gauge transformation (8.23) of the shadow field.

In a similar way, using (11.26) and (11.27), we learn that the leftover on shell gauge transformation of the massless normalizable AdS modes leads to the gauge transformation of the current.

Matching of bulk and boundary global symmetries

We finish our study of AdS/CFT correspondence with the comparison of bulk and boundary global symmetries. On the one hand, global symmetries of conformal currents and shadow fields are described by the conformal algebra $so(d,2)$. On the other hand, relativistic symmetries of the AdS$_{d+1}$ field dynamics are also described by the $so(d,2)$ algebra. For application to the study of AdS/CFT correspondence, it is convenient to realize the bulk $so(d,2)$ algebra symmetries by using the nomenclature of the conformal algebra. This is to say that to discuss the bulk

[21] We expect that use of the standard de Donder gauge condition leads to an isomorphic realization of conformal symmetries. At present time, it is difficult to check this statement explicitly because the standard de Donder gauge condition leads to coupled equations. Analysis of these equations is complicated and their solution is not known in closed form so far (see e.g. Ref. [41]).

$so(d,2)$ symmetries we use the basis of the $so(d,2)$ algebra which consists of translation generators P^a, conformal boost generators K^a, dilatation generator D, and generators of the $so(d-1,1)$ algebra, J^{ab}. In this basis, the $so(d,2)$ algebra transformations of the massless spin-s AdS$_{d+1}$ field $|\phi\rangle$ take the form $\delta_{\hat{G}}|\phi\rangle = \hat{G}|\phi\rangle$, where a realization of the $so(d,2)$ algebra generators $\hat{G}$ in terms of differential operators is given by

$$P^a = \partial^a, \tag{11.40}$$

$$J^{ab} = x^a\partial^b - x^b\partial^a + M^{ab}, \tag{11.41}$$

$$D = x\partial + \Delta, \qquad \Delta = z\partial_z + \frac{d-1}{2}, \tag{11.42}$$

$$K^a = K^a_{\Delta,M} + R^a, \tag{11.43}$$

$$R^a = R^a_{(0)} + R^a_{(1)}, \tag{11.44}$$

$$R^a_{(0)} = -z\tilde{C}^a\bar{e}_{1,1} + ze_{1,1}\bar{\alpha}^a, \tag{11.45}$$

$$R^a_{(1)} = -\frac{1}{2}z^2\partial^a, \tag{11.46}$$

and the operators M^{ab} and $\tilde{C}^a$ are given in (2.19) and (7.28), respectively, while $K^a_{\Delta,M}$ and $e_{1,1}$ are given in (2.16) and (11.13), respectively.

We note that the representation for generators given in (11.40)–(11.43) is valid for the gauge invariant theory of AdS fields. This to say that our modified Lorentz and de Donder gauges respect the Poincaré and dilatation symmetries, but break K^a symmetries. In other words, the expressions for generators P^a, J^{ab}, and D given in (11.40), (11.41), and (11.42) are still valid for the gauge-fixed AdS fields, while the expression for the generator K^a (11.43) should be modified to restore conformal boost symmetries for the gauge-fixed AdS fields. Therefore, let us first demonstrate matching of the Poincaré and dilatation symmetries. What is required is to demonstrate matching of the $so(d,2)$ algebra generators for bulk AdS fields given in (11.40), (11.41), and (11.42) and ones for boundary currents (or shadow fields) given in (2.12)–(2.14). As for generators of the Poincaré algebra, P^a, J^{ab}, they already coincide on both sides [see formulas (2.12) and (2.13) and the respective formulas (11.40) and (11.41)]. Next, consider the dilatation generator D. Here we need the explicit form of the solution to bulk theory equations of motion given in (11.22) and (11.23). Using the notation $D_{\rm AdS}$ and $D_{\rm CFT}$ to indicate the respective realizations of the dilatation generator D on the bulk fields (11.42) and the conformal currents and shadow fields (2.14), we obtain the relations

$$D_{\rm AdS}|\phi_{\rm norm}\rangle = U_\nu D_{\rm CFT}|\phi_{\rm cur}\rangle, \tag{11.47}$$

$$D_{\rm AdS}|\phi_{\text{non-norm}}\rangle = U_{-\nu} D_{\rm CFT}|\phi_{\rm sh}\rangle, \tag{11.48}$$

where the expressions for $D_{\rm CFT}$ corresponding to $|\phi_{\rm cur}\rangle$ and $|\phi_{\rm sh}\rangle$ can be obtained from (2.14) and the respective conformal dimension operators Δ given in (7.26) and (8.26). Thus, the generators $D_{\rm AdS}$ and $D_{\rm CFT}$ also match.

We now turn to matching the conformal boost K^a symmetries. Technically, this is the most difficult point of the analysis because matching the K^a symmetries requires analysis of some subtleties of our gauge fixing for the AdS field. We now discuss these subtleties.

As we have already said, our modified Lorentz and de Donder gauges break the K^a symmetries. This implies that generator K^a given in (11.43) should be modified to restore the conformal boost symmetries of the gauge-fixed AdS field theory. In order to restore these broken K^a symmetries we should, following standard procedure, add compensating gauge transformations to maintain the conformal boost K^a symmetries. Thus, in order to find improved K^a transformations of the gauge-fixed AdS field $|\phi\rangle$, we start with the generic global K^a transformations (11.43) supplemented by the appropriate compensating gauge transformation

$$K^a_{\rm impr}|\phi\rangle = K^a|\phi\rangle + \delta_{\xi^{K^a}}|\phi\rangle, \tag{11.49}$$

where the gauge transformation $\delta_{\xi^{K^a}}|\phi\rangle$ is obtained from (11.17) by substituting $|\xi\rangle \to |\xi^{K^a}\rangle$. The compensating gauge transformation parameter $|\xi^{K^a}\rangle$ can usually be found by requiring improved transformation (11.49) to maintain the gauge condition (11.8),

$$\bar{C}_{\rm mod}K^a_{\rm impr}|\phi\rangle = 0, \tag{11.50}$$

where the operator $\bar{C}_{\rm mod}$ is given in (11.9). Plugging (11.49) in (11.50), we find that Eq. (11.50) leads to the equation

$$\Big(\Box + \partial_z^2 - \frac{1}{z^2}\Big(\nu^2 - \frac{1}{4}\Big)\Big)|\xi^{K^a}\rangle - 2\bar{C}^a_\perp|\phi\rangle = 0, \tag{11.51}$$

where ν is given in (11.16) and $\bar{C}^a_\perp$ is defined in (7.29). Thus, we obtain the nonhomogeneous second-order differential equation for the compensating gauge transformation parameter $|\xi^{K^a}\rangle$. Plugging the normalizable solution (11.22) and the non-normalizable solution (11.23) in (11.51), we find the respective solutions to the compensating gauge transformation parameters,

$$|\xi^{K^a}_{\rm norm}(x,z)\rangle = zU_{\nu+1}\bar{C}^a_\perp(-)^{N_z}|\phi_{\rm cur}\rangle, \tag{11.52}$$

$$|\xi^{K^a}_{\text{non-norm}}(x,z)\rangle = zU_{-\nu+1}\bar{C}^a_\perp|\phi_{\rm sh}\rangle. \tag{11.53}$$

Making use of solutions (11.52) and (11.53) in (11.49), we obtain the improved K^a transformations. We then make sure that the improved K^a transformations of the normalizable/non-normalizable bulk AdS modes lead to the conformal boost transformations for the current/shadow fields obtained in Sec. VII/VIII. This can easily be proved by using relations for the operator U_ν given in (11.26)–(11.33). Details may be found in Appendix E.

The results presented here should have interesting generalizations to mixed-symmetry fields. In the case of mixed-symmetry fields we could, in principle, redo our analysis by using the equations of Ref. [24] given in Lorentz/de Donder gauge conditions. However, as in the case of totally symmetric fields, these gauge conditions lead to coupled equations. Analysis of these coupled equations is complicated and their solution is not known in closed form so far. On the other hand, a promising gauge invariant approach to mixed-symmetry AdS fields was recently developed in Ref. [34]. It would be interesting to generalize our modified de Donder gauge to the mixed-symmetry fields by using this approach. This will it make possible to extend our analysis to the case of mixed-symmetry fields.

XII. INTERRELATIONS BETWEEN GAUGE INVARIANT APPROACHES TO CURRENTS, SHADOW FIELDS AND MASSIVE FIELDS IN FLAT SPACE

The gauge invariant description of conformal currents and shadow fields involves Stueckelberg fields. As is well known, the gauge invariant description of a massive field is also formulated by using Stueckelberg fields. It is worth mentioning that the number of Stueckelberg fields in the gauge invariant approach to the spin-s current coincides with the number of Stueckelberg fields in the gauge invariant approach to the spin-s massive field. Moreover, there are other interesting interrelations between the gauge invariant approaches to conformal currents, shadow fields, and massive fields. These interrelations are realized by breaking the conformal symmetries and can be summarized as follows.

(i) The gauge transformations of the massive fields can be obtained from the ones of the conformal currents (or shadow fields) by making the replacement

$$\Box \to m^2 \tag{12.1}$$

in the gauge transformations of the conformal currents (or shadow fields) and by making the appropriate rescaling of the conformal currents (or shadow fields).

(ii) A Lorentz-like gauge for the massive spin-1 field and a de Donder-like gauge for the massive spin $s \geq 2$ fields can be obtained by making replacement (12.1) in the differential constraints of the conformal currents (or shadow fields) and by making the appropriate rescaling of the conformal currents (or shadow fields).

