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Noncommutative Q-lumps

Seçkin Kürkçüoğlu*

Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstraße 2, D-30167 Hannover, Germany

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Q-lumps associated with the noncommutative $\mathbb{C}P^N$ model in 2 + 1 dimensions are constructed. These are solitonic configurations which are time dependent and rotate with constant angular frequency. Energy of the Q-lumps is $E = 2\pi k + \alpha|Q|$, and we find that in a regime in which the noncommutativity parameter θ is related to the moduli determining the size of the lumps, it can be viewed to depend on θ via the Noether charge Q. We present a collective coordinate-type analysis signalling that $\mathbb{C}P^1$ Q-lumps remain stable under small radiative perturbations.

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I. INTRODUCTION

It was shown by Leese [1] quite some time ago that the $\mathbb{C}P^1$ model in 2 + 1 dimensions modified by the addition of a certain potential term admits Q-ball [2] type solutions. These are time dependent configurations with conserved topological charge (winding number) and Noether charge, and named Q-lumps in the literature [1]. Q-lumps of the $\mathbb{C}P^1$ model are constructed by finding a simple extension of the Bogomol'nyi-Prasad-Sommerfield (BPS) equations. Existence of this BPS-type bound ensures finite energy solutions, which are determined in terms of the Noether charge Q, the coupling constant of the potential and the winding number $k \in \mathbb{Z}$. For finiteness of the energy, it is necessary that $k \geq 2$, since the Noether charge of the $k = 1$ configurations diverges logarithmically. However, $k = 1$ lumps can exist as a part of a configuration of multilumps. Q-lump configurations in the $\mathbb{C}P^N$ models have also been studied in the literature [3]. They appear as stationary solutions of Kähler sigma models modified by a potential term which is left invariant under the transformations generated by a Killing vector of the target manifold. In general, the moduli space of the $\mathbb{C}P^N$ Q-lumps is smaller than that of the pure $\mathbb{C}P^N$ model lumps, since a solution with a given value of Q may be scaled to give another solution with a different value of Q [1,3]. It was also found that (4, 4)-supersymmetric 1 + 1-dimensional sigma models with hyper-Kähler target spaces admit Q-kink solutions. These are stationary configurations, which also saturate a BPS-type bound and carry $1/2$ of the supersymmetry [4]. More recent investigations indicated that $\mathcal{N} = 2$ supersymmetric four-dimensional hyper-Kähler sigma models have Q-lump configurations which are $1/4$ or $1/8$ BPS states [5,6]. Q-lumps of the $\mathcal{N} = 2$ supersymmetric $\mathbb{C}P^N$ model carrying half of the supersymmetries has appeared in [7].

Noncommutative (NC) field theories have been under investigation for about a decade now. Among them, field theories defined on the Groenewold-Moyal (GM) type deformations of spacetime [i.e. the noncommutative algebra $\mathcal{A}_\theta(\mathbb{R}^{(d+1)})$] hold a considerably large part of the literature. (See, for instance [8,9] for comprehensive reviews.) Formulation of instantons and solitons on the GM spacetime and other noncommutative spaces, such as the noncommutative tori and fuzzy spaces, have been extensively studied and found to present very rich mathematical structures [8–11]. $\mathbb{C}P^N$ models on the GM spacetime have been formulated and their BPS configurations were found in [12]. Stability properties of these models as well as the $U(N)$ chiral model on the GM spacetime have been studied in considerable detail in [13]. NC Q-balls were investigated in [14].

It is therefore desirable to explore the Q-lump configurations associated with the NC $\mathbb{C}P^N$ models. It appears to be rather straightforward to construct these configurations and it turns out that they strongly resemble their commutative cousins, and they too rotate with constant angular frequency in time. Nevertheless, we also find that in a regime in which the noncommutativity parameter θ is related to the moduli determining the size of the lumps, the energy of the NC Q-lump configurations can be viewed to depend on θ via the Noether charge Q.

