Classical solutions for the Lorentz-violating and *CPT*-even term of the standard model extension

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In this work, we calculate the classical solutions for the electrodynamics stemming from the Lorentz-violating (LV) and *CPT*-even term of the standard model extension. Static and stationary solutions for pointlike and extended charges are obtained from the wave equations by means of the Green method. A dipolar expansion is written for the field strengths. It is explicitly shown that charge and current generate LV first order effects for the magnetic and electric fields, respectively. Using the magnetic field generated by a macroscopic 1*C* charged sphere, we establish a stringent bound for the LV parameter: $\kappa \lesssim 10^{-16}$.

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I. INTRODUCTION

Einstein's principle of relativity sets up the Lorentz covariance as a fundamental symmetry of physics. The establishment of this principle as a truth of nature has been confirmed at a high level of precision by very sensitive experiments involving resonant cavities, masers [1], microwave resonators [2], new versions of the Michelson-Morley experiments [3], and *CPT* probing configurations [4]. The approximate or real exactness of Lorentz symmetry is an important issue with interesting consequences on the Planck scale physics. Indeed, since the demonstration about the possibility of Lorentz and *CPT* spontaneous breaking in the context of string theory [5], Lorentz-violating (LV) effects in the context of low-energy physical systems have been searched as a remanent outcome of Lorentz breakdown in the Planck scale. Such a question is of obvious interest for the development of a quantum theory of gravity. Actually, the main theoretical framework that governs such investigations is the standard model extension (SME) [6], which embodies Lorentz-violating coefficients in all sectors of interaction of the usual standard model. In the context of the SME, many authors have performed valuable contributions in several different respects [7–20].

In this work, we focus on the gauge sector of the standard model extension, whose full Lagrangian is composed of the terms

$$\mathcal{L} = -\frac{1}{4}F_{\alpha\nu}F^{\alpha\nu} - \frac{1}{4}\varepsilon_{\beta\alpha\rho\varphi}V^{\beta}A^{\alpha}F^{\rho\varphi} - \frac{1}{4}W_{\alpha\nu\rho\varphi}F^{\alpha\nu}F^{\rho\varphi} - J_{\alpha}A^{\alpha}. \tag{1}$$

Here, the second term is the well-known *CPT*-odd Carroll-Field-Jackiw term $\varepsilon_{\beta\alpha\rho\varphi}V^{\beta}A^{\alpha}F^{\rho\varphi}$ first proposed in 1990 [9]. The parameter V^{β} stands for the fixed background responsible for Lorentz and *CPT* violation and has mass dimension +1. It was very strongly constrained ($V^{\beta} \leq 10^{-33}$ eV) by birefringence data from the light of distant astronomical systems [9]. Since then the Carroll-Field-Jackiw electrodynamics has been examined in several distinct aspects, addressing consistency and quantization aspects [10], classical solutions [11,12], Cerenkov radiation [13], and induced corrections to the cosmic background radiation [14].

On the other hand, the *CPT*-even term $W_{\alpha\nu\rho\varphi}F^{\alpha\nu}F^{\rho\varphi}$ has not received the same attention, although already examined to some extent [15–20]. The tensor coefficient $W_{\alpha\nu\rho\varphi}$ is dimensionless and has the same symmetries of the Riemann tensor [$W_{\alpha\nu\rho\varphi} = -W_{\nu\alpha\rho\varphi}$, $W_{\alpha\nu\rho\varphi} = -W_{\alpha\nu\varphi\rho}$, $W_{\alpha\nu\rho\varphi} = W_{\rho\varphi\alpha\nu}$] and a double null trace which yields only 19 independent components.