We note that it is substitution (12.1) that breaks the conformal symmetries. Substitution (12.1) is similar to the one used in the procedure of the standard dimensional reduction from the *massless* field in $d+1$-dimensional flat space to the *massive* field in d-dimensional flat space. Note however that, in our approach, we break the conformal symmetries of d-dimensional space down to the d-dimensional Poincaré symmetries, while the standard procedure of dimensional reduction breaks the $d+1$-dimensional Poincaré symmetries down to the d-dimensional Poincaré symmetries.

We now demonstrate the interrelations for various spin fields in turn. In due course we present our de Donder-like gauge for massive spin-s, $s > 2$, fields. To our knowledge this gauge has not been discussed in the earlier literature.

Interrelations for spin-1 fields. In the gauge invariant approach, the massive spin-1 field is described by gauge fields ϕ_m^a, ϕ_m with Lagrangian

$$\mathcal{L} = -\frac{1}{4}F^{ab}F^{ab} - \frac{1}{2}(m\phi_m^a + \partial^a\phi_m)^2, \tag{12.2}$$

$F^{ab} = \partial^a\phi_m^b - \partial^b\phi_m^a$, which is invariant under the gauge transformations

$$\delta\phi_m^a = \partial^a\xi_m, \qquad \delta\phi_m = -m\xi_m. \tag{12.3}$$

It easy to see that gauge transformations (12.3) can be obtained by substituting

$$\phi_{\rm cur}^a \to \phi_m^a, \qquad \phi_{\rm cur} \to \frac{1}{m}\phi_m, \qquad \xi_{\rm cur} \to \xi_m \tag{12.4}$$

in gauge transformations of the spin-1 current (3.4) and (3.5). Also, it is easy to see that gauge transformations (12.3) can be obtained by substituting (12.1) and

$$\phi_{\rm sh}^a \to \phi_m^a, \qquad \phi_{\rm sh} \to m\phi_m, \qquad \xi_{\rm sh} \to \xi_m \tag{12.5}$$

in gauge transformations of the spin-1 shadow field (4.4) and (4.5).

We now consider the interrelations between the gauge condition for the massive field spin-1 field and the differential constraints for the current and shadow field. Let us consider the following well-known Lorentz-like gauge condition for the massive spin-1 gauge fields and the corresponding gauge-fixed equations

$$\partial^a\phi_m^a + m\phi_m = 0, \tag{12.6}$$

$$(\Box - m^2)\phi_m^a = 0, \tag{12.7}$$

which are invariant under leftover on shell gauge transformations (12.3), if the gauge transformation parameter satisfies the equation

$$(\Box - m^2)\xi_m = 0. \tag{12.8}$$

Note that gauge-fixed equations (12.7) can be obtained from the appropriate gauge-fixed Lagrangian. Namely, denoting the left-hand side of (12.6) by C_m, we obtain the well-known gauge-fixed Lagrangian

$$\mathcal{L}_{\rm total} \equiv \mathcal{L} - \frac{1}{2}C_mC_m, \tag{12.9}$$

$$\mathcal{L}_{\rm total} = \frac{1}{2}\phi_m^a(\Box - m^2)\phi_m^a, \qquad (12.10)$$

which leads to Eqs. (12.7).

We now note that the Lorentz-like gauge condition for massive gauge fields (12.6) can be obtained from the differential constraint for the conformal current (3.3) [or shadow field (4.3)] by making substitutions (12.1), (12.4), and (12.5).

Interrelations for spin-2 fields. In the gauge invariant approach, the massive spin-2 field is described by gauge fields ϕ_m^{ab}, ϕ_m^a, ϕ_m with Lagrangian [25]

$$\begin{aligned}\mathcal{L} =& \frac{1}{4}\phi_m^{ab}(E_{\rm EH}\phi_m)^{ab} + \frac{1}{2}\phi_m^a(E_{\rm Max}\phi_m)^a + \frac{1}{2}\phi_m\Box\phi_m \\ &+ m\phi_m^a(\partial^b\phi_m^{ba} - \partial^a\phi_m^{bb} - u\partial^a\phi_m) - \frac{m^2}{4}\phi_m^{ab}\phi_m^{ab} \\ &+ \frac{m^2}{4}\phi_m^{aa}\phi_m^{bb} + \frac{um^2}{2}\phi_m\phi_m^{aa} + \frac{dm^2}{2(d-2)}\phi_m^2,\end{aligned} \qquad (12.11)$$

where the respective second-derivative Einstein-Hilbert and Maxwell operators $E_{\rm EH}$, $E_{\rm Max}$ are given by

$$\begin{aligned}(E_{\rm EH}\phi)^{ab} =& \Box\phi^{ab} - \partial^a\partial^c\phi^{cb} - \partial^b\partial^c\phi^{ca} + \partial^a\partial^b\phi^{cc} \\ &+ \eta^{ab}(\partial^c\partial^e\phi^{ce} - \Box\phi^{cc}),\end{aligned} \qquad (12.12)$$

$$(E_{\rm Max}\phi)^a = \Box\phi^a - \partial^a\partial^b\phi^b, \qquad (12.13)$$

and u is defined in (5.5). Lagrangian (12.11) is invariant under the gauge transformations

$$\delta\phi_m^{ab} = \partial^a\xi_m^b + \partial^b\xi_m^a + \frac{2m}{d-2}\eta^{ab}\xi_m, \qquad (12.14)$$

$$\delta\phi_m^a = \partial^a\xi_m - m\xi_m^a, \qquad (12.15)$$

$$\delta\phi_m = -um\xi_m. \qquad (12.16)$$

It is easy to see that these transformations can be obtained by making substitutions (12.1) and

$$\phi_{\rm cur}^{ab} \to \phi_m^{ab}, \qquad \phi_{\rm cur}^a \to \frac{1}{m}\phi_m^a, \qquad \phi_{\rm cur} \to \frac{1}{m^2}\phi_m, \qquad (12.17)$$

$$\xi_{\rm cur}^a \to \xi_m^a, \qquad \xi_{\rm cur} \to \frac{1}{m}\xi_m, \qquad (12.18)$$

$$\phi_{\rm sh}^{ab} \to \phi_m^{ab}, \qquad \phi_{\rm sh}^a \to m\phi_m^a, \qquad \phi_{\rm sh} \to m^2\phi_m, \qquad (12.19)$$

$$\xi_{\rm sh}^a \to \xi_m^a, \qquad \xi_{\rm sh} \to m\xi_m, \qquad (12.20)$$

in gauge transformations of the current (5.6), (5.7), and (5.8) [or shadow field (6.5)–(6.7)].

Now let us consider interrelations between gauge conditions for the massive gauge fields and the differential constraints for the current and shadow field. We find the following de Donder-like gauge condition for the massive gauge fields:

$$\partial^b\phi_m^{ab} - \frac{1}{2}\partial^a\phi_m^{bb} + m\phi_m^a = 0, \qquad (12.21)$$

$$\partial^a\phi_m^a + \frac{1}{2}m\phi_m^{aa} + um\phi_m = 0. \qquad (12.22)$$

The surprise is that the gauge condition (12.21) and (12.22) leads to the *decoupled* equations of motion for the massive gauge fields,

$$\begin{aligned}(\Box - m^2)\phi_m^{ab} = 0, \qquad (\Box - m^2)\phi_m^a = 0, \\ (\Box - m^2)\phi_m = 0.\end{aligned} \qquad (12.23)$$

The gauge condition and equations of motion are invariant under leftover on shell gauge transformations (12.14)–(12.16), where the gauge transformation parameters satisfy the equations

$$(\Box - m^2)\xi_m^a = 0, \qquad (\Box - m^2)\xi_m = 0. \qquad (12.24)$$

Note that gauge-fixed equations (12.23) can be obtained from the appropriate gauge-fixed Lagrangian. Namely, if we denote the respective left-hand sides of (12.21) and (12.22) by C_m^a and C_m, and define the gauge-fixed Lagrangian as

$$\mathcal{L}_{\rm total} = \mathcal{L} - \frac{1}{2}C_m^aC_m^a - \frac{1}{2}C_mC_m, \qquad (12.25)$$

then we get the surprisingly simple gauge-fixed Lagrangian:

$$\begin{aligned}\mathcal{L}_{\rm total} =& \frac{1}{4}\phi_m^{ab}(\Box - m^2)\phi_m^{ab} - \frac{1}{8}\phi_m^{aa}(\Box - m^2)\phi_m^{bb} \\ &+ \frac{1}{2}\phi_m^a(\Box - m^2)\phi_m^a + \frac{1}{2}\phi_m(\Box - m^2)\phi_m,\end{aligned} \qquad (12.26)$$

which leads to Eqs. (12.23). To our knowledge, for $d > 4$, the gauge condition (12.21) and (12.22) and Lagrangian (12.26) have not been discussed in the earlier literature.

We now note that the de Donder-like gauge condition for the massive gauge fields (12.21) and (12.22) can simply be obtained by making substitutions (12.1), (12.17), and (12.19) in differential constraints for the current (5.3) and (5.4) [or shadow field (6.3) and (6.4)].