In the next section, Q-lump configurations of the NC $\mathbb{C}P^N$ model is presented. In Sec. III, we focus on the $\mathbb{C}P^1$ model Q-lumps and first see that at the elementary classical level their stability properties are similar to that of the commutative model. Subsequently, we present a collective-coordinate-type analysis signalling that NC $\mathbb{C}P^1$ model Q-lumps remain stable under small radiative perturbations. Contrary to the behavior of commutative Q-lumps, we find that the period of fluctuations around the radially symmetric Q-lump configurations depend, in addition to α, on a function $A(\kappa_0)$ of the ratio of the initial size of the lump to the scale of the noncommutativity parameter $\kappa_0 := \frac{\lambda_0}{\sqrt{2\theta}}$. We discuss this and other related findings in some detail and compare it with the properties of the commutative theory and show that our results go smoothly to those of the latter as θ tends to zero.

Throughout this paper, we work on the Groenewald-Moyal spacetime $\mathcal{A}_\theta(\mathbb{R}^{2+1})$ defined by the commutation

*seckin.kurkcuoglu@itp.uni-hannover.de

relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \qquad \mu, \nu = 0, 1, 2, \tag{1.1}$$

and assume that the spatial coordinates commute with time $t = x_0$, i.e. $\theta_{0i} = 0$.

II. Q-LUMPS OF THE NC $\mathbb{C}P^N$ MODEL

To facilitate the construction of the NC $\mathbb{C}P^N$ Q-lumps, we start with the Lagrangian

$$L = 2\pi\theta\tfrac{1}{2}\mathrm{Tr}\partial_\mu P\partial^\mu P + V(P), \tag{2.1}$$

where P is a projector living in the space $\mathcal{A}_\theta(\mathbb{R}^{2+1}) \otimes \mathrm{Mat}(N+1)$ and we also have that $Tr = \mathrm{Tr}_{\mathcal{F}} \otimes \mathrm{Tr}_{\mathrm{Mat}(N+1)}$, where $\mathcal{F}$ is the standard Fock space.

The $\mathbb{C}P^N$ manifold is defined through the $(N+1)$-component complex unit vector

$$\chi = \begin{pmatrix} u \\ 1 \end{pmatrix} \frac{1}{\sqrt{u^\dagger u + 1}}, \qquad u \equiv \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}, \qquad \chi^\dagger\chi = 1. \tag{2.2}$$

χ is the partial isometry associated with the projector P via $P = \chi\chi^\dagger$.

The potential term $V(P)$ in (2.1) may be given as

$$V(P) = -2\pi\theta\tfrac{1}{2}\alpha^2\mathrm{Tr}[\lambda_{(N+1)^2-1}, P]^2, \tag{2.3}$$

where $\lambda_{(N+1)^2-1}$ is the "hypercharge" generator of the global $U(N+1)$ symmetry of the NC $\mathbb{C}P^N$ model and α is a constant with dimensions of mass. We observe that $V(P)$ breaks the global $U(N+1)$ symmetry of the pure NC $\mathbb{C}P^N$ model down to $U(1) \times U(N)$. The absolute minimum for the potential occurs at $P = 0$, thus the global $U(1)$ symmetry is not spontaneously broken and Q-ball type solutions are possible [1,2]. Let us now see how they come about.

The energy, topological charge and the Noether charge for the model may be given by the expressions

$$E = 2\pi\theta\mathrm{Tr}P\partial_i P\partial_i P + 2\pi\theta\tfrac{1}{2}\mathrm{Tr}\partial_t P\partial_t P - 2\pi\theta\tfrac{1}{2}\alpha^2\mathrm{Tr}[\lambda_{(N+1)^2-1}, P]^2, \tag{2.4}$$

$$k = \frac{\theta}{i}\varepsilon_{ij}\mathrm{Tr}P\partial_i P\partial_j P, \tag{2.5}$$

$$Q = 2\pi\theta i\mathrm{Tr}\lambda_{(N+1)^2-1}[P, \partial_t P]. \tag{2.6}$$

Let us now consider the BPS-type inequality

$$2\pi\theta\tfrac{1}{2}\mathrm{Tr}(P\partial_\mu P \pm i\varepsilon_{\mu\nu}P\partial_\mu P)(\partial_\mu PP \mp i\varepsilon_{\mu\rho}\partial_\rho PP) + 2\pi\theta\tfrac{1}{2}\mathrm{Tr}(\partial_t P \pm i\alpha[\lambda_{(N+1)^2-1}, P])(\partial_t P \mp i\alpha[\lambda_{(N+1)^2-1}, P]^\dagger) \geq 0. \tag{2.7}$$

