Characteristic length of an AdS/CFT superconductor

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(Received 24 September 2008; published 13 November 2008)

We investigate in more detail the holographic model of a superconductor recently found by Hartnoll, Herzog, and Horowitz [Phys. Rev. Lett. **101**, 031601 (2008)], which is constructed from a condensate of a charged scalar field in AdS$_4$-Schwarzschild background. By analytically studying the perturbation of the gravitational system near the critical temperature T_c, we obtain the superconducting coherence length proportional to $1/\sqrt{1 - T/T_c}$ via AdS/CFT (anti–de Sitter/conformal field theory) correspondence. By adding a small external homogeneous magnetic field to the system, we find that a stationary diamagnetic current proportional to the square of the order parameter is induced by the magnetic field. These results agree with Ginzburg-Landau theory and strongly support the idea that a superconductor can be described by a charged scalar field on a black hole via AdS/CFT duality.

I. INTRODUCTION

The AdS/CFT (anti–de Sitter/conformal field theory) duality [1] gets into the limelight as a powerful tool to investigate the strongly coupled gauge theories. Motivated by the recent experimental data of quark-gluon plasma at the Relativistic Heavy Ion Collider [2,3], transport coefficients such as shear viscosity were calculated for various duality models. Interestingly, for a large class of dualities, it was found that the ratio of viscosity divided by entropy density is a universal constant compatible with the experimental data (see, for example, Refs. [4,5] for all references). This leads us to expect that strongly coupled phenomena such as quantum phase transition or superconducting phase transition in condensed matter systems are described by some kind of duality.

If a superconductor can be described by a gravitational model via the AdS/CFT duality, there should be a scalar hair on an anti–de Sitter (AdS) black hole which represents the condensation in the dual gauge theory. Gubser [6] has presented a counterexample to a no scalar hair theorem by giving a static solution of a charged scalar field coupled to an Abelian gauge field on the AdS$_4$-Reissner-Nortström black hole background if the charge of the black hole is large enough. Since the temperature of the black hole decreases as the charge increases, the black hole can support the scalar hair only for low temperature. This is because the scalar field condensation breaks the Abelian gauge symmetry spontaneously at a sufficiently low temperature. Hartnoll *et al.* [7] numerically showed that there is a critical temperature below which the charged scalar hair exists on the AdS$_4$-Schwarzschild black hole and the conductivity becomes infinite at the low frequency limit. It was also numerically shown that the scalar field condensa-

tion occurs below a critical temperature under the presence of an external magnetic field [8–10]. Quite recently, general properties of *p*-wave superconductors were investigated by Gubser and Pufu [11] and independently by Roberts and Hartnoll [12] in a model of a non-Abelian gauge field in the background of an AdS$_4$-Schwarzschild black hole.

The purpose in this paper is to explore a little further the model of the superconductor composed of the charged scalar field on an AdS$_4$-Schwarzschild background [7] by investigating perturbation of the system near the critical temperature. The order parameter of the superconductor is the scalar operator dual to the charged scalar field. So, the correlation length of the order parameter, or the superconducting coherence length ξ, is obtained by the perturbation of the scalar field. According to Ginzburg-Landau theory, a superconductor is characterized by only two parameters, ξ and the magnetic penetration length λ. Therefore, it will be of interest to determine the two parameters by investigating the perturbation. Motivated by this, we analytically investigate static fluctuation of the scalar field with nonzero spatial momentum along one spatial coordinate of the AdS boundary to obtain the superconducting coherence length ξ via AdS/CFT correspondence. Following [7], we take the probe limit where the fluctuation does not backreact on the original AdS$_4$-Schwarzschild geometry. Under the probe limit we also investigate static fluctuation of the Abelian gauge field forming a homogeneous magnetic field as a first step to derive the magnetic penetration length λ.

The plan of our paper is as follows: In Sec. II the charged scalar field solution obtained in the holographic model [7] is reconstructed by perturbation technique. In Sec. III we derive ξ by analyzing the equations for the perturbation. In Sec. IV we observe that the diamagnetic current can be induced by the small homogeneous magnetic field. Section V is devoted to conclusions and a discussion.