Also, we note that the gauge invariant field $T_{\rm cur}^{ab}$ (5.15) [or $T_{\rm sh}^{ab}$ (6.12)] can be related with the Pauli-Fierz field entering spin-2 massive field theory. Thus, in the gauge invariant approach, the Pauli-Fierz field has the following representation in terms of the massive gauge fields[22]:

[22]For $d = 4$, formula (12.27) was given in Ref. [46].

$$\Phi_{\rm PF}^{ab} = \phi_m^{ab} + \frac{1}{m}(\partial^a\phi_m^b + \partial^b\phi_m^a) + \frac{2}{um^2}\partial^a\partial^b\phi_m + \frac{2}{(d-2)u}\eta^{ab}\phi_m, \qquad (12.27)$$

where u is given in (5.5). One can make sure that

(i) $\Phi_{\rm PF}^{ab}$ is invariant under gauge transformations (12.14)–(12.16);

(ii) inserting the field $\Phi_{\rm PF}^{ab}$ into the Pauli-Fierz Lagrangian for the massive spin-2 field

$$\mathcal{L} = \frac{1}{4}\Phi_{\rm PF}^{ab}(E_{\rm EH}\Phi_{\rm PF})^{ab} - \frac{m^2}{4}(\Phi_{\rm PF}^{ab}\Phi_{\rm PF}^{ab} - \Phi_{\rm PF}^{aa}\Phi_{\rm PF}^{bb}), \qquad (12.28)$$

gives gauge invariant Lagrangian (12.11);

(iii) $\Phi_{\rm PF}^{ab}$ given in (12.27) can simply be obtained by making substitutions (12.1) and (12.17) in field $T_{\rm cur}^{ab}$ (5.15) [or by making substitutions (12.1) and (12.19) in field $T_{\rm sh}^{ab}$ (6.12)].

Interrelations for arbitrary spin fields. We begin with a presentation of the gauge invariant Lagrangian for the massive spin-s field in d-dimensional flat space. Gauge fields entering the gauge invariant Lagrangian can be collected in a ket-vector

$$|\phi_m\rangle \equiv \sum_{s'=0}^{s}\alpha_z^{s-s'}|\phi_{m,s'}\rangle, \qquad (12.29)$$

$$|\phi_{m,s'}\rangle \equiv \frac{\alpha^{a_1}\ldots\alpha^{a_{s'}}}{s'!\sqrt{(s-s')!}}\phi_{m,s'}^{a_1\ldots a_{s'}}|0\rangle. \qquad (12.30)$$

In terms of ket-vector (12.29), the Lagrangian of the massive gauge fields takes the form[23]

$$\mathcal{L} = \frac{1}{2}\langle\phi_m|E|\phi_m\rangle, \qquad (12.31)$$

where operator E is given by

$$E = E_{(2)} + E_{(1)} + E_{(0)}, \qquad (12.32)$$

$$E_{(2)} \equiv \Box - \alpha\partial\bar\alpha\partial + \frac{1}{2}(\alpha\partial)^2\bar\alpha^2 + \frac{1}{2}\alpha^2(\bar\alpha\partial)^2 - \frac{1}{2}\alpha^2\Box\bar\alpha^2 - \frac{1}{4}\alpha^2\alpha\partial\bar\alpha\partial\bar\alpha^2, \qquad (12.33)$$

$$E_{(1)} \equiv \bar e_{1m}\mathcal{A} + e_{1m}\bar{\mathcal{A}}, \qquad (12.34)$$

$$E_{(0)} \equiv m_1 + \alpha^2\bar\alpha^2 m_2 + \bar m_3\alpha^2 + m_3\bar\alpha^2, \qquad (12.35)$$

$$\mathcal{A} \equiv \alpha\partial - \alpha^2\bar\alpha\partial + \frac{1}{4}\alpha^2\alpha\partial\bar\alpha^2, \qquad (12.36)$$

$$\bar{\mathcal{A}} \equiv \bar\alpha\partial - \alpha\partial\bar\alpha^2 + \frac{1}{4}\alpha^2\bar\alpha\partial\bar\alpha^2, \qquad (12.37)$$

$$e_{1m} = m\alpha^z\tilde e_1, \qquad \bar e_{1m} = -m\tilde e_1\bar\alpha^z, \qquad (12.38)$$

$$m_1 = \frac{2s+d-2-N_z}{2s+d-2-2N_z}(N_z-1)m^2, \qquad (12.39)$$

$$m_2 = \frac{2(2s+d-2)+(2s+d-7)N_z-N_z^2}{4(2s+d-2-2N_z)}m^2, \qquad (12.40)$$

$$m_3 = \frac{1}{2}e_{1m}e_{1m}, \qquad \bar m_3 = \frac{1}{2}\bar e_{1m}\bar e_{1m}. \qquad (12.41)$$

The Lagrangian is invariant under the gauge transformation

$$\delta|\phi_m\rangle = \Big(\alpha\partial - e_{1m} - \frac{\alpha^2}{2s+d-6-2N_z}\bar e_{1m}\Big)|\xi_m\rangle, \qquad (12.42)$$

where the ket-vector of the gauge transformation parameter $|\xi_m\rangle$ is represented in terms of the $so(d-1,1)$ algebra tensor fields as

$$|\xi_m\rangle \equiv \sum_{s'=0}^{s-1}\alpha_z^{s-1-s'}|\xi_{m,s'}\rangle, \qquad (12.43)$$

$$|\xi_{m,s'}\rangle \equiv \frac{\alpha^{a_1}\ldots\alpha^{a_{s'}}}{s'!\sqrt{(s-1-s')!}}\xi_{m,s'}^{a_1\ldots a_{s'}}|0\rangle. \qquad (12.44)$$

The ket-vectors $|\phi_m\rangle$ and $|\xi_m\rangle$ satisfy the respective double-tracelessness and tracelessness constraints

$$(\bar\alpha^2)^2|\phi_m\rangle = 0, \qquad \bar\alpha^2|\xi_m\rangle = 0. \qquad (12.45)$$

Now let us consider the interrelations between the gauge invariant approaches to the massive field, conformal current, and shadow field. We begin with a comparison of the gauge transformations.

It is easy to see that the gauge transformation (12.42) can simply be obtained by making substitutions (12.1) and

$$|\phi_{\rm cur}\rangle \to m^{-N_z}|\phi_m\rangle, \qquad |\xi_{\rm cur}\rangle \to m^{-N_z}|\xi_m\rangle, \qquad (12.46)$$

$$|\phi_{\rm sh}\rangle \to m^{N_z}|\phi_m\rangle, \qquad |\xi_{\rm sh}\rangle \to m^{N_z}|\xi_m\rangle, \qquad (12.47)$$

in gauge transformation of the conformal current (7.23) [or shadow field (8.23)].

We now proceed with a comparison of the de Donder-like gauge for the massive gauge fields and the differential constraints for the currents and shadow fields. We find the following de Donder-like gauge condition for the massive arbitrary spin-s field

$$\bar C_m|\phi_m\rangle = 0, \qquad (12.48)$$

[23] In terms of the tensor fields $\phi_{m,s'}^{a_1\ldots a_{s'}}$, the Lagrangian (12.31) was found in [25].

$$\bar{C}_m \equiv \bar{\alpha}\partial - \frac{1}{2}\alpha\partial\bar{\alpha}^2 + \frac{1}{2}e_{1m}\bar{\alpha}^2 - \bar{e}_{1m}\Pi^{[1,2]}. \tag{12.49}$$

We note that gauge condition (12.48) leads to the decoupled gauge-fixed equations of motion for the massive gauge fields

$$(\Box - m^2)|\phi_m\rangle = 0. \tag{12.50}$$

These gauge-fixed equations of motion and the gauge condition (12.48) are invariant under leftover on shell gauge transformations (12.42) if the gauge transformation parameter satisfies the equation

$$(\Box - m^2)|\xi_m\rangle = 0. \tag{12.51}$$

Note that gauge-fixed equations (12.50) can be obtained from the appropriate gauge-fixed Lagrangian. Namely, if we define the gauge-fixed Lagrangian as

$$\mathcal{L}_{\rm total} = \mathcal{L} + \frac{1}{2}\langle\phi_m|C_m\bar{C}_m|\phi_m\rangle, \tag{12.52}$$

where $\bar{C}_m$ is given in (12.49), while C_m is defined by

$$C_m \equiv \alpha\partial - \frac{1}{2}\alpha^2\bar{\alpha}\partial + \frac{1}{2}\bar{e}_{1m}\alpha^2 - e_{1m}\Pi^{[1,2]}, \tag{12.53}$$

then we get the surprisingly simple gauge-fixed Lagrangian:

$$\mathcal{L}_{\rm total} = \frac{1}{2}\langle\phi_m|\Big(1 - \frac{1}{4}\alpha^2\bar{\alpha}^2\Big)(\Box - m^2)|\phi_m\rangle, \tag{12.54}$$

which leads to Eqs. (12.50). To our knowledge, the de Donder-like gauge condition (12.48) and gauge-fixed Lagrangian (12.54) have not been discussed in the earlier literature.

We now note that the de Donder-like gauge for the massive gauge fields (12.48) can simply be obtained by making substitutions (12.1), (12.46), and (12.47) in differential constraints for the currents (7.9) [or shadow fields (8.9)].

To summarize, we have obtained the gauge transformations and de Donder-like gauges of the massive fields from the gauge transformations and the differential constraints of the conformal currents (or shadow fields). It is clear that we can formally inverse our substitutions, i.e., we can obtain the gauge transformations and the differential constraints of the conformal currents (or shadow fields) from the gauge transformations and the de Donder-like gauge of the massive gauge fields by using formally the inverse substitution; i.e., first, by making the appropriate rescaling of the massive gauge fields and then making the substitution $m^2 \to \Box$. By now, in the literature, there are various approaches to gauge invariant formulations of massive fields. Obviously, use of the just mentioned interrelations between conformal currents (shadow fields) and massive fields might be helpful for a straightforward generalization of those approaches to the case of conformal currents and shadow fields.

