On matrix model formulations of noncommutative Yang-Mills theories

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We study the stability of noncommutative spaces in matrix models and discuss the continuum limit which leads to the noncommutative Yang-Mills theories. It turns out that most noncommutative spaces in bosonic models are unstable. This indicates perturbative instability of fuzzy $\mathbb{R}^D$ pointed out by Van Raamsdonk and Armoni *et al.* persists to nonperturbative level in these cases. In this sense, these bosonic noncommutative Yang-Mills theories are not well-defined, or at least their matrix model formulations studied in this paper do not work. We also show that noncommutative backgrounds are stable in a supersymmetric matrix model deformed by a cubic Myers term, though the deformation itself breaks supersymmetry.

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I. INTRODUCTION

Yang-Mills theory on a noncommutative space (noncommutative Yang-Mills theory, or simply NCYM) has attracted much interest in theoretical physics. It appears as an effective theory of string theory or its matrix models around certain flux backgrounds [1–6]. NCYM contains some interesting physical properties like spacetime uncertainty and peculiar solitonic solutions [7]. We also notice that it naturally contains gravity (for recent progress, see e.g. [8,9]). To understand the nonperturbative aspects of NCYM better, we need the nonperturbative formulation of it. Matrix models are expected to be the most promising approach. Using a matrix model, NCYM is realized as an effective theory of a matrix model around a certain background. However, such backgrounds are unstable for some cases and whether the theories are well-defined or not is a nontrivial question. It is well-defined only when the backgrounds are stable. In this note, we will discuss stability of noncommutative spaces and argue what kinds of NCYM can be realized using matrix models.

Realization of NCYM in matrix models is of interest also from *emergent geometry* point of view. The origin of this concept goes back to the early 1980s. The first example, as far as we know, is large-N reduction [10–12] which claims that large-N gauge theories are equivalent to their one point reduced models. In these models, spacetime is embedded in gauge fields [11–13]. We can also find it in the context of quantum theory of gravity. From this point of view, spacetime should emerge as a result of some dynamical mechanism. As nonperturbative formulations of string theory, various matrix models are proposed [2,3] and, especially in the Ishibashi-Kawai-Kitazawa-Tsuchiya (IKKT) matrix model [3], various interpretations are given to realize emergent geometry [9,14,15]. This concept is also discussed in the context of AdS/CFT [16,17].

NCYM is another example of emergent geometry. Let us briefly explain how it shows up and what kind of double-scaling limit is necessary. We only consider NCYM on a flat noncommutative space and mainly take the continuum limit in which the noncommutativity parameter θ is fixed. We set the gauge group to be $U(1)$ unless otherwise mentioned but generalization to $U(n)$ is straightforward.

For concreteness, let us consider zero-dimensional $SU(N)$ matrix models with a twisted boundary condition [12] or a Myers term added [18]. For these models, it is known that compact noncommutative spaces like fuzzy spheres are classical solutions. Once we fix θ, volume of the space and the UV cutoff are related to the matrix size N. Therefore, the gauge coupling g_{NC} runs with N. Strictly speaking, renormalizability of NCYM is a subtle problem. In principle, using numerical simulations, the scaling is determined nonperturbatively so that some renormalization condition is satisfied. For example, in [19], $D = 2$ case is discussed and renormalization is performed so that the expectation value of the Wilson loop with the same area in a physical unit is kept fixed. This result is equivalent to the one for the one-loop calculation. In principle, we can similarly perform renormalization for the case of $D = 4$, however, it is hard with current numerical resources. Therefore, we rely on the one-loop calculation for this case [20]. It is known that for non-Abelian gauge theory, the scaling of the gauge coupling turns out to be the same as that of the commutative case. On the other hand, the case of Abelian gauge theory is extremely different and it is known that the beta function is the same as that of non-Abelian gauge theory on commutative space. That is, for NCYM, Abelian gauge theory is also asymptotically free as a result of the existence of nonplanar diagrams.

In order for NCYM to be well-defined, noncommutative spaces must be stable in this double scaling limit. However, in some cases g_{NC} runs into a region where the space is not

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stable anymore. We show that this is the case for most of bosonic models. Therefore, as suspected for a long time [21], NCYM on fuzzy $\mathbb{R}^D$ is not well-defined nonperturbatively (At least matrix model formulations discussed in this paper do not work). Here we also notice that $D = 2$ pure NCYM is only one exception that we have found in this paper. In other words, NCYM describes a wrong vacuum and hence noncommutative spacetime is not an emergent background in this case. This is not necessarily a negative conclusion—we can say that NCYM correctly describes gravitational instability.

On the other hand, once supersymmetry is introduced we can expect that fuzzy spaces are stabilized because of the Bogomol'nyi-Prasad-Sommerfield monopoles (BPS) nature and noncommutative super Yang-Mills theory (NCSYM) on fuzzy $\mathbb{R}^4$ is realized. In order to formulate NCSYM on fuzzy $\mathbb{R}^4$, we add a cubic Myers term to the usual IKKT-like matrix models. One thing we notice here is the fact that these models themselves do not have supersymmetry but it recovers in the double scaling limit.

Organization of this paper is as follows. In Sec. II we study bosonic matrix models to formulate bosonic NCYM. We first discuss the twisted Eguchi-Kawai model [12] and explain that we cannot formulate $D = 4$ pure NCYM [22] while we can formulate $D = 2$ pure NCYM. Next, we discuss bosonic analogues of IKKT matrix models with a cubic Myers term and analyze the stability of solutions like fuzzy spheres. We show that we cannot formulate $D = 4$ and $D = 2$ NCYM with adjoint scalars using this formulation. We also demonstrate that pure $D = 2$ NCYM can be realized by adding a potential term to an adjoint scalar. In the end of this section, we comment on other scaling limits like the *commutative* limit. In Sec. III we study approximately supersymmetric matrix models with a cubic Myers term to formulate NCSYM and show that the approximate supersymmetry stabilizes fuzzy spaces.