It implies immediately the bound

$$E \geq 2\pi|k| + \alpha|Q|, \tag{2.8}$$

which is saturated by the configurations satisfying

$$P\partial_\mu P \pm i\varepsilon_{\mu\nu}P\partial_\mu P = 0, \qquad \partial_t P \pm i\alpha[\lambda_{(N+1)^2-1}, P] = 0. \tag{2.9}$$

We observe that the solutions of the above self-duality equations are given by the BPS configurations of the NC $\mathbb{C}P^N$ model [7], which now rotate with an angular frequency. The parameter α can be rescaled for a given $\mathbb{C}P^N$ model with fixed N such that the angular frequency of rotations are α. Then, the solutions can be specified by the partial isometry

$$\chi(z, t) = \begin{pmatrix} u(z)e^{\pm i\alpha t} \\ 1 \end{pmatrix} \frac{1}{\sqrt{u^\dagger u + 1}}, \tag{2.10}$$

where for self-dual solutions $u = u(z)$ is holomorphic in $z = x_1 + x_2$ with $[z, \bar{z}] = 2\theta$, and anti-self-dual solutions are antiholomorphic $u = u(\bar{z})$.

For the NC $\mathbb{C}P^1$ model we have

$$P(u, u^\dagger, t) = \begin{pmatrix} u\frac{1}{u^\dagger u+1}u^\dagger & u\frac{1}{u^\dagger u+1}e^{\mp i\alpha t} \\ \frac{1}{u^\dagger u+1}u^\dagger e^{\pm i\alpha t} & \frac{1}{u^\dagger u+1} \end{pmatrix}, \tag{2.11}$$

while in the $\mathbb{C}P^2$ model, Q-lumps are specified by the projector

$$P(u, u^\dagger, t) = \begin{pmatrix} u_1\frac{1}{\gamma}u_1^\dagger & u_1\frac{1}{\gamma}u_2^\dagger & u_1\frac{1}{\gamma}e^{\mp i\alpha t} \\ u_2\frac{1}{\gamma}u_1^\dagger & u_2\frac{1}{\gamma}u_2^\dagger & u_2\frac{1}{\gamma}e^{\mp i\alpha t} \\ \frac{1}{\gamma}u_1^\dagger e^{\pm i\alpha t} & \frac{1}{\gamma}u_2^\dagger e^{\pm i\alpha t} & \frac{1}{\gamma} \end{pmatrix} \tag{2.12}$$

where $\gamma = u_\alpha^\dagger u_\alpha + 1$.

The potential terms in these models may also be expressed as

$$V(u) := \pi\theta\alpha^2\mathrm{Tr}_{\mathcal{F}}\left(u\frac{1}{(u^\dagger u + 1)^2}u^\dagger + \frac{u^\dagger u}{(u^\dagger u + 1)^2}\right), \tag{2.13}$$

$$V(u_1, u_2) = 2\pi\theta\alpha^2\mathrm{Tr}_{\mathcal{F}}\left(u_1\frac{1}{\gamma^2}u_1^\dagger + u_2\frac{1}{\gamma^2}u_2^\dagger + \frac{\gamma - 1}{\gamma^2}\right). \tag{2.14}$$

These generalize the expression $V = \alpha^2\int d^2x g^{\alpha\beta}u_\alpha u_\beta$, $g^{\alpha\beta}$ being the Fubini-Study metric on $\mathbb{C}P^N$, given in [3] for the $\mathbb{C}P^N$ models for the values $N = 1$, 2 and they collapse to it in the commutative limit.