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APPENDIX A: RESTRICTIONS IMPOSED BY GAUGE INVARIANCE AND BY DILATATION SYMMETRY

Under the dilatation transformations, the currents and shadows transform as $\delta_D|\phi_{\rm cur}\rangle = D_{\rm cur}|\phi_{\rm cur}\rangle$, $\delta_D|\phi_{\rm sh}\rangle = D_{\rm sh}|\phi_{\rm sh}\rangle$, where $D_{\rm cur}$, $D_{\rm sh}$ are given by

$$D_{\rm cur} = x\partial + \Delta_{\rm cur}, \qquad \Delta_{\rm cur} = \Delta_{0\rm cur} - N_z, \tag{A1}$$

$$D_{\rm sh} = x\partial + \Delta_{\rm sh}, \qquad \Delta_{\rm sh} = \Delta_{0\rm sh} + N_z, \tag{A2}$$

and where $\Delta_{0\rm cur}$, $\Delta_{0\rm sh}$ are constants. We now demonstrate that the two-point current-shadow field interaction vertex

$$\mathcal{L} = \langle\phi_{\rm cur}|\boldsymbol{\mu}|\phi_{\rm sh}\rangle, \tag{A3}$$

is invariant under the dilatation transformations provided $\boldsymbol{\mu}$ takes the following form:

$$\boldsymbol{\mu} = 1 + g_1\alpha^2\bar{\alpha}^2 + g_2\Box\bar{\alpha}^2, \tag{A4}$$

$$g_2 = \alpha^z\alpha^z\tilde{g}_2, \tag{A5}$$

where g_1, $\tilde{g}_2$ depend only on N_z. To this end, we start with the general expression for $\boldsymbol{\mu}$:

$$\boldsymbol{\mu} = 1 + g_1\alpha^2\bar{\alpha}^2 + g_2'\bar{\alpha}^2 + g_3'\alpha^2, \tag{A6}$$

$$g_2' = \alpha^z\alpha^z g_2'', \qquad g_3' = g_3''\bar{\alpha}^z\bar{\alpha}^z, \tag{A7}$$

where g_1, g_2'', g_3'' depend only on N_z and $\Box$. Requiring vertex $\mathcal{L}$ (A3) to be invariant under the dilatation transformation, $\delta_D\mathcal{L} = 0$ (up to total derivative), gives the equation

$$D_{\rm cur}^\dagger\boldsymbol{\mu} + \boldsymbol{\mu}D_{\rm sh} = 0, \tag{A8}$$

which amounts to the following equations:

$$[x\partial + N_z, \boldsymbol{\mu}] = 0, \tag{A9}$$

$$\Delta_{0\rm cur} + \Delta_{0\rm sh} = d. \tag{A10}$$

It is easily seen that solution to Eq. (A9) is given by

$$g_2'' = \Box\tilde{g}_2, \qquad g_3'' = 0, \tag{A11}$$

where $\tilde{g}_2$ depends only on N_z. Plugging this solution in (A6), we see that $\boldsymbol{\mu}$ takes the form given in (A4) and (A5).

We now find the restrictions imposed on the gauge transformations of $|\phi_{\rm cur}\rangle$ and $|\phi_{\rm sh}\rangle$ by the dilatation symmetry. We are going to demonstrate that the dilatation symmetry leads to the following gauge transformations of the currents and shadows:

$$\delta|\phi_{\rm cur}\rangle = G_{\rm cur}|\xi_{\rm cur}\rangle, \qquad \text{(A12)}$$

$$\delta|\phi_{\rm sh}\rangle = G_{\rm sh}|\xi_{\rm sh}\rangle, \qquad \text{(A13)}$$

$$G_{\rm cur} = \alpha\partial + b_{1\rm cur} + b_{2\rm cur}\alpha^2\Box, \qquad \text{(A14)}$$

$$G_{\rm sh} = \alpha\partial + b_{1\rm sh}\Box + b_{2\rm sh}\alpha^2, \qquad \text{(A15)}$$

$$b_{1\rm cur} = \alpha^z\tilde{b}_{1\rm cur}, \qquad b_{2\rm cur} = \tilde{b}_{2\rm cur}\bar{\alpha}^z, \qquad \text{(A16)}$$

$$b_{1\rm sh} = \alpha^z\tilde{b}_{1\rm sh}, \qquad b_{2\rm sh} = \tilde{b}_{2\rm sh}\bar{\alpha}^z, \qquad \text{(A17)}$$

where $\tilde{b}_{1\rm cur}$, $\tilde{b}_{2\rm cur}$, $\tilde{b}_{1\rm sh}$, $\tilde{b}_{2\rm sh}$ depend only on N_z. To this end, we note that under the dilatation transformations the gauge transformation parameters $|\xi_{\rm cur}\rangle$ and $|\xi_{\rm sh}\rangle$ transform as $\delta_D|\xi_{\rm cur}\rangle = D_{\xi_{\rm cur}}|\xi_{\rm cur}\rangle$, $\delta_D|\xi_{\rm sh}\rangle = D_{\xi_{\rm sh}}|\xi_{\rm sh}\rangle$, where $D_{\xi_{\rm cur}}$, $D_{\xi_{\rm sh}}$ are given by

$$D_{\xi_{\rm cur}} = D_{\rm cur} - 1, \qquad D_{\xi_{\rm sh}} = D_{\rm sh} - 1, \qquad \text{(A18)}$$

and $D_{\rm cur}$, $D_{\rm sh}$ are defined in (A1) and (A2). To avoid repetition, we restrict our attention to the gauge transformation of the current. We note that the general form of the gauge transformation operator $G_{\rm cur}$ (A12) is given by

$$G_{\rm cur} = \alpha\partial + b_1' + b_2'\alpha^2, \qquad \text{(A19)}$$

where b_1', b_2' depend on α^z, $\bar{\alpha}^z$, and $\Box$. Requiring the gauge symmetry to respect the dilatation transformation, gives the equation

$$D_{\rm cur}G_{\rm cur} = G_{\rm cur}D_{\xi_{\rm cur}}. \qquad \text{(A20)}$$

Plugging $G_{\rm cur}$ (A19) in (A20) and using (A1) and (A18), we see that Eq. (A20) leads to the following solution for b_1', b_2':

$$b_1' = \alpha^z\tilde{b}_{1\rm cur} \qquad b_2' = \Box\tilde{b}_{2\rm cur}\bar{\alpha}^z, \qquad \text{(A21)}$$

where $\tilde{b}_{1\rm cur}$, $\tilde{b}_{2\rm cur}$ depend only on N_z, i.e., we arrive at $G_{\rm cur}$ given in (A14) and (A16).

In a quite similar way, one can obtain the representation for $G_{\rm sh}$ given in (A15) and (A17).

We now demonstrate that requiring the vertex $\mathcal{L}$ to be invariant under gauge transformations (A12) and (A13) leads to the following results:

(i) The operators $\bar{C}_{\rm cur}$, $\bar{C}_{\rm sh}$ take form:

$$\bar{C}_{\rm cur} = \bar{\alpha}\partial - \frac{1}{2}\alpha\partial\bar{\alpha}^2 + c_{1\rm cur}\bar{\alpha}^2 + c_{2\rm cur}\Box\Pi^{[1,2]}, \qquad \text{(A22)}$$

$$\bar{C}_{\rm sh} = \bar{\alpha}\partial - \frac{1}{2}\alpha\partial\bar{\alpha}^2 + c_{1\rm sh}\Box\bar{\alpha}^2 + c_{2\rm sh}\Pi^{[1,2]}. \qquad \text{(A23)}$$

(ii) The c operators and b operators are related as

$$c_{1\rm cur} = -\frac{1}{2}b_{1\rm cur}, \qquad \text{(A24)}$$

$$c_{2\rm sh} = (2s + d - 6 - 2N_z)b_{2\rm cur}, \qquad \text{(A25)}$$

$$c_{1\rm cur} = \frac{1}{2}b_{2\rm sh}^\dagger(2s + d - 6 - 2N_z), \qquad \text{(A26)}$$

$$c_{2\rm cur} = -b_{1\rm sh}^\dagger, \qquad \text{(A27)}$$

$$c_{1\rm sh} = -\frac{1}{2}b_{1\rm sh}, \qquad \text{(A28)}$$

$$c_{2\rm sh} = (2s + d - 6 - 2N_z)b_{2\rm sh}. \qquad \text{(A29)}$$

$$c_{1\rm sh} = \frac{1}{2}b_{2\rm cur}^\dagger(2s + d - 6 - 2N_z), \qquad \text{(A30)}$$

$$c_{2\rm sh} = -b_{1\rm cur}^\dagger; \qquad \text{(A31)}$$

i.e., the c operators are represented similarly to the b operators [see (A16) and (A17)]

$$c_{1\rm cur} = \alpha^z\tilde{c}_{1\rm cur}, \qquad c_{2\rm cur} = \tilde{c}_{2\rm cur}\bar{\alpha}^z, \qquad \text{(A32)}$$

$$c_{1\rm sh} = \alpha^z\tilde{c}_{1\rm sh}, \qquad c_{2\rm sh} = \tilde{c}_{2\rm sh}\bar{\alpha}^z, \qquad \text{(A33)}$$

where $\tilde{c}$-operators depend only on N_z.

(iii) The $\tilde{c}$ operators satisfy the relations:

$$\tilde{c}_{1\rm cur}\tilde{c}_{2\rm cur} = \frac{1}{2}\tilde{e}_1{}^2, \qquad \tilde{c}_{1\rm sh}\tilde{c}_{2\rm sh} = \frac{1}{2}\tilde{e}_1{}^2, \qquad \text{(A34)}$$

where $\tilde{e}_1$ is defined in (7.14).

(iv) g_1 and g_2 are determined to be

$$g_1 = -\frac{1}{4}, \qquad g_2 = 0. \qquad \text{(A35)}$$

Before proving these results, we note that the methods for finding the operators $\bar{C}_{\rm cur}$ and $\bar{C}_{\rm sh}$ are quite similar. Therefore to avoid repetition we present details of the derivations of the operator $\bar{C}_{\rm sh}$.