II. BOSONIC MATRIX MODELS AND BOSONIC NCYM ON FUZZY $\mathbb{R}^D$

In this section, we study bosonic matrix models and their double scaling limit which leads to bosonic NCYM on fuzzy $\mathbb{R}^D$. In Sec. II A we briefly review the twisted Eguchi-Kawai model (TEK) [12] and discuss the stability of the ground state [22,23]. In Sec. II B we explain the formulation of NCYM using TEK [5,24] and explain the double scaling limit. It turns out that NCYM on fuzzy $\mathbb{R}^4$ cannot be realized using it [22]. In Sec. II C we introduce bosonic analogue to IKKT matrix model with a cubic Myers term, which has fuzzy $S^2 \times S^2$ as a classical solution. We show that this background is unstable in the double scaling limit. Discussion in this subsection applies also to other deformations with a cubic Myers term. In Sec. II D we study other possible limits including commutative limit.

A. Twisted Eguchi-Kawai model

Twisted Eguchi-Kawai model [12] is a unitary matrix model defined by the action

$$S_{\rm TEK} = -\beta N \sum_{\mu\neq\nu} Z_{\mu\nu}\, {\rm Tr}(U_\mu U_\nu U_\mu^\dagger U_\nu^\dagger), \tag{2.1}$$

where U_μ are $N \times N$ unitary matrices with the Greek indices run from 1 to D and β is the inverse of the 't Hooft coupling. We mainly concentrate on the case of $D = 4$. We comment on the case of $D = 2$ in the end of the next subsection where we discuss matrix formulation of NCYM on fuzzy $\mathbb{R}^2$.

The phase factors $Z_{\mu\nu}$ are defined by

$$Z_{\mu\nu} = \exp(2\pi i n_{\mu\nu}/N), \qquad n_{\mu\nu} = -n_{\nu\mu} \in \mathbb{Z}_N.$$

In this paper, we use the skew-diagonal twist which is written as

$$(n_{\mu\nu}) = \begin{pmatrix} 0 & L & 0 & 0 \\ -L & 0 & 0 & 0 \\ 0 & 0 & 0 & L \\ 0 & 0 & -L & 0 \end{pmatrix}, \tag{2.2}$$

where $L = \sqrt{N}$ corresponds to the lattice size [12]. There are other ways of twisting, but the discussion is completely parallel and the conclusion is the same as far as the double scaling limit which leads to NCYM is concerned.

In the weak coupling limit ($\beta \to \infty$), the path-integral is dominated by configurations with the minimum value of the action. This configuration $U_\mu^{(0)} = \Gamma_\mu$ is called "twist eater" and satisfies the 't Hooft algebra

$$\Gamma_\mu \Gamma_\nu = Z_{\nu\mu} \Gamma_\nu \Gamma_\mu. \tag{2.3}$$

For the skew-diagonal twist, we can easily construct a twist eater configuration by introducing $L \times L$ "shift" matrix $\hat{S}_L$ and "clock" matrix $\hat{C}_L$

$$\hat{S}_L = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \hat{C}_L = \begin{pmatrix} 1 & & & & \\ & e^{2\pi i/L} & & & \\ & & e^{2\pi i\cdot 2/L} & & \\ & & & \ddots & \\ & & & & e^{2\pi i(L-1)/L} \end{pmatrix}. \tag{2.4}$$

These matrices satisfy

$$\hat{C}_L \hat{S}_L = e^{-2\pi i/L} \hat{S}_L \hat{C}_L, \tag{2.5}$$

and then we can construct a twist eater configuration for the above skew-diagonal twist as

$$\Gamma_1 = \hat{C}_L \otimes \mathbf{1}_L, \qquad \Gamma_2 = \hat{S}_L \otimes \mathbf{1}_L, \qquad \Gamma_3 = \mathbf{1}_L \otimes \hat{C}_L, \qquad \Gamma_4 = \mathbf{1}_L \otimes \hat{S}_L. \tag{2.6}$$

This twist eater configuration is nothing but fuzzy T^4 in the context of NCYM. We will explain the relation between fuzzy T^4 and fuzzy $\mathbb{R}^4$ when we use TEK as a potential nonperturbative formulation of NCYM on fuzzy $\mathbb{R}^4$ in the next subsection.

In [22], it was shown, by the Monte-Carlo study of TEK, that the configuration deviates from the Γ_μ and the fuzzy torus collapses in a certain range of the inverse 't Hooft coupling β. The upper boundary of this region scales as

$$\beta_c \simeq 0.0034N + 0.25. \tag{2.7}$$

We can estimate this behavior easily and somehow roughly as follows. For simplicity, we assume that the fuzzy torus $U_\mu = \Gamma_\mu$ collapses to the identity configuration $U_\mu = \mathbf{1}_N$. The difference of energy between these configurations is

$$\Delta S = S_{\mathrm{TEK}}(U_\mu = \mathbf{1}_N) - S_{\mathrm{TEK}}(U_\mu = \Gamma_\mu) = 8\pi^2 \beta N. \tag{2.8}$$

Far from the weak coupling limit, the system has quantum fluctuations. Especially quantum fluctuations about twist eater is known to be $O(N^2)$ [25]. Roughly expecting that the fuzzy torus collapses if the fluctuation around twist-eater configuration exceeds the energy difference ΔS, we can estimate the critical point β_c^L on which the torus begins to collapse as[1]

$$\beta_c \sim N, \tag{2.9}$$

which is consistent with the numerical results (2.7).

B. TEK and NCYM on fuzzy $\mathbb{R}^D$

TEK is a potential nonperturbative formulation of pure NCYM on fuzzy $\mathbb{R}^D$. [5,24]. In order to realize the formulation, we notice that fuzzy $\mathbb{R}^D$ is realized as a tangent space of fuzzy T^D. We can determine whether we can formulate the NCYM or not by analyzing the stability of the torus in the double scaling limit. Here we review the formulation of NCYM on $\mathbb{R}^4$ using TEK and especially discuss the double scaling limit [22] and the stability of the fuzzy T^4. We also comment on the case of $D = 2$.