It is apparent that the time dependence of these solutions is exactly the same as that obtained in [1,3] for the Q-lumps of the commutative $\mathbb{C}P^N$ models. Likewise, these solutions may have finite energy only for winding numbers $k \geq 2$. This is due to the fact that the Noether charge Q diverges for configurations with $k = 1$, as can be seen by inspecting the trace involved. We will return to the detailed study of these traces shortly.

The contribution to the energy due the Noether charge lifts the degeneracy of a class of solutions in the solution space of solitons of arbitrary sizes. This is quite expected as the addition of the potential term breaks the scaling invariance of the NC $\mathbb{C}P^N$ model even at the level of solutions.[1] For instance, this is so for the radially symmetric configurations of the $\mathbb{C}P^1$ model:

$$u = \frac{(2\theta)^{k/2}}{\lambda^k} a^k e^{i\alpha t}, \qquad [a, a^\dagger] = 1, \qquad k \neq 1, \qquad \lambda \neq 0, \tag{2.15}$$

where λ characterizes the size of the soliton. These features are essentially the same as those found for the commutative Q-lumps.

However, it is important to remark that in contrast to the commutative theory, computing the Noether charge or the energy for a given configuration with a generic winding number k is not an easy task. Even for the class of winding number k configurations specified by (2.15), it is rather difficult to compute the traces involved in Q. Explicitly, we have

$$|Q| = 2\pi\theta 2\alpha\kappa^{2k}\Sigma_k(\kappa) = \pi\lambda^2 2\alpha\kappa^{2k-2}\Sigma_k(\kappa), \qquad \kappa = \frac{\lambda}{\sqrt{2\theta}}, \tag{2.16}$$

where

$$\Sigma_k(\kappa) := \frac{1}{2}\,\mathrm{Tr}_{\mathcal{F}}\left(\frac{a^{\dagger k}a^k}{[a^{\dagger k}a^k + \kappa^{2k}]^2} + a^k \frac{1}{[a^{\dagger k}a^k + \kappa^{2k}]^2} a^{\dagger k}\right) = \sum_{j=0}^{\infty} \frac{\frac{(j+k)!}{j!}}{(\frac{(j+k)!}{j!} + \kappa^{2k})^2}. \tag{2.17}$$

The series $\Sigma_k(\kappa)$ converges for $k \geq 2$ as can be verified by applying Raabe's test, while it diverges for $k = 1$. For small values of k, the series $\Sigma_k(\kappa)$ may be summed by using Mathematica or formulas from [15]. For $k = 2$ we have[2]

$$\Sigma_2(\kappa) = \frac{\pi \sec^2(\frac{1}{2}\pi\sqrt{1-4\kappa^4})(-2\pi\kappa^4\sqrt{1-4\kappa^4} + (1-2\kappa^4)\sin(\pi\sqrt{1-4\kappa^4}))}{2(1-4\kappa^4)^{3/2}}. \tag{2.18}$$

This result will be made use of in the next section to concretely demonstrate the new features encountered in the stability properties of NC Q-lumps under small radiative perturbations.

The expression in (2.16) suggests that the Noether charge Q is determined by k, α, θ, and λ. In the commutative theory, Q depends on k, α, and λ already [1], thus it is important to assess if and how the new alleged dependence of Q on θ is genuine. For this purpose, let us first observe that for the solutions of the form (2.15) the moduli space metric is given by

$$ds^2 = -\pi k^2 2^k \kappa^{2k-2}(t)\Sigma_k(\kappa(t))(d\lambda^2) =: g_k(\kappa(t))(d\lambda^2). \tag{2.19}$$

Thus, it depends on k and $\kappa(t)$ only. In order to claim that Q indeed depends on θ, there should be a way to fix our position in the moduli space while θ is still allowed to vary. From (2.19) it is clear how this could be achieved. Namely, allowing θ and λ to vary while keeping κ fixed, the moduli space metric does not change; however, Q continues to vary with θ. In other words, taking θ proportional to λ^2, we can think of Q as a function of either θ or λ. It is only in this regime that Q (and consequently the energy) may be viewed to depend on θ.