To find the restrictions imposed on $\bar{C}_{\rm sh}$ by requiring that $\mathcal{L}$ be invariant under the gauge transformation of $|\phi_{\rm cur}\rangle$ we note the relation (up to total derivative)

$$-\langle G_{\rm cur}\xi_{\rm cur}|\boldsymbol{\mu}|\phi_{\rm sh}\rangle = \langle\xi_{\rm cur}|(\bar\alpha\partial + 2g_1\alpha\partial\bar\alpha^2 + g_2\Box\bar\alpha\partial\bar\alpha^2 - b_{1\rm cur}^\dagger - (b_{1\rm cur}^\dagger g_2 + b_{2\rm cur}^\dagger + 2b_{2\rm cur}^\dagger g_1(2N_\alpha + d))\Box\bar\alpha^2)|\phi_{\rm sh}\rangle, \qquad (A36)$$

which implies that the requirement of invariance of $\mathcal{L}$ under the gauge transformation of $|\phi_{\rm cur}\rangle$,

$$\langle G_{\rm cur}\xi_{\rm cur}|\boldsymbol{\mu}|\phi_{\rm sh}\rangle = 0, \qquad (A37)$$

leads to the constraint

$$\bar C_{\rm sh}|\phi_{\rm sh}\rangle = 0, \qquad (A38)$$

with the following $\bar C_{\rm sh}$:

$$\bar C_{\rm sh} = \Pi^{[1,2]}(\bar\alpha\partial + 2g_1\alpha\partial\bar\alpha^2) + g_2\Box\bar\alpha\partial\bar\alpha^2 - (b_{1\rm cur}^\dagger g_2 + b_{2\rm cur}^\dagger + 2b_{2\rm cur}^\dagger g_1(2N_\alpha + d))\Box\bar\alpha^2 - b_{1\rm cur}^\dagger\Pi^{[1,2]}. \qquad (A39)$$

We now find the restrictions on $\bar C_{\rm sh}$ which are obtained by requiring that the constraint (A38) be invariant under the gauge transformation of $|\phi_{\rm sh}\rangle$, i.e. we consider the equation

$$\bar C_{\rm sh}G_{\rm sh}|\xi_{\rm sh}\rangle = 0, \qquad (A40)$$

where $G_{\rm sh}$ and $\bar C_{\rm sh}$ are given in (A15) and (A39), respectively. Before studying all restrictions on $\bar C_{\rm sh}$ which are obtainable from (A40) we note that the requirement for cancellation of $\alpha\partial\bar\alpha\partial$ and $(\bar\alpha\partial)^2$ terms in (A40) leads to g_1, g_2 given in (A35). Plugging g_1, g_2 in (A39), we obtain (A23) with $c_{1\rm sh}$, $c_{2\rm sh}$ given in (A30) and (A31). Now we are ready to find all restrictions on $\bar C_{\rm sh}$ which are obtainable from (A40). Thus, using (A23) we represent the left-hand side of (A40) as

$$\bar C_{\rm sh}G_{\rm sh}|\xi_{\rm sh}\rangle = (\Box X_1 + \Box\bar\alpha\partial X_2 + CX_3)|\xi_{\rm sh}\rangle, \qquad (A41)$$

$$C \equiv \alpha\partial - \alpha^2\frac{1}{2N_\alpha + d}\bar\alpha\partial, \qquad (A42)$$

$$X_1 \equiv 1 + c_{2\rm sh}b_{1\rm sh} + 2(2s + d - 2 - 2N_z)c_{1\rm sh}b_{2\rm sh}, \qquad (A43)$$

$$X_2 \equiv b_{1\rm sh} + 2c_{1\rm sh}, \qquad (A44)$$

$$X_3 \equiv c_{2\rm sh} - (2s + d - 6 - 2N_z)b_{2\rm sh}. \qquad (A45)$$

From (A41), we see that Eq. (A40) amounts to the equations $X_i|\xi_{\rm sh}\rangle = 0$, $i = 1, 2, 3$. The solution to equations $X_2|\xi_{\rm sh}\rangle = 0$, $X_3|\xi_{\rm sh}\rangle = 0$ is given by (A28) and (A29). Making use of (A28) and (A29) in Eq. $X_1|\xi_{\rm sh}\rangle = 0$ gives the equation

$$\left(c_{2\rm sh}c_{1\rm sh} - \frac{2s + d - 2 - 2N_z}{2s + d - 4 - 2N_z}c_{1\rm sh}c_{2\rm sh} - \frac{1}{2}\right)|\xi_{\rm sh}\rangle = 0. \qquad (A46)$$

Using a representation for the c operators given in (A33), we find that Eq. (A46) allows us to determine the quantity $\tilde c_{1\rm sh}\tilde c_{2\rm sh}$ uniquely. The result is given in (A34).

We finish the discussion in this appendix by remarking on the similarity transformation of the currents and shadows. As we have demonstrated, requiring the differential constraints for the currents and shadows to be invariant under the gauge transformations gives a unique solution for the products $\tilde c_{1\rm cur}\tilde c_{2\rm cur}$, $\tilde c_{1\rm sh}\tilde c_{2\rm sh}$ (A34). From (A24), (A27), (A28), and (A31), it is seen that the $\tilde c$ operators are related as

$$\tilde c_{1,\rm cur} = \frac{1}{2}\tilde c_{2,\rm sh}, \qquad \tilde c_{1,\rm sh} = \frac{1}{2}\tilde c_{2\rm cur}. \qquad (A47)$$

It turns out that there are no additional restrictions on the $\tilde c$ operators. This implies that there is an arbitrariness in the choice of the $\tilde c$ operators. We note that this arbitrariness is related with the similarity transformation of the currents and shadows,

$$|\phi_{\rm cur}\rangle \to U|\phi_{\rm cur}\rangle, \qquad |\phi_{\rm sh}\rangle \to U^{-1}|\phi_{\rm sh}\rangle, \qquad (A48)$$

where U is an arbitrary function of N_z with the restriction that U is not equal to zero for the allowed eigenvalues of N_z equal to $0, 1, \ldots, s$. It is seen that transformation (A48) leaves the vertex $\mathcal{L}$ invariant, but changes the $\tilde c$ operators. Using this transformation one of the $\tilde c$ operators can be made the arbitrary function of N_z with the restriction that this function is not equal to zero for allowed eigenvalues of N_z equal to $0, 1, \ldots, s$. The remaining $\tilde c$ operators are then determined uniquely by relations (A34) and (A47). In this paper, we use the following choice of $\tilde c$ operators:

$$\tilde c_{1\rm cur} = \frac{1}{2}\tilde e_1, \qquad \tilde c_{2\rm cur} = \tilde e_1, \qquad (A49)$$

$$\tilde c_{1\rm sh} = \frac{1}{2}\tilde e_1, \qquad \tilde c_{2\rm sh} = \tilde e_1. \qquad (A50)$$

This choice turns out to be convenient for the study of AdS/CFT correspondence.

APPENDIX B: RESTRICTIONS IMPOSED BY CONFORMAL BOOST SYMMETRIES

In this appendix, we use the notation $R_{\rm cur}^a$ and $R_{\rm sh}^a$ to indicate the respective realizations of operator R^a on the current $|\phi_{\rm cur}\rangle$ and the shadow field $|\phi_{\rm sh}\rangle$. Because the methods for finding the operators $R_{\rm cur}^a$ and $R_{\rm sh}^a$ are quite similar we present details of the derivation of the operator $R_{\rm sh}^a$ and outline a procedure for the derivation of the operator $R_{\rm cur}^a$.

We find the operator $R_{\rm sh}^a$ by requiring that

(i) the differential constraint for $|\phi_{\rm sh}\rangle$ be invariant under conformal boost transformations;

(ii) the operator $R_{\rm sh}^a$ be independent of the derivatives ∂^a.

Before analyzing restrictions imposed on $R_{\rm sh}^a$ by the conformal boost symmetries we find the general expression for the operator $R_{\rm sh}^a$ that respects the dilatation symmetry and algebraic constraint (8.6). Requiring that the operator R^a respects algebraic constraint (8.6) and the commutation relation $[D, K^a] = K^a$ gives

$$[N_\alpha + N_z, R_{\rm sh}^a] = 0 \tag{B1}$$

$$[\Delta_{\rm sh}, R_{\rm sh}^a] = R_{\rm sh}^a. \tag{B2}$$

To derive (B2) we take into account that the operator $R_{\rm sh}^a$ is independent of the space coordinates x^a because of commutator (2.9). Also, we use our assumption that $R_{\rm sh}^a$ is independent of the derivatives ∂^a. Taking into account the expression for $\Delta_{\rm sh}$ in (A2), it is easy to see that (B1) and (B2) amount to the commutators

$$[N_\alpha, R_{\rm sh}^a] = -R_{\rm sh}^a, \qquad [N_z, R_{\rm sh}^a] = R_{\rm sh}^a. \tag{B3}$$

The general solution to (B3) is obvious:

$$R_{\rm sh}^a = r_{0,1,{\rm sh}}\bar\alpha^a + r_{0,2,{\rm sh}}\alpha^a\bar\alpha^2 + r_{0,3,{\rm sh}}\alpha^2\bar\alpha^a\bar\alpha^2, \tag{B4}$$

$$r_{0,k,{\rm sh}} = \alpha^z\tilde r_{0,k,{\rm sh}}, \qquad k = 1, 2, 3, \tag{B5}$$

where the operators $\tilde r_{0,k,{\rm sh}}$ depend only on N_z. Note that to derive (B4) we take into account constraint (8.8) which tells us that the contribution of $(\bar\alpha^2)^n$ terms to $R_{\rm sh}^a$ is irrelevant when $n \geq 2$.