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II. A MODEL OF SUPERCONDUCTOR IN ADS/CFT

In this section we reconstruct the charged scalar field solution numerically obtained in [7] by using the regular perturbation theory technique. The background spacetime is AdS$_4$-Schwarzschild black hole with the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + \frac{r^2}{L^2}(dx^2 + dy^2),$$

$$f(r) = \frac{r^2}{L^2}\left(1 - \frac{r_0^3}{r^3}\right), \tag{2.1}$$

where r_0 is the horizon radius and L is the AdS radius. The Hawking temperature T is given by

$$T = \frac{3}{4\pi}\frac{r_0}{L^2}. \tag{2.2}$$

It is convenient to introducing a new coordinate $u := r_0/r$, and the metric (2.1) is written as

$$ds^2 = \frac{L^2\alpha^2(T)}{u^2}(-h(u)dt^2 + dx^2 + dy^2) + \frac{L^2 du^2}{u^2 h(u)},$$

$$h(u) = 1 - u^3, \tag{2.3}$$

where $\alpha(T) := 4\pi T/3 = r_0/L^2$.

We consider the matter fields on the background spacetime which consist of a Maxwell field and a charged complex scalar field with charge e and mass m. The Lagrangian density is given by

$$\mathcal{L} = \frac{L^2}{2\kappa_4^2 e^2}\sqrt{-g}\left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - |D\Psi|^2 - m^2|\Psi|^2\right),$$

$$D_\mu := \partial_\mu - iA_\mu. \tag{2.4}$$

Following [7], we shall confine our attention to the case $L^2 m^2 = -2$ and consider the probe limit in which the gauge field and scalar field do not backreact on the original metric (2.1). This limit is realized by taking the limit $e \to \infty$, keeping A_μ and Ψ fixed. So, the equations of motion for A_μ and Ψ are decoupled from Einstein's equations and we obtain the following equations:

$$0 = D^2\Psi - m^2\Psi,$$

$$0 = \nabla_\nu F_\mu{}^\nu - i[(D_\mu\Psi)^\dagger\Psi - \Psi^\dagger(D_\mu\Psi)]. \tag{2.5}$$

Under the ansatz

$$\Psi = \Psi(u), \qquad A_\mu = \Phi(u)(dt)_\mu, \tag{2.6}$$

the equations of motion (2.5) are reduced to

$$0 = \left(u^2\frac{d}{du}\frac{h(u)}{u^2}\frac{d}{du} - \frac{L^2 m^2}{u^2}\right)\tilde{\Psi} + \frac{\tilde{\Phi}^2}{h(u)}\tilde{\Psi}, \tag{2.7}$$

$$0 = h(u)\frac{d^2\tilde{\Phi}}{du^2} - \frac{2|\tilde{\Psi}|^2}{u^2}\tilde{\Phi}, \tag{2.8}$$

where new variables $\tilde{\Phi} := \Phi/\alpha(T)$ and $\tilde{\Psi} := L\Psi$ are

dimensionless quantities. Without loss of generality, we can set $\tilde{\Psi}$ to be real.

The trivial solution is easily found as

$$\tilde{\Psi} = 0, \qquad \tilde{\Phi} = \mu/\alpha(T) - qu = q(1 - u), \tag{2.9}$$

where μ is interpreted as the external source in the dual $(2 + 1)$-dimensional gauge theory and it is determined by the condition $A_\mu dx^\mu$ to be well defined at the horizon, i.e., $\Phi(u = 1) = 0$ [13]. Dimensionless constant q is related to the dual charge density coupled to μ as

$$\langle J^t(x)\rangle = \frac{\delta S_{\text{on-shell boundary}}}{\delta A_t(x)}\bigg|_{u=0} = \frac{L^2}{2\kappa_4^2 e^2}\left(\frac{4\pi T}{3}\right)^2 q. \tag{2.10}$$

The nontrivial solution asymptotically behaves near the AdS boundary as

$$\tilde{\Psi} = \tilde{\Psi}^{(-)}u^{\Delta_-} + \tilde{\Psi}^{(+)}u^{\Delta_+} + \cdots,$$

$$\tilde{\Phi} = \mu/\alpha(T) - qu + \cdots, \tag{2.11}$$

where $\Delta_\pm := (3 \pm \sqrt{9 + 4L^2 m^2})/2$. For the $L^2 m^2 = -2$ case, we obtain $\Delta_- = 1$ and $\Delta_+ = 2$, where both falloffs of $\tilde{\Psi}$ are normalizable.