It is also important to study the case when θ is fixed and λ is allowed to vary. In this situation, although both the metric $g_k(\kappa)$ and Q continue to vary with λ, we notice from (2.16) and (2.19) that we can always write $Q = -\frac{\alpha}{2^{k-1}k^2}\lambda^2 g_k(\kappa)$, and thus the factor $g_k(\kappa)$ entirely compensates for the change in the moduli space metric as λ is varied and consequently we can view Q as a function of λ.

Finally, we note that the commutative limit is recovered by taking $\kappa \to \infty$, keeping λ fixed and taking $\theta \to 0$.

III. STABILITY OF NC $\mathbb{C}P^1$ Q-LUMPS

Elementary classical stability properties of the NC Q-lumps are also quite similar to their commutative counterparts. As the NC Q-lumps saturate the BPS-type bound, they are automatically classically stable configurations. The quantum stability of Q-lumps requires the energy-charge ratio to be smaller than the meson mass in the theory [1]. For the present case, the maximum energy transferable to radiating mesons is $E' = E - 2\pi N$ due to the topological stability of the NC Q-lump configuration, and hence the energy-charge ratio is given as $\frac{E'}{Q} = \alpha$, due to

[1] Recall that the $\mathbb{C}P^N$ model action in NC spacetime is not scale invariant due to the noncommutativity; however, its solitonic solutions retain this feature.

[2] $\Sigma_3(\kappa)$ is also available through Mathematica but significantly more complicated then $\Sigma_2(\kappa)$.

the BPS-type bound. This is precisely the same situation encountered for the commutative Q-lumps, which appear to be at the threshold of quantum stability. Consequently, the same crucial question regarding the radiative stability of the Q-lumps, which was analyzed in detail by Leese [1], is also of essential interest here. To be more precise, although the NC Q-lumps appear to be quantum mechanically stable (more accurately at the threshold of quantum stability), it is easily seen that any small perturbation could start a continuous emission of radiating mesons as the ratio $\frac{E'}{Q}$ grows larger once such a process is initiated, and this would eventually lead the Q-lump to shrink to a spike. Therefore, the question which needs to be answered is whether such a continuous emission of radiating mesons is classically possible. Leese analyzed this problem using numerical techniques and also provided a rather simple analytic discussion that corroborates with his numerical findings that such radiative instabilities are not present as long as there is a potential barrier between Q-lumps with winding number k and configurations consisting of Q-lumps with winding number $k' < k$ present together with some mesons at larger distances. The situation in the noncommutative setting appears to be somewhat more complicated, and at present we will not attempt to give a full result using numerical techniques. In what follows, we apply the aforementioned analytical procedure to the NC Q-lumps. This will help us to see some new features of these configurations and also allow us to show that they remain stable under small radiative perturbations.

The most general radially symmetric configuration (not necessarily a solution) with winding number k may be given by

$$u = \frac{(2\theta)^{k/2}}{\lambda^k(\hat{N}, t)} a^k e^{i\psi(\hat{N},t)}, \qquad \lambda(\hat{N}, t) \neq 0. \tag{3.1}$$

In (3.1), $\hat{N} = a^\dagger a$ is the number operator, which maps to the square of the radial coordinate under the diagonal coherent states map. Assuming that the system remains approximately radially symmetric during the time evolution, we can drop the $\hat{N}$ dependence in $\lambda(\hat{N}, t)$ and $\psi(\hat{N}, t)$. Then, the associated Lagrangian becomes

$$L_k = 2\pi k + 2\pi\theta \frac{\lambda^{2k}(t)}{\theta^k}\left(\alpha^2 - \left(\frac{\lambda(t)'^2}{\lambda(t)^2} k^2 + \psi'^2(t)\right)\right) \times \Sigma_k\left(\frac{\lambda(t)}{\sqrt{2\theta}}\right), \tag{3.2}$$

where $'$ denotes the derivatives with respect to t.