We now consider restrictions imposed on $R_{\rm sh}^a$ by the conformal boost symmetries. Consider the differential constraint for the shadow filed $|\phi_{\rm sh}\rangle$,

$$\bar C_{\rm sh}|\phi_{\rm sh}\rangle = 0, \tag{B6}$$

where $\bar C_{\rm sh}$ is given in (A23). Requiring this constraint to be invariant under the conformal boost transformations gives the equations

$$\bar C_{\rm sh}K^a|\phi_{\rm sh}\rangle = 0, \tag{B7}$$

where the conformal boost operator K^a takes the form given in (2.15), $K^a = K^a_{\Delta_{\rm sh},M} + R^a_{\rm sh}$. To analyze Eqs. (B7) we note the following helpful formulas:

$$[\bar C_{\rm sh}, K^a_{\Delta_{\rm sh},M}] = x^a\bar C_{\rm sh} + \bar C^a_{{\rm sh}(0)} + \bar C^a_{{\rm sh}(1)}, \tag{B8}$$

$$\bar C^a_{{\rm sh}(0)} \equiv (\Delta_{\rm sh} - N_\alpha - d + 1)\bar C^a_\perp - \frac{1}{2}(2N_\alpha + d - 4)C^a\bar\alpha^2, \tag{B9}$$

$$\bar C^a_{{\rm sh}(1)} \equiv (2\Delta_{\rm sh} - d)c_1\bar\alpha^2\partial^a + 2c_1M^{ab}\partial^b\bar\alpha^2, \tag{B10}$$

$$C^a \equiv \alpha^a - \alpha^2\frac{1}{2N_\alpha + d}\bar\alpha^a, \tag{B11}$$

$$\bar C_{\rm sh}R^a|\phi_{\rm sh}\rangle = \mathbf{Y}^a|\phi_{\rm sh}\rangle, \tag{B12}$$

$$\mathbf{Y}^a \equiv Y_1C^a\bar\alpha^2 + Y_2\Box\bar\alpha^a\bar\alpha^2 + Y_3\bar C^a_\perp + Y_4\partial^a\bar\alpha^2 + Y_5M^{ab}\partial^b\bar\alpha^2 + Y_6C\bar\alpha^a\bar\alpha^2, \tag{B13}$$

$$Y_1 \equiv \frac{1}{2}c_{2{\rm sh}}r_{0,1,{\rm sh}} + c_{2{\rm sh}}r_{0,2,{\rm sh}} - \frac{2N_\alpha + d - 4}{2(2N_\alpha + d - 2)}r_{0,1,{\rm sh}}c_{2{\rm sh}}, \tag{B14}$$

$$Y_2 \equiv [c_{1{\rm sh}}, r_{0,1,{\rm sh}}] + 2c_{1{\rm sh}}r_{0,2,{\rm sh}} + 2(2N_\alpha + d)c_{1{\rm sh}}r_{0,3,{\rm sh}}, \tag{B15}$$

$$Y_3 \equiv [c_{2{\rm sh}}, r_{0,1,{\rm sh}}], \tag{B16}$$

$$Y_4 \equiv \frac{1}{2}r_{0,1,{\rm sh}} + r_{0,2,{\rm sh}}, \tag{B17}$$

$$Y_5 \equiv r_{0,2,{\rm sh}}, \tag{B18}$$

$$Y_6 \equiv -r_{0,3,{\rm sh}}(2N_\alpha + d - 4). \tag{B19}$$

Also, we note that to derive (B12) we use constraint (B6). Using (B6), (B8), and (B12) it is easy to see that Eqs. (B7) lead to the equations,

$$(\mathbf{Y}^a + \bar C^a_{{\rm sh}(0)} + \bar C^a_{{\rm sh}(1)})|\phi_{\rm sh}\rangle = 0. \tag{B20}$$

Taking into account (B9)–(B13), we see that Eqs. (B20) amount to the following equations:

$$\Big(Y_1 - \frac{1}{2}(2N_\alpha + d - 4)\Big)C^a\bar\alpha^2|\phi_{\rm sh}\rangle = 0, \tag{B21}$$

$$Y_2\bar\alpha^a\bar\alpha^2|\phi_{\rm sh}\rangle = 0, \tag{B22}$$

$$(Y_3 + \Delta_{\rm sh} - N_\alpha - d + 1)\bar C^a_\perp|\phi_{\rm sh}\rangle = 0, \tag{B23}$$

$$(Y_4 + (2\Delta_{\rm sh} - d)c_1)\bar\alpha^2|\phi_{\rm sh}\rangle = 0, \tag{B24}$$

$$(Y_5 + 2c_1)M^{ab}\bar\alpha^2|\phi_{\rm sh}\rangle = 0, \tag{B25}$$

$$Y_6C\bar\alpha^a\bar\alpha^2|\phi_{\rm sh}\rangle = 0. \tag{B26}$$

Analysis of Eqs. (B21)–(B26) is straightforward. From (B25) and (B26), we obtain

$$r_{0,2,{\rm sh}} = -2c_{1{\rm sh}}, \tag{B27}$$

$$r_{0,3,{\rm sh}} = 0. \tag{B28}$$

From (B24) and (B27), we find

$$r_{0,1,{\rm sh}} = 2(d + 2 - 2\Delta_{\rm sh})c_{1{\rm sh}}. \tag{B29}$$

Using (B27)–(B29), we find that Eq. (B22) is satisfied automatically. Using (B29), we represent Eq. (B23) as

$$(2(d-2\Delta_{\rm sh})c_{2\rm sh}c_{1\rm sh}-2(d+2-2\Delta_{\rm sh})c_{1\rm sh}c_{2\rm sh} + \Delta_{\rm sh}+N_z-s-d+2)\bar{C}^a_\perp|\phi_{\rm sh}\rangle=0. \qquad (B30)$$

Using the solution for the c operators given in (A33) and (A34), we find that Eq. (B30) is solved by

$$\Delta_{0\rm sh}=2-s. \qquad (B31)$$

Finally, using (B27), (B29), and (B31) and the solution for the c operators given in (A33) and (A34), we check that Eq. (B21) is satisfied automatically.

To summarize, taking into account the solution for the r operators given in (B27)–(B29) and (B31) and using (8.6), we cast the operator $R^a_{\rm sh}$ into the following form:

$$R^a_{\rm sh}=r_{0,1,\rm sh}\Big(\bar{\alpha}^a-\alpha^a\frac{1}{2N_\alpha+d}\bar{\alpha}^2\Big), \qquad (B32)$$

$$r_{0,1,\rm sh}=2c_{1\rm sh}(2s+d-4-2N_z). \qquad (B33)$$

Inserting $\tilde{c}_{1\rm sh}$ (A50) in (B32) and (B33) gives $R^a_{\rm sh}$ (8.27).

In a similar way, we can find the operator $R^a_{\rm cur}$. Requiring that the operator $R^a_{\rm cur}$ respects algebraic constraints (7.6) and (7.8) and the commutation relation $[D,K^a]=K^a$ gives

$$R^a_{\rm cur}=r_{0,1,\rm cur}\tilde{C}^a+r_{0,2,\rm cur}\alpha^2\bar{C}^a_\perp+r_{0,3,\rm cur}\alpha^2C^a\bar{\alpha}^2, \qquad (B34)$$

$$r_{0,k,\rm cur}=\tilde{r}_{0,k,\rm cur}\bar{\alpha}^z, \qquad k=1,2,3, \qquad (B35)$$

where the operators $\tilde{r}_{0,k,\rm cur}$ depend only on N_z. The operators $\tilde{C}^a$, $\bar{C}^a_\perp$, C^a are defined in (7.28), (7.29), and (B11), respectively. Requiring the constraint $\bar{C}_{\rm cur}|\phi_{\rm cur}\rangle=0$ to be invariant under the conformal boost symmetries leads to the following solution for the r operators:

$$r_{0,1,\rm cur}=-(2s+d-4-2N_z)c_{2\rm cur}, \qquad (B36)$$

$$r_{0,2,\rm cur}=-\frac{2}{2s+d-6-2N_z}c_{2\rm cur}, \qquad (B37)$$

$$r_{0,3,\rm cur}=0. \qquad (B38)$$

Inserting these r operators in (B34) and using (7.6), we cast the operator $R^a_{\rm cur}$ into the following form:

$$R^a_{\rm cur}=r_{0,1,\rm cur}\Big(\tilde{C}^a+\alpha^2\frac{2}{(2N_\alpha+d-2)(2N_\alpha+d)}\bar{C}^a_\perp\Big). \qquad (B39)$$

With the choice of the $\tilde{c}_{2\rm cur}$ operator made in (A49), the operator $R^a_{\rm cur}$ (B39) takes the form given in (7.27).

Alternatively, the operator $R^a_{\rm cur}$ can be evaluated by using $R^a_{\rm sh}$ (B32) and requiring the vertex $\mathcal{L}$,

$$\mathcal{L}=\langle\phi_{\rm cur}|\boldsymbol{\mu}|\phi_{\rm sh}\rangle, \qquad \boldsymbol{\mu}\equiv1-\frac{1}{4}\alpha^2\bar{\alpha}^2, \qquad (B40)$$

to be invariant under the conformal boost transformations. To this end, let us use the notation $K^a_{\rm cur}$ and $K^a_{\rm sh}$ to indicate the respective realizations of the operator K^a on the current $|\phi_{\rm cur}\rangle$ and the shadow field $|\phi_{\rm sh}\rangle$. Requiring vertex $\mathcal{L}$ (B40) to be invariant under the conformal boost transformations gives the relation (up to total derivative)

$$\langle\phi_{\rm cur}|\boldsymbol{\mu}K^a_{\rm sh}|\phi_{\rm sh}\rangle=-\langle K^a_{\rm cur}\phi_{\rm cur}|\boldsymbol{\mu}|\phi_{\rm sh}\rangle. \qquad (B41)$$

Taking into account that the operators $K^a_{\Delta_{\rm cur},M}$, $K^a_{\Delta_{\rm sh},M}$ satisfy the relation (up to total derivative)

$$\langle\phi_{\rm cur}|\boldsymbol{\mu}K^a_{\Delta_{\rm sh},M}|\phi_{\rm sh}\rangle=-\langle K^a_{\Delta_{\rm cur},M}\phi_{\rm cur}|\boldsymbol{\mu}|\phi_{\rm sh}\rangle, \qquad (B42)$$

we conclude that the operators $R^a_{\rm cur}$ and $R^a_{\rm sh}$ should satisfy the relation

$$\langle\phi_{\rm cur}|\boldsymbol{\mu}R^a_{\rm sh}|\phi_{\rm sh}\rangle=-\langle R^a_{\rm cur}\phi_{\rm cur}|\boldsymbol{\mu}|\phi_{\rm sh}\rangle. \qquad (B43)$$

Using (A47) and (B32), we make sure that relation (B43) leads to $R^a_{\rm cur}$ given in (B39). This provides an additional check to our calculations.