In the present work, we follow the prescription stated in Ref. [15], where the background tensor $W_{\alpha\nu\rho\varphi}$ is written in terms of four 3×3 matrices κ_{DE}, κ_{DB}, κ_{HE}, κ_{HB}, defined as

$$(\kappa_{DE})^{jk} = -2W^{0j0k}, \qquad (\kappa_{HB})^{jk} = \frac{1}{2}\epsilon^{jpq}\epsilon^{klm}W^{pqlm},$$
$$(\kappa_{DB})^{jk} = -(\kappa_{HE})^{kj} = \epsilon^{kpq}W^{0jpq}. \tag{2}$$

The matrices κ_{DE}, κ_{HB} contain together 11 independent components while κ_{DB}, κ_{HE} possess together 8 components, which sums the 19 independent elements of the tensor $W_{\alpha\nu\rho\varphi}$. Such coefficients can be parametrized in terms of four tilde matrices and one trace element, written as suitable combinations of κ_{DE}, κ_{DB}, κ_{HE}, κ_{HB}, namely

$$(\tilde{\kappa}_{e+})^{jk} = \frac{1}{2}(\kappa_{DE} + \kappa_{HB})^{jk},$$
$$(\tilde{\kappa}_{e-})^{jk} = \frac{1}{2}(\kappa_{DE} - \kappa_{HB})^{jk} - \frac{1}{3}\delta^{jk}(\kappa_{DE})^{ii}, \tag{3}$$

$$(\tilde{\kappa}_{o+})^{jk} = \frac{1}{2}(\kappa_{DB} + \kappa_{HE})^{jk},$$
$$(\tilde{\kappa}_{o-})^{jk} = -\frac{1}{2}(\kappa_{DB} - \kappa_{HE})^{jk}, \qquad (\tilde{\kappa}_{\mathrm{tr}})^{jk} = \frac{1}{3}(\kappa_{DE})^{ii}. \tag{4}$$

Ten of the 19 elements of the tensor $W_{\alpha\nu\rho\varphi}$ (belonging to the matrices $\tilde{\kappa}_{e+}$ and $\tilde{\kappa}_{o-}$) are strongly constrained (1 part in 10^{32}) by birefringence data (see Refs. [15–17]). From the 11 independent components of the matrices $\tilde{\kappa}_{e+}$, $\tilde{\kappa}_{e-}$, five are constrained by birefringence. For calculation purposes, we will suppose that the other six nonbirefringent coefficients are null, which is equivalent to choosing $\kappa_{DE} = \kappa_{HB} = 0$ (including $\tilde{\kappa}_{\rm tr} = 0$). On the other hand, the matrices $\tilde{\kappa}_{o-}$ and $\tilde{\kappa}_{o+}$ comprise eight elements, from which five are bounded by birefringence. The three remaining coefficients are our object of investigation in this work. The birefringence limitation over $\tilde{\kappa}_{o-}$ can be read as

$$(\kappa_{DB} - \kappa_{HE}) \leq 10^{-32}, \tag{5}$$

which is compatible with $\kappa_{DB} = \kappa_{HE} \neq 0$, as proposed in Ref. [17]. The conditions $(\kappa_{DB}) = -(\kappa_{HE})^T$ and $\kappa_{DB} = \kappa_{HE}$ imply together that the matrix $\kappa_{DB} = \kappa_{HE}$ is antisymmetric, presenting only three non-null elements (the nonbirefringent ones). These are the only nonvanishing LV coefficients of the tensor $W_{\alpha\nu\rho\varphi}$ to be regarded from now on, and can be expressed in terms of a vector $\kappa^j = \frac{1}{2}\epsilon^{jpq}(\kappa_{DB})^{pq}$. The present approach is equivalent to considering $(\tilde{\kappa}_{e+})^{jk} = (\tilde{\kappa}_{e-})^{jk} = (\tilde{\kappa}_{o-})^{jk} = \tilde{\kappa}_{\rm tr} = 0$, $(\tilde{\kappa}_{o+})^{jk} = (\kappa_{DB})^{jk}$, which means that we are regarding as null the parity-even sector of $W_{\alpha\nu\rho\varphi}$ (due to the assumption $\kappa_{DE} = \kappa_{HB} = 0$), while the parity-odd sector is reduced to three elements. The possibility of adopting different choices of parameters, as it is discussed in Ref. [19], should be mentioned. Nowadays, the κ^j nonbirefringent coefficients are constrained by microwave cavity experiments [15], which impose $\kappa^j \leq 10^{-12}$, and by the absence of vacuum Cerenkov radiation for ultrahigh-energy cosmic rays (UHECRs) [18], which state $\kappa^j < 10^{-17} - 10^{-18}$.