$L_k(\lambda(t), \psi(t))$ specifies a dynamical system in $\lambda(t)$ and $\psi(t)$. Let us now consider a small perturbation around the Q-lump configuration (2.11)

$$\lambda(t) = \lambda_0 + \varepsilon(t), \qquad \psi(t) = t(\alpha + \delta(t)), \tag{3.3}$$

where λ_0 stands for $\lambda(t = 0)$ for short. Using the equation of motion for $\psi(t)$ and (3.3) we have

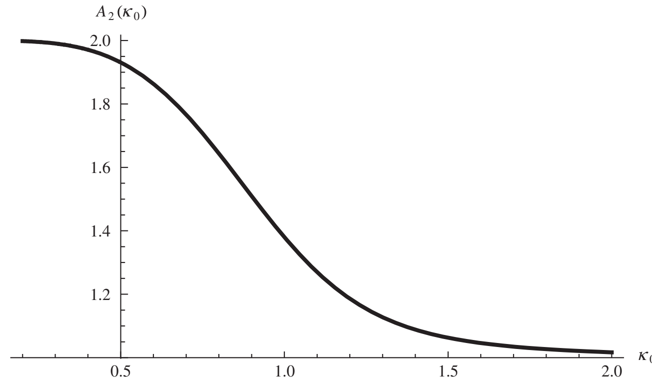

FIG. 1. Plot of $A_2(\kappa_0)$ as a function of κ_0.

$$\psi'(t) = \alpha \frac{\lambda_0^{2k}\Sigma_k(\frac{\lambda_0}{\sqrt{2\theta}})}{\lambda^{2k}(t)\Sigma_k(\frac{\lambda(t)}{\sqrt{2\theta}})}. \tag{3.4}$$

For our purposes, we only need the equation of motion[3] for $\lambda(t)$ at first order in $\varepsilon(t)$. This takes the form

$$\varepsilon''(t) + \frac{4\alpha^2}{k^2} A_k^2(\kappa_0)\varepsilon(t) + O(\varepsilon^2(t)) = 0, \tag{3.5}$$

where

$$A_k(\kappa) := \left(k + \frac{1}{2}\kappa_0 \frac{\partial_\kappa \Sigma_k(\kappa)|_{\kappa=\kappa_0}}{\Sigma_k(\kappa_0)}\right), \qquad \kappa_0 = \frac{\lambda_0}{\sqrt{2\theta}}. \tag{3.6}$$

Thus, the width of the radially symmetric lumps oscillate under small perturbations albeit with a period $\tau = \frac{\pi k}{\alpha A_k(\kappa_0)}$, which now depends on the function $A_k(\kappa_0)$ of $\kappa_0 = \frac{\lambda_0}{\sqrt{2\theta}}$ in contrast to the commutative theory. We conclude that the Q-lump configurations remain stable under small radiative perturbations. As κ_0 gets larger (i.e. $\theta \ll \lambda_0$), we have that $\Sigma_k(\kappa_0 \to \infty) \approx \kappa_0^{2-2k}$ and hence $A_k(\kappa_0 \to \infty) \to 1$. The commutative limit is, therefore, smoothly recovered.

It is possible to investigate the behavior of $A_{k=2}(\kappa_0)$ in more detail. In Fig. 1, we have plotted the response of $A_2(\kappa_0)$. We observe that $A_2(\kappa_0)$ smoothly approaches to the value 1 as $\kappa_0 \to \infty$ and hence τ approaches its commutative value. Put another way, we could state that the noncommutativity leads to more rapid oscillations around the Q-lump configuration.