APPENDIX C: MODIFIED LORENTZ AND DE DONDER GAUGE CONDITIONS

In this appendix, we explain some details of the derivation of the modified Lorentz and de Donder gauge conditions.

Spin-1 field. We use field Φ^A carrying flat Lorentz algebra $so(d,1)$ vector indices $A,B=0,1,\ldots,d-1,d$. The field Φ^A is related with the field carrying the base manifold indices Φ^μ, $\mu=0,1,\ldots,d$, in a standard way $\Phi^A=e^A_\mu\Phi^\mu$, where e^A_μ is the vierbein of AdS_{d+1} space. For the Poincaré parametrization of AdS_{d+1} space (10.1), the vierbein $e^A=e^A_\mu dx^\mu$ and Lorentz connection $de^A+\omega^{AB}\wedge e^B=0$ are given by

$$e^A_\mu=\frac{1}{z}\delta^A_\mu, \qquad \omega^{AB}_\mu=\frac{1}{z}(\delta^A_z\delta^B_\mu-\delta^B_z\delta^A_\mu), \qquad (C1)$$

where δ^A_μ is the Kronecker delta symbol. We use a covariant derivative with the flat indices $\mathcal{D}^A$,

$$\mathcal{D}_A\equiv e^\mu_A\mathcal{D}_\mu, \qquad \mathcal{D}^A=\eta^{AB}\mathcal{D}_B, \qquad (C2)$$

where e^μ_A is inverse of the AdS vielbein, $e^A_\mu e^\mu_B=\delta^A_B$ and η^{AB} is the flat metric tensor. With the choice made in (C1), the covariant derivative takes the form

$$\mathcal{D}^A\Phi^B=\hat{\partial}^A\Phi^B+\omega^{ABC}\Phi^C, \qquad (C3)$$

$$\hat{\partial}^A=z\partial^A, \qquad \omega^{ABC}=\eta^{AC}\delta^B_z-\eta^{AB}\delta^C_z, \qquad (C4)$$

where we adapt the following conventions for the derivatives and coordinates: $\partial^A=\eta^{AB}\partial_B$, $\partial_A=\partial/\partial x^A$, $x^A\equiv\delta^A_\mu x^\mu$, $x^A=x^a,x^d$, $x^d\equiv z$.

With these conventions, the equations of motion of the massless spin-1 AdS field $\Phi^A=\Phi^a,\Phi^z$,

$$\mathcal{D}^AF^{AB}=0, \qquad F^{AB}=\mathcal{D}^A\Phi^B-\mathcal{D}^B\Phi^A, \qquad (C5)$$

can be represented as

$$(\hat{\partial}^2 - d\hat{\partial}_z + d - 1)\Phi^A - \hat{\partial}^A(\mathcal{D}\Phi + 2\Phi^z) + 2\delta_z^A\mathcal{D}\Phi + (d+1)\delta_z^A\Phi^z = 0, \qquad (C6)$$

where $\hat{\partial}^2 \equiv \hat{\partial}^A\hat{\partial}^A$, $\mathcal{D}\Phi \equiv \mathcal{D}^A\Phi^A$. Our modified Lorentz gauge condition is defined by the relation [11]

$$\mathcal{D}^A\Phi^A + 2\Phi^z = 0, \qquad (C7)$$

which, in the Poincaré coordinates, can be represented as

$$\hat{\partial}^A\Phi^A + (2-d)\Phi^z = 0. \qquad (C8)$$

Using (C7) in gauge invariant equations of motion (C6) leads to the decoupled gauge-fixed equations of motion

$$(\hat{\partial}^2 - d\hat{\partial}_z + d - 1)\Phi^A + (d-3)\delta_z^A\Phi^z = 0, \qquad (C9)$$

which can be represented as

$$(z^2(\Box + \partial_z^2) + (1-d)z\partial_z + d - 1)\Phi^a = 0, \qquad (C10)$$

$$(z^2(\Box + \partial_z^2) + (1-d)z\partial_z + 2d - 4)\Phi^z = 0. \qquad (C11)$$

Introducing the canonically normalized field ϕ^A,

$$\Phi^A = z^{(d-1)/2}\phi^A, \qquad (C12)$$

and using the identification $\phi^z = \phi$, we make sure that Eqs. (C10) and (C11) amount to the respective Eqs. (10.3) and (10.4), while the modified Lorentz gauge condition (C7) takes the form given in (10.2).

Equations of motion (C5) are invariant under the gauge transformations

$$\delta\Phi^A = \hat{\partial}^A\Xi. \qquad (C13)$$

Making the rescaling

$$\Xi = z^{(d-3)/2}\xi, \qquad (C14)$$

we check that the gauge transformations (C13) lead to the ones given in (10.6) and (10.7).

Spin-2 field. Einstein equations of motion for the massless spin-2 field in AdS_{d+1} can be represented as

$$\mathcal{D}^2h^{AB} - \mathcal{D}^A\mathcal{D}^Ch^{CB} - \mathcal{D}^B\mathcal{D}^Ch^{CA} + \mathcal{D}^A\mathcal{D}^Bh + 2h^{AB} - 2\eta^{AB}h = 0, \qquad (C15)$$

$$h \equiv h^{AA}, \qquad (C16)$$

where the field with the flat indices, h^{AB}, is related with the field carrying the base manifold indices in a standard way $h^{AB} = e_\mu^A e_\nu^B h^{\mu\nu}$. Gauge transformations of h^{AB} take the form

$$\delta h^{AB} = \mathcal{D}^A\Xi^B + \mathcal{D}^B\Xi^A. \qquad (C17)$$

In terms of $h^{AB} = h^{ab}, h^{za}, h^{zz}$, our modified de Donder gauge condition is defined to be

$$\mathcal{D}^Bh^{AB} - \frac{1}{2}\mathcal{D}^Ah + 2h^{zA} - \eta^{zA}h = 0. \qquad (C18)$$

In the Poincaré coordinates, this gauge condition can be represented as

$$z\partial^Bh^{AB} - \frac{1}{2}z\partial^Ah + (1-d)h^{zA} = 0. \qquad (C19)$$

Introducing the canonically normalized fields $\tilde{\phi}^{AB}$,

$$h^{AB} = z^{(d-1)/2}\tilde{\phi}^{AB}, \qquad (C20)$$

and using (C19) we represent Eqs. (C15) as

$$\Big(\Box + \partial_z^2 - \frac{d^2-1)}{4z^2}\Big)\tilde{\phi}^{ab} - \frac{2}{z^2}\eta^{ab}\tilde{\phi}^{zz} = 0, \qquad (C21)$$

$$\Big(\Box + \partial_z^2 - \frac{(d-1)(d-3)}{4z^2}\Big)\tilde{\phi}^{za} = 0, \qquad (C22)$$

$$\Big(\Box + \partial_z^2 - \frac{(d-3)(d-5)}{4z^2}\Big)\tilde{\phi}^{zz} = 0. \qquad (C23)$$

From these equations, we see that the modified de Donder gauge itself does not lead automatically to decoupled equations. In order to get the decoupled equations, we introduce our fields ϕ^{ab}, ϕ^a, ϕ defined by

$$\phi^{ab} = \tilde{\phi}^{ab} + \frac{1}{d-2}\eta^{ab}\tilde{\phi}^{zz}, \qquad (C24)$$

$$\phi^a = \tilde{\phi}^{za}, \qquad (C25)$$

$$\phi = \frac{1}{2}u\tilde{\phi}^{zz}, \qquad (C26)$$

where u is defined in (5.5). In terms of our fields (C24)–(C26), the gauge-fixed equations of motion (C21)–(C23) take the decoupled form given in (10.35), (10.36), and (10.37).

The gauge transformations we use in Sec. X are obtained from (C17) by introducing

$$\Xi^A = z^{(d-3)/2}\xi^A, \qquad (C27)$$

and making the identification for the $so(d-1,1)$ algebra scalar mode $\xi \equiv \xi^z$.

Arbitrary spin field. For the massless arbitrary spin-s field in AdS_{d+1}, we define our modified de Donder gauge condition as follows. Consider the totally symmetric double-traceless $so(d,1)$ algebra tensor field $\Phi^{A_1\ldots A_s}$, $\Phi^{AABBA_5\ldots A_s} = 0$. The modified de Donder gauge condition, found in Ref. [16], is defined as

$$\mathcal{D}^B\Phi^{A_1\ldots A_{s-1}B} - \frac{s-1}{2}\mathcal{D}^{(A_1}\Phi^{A_2A_3\ldots A_{s-1})BB} + 2\Phi^{A_1\ldots A_{s-1}z} - (s-1)\eta^{z(A_1}\Phi^{A_2\ldots A_{s-1})BB} = 0, \qquad (C28)$$

where the symmetrization of the indices $A_1\ldots A_{s-1}$ is normalized as $(A_1\ldots A_n) = \frac{1}{n!}(A_1\ldots A_n + (n!-1)\text{terms})$.