By taking $U_\mu = e^{iaA_\mu}$, where a corresponds to the lattice spacing, and expanding the action of TEK (2.1), we have its continuum version as

$$S_{\mathrm{TEK}} = -\frac{1}{4g^2}\sum_{\mu\neq\nu} \mathrm{Tr}([A_\mu, A_\nu] - i\theta_{\mu\nu})^2, \tag{2.10}$$

up to higher order terms in a, where

$$\theta_{\mu\nu} = \frac{2\pi n_{\mu\nu}}{Na^2}, \qquad \frac{1}{4g^2} = a^4 \beta N. \tag{2.11}$$

Then, by expanding the action around a classical solution (2.10)

$$A_\mu^{(0)} = \hat{p}_\mu, \qquad [\hat{p}_\mu, \hat{p}_\nu] = i\theta_{\mu\nu}, \tag{2.12}$$

we obtain the $U(1)$ NCYM on fuzzy $\mathbb{R}^4$ as follows. Let us define the "noncommutative coordinate" $\hat{x}^\mu = (\theta^{-1})^{\mu\nu}\hat{p}_\nu$. Then we have

$$[\hat{x}^\mu, \hat{x}^\nu] = -i(\theta^{-1})^{\mu\nu}. \tag{2.13}$$

This commutation relation is the same as that of coordinates on fuzzy $\mathbb{R}^4$ with noncommutativity parameter θ, and hence functions of $\hat{x}$ can be mapped to functions on fuzzy $\mathbb{R}^4$. More precisely, we have the following mapping rule:

$$f(\hat{x}) = \sum_k \tilde{f}(k) e^{ik\hat{x}} \leftrightarrow f(x) = \sum_k \tilde{f}(k) e^{ikx},$$
$$f(\hat{x})g(\hat{x}) \leftrightarrow f(x) \star g(x), \qquad i[\hat{p}_\mu, \cdot] \leftrightarrow \partial_\mu, \tag{2.14}$$
$$\mathrm{Tr} \leftrightarrow \frac{\sqrt{\det\theta}}{4\pi^2}\int d^4x,$$

where $\star$ represents the noncommutative product,

$$f(x) \star g(x) = f(x)\exp\left(-\frac{i}{2}\overleftarrow{\partial}_\mu(\theta^{-1})^{\mu\nu}\vec{\partial}_\nu\right)g(x), \tag{2.15}$$

and we obtain $U(1)$ NCYM with coupling constant

$$g_{\mathrm{NC}}^2 = 4\pi^2 g^2/\sqrt{\det\theta}. \tag{2.16}$$

In order to keep the noncommutative scale θ finite, we should take the double scaling limit with

$$a^{-1} \sim \Lambda \sim N^{1/4}. \tag{2.17}$$

As we have explained the identification to formulate pure NCYM using TEK, we next determine the double scaling limit explicitly and discuss the stability of the fuzzy T^4. The one-loop beta function for $D = 4$ $U(1)$ NCYM is given by [20]

$$\beta_{1-\mathrm{loop}}(g_{\mathrm{NC}}) = -\frac{1}{(4\pi)^2}\frac{11}{3}g_{\mathrm{NC}}^3 + O(g_{\mathrm{NC}}^5). \tag{2.18}$$

Therefore, the inverse 't Hooft coupling β scales as

$$\beta \sim \frac{1}{g_{\mathrm{NC}}^2} \sim \log\Lambda \sim \log N. \tag{2.19}$$

Since we know that the torus collapses below the critical point β_c which scales as (2.7), we can see that the fuzzy T^4 collapses in the double scaling limit. Therefore we finally see that we cannot formulate $D = 4$ pure NCYM using TEK.

Before closing this subsection we comment on the results for $D = 2$. In this case, Eqs. (2.2) and (2.6) are replaced by

[1]In this paper, we often estimate the power of N only and we use "$A \sim B$" (resp. "$A \lesssim B$") to represent that the order of A is equal to (resp. equal to or less than) that of B.

$$(n_{\mu\nu}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.20}$$

and

$$\Gamma_1 = \hat{C}_N, \qquad \Gamma_2 = \hat{S}_N. \tag{2.21}$$

This corresponds to a fuzzy T^2. In order to take the double scaling limit with the noncommutative parameter $\theta \sim (a^2 N)^{-1}$ fixed, we must scale the lattice spacing as $a \sim N^{-1/2}$. For $D = 2$, the double scaling limit is determined by numerical simulations as [19]

$$\beta \sim N. \tag{2.22}$$

In this scaling, $g^2 \sim g_{\rm NC}^2$ does not run, which is consistent with the one-loop beta function for $D = 2$. Since we know that fuzzy T^2 does not collapse in $D = 2$ TEK (This case is exceptional because there are no physical degrees of freedom.), we see that we can formulate $D = 2$ pure NCYM on fuzzy $\mathbb{R}^2$. For detailed simulations and renormalizability see [19].

C. Matrix model with a cubic Myers term

In this section, we use an bosonic analog to IKKT-type matrix model with a cubic Myers term. More concretely, we consider a matrix model which has $S^2 \times S^2$ as a classical solution by choosing the cubic term coupling appropriately. Then we discuss the stability of fuzzy $S^2 \times S^2$. Although we use the specific solution, this argument itself can be applied to the case of other compact noncommutative manifolds like fuzzy S^2 and fuzzy $\mathbb{C}P^2$.[2]

Let us start with the $d = 6$ bosonic analog to IKKT model with a cubic Myers term. The action is written as

$$S = \frac{1}{g^2} \mathrm{Tr}\left(-\frac{1}{4}[A_\mu, A_\nu]^2 + \frac{2i}{3}\alpha f^{\mu\nu\rho} A_\mu A_\nu A_\rho \right), \tag{2.23}$$

where A_μ is $N \times N$ hermitian matrix and the Greek indices run from 1 to 6. In the cubic term, $f^{\mu\nu\rho}$ is the structure constant of $SU(2) \times SU(2)$ and α is a constant which characterizes the radii of fuzzy spheres. We choose the totally antisymmetric tensor $f^{\mu\nu\rho}$ such that the only nonzero components are $f^{123} = f^{456} = 1$ and their permutations.