Each coefficient $\tilde{\Psi}^{(\pm)}$ is proportional to the condensate thermal expectation value of the scalar operator $\langle\mathcal{O}_\pm\rangle$ of dimension $\Delta_\pm$. To obtain a stable solution, we must impose either $\tilde{\Psi}^{(-)} = 0$ or $\tilde{\Psi}^{(+)} = 0$. So, the asymptotic boundary condition of the scalar field $\tilde{\Psi}$ dual to the scalar operator $\langle\mathcal{O}_-\rangle$ ($\langle\mathcal{O}_+\rangle$) is $\tilde{\Psi}^{(+)} = 0$ ($\tilde{\Psi}^{(-)} = 0$), and $\tilde{\Psi}$ has the asymptotic behavior near the AdS boundary as

$$\tilde{\Psi} = \tilde{\Psi}^{(I)}u^{\Delta_I}[1 + O(u^2)], \tag{2.12}$$

where $I = \pm$ for $\langle\mathcal{O}_\pm\rangle$.

Since the trivial solution $\tilde{\Phi}$ in Eqs. (2.7) and (2.8) is parametrized by the dimensionless constant $q \propto \langle J^t\rangle/T^2$ only, the nontrivial solution $\tilde{\Psi}$ emerges above a critical value q_c under the boundary condition. According to the numerical calculation [7], the thermal expectation value $\langle\mathcal{O}_I\rangle$ behaves as

$$\tilde{\Psi}^{(I)} \propto \langle\mathcal{O}_I\rangle \sim (1 - T/T_c)^{1/2} \tag{2.13}$$

for a given μ (or $\langle J^t\rangle$), or equivalently

$$\langle\mathcal{O}_I\rangle \sim (q/q_c - 1)^{1/2} \tag{2.14}$$

near the critical temperature T_c. In the limit $T \to T_c$, $\epsilon := q/q_c - 1(>0)$ is a small parameter, and the nontrivial solution to Eqs. (2.7) and (2.8) can be obtained as a series in ϵ. From the continuity, the solution at the critical temperature should be

$$\tilde{\Psi}_c = 0, \qquad \tilde{\Phi}_c = q_c(1 - u). \tag{2.15}$$

So, we can expand $\tilde{\Psi}$ and $\tilde{\Phi}$ as

$$\tilde{\Psi}(u) = \epsilon^{1/2}\tilde{\Psi}_1(u) + \epsilon^{3/2}\tilde{\Psi}_2(u) + \cdots,$$
$$\tilde{\Phi}(u) = \tilde{\Phi}_c(u) + \epsilon\tilde{\Phi}_1(u) + \cdots. \tag{2.16}$$

Here, we should note that the difference of ϵ-behavior between $\tilde{\Psi}$ and $\tilde{\Phi}$ comes from Eqs. (2.7) and (2.8).

Substituting Eq. (2.16) into Eqs. (2.7) and (2.8) we obtain equations for $\tilde{\Psi}_1$ and $\tilde{\Phi}_1$:

$$0 = \mathcal{L}_\psi \tilde{\Psi}_1, \qquad 0 = \frac{d^2\tilde{\Phi}_1}{du^2} - \frac{2|\tilde{\Psi}_1|^2\tilde{\Phi}_c}{u^2 h(u)}, \tag{2.17}$$

where the differential operators $\mathcal{L}_\psi$ is defined by

$$\mathcal{L}_\psi := -\left(u^2 \frac{d}{du}\frac{h(u)}{u^2}\frac{d}{du} - \frac{L^2 m^2}{u^2} + \frac{\tilde{\Phi}_c^2}{h(u)}\right). \tag{2.18}$$

By imposing the regularity condition at the horizon

$$\frac{1}{\tilde{\Psi}_1}\frac{d\tilde{\Psi}_1}{du}\bigg|_{u=1} = -\frac{L^2 m^2}{3} = \frac{2}{3}, \tag{2.19}$$

we find the constant q_c for which there is a unique regular solution satisfying the asymptotic boundary condition mentioned above. In the $I = +$ case, for example, $q_c \sim 4.07$, which is consistent with the numerical result in [7].