Note however that gauge condition (C28) itself does not lead automatically to decoupled equations. One needs to make a transformation similar to the one in (C24)–(C26). A discussion of the transformation and the field variables which lead to the decoupled equations of motion in Sec. XI may be found in Ref. [16].

APPENDIX D: MODIFIED LORENTZ AND DE DONDER GAUGE CONDITIONS IN CONFORMAL FLAT SPACE

We now generalize from the modified Lorentz and de Donder gauge conditions to the case of massless arbitrary spin fields propagating in conformal flat space.

The line element of conformal flat space takes the form

$$ds^2 = \frac{1}{Z^2} dx^A dx^A, \tag{D1}$$

where the conformal factor $Z = Z(x)$ depends on coordinates x^A. For parametrization of conformal space (D1), the vierbein $e^A = e^A_\mu dx^\mu$ and Lorentz connection ω^{BC}_μ are given by

$$e^A_\mu = \frac{1}{Z}\delta^A_\mu, \qquad \omega^{BC}_\mu = \frac{1}{Z}(\delta^C_\mu Z^B - \delta^B_\mu Z^C), \tag{D2}$$

$$Z^A \equiv \partial^A Z. \tag{D3}$$

We note that AdS_{d+1} space is obtained by requiring the conformal factor Z to satisfy the equation

$$Z\partial^A\partial^B Z = \frac{1}{2}\eta^{AB}(Z^C Z^C - 1), \qquad d > 1, \tag{D4}$$

$$Z\partial^A\partial^A Z = Z^A Z^A - 1, \qquad d = 1. \tag{D5}$$

With the choice made in (D2), the covariant derivative takes the form

$$\mathcal{D}^A\Phi^B = \hat{\partial}^A\Phi^B + \omega^{ABC}\Phi^C, \tag{D6}$$

$$\hat{\partial}^A = Z\partial^A, \qquad \omega^{ABC} = \eta^{AC}Z^B - \eta^{AB}Z^C. \tag{D7}$$

We note that various conformal flat geometries are specialized by appropriate choice of the conformal factor Z. This is to say that the Poincaré parametrization of AdS_{d+1} space with coordinates $x^A = x^a$, z, $a = 0, 1, \ldots, d-1$, is specialized by

$$Z(x) = z. \tag{D8}$$

Also we note that the stereographic parametrization of AdS_{d+1} space with coordinates x^A, $A = 0, 1, \ldots, d$, is specialized by

$$Z(x) = 1 - \frac{1}{4}x^A x^A. \tag{D9}$$

Famous $\mathrm{AdS}_{d+1} \times S^{d+1}$ space is also conformal flat. Thus, the $\mathrm{AdS}_{d+1} \times S^{d+1}$ space can be described by coordinates $x^A = x^a$, x^M, $a = 0, 1, \ldots, d-1$, $M = d, \ldots, 2d+1$, with the conformal factor given by

$$Z(x) = \sqrt{x^M x^M}. \tag{D10}$$

Now let us describe modified gauge conditions for massless fields in conformal flat space. For the massless spin-1 field, our modified Lorentz gauge condition takes the form

$$\mathcal{D}^A\Phi^A + 2Z^A\Phi^A = 0, \tag{D11}$$

while for the massless spin-2 field the modified de Donder gauge is defined to be

$$\mathcal{D}^B h^{AB} - \frac{1}{2}\mathcal{D}^A h + 2Z^B h^{AB} - Z^A h = 0. \tag{D12}$$

For the massless arbitrary spin-s field propagating in conformal flat space, the modified de Donder gauge condition takes the form

$$\mathcal{D}^B\Phi^{A_1\ldots A_{s-1}B} - \frac{s-1}{2}\mathcal{D}^{(A_1}\Phi^{A_2A_3\ldots A_{s-1})BB} + 2Z^B\Phi^{A_1\ldots A_{s-1}B} - (s-1)Z^{(A_1}\Phi^{A_2\ldots A_{s-1})BB} = 0. \tag{D13}$$

It is easy to see that by choosing Z corresponding to Poincaré parametrization (D8) gauge conditions (D11)–(D13) reduce to the respective gauge conditions given in (C7), (C18), and (C28).

APPENDIX E: MATCHING OF CONFORMAL BOOST SYMMETRIES

We now demonstrate matching of the improved K^a transformations of the non-normalizable bulk AdS modes and the conformal boost transformations of the boundary shadow fields. Matching of conformal boost symmetries of bulk normalizable AdS modes and boundary currents can be demonstrated in a quite similar way.

Improved K^a transformations of AdS field take the form

$$K^a_{\rm impr}|\phi\rangle = K^a_{\rm AdS}|\phi\rangle + G_{\rm AdS}|\xi^{K^a}\rangle, \tag{E1}$$

where the compensating gauge transformation parameter $|\xi^{K^a}\rangle$ corresponding to the non-normalizable solution is given in (11.53). The generic generator of K^a symmetries, denoted by $K^a_{\rm AdS}$ in this appendix, is given in (11.43), while the gauge transformation operator $G_{\rm AdS}$ can be read from (11.17),

$$G_{\rm AdS} \equiv \alpha\partial - e_1 - \frac{\alpha^2}{2s + d - 6 - 2N_z}\bar{e}_1. \tag{E2}$$

Now we are going to demonstrate that the improved K^a transformations of the non-normalizable massless spin-s AdS_{d+1} modes become K^a transformations of the shadow field. Thus, we are going to prove the following relation

$$K^a_{\rm impr}|\phi_{\rm non\text{-}norm}\rangle = U_{-\nu}K^a_{\rm CFT}|\phi_{\rm sh}\rangle, \tag{E3}$$

where $K^a_{\rm CFT}$ stands for representation of the conformal

boost generator on space of the shadow field given in (2.15)
.

To prove relation (E3) we represent the operator $K^a_{\rm AdS}$ as

$$K^a_{\rm AdS} = K^a_{\Delta_{\rm AdS}} + R^a_{(1)} + M^{ab}x^b + R^a_{(0)}, \qquad (E4)$$

$$K^a_{\Delta_{\rm AdS}} \equiv -\frac{1}{2}x^2\partial^a + x^a D_{\rm AdS}, \qquad (E5)$$

where $D_{\rm AdS}$ takes the form given in (11.42), while operators $R^a_{(0)}$, $R^a_{(1)}$ are given in (11.45),(11.46). Then, we note the relations

$$(K^a_{\Delta_{\rm AdS}} + R^a_{(1)})|\phi_{\rm non\text{-}norm}\rangle = U_{-\nu}K^a_{\Delta_{\rm sh}}|\phi_{\rm sh}\rangle, \qquad (E6)$$

$$(M^{ab}x^b + R^a_{(0)})|\phi_{\rm non\text{-}norm}\rangle + G_{\rm AdS}|\xi^{K^a}_{\rm non\text{-}norm}\rangle = U_{-\nu}(M^{ab}x^b + R^a_{\rm sh})|\phi_{\rm sh}\rangle, \qquad (E7)$$

where

$$K^a_{\Delta_{\rm sh}} \equiv -\frac{1}{2}x^2\partial^a + x^a D_{\rm sh}, \qquad (E8)$$

and $D_{\rm sh}$ takes the form given in (2.14) with Δ in (8.26), while $R^a_{\rm sh}$ takes the form given in (8.27). Using (E6) and (E7), we see that relation (E3) holds.

We now comment on the derivation of relations (E6) and (E7). These relations are obtained by using the following general formulas

$$(K^a_{\Delta_{\rm AdS}} + R^a_{(1)})U_{-\nu} = U_{-\nu}(K^a_{\Delta_{\rm sh}} + x^a z\partial_z) - q^{\nu-(3/2)}\partial^a(\partial_q Z_{-\nu}(qz))z\partial_z, \qquad (E9)$$

$$(M^{ab}x^b + R^a_{(0)})U_{-\nu} + G_{\rm AdS}(zU_{-\nu+1}\bar{C}^a_\perp) \approx U_{-\nu}(M^{ab}x^b + R^a_{\rm sh}), \qquad (E10)$$

where q is defined in (10.13) and we use the notation $Z_\nu(z) \equiv \sqrt{z}J_\nu(z)$. In (E10) and in some relations given below, the signs $\approx$ indicate that these relations are valid by applying to the ket-vector $|\phi_{\rm sh}\rangle$ subject to differential constraint (8.9). We now see that by applying relations (E9) and (E10) to $|\phi_{\rm sh}\rangle$ we obtain the respective relations (E6) and (E7).

Finally, we note the helpful formulas for deriving relation (E10),

$$M^{ab}x^bU_{-\nu} \approx U_{-\nu}M^{ab}x^b - zU_{-\nu+1} \times (G_{\rm sh}\bar{C}^a_\perp + c_2\tilde{C}^a + 2c_1\bar{\alpha}^a\Box), \qquad (E11)$$

$$R^a_{(0)}U_{-\nu} = -zU_{-\nu+1}\bar{e}_{1,1}\tilde{C}^a + U_{-\nu-1}e_{1,1}\bar{\alpha}^a, \qquad (E12)$$

$$G_{\rm AdS}(zU_{-\nu+1}\bar{C}^a_\perp) \approx zU_{-\nu+1}G_{\rm sh}\bar{C}^a_\perp - U_{-\nu}2e_{1,1}\bar{C}^a_\perp. \qquad (E13)$$

These formulas can be obtained by using differential constraint (8.9) and relations for the operator U_ν given in (11.26)–(11.33). Also, to derive $R^a_{\rm sh}$ term in (E10) we use the formula $zU_{-\nu-1} + z\Box U_{-\nu+1} = -2\nu U_{-\nu}$.

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