The equation of motion for this model is

$$[[A_\mu, A_\nu], A_\nu] + 2i\alpha f^{\mu\nu\rho} A_\nu A_\rho = 0. \tag{2.24}$$

A classical solution called fuzzy $S^2 \times S^2$ is given by

$$A_\mu^{(0)} = \alpha J_\mu, \tag{2.25}$$

where J_μ is a generator of $SU(2) \times SU(2)$ which satisfies

$$[J_\mu, J_\nu] = i f^{\mu\nu\rho} J_\rho. \tag{2.26}$$

J_μ can be expressed as[3]

$$J_{1,2,3} = J_{1,2,3}^{(s)} \otimes \mathbf{1}_{2s+1}, \qquad J_{4,5,6} = \mathbf{1}_{2s+1} \otimes J_{1,2,3}^{(s)} \tag{2.27}$$

where $J^{(s)}$ is the spin-s generator and $N = (2s+1)^2$. In this case two fuzzy spheres have the same radius and square of the radius R of the fuzzy sphere is given by

$$R^2 = \sum_{i=1}^{3} (A_i^{(0)})^2 = \alpha^2 s(s+1). \tag{2.28}$$

Expanding the matrix model (2.23) about (2.27), we obtain NCYM on fuzzy $S^2 \times S^2$ coupled to two adjoint scalars. By zoomingup the north pole, i.e. considering only states with $J_3 \sim J_6 \sim s$, we formally obtain NCYM on fuzzy $\mathbb{R}^4$ with two adjoint scalars, which originate from transverse directions of the fuzzy $S^2 \times S^2$. Because

$$[A_1^{(0)}, A_2^{(0)}] = [A_4^{(0)}, A_5^{(0)}] = i\alpha^2 J_3 \sim i\alpha^2 s, \tag{2.29}$$

the noncommutativity parameter θ is

$$\theta \sim \alpha^2 s \sim \alpha^2 \sqrt{N}. \tag{2.30}$$

In order to keep θ fixed, we must scale α as

$$\alpha \sim N^{-1/4}, \tag{2.31}$$

and therefore the momentum cutoff scales as

$$\Lambda \sim \alpha s \sim N^{1/4}. \tag{2.32}$$

As a result, in order to take the continuum limit with θ fixed, we have to scale g^2 as [20]

$$g^{-2} = \frac{4\pi^2}{\theta^2} g_{\rm NC}^{-2} \sim \log\Lambda \sim \log N. \tag{2.33}$$

We can easily see that fuzzy $S^2 \times S^2$ collapses when $\frac{1}{g^2} \lesssim N$, because the energy difference between fuzzy $S^2 \times S^2$ and $A_\mu = 0$ is of order $\frac{\alpha^4 N^2}{g^2} \sim \frac{N}{g^2}$, while quantum fluctuations are of order N^2. Therefore, fuzzy $S^2 \times S^2$ collapses when we take the double scaling limit (2.31) and (2.33) and then we cannot take the continuum limit.

This bound was derived more rigorously using a Monte-Carlo simulation. Interestingly, this bound can also be derived through perturbative calculations of the matrix model [27]. First, notice that eigenvalues are concentrated around the origin after the collapse of fuzzy sphere. This can be confirmed by numerical simulations. Then it is reasonable to assume that, in the perturbative analysis, such instability can be detected by considering only "re-

[2]The argument below can be parallelly applied to the case of fuzzy S^2 and fuzzy $\mathbb{C}P^2$ because they are classical solutions of (2.23) with $f^{\mu\nu\rho}$ appropriately chosen. Fuzzy S^4, however, is a classical solution to a bosonic matrix model with a quintic Myers term. In this case, perturbative calculation is not valid. In [26], it is numerically shown that S^4 is unstable in a bosonic matrix model.

[3]We can also combine generators with different spins, but the argument does not change qualitatively.

scaled fuzzy sphere'' $A_\mu^{\rm rescaled} = A_\mu^{(0)} \times$ const and by calculating its free energy as a function of the radius. At large enough coupling, there is a minimum to the free energy, which indicates that the background is stable. However, below some critical point, the minimum disappears and we can expect that the background is not stable anymore. This is actually the case and the critical value obtained in this way agrees with the numerical result very accurately. For details, see [27]. In the next section, we assume the validity of the perturbative calculation and use it to justify the matrix formulation of supersymmetric noncommutative Yang-Mills theory.

Here we comment on results for the formulation of $D = 2$ NCYM with an adjoint scalar. Let us take $f^{\mu\nu\rho}$ to be the structure constant $\epsilon_{\mu\nu\rho}$ of $SU(2)$ where the Greek indices run from 1 to 3. As a classical solution of this matrix model we can obtain fuzzy S^2. By zooming up the north pole as we did above, we obtain the NCYM on fuzzy $\mathbb{R}^2$. In order for the noncommutativity parameter to be fixed, we have to scale the coupling constant for the cubic Myers term as

$$\alpha \sim N^{-1/2}. \tag{2.34}$$

Because the potential difference between the fuzzy S^2 and $A_\mu = 0$ is of order $\frac{\alpha^4 N^3}{g^2} \sim \frac{N}{g^2}$ while one-loop fluctuation is of order N^2, fuzzy S^2 collapses when $\frac{1}{g^2} \lesssim N$. On the other hand, we can see the gauge coupling constant g^2 does not run similarly to the case of Sec. II B. Therefore, we cannot take the continuum limit as $D = 2$ NCYM with an adjoint scalar.

1. *Adding potential terms for adjoint scalars*

In [28], another matrix model formulation of NCYM is introduced. This matrix model has fuzzy S^2 as a classical solution. In the original paper above, the commutative limit $\theta \to \infty$ was studied. In this section, we rather discuss the double scaling limit with θ fixed and see whether we can use this matrix model to formulate NCYM.

For this model the action is given by

$$S = \frac{1}{4g^2} \mathrm{Tr}\left\{(\alpha A_i + i\epsilon_{ijk}A_jA_k)^2 + \left(A_i^2 - \frac{\alpha^2}{4}(N^2-1)\right)^2\right\}. \tag{2.35}$$

By expanding the action about a classical solution

$$A_i = \alpha J_i, \tag{2.36}$$

where J_i are $SU(2)$ generators with spin $s = \frac{N-1}{2}$, NCYM on fuzzy S^2 is realized. (The second term in (2.35) gives potential for adjoint scalar.) To take a continuum limit with a fixed noncommutativity parameter, we should take large-N limit with g^2 fixed and $\alpha \sim \frac{1}{\sqrt{N}}$.

However, we can easily see that this background can collapse to a point e.g.

$$A_1 = A_2 = 0, \qquad A_3 = \frac{\alpha\sqrt{N^2-1}}{2}. \tag{2.37}$$

We can easily see the difference of tree-level potential at (2.36) and a (2.37) is of order $\frac{\alpha^4 N^3}{g^2}$, while quantum fluctuations are of order N^2 in the double scaling limit. Therefore we can see that the critical coupling is $\frac{1}{g_c^2} \sim N$ and fuzzy S^2 collapses in the limit with $\frac{1}{g^2} \lesssim \frac{1}{g_c^2}$.