In this work, we aim at evaluating the classical solutions of the Maxwell electrodynamics supplemented by the LV $\boldsymbol{\kappa}$-vector, in an extension to the developments of Ref. [20]. We take as a starting point the modified Maxwell equations and the wave equations for the potentials and field strengths. Such equations show that charges contribute to the magnetic sector and that currents contribute to the electric field. Such contributions are explicitly carried out by means of the Green method, which provides solutions for pointlike and spatially extended sources. The key-point is the expression for the scalar potential, written for a general source $(\rho, \mathbf{j})$. From it, we obtain the electric and magnetic field strength at second order in κ. A dipolar expansion is evaluated for these fields, revealing that the current contributions for the electric field and the charge contributions for the magnetic field are first order ones. We finalize establishing upper bounds for the LV parameter as stringent as $k < 10^{-16}$, a nice value for an Earth based laboratory experiment.

II. CLASSICAL ELECTRODYNAMICS STEMMING FROM THE *CPT*-EVEN TERM

Focusing specifically on the *CPT*-even term ($V^\beta = 0$), the Euler-Lagrange equation leads to the following motion equation:

$$\partial_\nu F^{\nu\alpha} - W^{\alpha\nu\rho\lambda}\partial_\nu F_{\rho\lambda} = J^\alpha, \tag{6}$$

which contains the two modified inhomogeneous Maxwell equations, while the two homogeneous ones come from the Bianchi identity ($\partial_\nu F^{\nu\alpha *} = 0$), with $F^{\alpha\beta *} = \frac{1}{2}\epsilon^{\alpha\beta\lambda\mu}F_{\lambda\mu}$ being the dual tensor. The Maxwell equations

$$\nabla \cdot \mathbf{E} + \boldsymbol{\kappa} \cdot (\nabla \times \mathbf{B}) = +\rho, \tag{7}$$

$$\nabla \times \mathbf{B} - \partial_t(\mathbf{B} \times \boldsymbol{\kappa}) - \partial_t \mathbf{E} + \nabla \times (\mathbf{E} \times \boldsymbol{\kappa}) = \mathbf{j}, \tag{8}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{9}$$

$$\nabla \times \mathbf{E} + \partial_t \mathbf{B} = \mathbf{0}, \tag{10}$$

are the starting point for searching classical solutions. In order to solve such equations, we should achieve wave equations for the vector and scalar potentials and field strengths. Working at the stationary regime, we attain the following equations:

$$\nabla^2 A_0 - \boldsymbol{\kappa} \cdot (\nabla \times \mathbf{B}) = -\rho, \tag{11}$$

$$\nabla^2 \mathbf{A} + [(\boldsymbol{\kappa} \cdot \nabla)\nabla - \boldsymbol{\kappa}\nabla^2]A_0 = -\mathbf{j}. \tag{12}$$

Using $\nabla \cdot \mathbf{A} = 0$, a consequence of the stationary condition on the Lorentz condition ($\partial_\mu A^\mu = 0$), Eq. (11) takes the form

$$\nabla^2 A_0 + \boldsymbol{\kappa} \cdot (\nabla^2 \mathbf{A}) = -\rho. \tag{13}$$

The curl operator, when applied on Eq. (12), implies

$$\nabla^2 \mathbf{B} + (\boldsymbol{\kappa} \times \nabla)\nabla^2 A_0 = -\nabla \times \mathbf{j}. \tag{14}$$

Taking the scalar product of the vector $\boldsymbol{\kappa}$ with the entire expression (12) and replacing it on Eq. (13), we attain a wave equation for the scalar potential, namely

$$[(1 + \boldsymbol{\kappa}^2)\nabla^2 - (\boldsymbol{\kappa} \cdot \nabla)^2]A_0 = -\rho + \boldsymbol{\kappa} \cdot \mathbf{j}. \tag{15}$$