III. SUPERCONDUCTING COHERENCE LENGTH

In this section, we will determine the superconducting coherence length ξ by investigating fluctuations around the background field (2.6) $(\tilde{\Psi}(u), \tilde{\Phi}(u))$. It is enough to consider static perturbations for the purpose, so let us confine our attention to the fluctuations with only spatial momentum along the x direction:

$$\delta A_\mu(u, x)dx^\mu = [A_x(u, k)dx + A_y(u, k)dy$$
$$+ \phi(u, k)dt]e^{ikx},$$
$$\delta\Psi(u, x) = \frac{1}{L\alpha(T)}[\psi(u, k) + i\hat{\psi}(u, k)]e^{ikx}, \tag{3.1}$$

where both functions ψ and $\hat{\psi}$ are real and metric fluctuations of the order of the gauge and scalar fluctuations can be consistently set to zero under the probe limit. From the perturbed equations derived from Eq. (2.5), we find the following three linearized equations for ϕ, ψ, and A_y decoupled from the other variables:

$$\tilde{k}^2\psi = \left(u^2 \frac{d}{du}\frac{h(u)}{u^2}\frac{d}{du} - \frac{L^2 m^2}{u^2} + \frac{\tilde{\Phi}^2}{h(u)}\right)\psi + \frac{2\tilde{\Phi}\tilde{\Psi}}{h(u)}\phi, \tag{3.2}$$

$$\tilde{k}^2\phi = \left(h(u)\frac{d^2}{du^2} - \frac{2\tilde{\Psi}^2}{u^2}\right)\phi - \frac{4\tilde{\Phi}\tilde{\Psi}}{u^2}\psi, \tag{3.3}$$

$$\tilde{k}^2 A_y = \left(\frac{d}{du}h(u)\frac{d}{du} - \frac{2\tilde{\Psi}^2}{u^2}\right)A_y, \tag{3.4}$$

where $\tilde{k} := k/\alpha(T)$.

The superconducting coherence length ξ is nothing but the correlation length of the order parameter, and ξ appears as the pole of the static correlation function of the order parameter in the Fourier space

$$\langle\tilde{\mathcal{O}}(\vec{k})\tilde{\mathcal{O}}(-\vec{k})\rangle \sim \frac{1}{|\vec{k}|^2 + 1/\xi^2}. \tag{3.5}$$

Since the complex scalar field Ψ plays a role of the order parameter in our model and the background $\tilde{\Psi}$ is real, the real part of the order parameter fluctuation ψ gives the superconducting coherence length.

The pole of the static correlation function of a dual field operator is obtained by solving the eigenvalue problem for the static perturbation with wave number k of the corresponding bulk field as $1/\xi^2 = -k_*^2$, where k_* is a wave number permitted as eigenvalues. In the present case, our task is to evaluate eigenvalues $\tilde{k}^2$ for Eqs. (3.2) and (3.3) under the appropriate boundary conditions.

Since it is difficult to solve the eigenvalue equations (3.2) and (3.3) analytically, we solve the equations as a series in ϵ near the critical temperature T_c. According to Ginzburg-Landau theory, ξ diverges to infinity as $T \to T_c$. This implies that there exists zero eigenvalue $k_* = 0$ solution at the critical temperature T_c. Hereafter, we shall confine our attention to the eigensystem with $\lim_{\epsilon\to 0} k_* = 0$.

From the behavior (2.16), Eqs. (3.2) and (3.3) are expanded in ϵ as

$$-\tilde{k}^2\psi = \left(\mathcal{L}_\psi - \frac{2\epsilon\tilde{\Phi}_c\tilde{\Phi}_1}{h}\right)\psi - \frac{2\epsilon^{1/2}\tilde{\Phi}_c\tilde{\Psi}_1}{h}\phi, \tag{3.6}$$

$$-\tilde{k}^2\phi = \left(-h\frac{d^2}{du^2} + \frac{2\epsilon\tilde{\Psi}_1^2}{u^2}\right)\phi + \frac{4\epsilon^{1/2}\tilde{\Phi}_c\tilde{\Psi}_1}{u^2}\psi. \tag{3.7}$$