IV. CONCLUSIONS AND OUTLOOK

In this paper we have constructed the Q-lump configurations associated with the NC $\mathbb{C}P^N$ models. We have found that, similar to their commutative counterparts, they too appear as extended field configurations, which rotate with a fixed angular frequency in time, and saturate

[3]The full equation of motion is given in the appendix for completeness.

a BPS-type bound. Quite interestingly, it was also found that in a regime in which θ is taken to be proportional to λ^2, the energy of the NC Q-lump configurations can be viewed to depend on θ via the Noether charge Q. A collective coordinate-type analysis helped us to show that $\mathbb{C}P^1$ model Q-lumps remain stable under small radiative perturbations. Contrary to the behavior of the commutative Q-lumps, we have also seen that, due to the noncommutativity the period of fluctuations around the radially symmetric Q-lump configurations depend on the function $A(\kappa_0)$ leading to more rapid oscillations around the Q-lump configuration.

It seems rather straightforward to obtain the supersymmetric extensions of the NC Q-lumps following the ideas of [7,16]. To be more concrete, focusing on the $\mathcal{N} = 2$ superspace $\mathcal{A}_\theta(\mathbb{R}^{2+1|4})$ with only the Moyal-type noncommutativity, (i.e. Grassmann coordinates are undeformed and they anticommute), it is possible to consider the supersymmetric Lagrangian

$$L = \int d^2\theta \mathrm{Tr} D\mathcal{P}\bar{D}\mathcal{P} + \mathcal{W}(\chi), \tag{4.1}$$

where $\mathcal{P} \equiv \mathcal{P}(\hat{x}_\mu, \theta_\alpha)$ is a projector in $\mathcal{A}_\theta(\mathbb{R}^{2+1|4}) \otimes \mathrm{Mat}(N+1)$, $\chi \equiv \chi(\hat{x}_\mu, \theta_\alpha)$ is the partial isometry fulfilling $\mathcal{P} = \chi\chi^\dagger$ and $\chi^\dagger\chi = 1$, and $\mathcal{W}(\chi)$ is a superpotential. Following [7], the superpotential can be taken to be of the form

$$\mathcal{W}(\chi) = \beta \mathrm{Tr} \chi^\dagger \mathcal{K} \chi, \tag{4.2}$$

where $\mathcal{K} = \mathrm{diag}(1, 1, 1, \cdots, 0) \in \mathrm{Mat}(N+1)$ is a projector. It may be shown that this system leads to the same bosonic NC $\mathbb{C}P^N$ Q-lump configurations with half of the supersymmetries, while the fermionic part is assumed to vanish. Clearly, a comprehensive study of noncommutative deformations of massive supersymmetric sigma models in various dimensions still has to be made to shed more light into their detailed structure.

There are several other issues which remain to be investigated. First of all, it should be possible to explore the scattering of NC Q-lumps and compare it with those of the commutative theory, as well as with those in the pure NC $\mathbb{C}P^N$ models. It may also be possible to explore the addition of topological terms, such as the Chern-Simons term or a Berry phaselike term into the action. It appears that the latter of these lead to divergent contributions in general [17]. Nevertheless, it may be possible to regulate the contribution of divergent traces by restricting the configuration space to an infinite strip [18] or a disc [19] on the GM plane. Such a regularization, however, also alters the structure of the pure sigma model lumps as well as the Q-lumps, and further investigation is necessary to understand the behavior of these configurations. We hope to report on the progress on these topics elsewhere.

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APPENDIX

The equation for $\lambda(t)$ is

$$\lambda''(t) + \frac{1}{2}\left(2(k-1) + \kappa(t)\frac{\partial_\kappa \Sigma_k(\kappa)}{\Sigma_k(\kappa)}\right)\frac{\lambda'^2(t)}{\lambda(t)} + \frac{1}{k^2}(\alpha^2 - \psi'^2(t))\left(k + \frac{1}{2}\kappa(t)\frac{\partial_\kappa \Sigma_k(\kappa)}{\Sigma_k(\kappa)}\right)\lambda(t) = 0. \tag{A1}$$

As $\theta \to 0$, $\kappa(t)\frac{\partial_\kappa \Sigma_k(\kappa)}{\Sigma_k(\kappa)} \to 2(1-k)$ and the result of the commutative theory is recovered.

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