In [29] a slightly generalized version of (2.35),

$$S = N\,\mathrm{Tr}\left\{-\frac{1}{4}[X_i, X_j]^2 + \frac{2i\rho}{3}\epsilon^{ijk}X_iX_jX_k - m^2\rho^2X_i^2 + \frac{2m^2}{N^2-1}(X_i^2)^2\right\}, \tag{2.38}$$

was studied both numerically and perturbative and the critical point is found to be

$$\rho_c = \left(\frac{8}{m^2+\sqrt{2}-1}\right)^{1/4}. \tag{2.39}$$

By redefining the field and by identifying parameters as

$$A_i = g^{1/2}N^{1/4}X_i, \qquad m^2 \sim N^2, \qquad \alpha \sim g^{1/2}N^{1/4}\rho, \tag{2.40}$$

we obtain (2.35) from (2.38) up to $O(1)$ factors. With this identification, the scaling of the critical coupling becomes

$$\frac{1}{g_c^2} \sim (N^{1/4}\rho_c\alpha^{-1})^4 \sim N, \tag{2.41}$$

which agrees with the rough estimation just below (2.37). Therefore it finally follows that we cannot formulate $D = 2$ NCYM with an adjoint scalar using (2.35).

However, the generalized model (2.38) has another NCYM limit. To prevent the fuzzy sphere from collapsing in the continuum limit (g^2 fixed and $\alpha \sim 1/\sqrt{N}$) we have to scale $\frac{1}{g_c^2} \lesssim O(1)$. To realize this scaling with the redefinition of the field and identification of α shown in (2.40), we have to scale m as

$$m^2 \gtrsim N^3, \tag{2.42}$$

instead of N^2. Since last two terms in (2.38) are rewritten as

$$N \cdot \frac{2m^2}{N^2-1}\mathrm{Tr}\left(X_i^2 - \frac{N^2-1}{4}\rho^2\right)^2 + \text{const}, \tag{2.43}$$

they suppress the fluctuation perpendicular to fuzzy sphere. Therefore, an adjoint scalar, which corresponds to this direction decouples and we obtain $D = 2$ pure NCYM with the scaling (2.42).

Before closing this subsection let us remark on the subtlety in the above argument. The bound (2.39) is obtained by calculating the free energy of the rescaled fuzzy sphere. However, if the value of m is extremely large, the instability (if exists) cannot be captured in this way, because collapse without changing the value of A_i^2 is more economical. If adjoint scalars decouple, the situation is analogous to the case of TEK. In $D = 2$ TEK, fuzzy T^2 does not collapse. Therefore in the case of fuzzy S^2, we do not expect this kind of instability. On the other hand, in the case of fuzzy $S^2 \times S^2$ or fuzzy $\mathbb{C}P^2$ with adjoint scalar potentials [30,31], we expect this instability similarly to fuzzy T^4 in $D = 4$ TEK and hence NCYM on fuzzy $\mathbb{R}^4$ cannot be obtained.[4] It is desirable to check it directly with Monte-Carlo simulation.

D. Other limits

So far, we considered the continuum limit with the noncommutativity parameter fixed and showed that most of bosonic models have instability. In this subsection, we discuss other possible limits. For concreteness we consider the $D = 4$ TEK model.

First, let us consider the case in which the fuzzy torus does not collapse. The noncommutativity parameter is expressed as

$$\theta \sim \frac{1}{\sqrt{N}a^2} \sim \frac{\Lambda^2}{\sqrt{N}}. \qquad (2.44)$$

To prevent the fuzzy torus from collapsing, the momentum cutoff must be large enough so that

$$\log\Lambda \sim \frac{1}{g^2} \sim \Lambda^{-4}\beta N \gtrsim \Lambda^{-4}N^2. \qquad (2.45)$$

On the other hand, to keep the volume of noncommutative space $a\sqrt{N} \sim \sqrt{N}/\Lambda$ nonzero, Λ cannot be so large:

$$\Lambda \lesssim \sqrt{N}. \qquad (2.46)$$

The only solution to the above constraints (2.45) and (2.46) is

$$\theta \sim \sqrt{N}, \qquad \Lambda \sim \sqrt{N}, \qquad (2.47)$$

up to $\log\Lambda$ corrections. In this limit, noncommutativity length θ^{-1} goes to zero and the spacetime volume is fixed. This limit has been studied in many references. This limit is of interest as an alternative to the lattice gauge theory, because it might provide a simpler way to introduce chiral fermions [33].

Next let us consider the case that fuzzy torus does collapse. From D-brane point of view, it just means D-brane collapses to lower dimensional configuration. From NCYM perspective this limit seems not to have a sensible continuum limit because there is no extended direction. In [34] a slightly different model with two commutative and two noncommutative dimensions has been studied numerically. In that case two noncommutative dimensions collapse similarly to our case, but numerical results suggest that there is a continuum limit with two commutative noncompact directions and two compact, finite size ‘‘noncommutative’’ directions. Such models would be interesting as a toy model for compactification mechanism in matrix models.

[4]There are models in which adjoint scalars are dropped by hand. We expect the situation is the same [31,32].

III. SUPERSYMMETRIC MATRIX MODEL AND NONCOMMUTATIVE SUPER YANG-MILLS

In the previous section, we have discussed various matrix model formulations of bosonic NCYM. In this section we explain the formulation of noncommutative *super* Yang-Mills (NCSYM). For this purpose we introduce matrix models with an approximate supersymmetry and perturbatively discuss stability of noncommutative spaces.

Let us consider the IKKT-like matrix model [3] with a cubic Myers term

$$S = \frac{1}{g^2}\,\mathrm{Tr}\Big(-\frac{1}{4}[A_\mu, A_\nu]^2 + \frac{2i}{3}\alpha f^{\mu\nu\rho}A_\mu A_\nu A_\rho - \frac{1}{2}\bar{\psi}\Gamma^\mu[A_\mu, \psi]\Big), \qquad (3.1)$$

where A_μ and ψ are bosonic and fermionic Hermitian $SU(N)$ matrices, Greek indices run from 1 to d ($d = 4$, 6, 10), ψ has a spinor index and Γ^μ is the $SO(d)$ Gamma matrix. $f^{\mu\nu\rho}$ is the structure constant of a Lie group whose rank r is less than d. Except for the cubic Myers term, we can obtain this action from D-dimensional $\mathcal{N} = 1$ $SU(N)$ super Yang-Mills by dimensional reduction.