The boundary conditions for Eqs. (3.6) and (3.7) are as follows: at the horizon,

$$\psi(1) = \text{regular}, \qquad \phi(1) = 0, \tag{3.8}$$

and near the AdS boundary

$$\psi(u) = (\text{const}) \times u^{\Delta_I}[1 + O(u^2)],$$
$$\phi(u) = (\text{const}) \times u + O(u^2). \tag{3.9}$$

In the eigenvalue equations (3.6) and (3.7) with the boundary conditions (3.8) and (3.9), the infinitesimal expansion parameter is $\epsilon^{1/2}$, so one may expect that we have eigenvalue $\tilde{k}_*^2 = O(\epsilon^{1/2})$. However, we have $\tilde{k}_*^2 = O(\epsilon)$ as seen later.

It is easy to show that the zeroth order solution, ψ_0 and ϕ_0, for the eigenvalue equation (3.6) and (3.7) satisfying the boundary conditions (3.8) and (3.9) is given by

$$\psi_0 = \tilde{\Psi}_1, \qquad \phi_0 = 0, \qquad (3.10)$$

where we use $\mathcal{L}_\psi \tilde{\Psi}_1 = 0$. This means that $\phi = O(\epsilon^{1/2})$ at most from Eq. (3.7). So we put $\phi =: \epsilon^{1/2}\varphi$, and Eqs. (3.6) and (3.7) are rewritten by

$$-\tilde{k}^2 \psi = \mathcal{L}_\psi \psi - \epsilon \frac{2\tilde{\Phi}_c}{h}(\tilde{\Phi}_1 \psi + \tilde{\Psi}_1 \varphi), \qquad (3.11)$$

$$-\tilde{k}^2 \varphi = \left(-h\frac{d^2}{du^2}\varphi + \frac{4\tilde{\Phi}_c\tilde{\Psi}_1}{u^2}\psi\right) + \epsilon\frac{2\tilde{\Psi}_1^2}{u^2}\varphi. \quad (3.12)$$

Thus, the expansion parameter on the right-hand side in Eqs. (3.11) and (3.12) is ϵ, implying $\tilde{k}_*^2 = O(\epsilon)$. The temperature dependence of the coherence length, $\xi \propto (-\tilde{k}_*^2)^{-1/2} \propto (1 - T/T_c)^{-1/2}$, is an expected behavior in Ginzburg-Landau theory.

Now let us evaluate the superconducting coherence length ξ at the leading order in ϵ. We expand ψ, φ, and $\tilde{k}_*^2$ as

$$\psi = \tilde{\Psi}_1 + \epsilon\psi_1 + O(\epsilon^2), \qquad \varphi = \varphi_0 + O(\epsilon),$$
$$\tilde{k}_*^2 = \epsilon(\tilde{k}^2)_1 + O(\epsilon^2). \qquad (3.13)$$

Then, Eq. (3.11) is rewritten up to $O(\epsilon)$ as

$$-(\tilde{k}^2)_1\tilde{\Psi}_1 = \mathcal{L}_\psi\psi_1 - \frac{2\tilde{\Phi}_c\tilde{\Psi}_1}{h}(\tilde{\Phi}_1 + \varphi_0), \qquad (3.14)$$

and the equation of motion for φ_0 is given by

$$\frac{d^2\varphi_0}{du^2} = \frac{4\tilde{\Phi}_c\tilde{\Psi}_1^2}{u^2 h} = 2\frac{d^2\tilde{\Phi}_1}{du^2}, \qquad (3.15)$$

where we use Eq. (2.17). The solution of Eq. (3.15) with the boundary conditions (3.8) and (3.9) is given by

$$\varphi_0(u) = 2[\tilde{\Phi}_1(u) - \tilde{\Phi}_1(0)(1 - u)] \in \mathbb{R}. \qquad (3.16)$$

For states ψ_I, ψ_{II} with the boundary conditions (3.8) and (3.9), let us introduce an inner product

$$\langle\psi_I|\psi_{II}\rangle := \int_0^1 \frac{du}{u^2}\psi_I^*(u)\psi_{II}(u). \qquad (3.17)$$

It is easily checked that $\mathcal{L}_\psi$ is Hermitian for the inner product (3.17).

Making use of $\mathcal{L}_\psi\tilde{\Psi}_1 = 0$ and Hermiticity of $\mathcal{L}_\psi$, the inner product between $\tilde{\Psi}_1$ and Eq. (3.14) gives us

$$-(\tilde{k}^2)_1\langle\tilde{\Psi}_1|\tilde{\Psi}_1\rangle = \langle\tilde{\Psi}_1|\mathcal{L}_\psi|\psi_1\rangle - \left\langle\tilde{\Psi}_1\left|\frac{2\tilde{\Phi}_c\tilde{\Psi}_1}{h}(\tilde{\Phi}_1 + \varphi_0)\right.\right\rangle$$
$$= -\left\langle\tilde{\Psi}_1\left|\frac{2\tilde{\Phi}_c\tilde{\Psi}_1}{h}\tilde{\Phi}_1\right.\right\rangle - \int_0^1 du\frac{2\tilde{\Phi}_c\tilde{\Psi}_1^2}{u^2 h}\varphi_0$$
$$= -\left\langle\tilde{\Psi}_1\left|\frac{2\tilde{\Phi}_c\tilde{\Psi}_1}{h}\tilde{\Phi}_1\right.\right\rangle + \frac{1}{2}\int_0^1 du\left(\frac{d\varphi_0}{du}\right)^2,$$
$$(3.18)$$

where we used Eq. (3.15) and the boundary conditions (3.8) and (3.9) in the third equality.

We can show that the first term in Eq. (3.18) vanishes as follows: From Eq. (2.7), we have the equation of motion for $\tilde{\Psi}_2$ defined by Eq. (2.16) as

$$\mathcal{L}_\psi\tilde{\Psi}_2 = \frac{2\tilde{\Phi}_c\tilde{\Psi}_1}{h}\tilde{\Phi}_1, \qquad \tilde{\Psi}_2(1) = \text{regular},$$
$$\tilde{\Psi}_2(0) = (\text{const}) \times u^{\Delta_I}. \qquad (3.19)$$

So the inner product (3.17) is well defined for $\tilde{\Psi}_2$, and Eq. (3.19) gives us

$$0 = -2\langle\mathcal{L}_\psi\tilde{\Psi}_1|\tilde{\Psi}_2\rangle = -2\langle\tilde{\Psi}_1|\mathcal{L}_\psi\tilde{\Psi}_2\rangle$$
$$= -2\left\langle\tilde{\Psi}_1\left|\frac{2\tilde{\Phi}_c\tilde{\Psi}_1}{h}\tilde{\Phi}_1\right.\right\rangle, \qquad (3.20)$$

where we use the fact that $\mathcal{L}_\psi$ is Hermitian and $\mathcal{L}_\psi\tilde{\Psi}_1 = 0$.

Therefore, up to $O(\epsilon)$, the eigenvalue is given by

$$-\tilde{k}_*^2 = \epsilon N/D + O(\epsilon^2), \qquad (3.21)$$

$$N = 2\int_0^1 du(\tilde{\Phi}_1'(u) + \tilde{\Phi}_1(0))^2 > 0,$$
$$D := \int_0^1 du\frac{\tilde{\Psi}_1^2(u)}{u^2} > 0, \qquad (3.22)$$

and we finally obtain the superconducting coherence length as

$$\xi = \frac{\epsilon^{-1/2}}{\alpha(T_c)}\sqrt{\frac{D}{N}} + O(\epsilon^{1/2}) \propto \frac{1}{T_c}\left(1 - \frac{T}{T_c}\right)^{-1/2}. \qquad (3.23)$$

We note that since D and N are dimensionless quantities, they do not depend on T_c directly, but depend on q_c only.

IV. DIAMAGNETIC CURRENT

In this section, we calculate the diamagnetic current induced by a homogeneous external magnetic field perpendicular to the surface of the superconductor. As mentioned before, in the probe limit $e \to \infty$, the magnetic field does not backreact to the background spacetime (2.1). Under the ansatz $\delta A_y(u, x) = b(u)x$ (the bulk magnetic field $F_{xy} = \partial_x\delta A_y = b(u)$), the equation of motion for $b(u)$ is decoupled from the other ones for ψ and ϕ,[1] and it is equivalent to Eq. (3.4) for $k = 0$:

$$\left(\frac{d}{du}h(u)\frac{d}{du} - \frac{2\tilde{\Psi}^2(u)}{u^2}\right)b(u) = 0, \qquad (4.1)$$

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[1]At the nonlinear regime, the fluctuation of the gauge field is coupled to the one of the scalar field. This effect is important at large x. As far as we are concerned with the neighborhood of the origin $x = 0$, this effect is negligible. The nonlinear effect has been considered in [9].

with the regularity boundary condition at the horizon $u = 1$.