Since numerical simulations for these matrix models are difficult, except for $d = 4$ case [35] due to the notorious sign problem, it is difficult to discuss stability of backgrounds nonperturbatively. Hence, we provide only perturbative arguments, which works perfectly well for bosonic models. The perturbative argument is carried out similarly to the case of bosonic analog of IKKT-like matrix model with a cubic Myers term.

Although this model is not supersymmetric,[5] the noncommutative background can be stabilized for any value of α. At large α it is stable because potential barrier is very high and, furthermore, fluctuations are suppressed due to approximate supersymmetry. At small α, it can be stabilized since this model is almost supersymmetric (at $\alpha = 0$ the supersymmetry recovers) and this background is almost BPS.

As a concrete example, let us take $d = 10$ and $f_{\mu\nu\rho}$ to be the structure constant of $SU(2) \times SU(2)$. Fuzzy $S^2 \times S^2$ (2.27) is one of the classical solutions for it. (Indeed, there is a subtlety for this background. We will discuss it in Sec. III A). Quantum corrections to this background is calculated in [38]. Here we consider the deformation in radial direction only as we have explained in Sec. II C. Up

[5]$d = r = 3$ model with the cubic term is supersymmetric [36]. However, in this case the finiteness of partition function is not known for generic N [37].

to the leading order of $1/N$, the tree-level action $\Gamma_{\rm tree}$ and the one-loop correction $\Gamma_{1-\rm loop}$ for rescaled fuzzy sphere $P_\mu = (1+\epsilon)\alpha J_\mu$ are calculated as

$$\Gamma_{\rm tree} = \frac{\alpha^4 N^2}{4g^2}\left\{(1+\epsilon)^4 - \frac{4}{3}(1+\epsilon)^3\right\},$$
$$\Gamma_{1-\rm loop} = N\cdot 2\log 2\cdot\left(2+\frac{\epsilon^2}{(1+\epsilon)^2}\right). \qquad (3.2)$$

(See Appendix A for derivation.) If we scale $\alpha\sim N^{-1/4}$, we have

$$\Gamma_{\rm tree}\sim -Ng^{-2}, \qquad \Gamma_{1-\rm loop}\sim N. \qquad (3.3)$$

The matrix model we are considering here is expected to realize $D=4$ $\mathcal{N}=4$ NCSYM in the continuum limit and then the coupling g does not run at one-loop level in this limit.

From (3.3), the one-loop correction is smaller than tree-level action provided that g^2 is sufficiently small. We also notice that n-loop effect is $O(N(g^2/(\alpha^4 N))^{n-1}) = Ng^{2(n-1)}$ as a result of approximate supersymmetry (SUSY) and higher loop effects are negligible in this case [38]. We therefore see that the classical minimum $\epsilon=0$ survives after taking into account quantum corrections and we can expect that the fuzzy $S^2\times S^2$ does not collapse. On the other hand, at strong coupling it might collapse. To overcome this difficulty, it is probably useful to consider a supersymmetric deformation in [39,40].

In the above construction using the matrix model with the cubic Myers term, only extended supersymmetry can be realized. In order to construct $\mathcal{N}=1$ NCSYM, supersymmetric generalization of TEK would be necessary.

A. Subtlety for $S^2\times S^2$ case

In this subsection we discuss a subtlety for fuzzy $S^2\times S^2$ background.

Because fuzzy S^2 has smaller free energy, fuzzy $S^2\times S^2$ is not stable; one of the S^2 can shrink, while the other expands [41]. We notice that $SU(2)\times SU(2)$ is preserved in this process and that this instability cannot be read off from the one-loop effective action (3.2). To avoid this instability, we should use four-dimensional fuzzy manifolds with higher symmetry, e.g. the fuzzy $\mathbb{C}P^2$. $\mathbb{C}P^2$ can be stable since the symmetry must be broken during the transition to S^2. The effective action does not change qualitatively [42] and we can realize $D=4$ $\mathcal{N}=4$ NCSYM using $\mathbb{C}P^2$.

It is difficult to realize $D=4$ $\mathcal{N}=2$ NCSYM using matrix model formulation because of the instability of fuzzy $S^2\times S^2$. Naively, if we add a cubic Myers term to $d=6$ supersymmetric matrix model as above, four-dimensional $\mathcal{N}=2$ NCSYM is expected to be realized in the continuum limit. In this case, the coupling runs as $g^{-2}\sim\log N$, and hence the background is stable. However, to realize four-dimensional compact fuzzy space with 6 matrices in this model, we need to use $S^2\times S^2$. It is necessary to fix the radii somehow, for example, by quenching the background or adding a small potential term to the adjoint scalars, while keeping the continuum theory unchanged. Instead of fuzzy $S^2\times S^2$ the fuzzy S^4 might be useful. To make fuzzy S^4 a classical solution, we have to add the quintic Myers term. However, it is difficult to discuss the stability because perturbative calculation is not valid.

IV. CONCLUSIONS AND DISCUSSIONS

In this paper, we studied the stability of noncommutative spaces in several matrix models and discussed whether or not they provide nonperturbative formulation of noncommutative Yang-Mills theory (NCYM). It turns out that most of matrix model formulations of *bosonic* NCYM on fuzzy $\mathbb{R}^D$ do not work. The only exception we found is $D=2$ pure NCYM. In the context of D-branes dynamics, those not realized correspond to false vacua. This might be a negative conclusion if one regards NCYM itself as a UV complete theory. However, as an effective description for a D-brane system, these bosonic NCYM correctly reproduce the instability of the system. According to [21], large one-loop correction to free energy, which leads instability of NCYM, is due to UV/IR mixing. Hence by eliminating UV/IR mixing somehow, we expect that NCYM be stabilized.