As seen in the previous section, the equation can be solved as a series in ϵ. Expanding b as

$$b(u) = b_0(u) + \epsilon b_1(u) + O(\epsilon^2), \qquad (4.2)$$

and using Eq. (2.16), we obtain equations for b_0 and b_1 as

$$0 = \frac{d}{du} h \frac{d}{du} b_0(u), \qquad (4.3)$$

$$0 = \frac{d}{du} h \frac{d}{du} b_1(u) - \frac{2\tilde{\Psi}_1^2}{u^2} b_0(u). \qquad (4.4)$$

The solution of Eq. (4.3) satisfying the regularity condition is

$$b_0(u) = C = (\text{const}). \qquad (4.5)$$

So, the regularity solution of Eq. (4.4) should satisfy

$$\frac{db_1}{du} = -\frac{2C}{h(u)} \int_u^1 du_0 \frac{\tilde{\Psi}_1^2(u_0)}{u_0^2}, \qquad (4.6)$$

and

$$b_1(u) = D - 2C \int_0^u \frac{du_1}{h(u_1)} \int_{u_1}^1 du_0 \frac{\tilde{\Psi}_1^2(u_0)}{u_0^2}. \qquad (4.7)$$

Thus, we obtain

$$b(u) = C + \epsilon D - 2\epsilon C \int_0^u \frac{du_1}{h(u_1)} \int_{u_1}^1 du_0 \frac{\tilde{\Psi}_1^2(u_0)}{u_0^2} + O(\epsilon^2)$$

$$= B - 2\epsilon B \int_0^u \frac{du_1}{h(u_1)} \int_{u_1}^1 du_0 \frac{\tilde{\Psi}_1^2(u_0)}{u_0^2} + O(\epsilon^2), \qquad (4.8)$$

and

$$\delta A_y(u, x) = \delta A_y^{(0)}(x) \left(1 - 2\epsilon \int_0^u \frac{du_1}{h(u_1)} \int_{u_1}^1 du_0 \frac{\tilde{\Psi}_1^2(u_0)}{u_0^2} \right) + O(\epsilon^2), \qquad (4.9)$$

where we define $B := \lim_{u \to 0} b(u)$ and $\delta A_y^{(0)}(x) := \lim_{u \to 0} \delta A_y(u, x)$.

From the asymptotic behavior of $\delta A_y(u, x)$ near the AdS boundary, we can read out the dual source $\delta A_y^{(0)}$ and the thermal expectation value of the current $\langle J_y \rangle$ as

$$\delta A_y(u, x) = \delta A_y^{(0)}(x) + \frac{2\kappa_4^2 e^2}{L^2} \frac{3}{4\pi T} \langle J_y(x) \rangle u + O(u^2). \qquad (4.10)$$

From Eq. (4.9), we obtain

$$\langle J_y(x) \rangle = \frac{L^2}{2\kappa_4^2 e^2} \frac{4\pi T_c}{3} \left(-2\epsilon \int_0^1 du \frac{\tilde{\Psi}_1^2(u)}{u^2} \right) \delta A_y^{(0)}(x)$$

$$+ O(\epsilon^2)$$

$$= -\frac{L^2}{2\kappa_4^2 e^2} \frac{8\pi T_c}{3} [L\Psi^{(I)}]^2$$

$$\times \left(\frac{\int_0^1 du \Psi^2(u)/u^2}{[\Psi^{(I)}]^2} \right) \delta A_y^{(0)}(x) + O(\epsilon^2), \qquad (4.11)$$

where we use $L\Psi = \tilde{\Psi} = \epsilon^{1/2}(\tilde{\Psi}_1 + O(\epsilon))$. Because $\tilde{\Psi}_1$ (or Ψ at the leading order in ϵ) is the solution of the linear equation (2.17), we can express $\Psi(u)$ as

$$\Psi(u) = \Psi^{(I)} F(u), \qquad (4.12)$$

where $F(u)$ is the solution of Eq. (2.17) satisfying $\lim_{u \to 0} F(u) = u^{\Delta_I}$ and the regularity boundary condition at the horizon. So, the parenthesis in the last term of Eq. (4.11) depends on q_c only, not on $\Psi^{(I)}$ and Eq. (4.11) is simplified as

$$\langle J_y(x) \rangle \sim -T_c \epsilon \delta A_y^{(0)}(x) \propto -|\langle \mathcal{O}_I \rangle|^2 \delta A_y^{(0)}(x). \qquad (4.13)$$

Thus, the stationary current is induced only when condensation occurs.

Interestingly, Eq. (4.13) is very similar to the expression expected by Ginzburg-Landau theory. In the theory, when the phase of the order parameter ψ coupled to the U(1) gauge field A is constant, the current J is described by the London equation

$$J = -\frac{e_*^2}{m_*} \psi^2 A = -e_* n_s A, \qquad (4.14)$$

where e_* and m_* are effective charge and mass of the order parameter, and n_s is the superfluid number density.

While $\delta A_y^{(0)}$ in Eq. (4.11) is an external source, the macroscopic gauge field A in Eq. (4.14) is composed of the spatial average of the microscopic field and external field. Since we have no dynamical photon in our holographic superconductor model, the current does not produce its own microscopic magnetic fields. This means that the external gauge field $\delta A_y^{(0)}$ is equal to the macroscopic gauge field A in the AdS/CFT superconductor. So, comparing Eq. (4.14) with Eq. (4.13), we obtain the superfluid number density[2] n_s which behaves as

$$n_s \sim \epsilon T_c \sim T_c - T, \qquad (4.15)$$

near the critical temperature.

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[2]Our superfluid number density n_s is equal to the one obtained from the electric conductivity $\Im[\sigma(\omega)]$ by Ref. [7].

V. CONCLUSIONS AND DISCUSSION

We have investigated linear fluctuations of the scalar field solution in the holographic model of a superconductor in Ref. [7] under the probe limit, where the fluctuations do not backreact on the geometry. By solving analytically the linearized equations with only spatial momentum along one spatial coordinate of the AdS boundary, we find that the superconducting coherence length ξ diverges at the critical temperature T_c as $\xi \sim (1 - T/T_c)^{-1/2}/T_c$. We also find a diamagnetic current induced by an external small homogeneous magnetic field. The current is proportional to the external gauge field and goes to zero as $T_c - T$ at the critical temperature. These results are in agreement with the behaviors predicted by Ginzburg-Landau theory and Eq. (4.14) is the London equation in the AdS/CFT superconductor.

If we would have a dynamical photon, then according to Ginzburg-Landau theory, the magnetic penetration depth λ would be related to the superfluid density n_s as

$$\lambda \sim 1/\sqrt{n_s}. \qquad (5.1)$$

In Ginzburg-Landau theory, the coefficient $\kappa = \lambda/\xi$ classifies the superconductors into two types, i.e. $\kappa < 1/\sqrt{2}$ for type I superconductors and $\kappa > 1/\sqrt{2}$ for type II superconductors. Using Eq. (5.1) formally, from Eqs. (3.23) and (4.15) we obtain

$$\kappa = \frac{\lambda}{\xi} \sim T_c^{1/2}. \qquad (5.2)$$

This may suggest that for a sufficiently small critical temperature T_c, the AdS/CFT superconductor behaves as type I, while for a sufficiently large critical temperature T_c, it behaves as type II. This simple classification calls for further investigation.

Note added.—After having submitted this article, we learned of a work by S. A. Hartnoll, C. P. Herzog, and G. T. Horowitz, which argued that AdS/CFT superconductor should be type II [14]. We also learned of a work by G. T. Horowitz and M. M. Roberts, which argued the dependence of the AdS/CFT superconductor on the scalar field mass [15]. As easily seen in the derivation of ξ in Sec. III, the mass dependence only appears via the background scalar field solution $\tilde{\Psi}_1$ of the differential equation (2.7). Since $\mathcal{L}_\psi$ is still Hermitian for the general mass satisfying the Breitenlohner-Freedman bound $L^2 m^2 > L^2 m_{\mathrm{BF}}^2 = -9/4$, we can extend our calculation to the general mass case.

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