On the other hand, noncommutative *super* Yang-Mills (NCSYM) on fuzzy $\mathbb{R}^4$ with extended supersymmetry can be formulated using a supersymmetric matrix model deformed by a cubic Myers term. At least, as we have see above, $D=4$ $\mathcal{N}=4$ NCSYM in weak coupling is realized using this formulation. Also in certain nonsupersymmetric model with adjoint fermions, $\mathbb{Z}_N$ symmetry is not broken [43]. Then combining it with a twist prescription a certain nonsupersymmetric NCYM will be obtained.

Here we comment on the formulations of NCSYM at finite temperature. For this purpose, we consider supersymmetric matrix quantum mechanics[6] with Euclidean time direction compactified and antiperiodic boundary condition for fermionic variables imposed. At high temperature, fermionic modes decouples and the theory becomes essentially bosonic. Therefore, we can expect that noncompact fuzzy space cannot be constructed in the high temperature limit. Whether NCSYM at nonzero temperature exists or not is a subtle problem and numerical simulation along the line of [44,45] will be necessary.

Though we have discussed matrix models formulation only in this paper, there is another candidate for nonper-

[6]Monte Carlo simulation for supersymmetric matrix quantum mechanics without cubic term has been performed recently [44,45], and incorporation of a cubic term [46] will be straightforward. Thermodynamical property of fuzzy sphere in bosonic model is studied in [47].

turbative formulation of bosonic NCYM [48]. However, it seems to share the same problem with the matrix model formulation considered in this paper. In [48] NCYM is mapped to a lattice gauge theory with twisted boundary condition. In the continuum limit with noncommutativity parameter fixed, however, corresponding lattice gauge theory goes to zero volume and essentially reduces to the TEK model (see Appendix B.)

Of course, the pathology discussed above does not prevent us from nonperturbative formulations of nongauge theories on noncommutative spaces using matrix models. For example, scalar field theories are well-defined and we can numerically analyze them using matrix model formulations [49,50]. We also notice, as explained in Sec. II D, we can take the "commutative" limit of NCYM, in which the noncommutativity length $\theta^{-1/2}$ goes to zero. Therefore, one may still regard NCYM as an alternative to the lattice construction for gauge theories on commutative spaces.

In the end, we comment on the recent progress in TEK and its relation to the matrix formulation of bosonic NCYM. Since the collapse of the fuzzy sphere in TEK model is nothing but the breakdown of $\mathbb{Z}_N$ symmetry (original motivation for TEK is to keep this symmetry unbroken), construction for bosonic NCYM is tightly related to a modification of Eguchi-Kawai model [10] such that $\mathbb{Z}_N$ does not break and large-N reduction works. Historically two options have been studied. One is TEK, which works fine at $D = 2$ but turns out to fail at $D = 4$. Another one is the quenched Eguchi-Kawai model (QEK) [11], in which commutative and extended background is "quenched" by hand. Naively by combining twist and quench prescriptions, i.e. by fixing *noncommutative* background by hand, NCYM seems to be realized. However, it does not seem to work. Indeed, recently it was argued that QEK does not work due to the following reason [51]. In QEK, unitary link variables U_μ's are constrained to be $V_\mu e^{iP_\mu} V_\mu^\dagger$, where $P_\mu = \mathrm{diag}(p_\mu^1, \cdots, p_\mu^N)$ is fixed, V_μ's are unitary matrices and p_μ^i's are distributed uniformly in $\mathbb{R}^4$. Naively one expects V_μ's fluctuate around $\mathbf{1}_N$ and, therefore, $\mathbb{Z}_N$ is not broken. However, what actually happens is that V_μ's become certain permutation matrices, so that quenched momenta are "locked" [51] and free energy becomes smaller. Intuitively, this result implies, even if the background is quenched by hand, V_μ can get a nontrivial VEV and an essentially different background emerges.

The same can take place also when we quench the noncommutative background. Such a subtlety does not exist in a supersymmetric case, and $D = 4$ $\mathcal{N} = 2$ NCSYM would be realized by quenching fizzy $S^2 \times S^2$ background.

Recently a new deformation to Eguchi-Kawai model was proposed in [52]. They added potential terms for Wilson lines to prevent $\mathbb{Z}_N$ from breakdown and argued that the additional terms do not contribute in the continuum limit. If it really works, by combining this method with the twist prescription, bosonic NCYM might be realized. Then, it would be interesting to understand the meaning of the deformation in the context of D-brane dynamics.

In this paper, we assumed the running of the coupling constant is determined by one-loop beta function when we discuss the case of $D = 4$. However, renormalizability of the NCYM is of course controversial. It will be better if we can determine the running more rigorously, for example, by calculating correlation functions using numerical simulations.

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APPENDIX A: DERIVATION OF ONE-LOOP EFFECTIVE ACTION IN SUPERSYMMETRIC MATRIX MODEL

Let us expand the action

$$S = \frac{1}{g^2} \mathrm{Tr}\Big(-\frac{1}{4}[A_\mu, A_\nu]^2 + \frac{2i}{3}\alpha f^{\mu\nu\rho} A_\mu A_\nu A_\rho - \frac{1}{2}\bar{\psi}\Gamma^\mu[A_\mu, \psi]\Big) \tag{A1}$$

about the rescaled fuzzy sphere

$$P_\mu = (1 + \epsilon)\alpha J_\mu. \tag{A2}$$

At tree level, we have

$$\Gamma_{\mathrm{tree}} = \frac{\alpha^4}{g^2} N s(s+1)\Big\{(1+\epsilon)^4 - \frac{4}{3}(1+\epsilon)^3\Big\} \sim \frac{\alpha^4 N^2}{4g^2}\Big\{(1+\epsilon)^4 - \frac{4}{3}(1+\epsilon)^3\Big\}, \tag{A3}$$

where $N = (2s+1)^2$. Then, the one-loop effective action is [3]

$$\Gamma_{\mathrm{1loop}} = \frac{1}{2}\mathrm{Tr}\log\Big(\delta_{\mu\nu} - \frac{\epsilon f^{\mu\nu\rho}}{1+\epsilon}\frac{adJ_\rho}{(adJ)^2}\Big) - \frac{1}{4}\mathrm{Tr}\log\Big\{\Big(1 + \frac{i}{2}\Gamma^{\mu\nu} f^{\mu\nu\rho}\frac{adJ_\rho}{(adJ)^2}\Big)\frac{1+\Gamma^{11}}{2}\Big\}. \tag{A4}$$

To leading order in N, we have

$$\Gamma_{1\text{loop}} = \frac{1}{2}\,\mathrm{Tr}\Big\{\frac{1}{2}\Big(\frac{\epsilon f^{\mu\nu\rho}}{1+\epsilon}\frac{adJ_\rho}{(adJ)^2}\Big)^2\Big\} - \frac{1}{4}\,\mathrm{Tr}\Big\{-\frac{1}{2}\Big(\frac{i}{2}\Gamma^{\mu\nu}f^{\mu\nu\rho}\frac{adJ_\rho}{(adJ)^2}\Big)^2\frac{1+\Gamma^{11}}{2}\Big\} = \Big(2+\frac{\epsilon^2}{(1+\epsilon)^2}\Big)\mathrm{Tr}\frac{1}{(adJ)^2} = N\cdot 2\log 2\cdot\Big(2+\frac{\epsilon^2}{(1+\epsilon)^2}\Big), \tag{A5}$$

where we have used [38]

$$\mathrm{Tr}\,\frac{1}{(adJ)^2} = \mathrm{Tr}\frac{1}{(adJ^{(s)}\otimes 1)^2 + (1\otimes adJ^{(s)})^2} = \sum_{j=1}^{2s}\sum_{j'=1}^{2s}\frac{(2j+1)(2j'+1)}{j(j+1)+j'(j'+1)} \simeq 2N\log 2. \tag{A6}$$

APPENDIX B: LATTICE FORMULATION

Lattice regularization [48] relates commutative $U(N)$ lattice gauge theory on twisted torus to a ‘‘lattice regularization’’ of $U(1)$ NCYM on periodic fuzzy torus. Basically this relation is as a result of the fact that the Morita equivalence holds at lattice level.

For simplicity, we consider the $D = 4U(N)$ gauge theory on a rectangular four-torus with period L. The action is

$$S = -\frac{1}{g^2}\sum_x\sum_{\mu\neq\nu}\mathrm{tr}[U_\mu(x)U_\nu(x+a\hat{\mu})U_\mu(x+a\hat{\nu})^\dagger \times U_\nu(x)^\dagger], \tag{B1}$$

where U_μ are unitary matrices which correspond to $U(N)$ gauge fields. They satisfy twisted boundary condition

$$U_\mu(x+l\hat{\nu}) = \Gamma_\nu U_\mu \Gamma_\nu^\dagger, \tag{B2}$$

where Γ_ν are twist eaters appeared in Sec. II A.

We now introduce a map $\hat{\Delta}(x)$ between lattice fields $U_\mu(x)$ and operators $\hat{U}_\mu$ as

$$\hat{U}_\mu = \sum_x \hat{\Delta}(x)U_\mu(x) \tag{B3}$$

where the mapping function $\hat{\Delta}(x)$ is defined as

$$\hat{\Delta}(x) = \Big(\frac{l}{a}\Big)^N \sum_{m^i\in\mathbb{Z}/n}\Big(\prod_{i=1}^4 e^{ik_a(\hat{x}_a - x_a)}\Big), \tag{B4}$$

where k_a is a momentum $k_a = 2\pi m_a/l$ and n is a integer $n = l/a$.

In order to relate operators $\hat{U}_\mu$ to noncommutative $U(1)$ gauge fields, we now introduce another mapping function $\hat{\Delta}'(x')$ defined as

$$\hat{\Delta}'(x') = \Big(\frac{l'}{\epsilon}\Big)^N e^{-\pi i\sum_{a<b} m_a\Theta_{ab}m_b}\sum_{m^a\in\mathbb{Z}/n'}\Big(\prod_{a=1}^4 e^{ik'_a(\hat{x}_a - x'_a)}\Big), \tag{B5}$$

where $l' = l\sqrt{N}$, $k'_a = 2\pi m_a/l'$, $n' = l'/a$ and

$$\Theta_{ab} = \begin{pmatrix} 0 & \Theta & 0 & 0 \\ -\Theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \Theta \\ 0 & 0 & -\Theta & 0 \end{pmatrix}_{ab}, \qquad \Theta = \frac{1}{\sqrt{N}}. \tag{B6}$$

We have used primed quantities to represent those on a lattice corresponding to Δ'. This $\hat{\Delta}'(x')$ maps the noncommutative lattice fields to operators whose dimensionless noncommutativity parameters is Θ. Because of the twist boundary condition of $U_\mu(x)$ the operator $\hat{U}_\mu$ have another expansion using $\hat{\Delta}'(x')$,

$$\hat{U}_\mu = \sum_{x'}\hat{\Delta}'(x')U'_\mu(x') \tag{B7}$$

where $U'_\mu(x')$ are noncommutative $U(1)$ gauge fields which live in periodic torus whose size is l' and the dimensionless noncommutativity parameter is Θ.

Now we gain a map from $U(N)$ gauge fields $U_\mu(x)$ on a twisted commutative torus to the noncommutative $U(1)$ gauge fields $U'_\mu(x')$ on a periodic fuzzy torus. Indeed the action (B1) is rewritten in terms of $U'_\mu(x')$ as

$$S = -\frac{1}{g'^2}\sum_{x'}\sum_{\mu\neq\nu}\mathrm{tr}[U'_\mu(x')\star U'_\nu(x'+a\hat{\mu}) \star U'_\mu(x'+a\hat{\nu})^\dagger \star U'_\nu(x')^\dagger], \tag{B8}$$

where

$$g'^2 = Ng^2. \tag{B9}$$

The dimensionful noncommutativity parameter, which appears in commutators of coordinates is given by

$$\theta = \Theta\cdot\frac{l'^2}{2\pi} = \frac{l^2\sqrt{N}}{2\pi}. \tag{B10}$$

Now let us consider the limit which leads to fuzzy $\mathbb{R}^4$ with finite value of θ. To fix θ, we have to take

$$l \sim N^{-1/4}, \tag{B11}$$

that is, we have to take infinitely small twisted torus and the model essentially reduces to TEK. Therefore it is plausible that the center symmetry $U(1)$ breaks down. This means that the fuzzy torus collapses and we cannot realize fuzzy $\mathbb{R}^4$ which is expected to appear as a tangent space of the torus.

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