Now, applying the full differential operator $[(1 + \boldsymbol{\kappa}^2)\nabla^2 - (\boldsymbol{\kappa} \cdot \nabla)^2]$ on Eqs. (12) and (14) leads to intricate wave equations for the vector potential and magnetic field strength

$$\begin{aligned}\nabla^2[(1 + \boldsymbol{\kappa}^2)\nabla^2 - (\boldsymbol{\kappa} \cdot \nabla)^2]\mathbf{A} &= [(\boldsymbol{\kappa} \cdot \nabla)\nabla - \boldsymbol{\kappa}\nabla^2][\rho - \boldsymbol{\kappa} \cdot \mathbf{j}] \\ &\quad - [(1 + \boldsymbol{\kappa}^2)\nabla^2 - (\boldsymbol{\kappa} \cdot \nabla)^2]\mathbf{j},\end{aligned} \tag{16}$$

$$\nabla^2[(1+\boldsymbol{\kappa}^2)\nabla^2-(\boldsymbol{\kappa}\cdot\nabla)^2]\mathbf{B} = (\boldsymbol{\kappa}\times\nabla)\nabla^2[\rho-(\boldsymbol{\kappa}\cdot\mathbf{j})] - [(1+\boldsymbol{\kappa}^2)\nabla^2-(\boldsymbol{\kappa}\cdot\nabla)^2]\nabla\times\mathbf{j}. \tag{17}$$

An alternative and simpler relation for the magnetic field can be derived from Eq. (14), which implies

$$\mathbf{B} = \boldsymbol{\kappa}\times\mathbf{E} + \frac{1}{4\pi}\nabla\times\int d^3\mathbf{r}'\,\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \tag{18}$$

which relates the magnetic field with the electric field and the current. The term $-|\mathbf{r}-\mathbf{r}'|^{-1}/4\pi$ is the usual Green function of the Laplacian operator ∇^2. It gives an easy way to evaluate the magnetic field generated by a generic source $(\rho, \mathbf{j})$, once the electric field is known.

All these wave equations reveal that the electric and magnetic sectors are closely entwined in the sense that both charge and current generate both magnetic and electric field strengths. Such connection in this model was discussed in Refs. [17,20] and also appears in the case of the Carroll-Field-Jackiw (CFJ) electrodynamics [9,12] for a pure spacelike background. The difference is that in the present case this connection is really manifest in the solutions for any background configuration, whereas the electric and magnetic CFJ solutions remain uncoupled for the case of a purely timelike background.

A general solution for Eq. (15) can be given by the integral expression

$$A_0(\mathbf{r}) = \int G(\mathbf{r}-\mathbf{r}')[-\rho(\mathbf{r}')+\boldsymbol{\kappa}\cdot\mathbf{j}(\mathbf{r}')]d^3\mathbf{r}', \tag{19}$$

where $G(\mathbf{r}-\mathbf{r}')$ is the associated Green function which fulfills the differential equation

$$[(1+\boldsymbol{\kappa}^2)\nabla^2-(\boldsymbol{\kappa}\cdot\nabla)^2]G(\mathbf{r}-\mathbf{r}') = \delta^3(\mathbf{r}-\mathbf{r}'). \tag{20}$$

In order to achieve $G(\mathbf{r})$, we use the Fourier transform $G(\mathbf{r}-\mathbf{r}') = (2\pi)^{-3}\int d^3\mathbf{p}\tilde{G}(\mathbf{p})\exp[-i(\mathbf{r}-\mathbf{r}')]$, so that $\tilde{G}(\mathbf{p}) = -[\mathbf{p}^2(1+\boldsymbol{\kappa}^2\sin^2\alpha)]^{-1}$, with α being the angle defined by the background vector $(\boldsymbol{\kappa})$ and the vector $\mathbf{p}$, so that $\boldsymbol{\kappa}\cdot\mathbf{p} = \kappa p\cos\alpha$. Here, we need to define the spherical coordinates of the momentum vector $\mathbf{p} = (p, \theta, \phi)$ and the coordinates of the fixed background $\boldsymbol{\kappa} = (\kappa, \theta_1, \phi_1)$. For calculation purposes, we align the vector $(\mathbf{r}-\mathbf{r}')$ with the z-axis, so that θ_1 is the angle defined by the vectors $\boldsymbol{\kappa}$ and $(\mathbf{r}-\mathbf{r}')$ $[\boldsymbol{\kappa}\cdot(\mathbf{r}-\mathbf{r}') = \kappa|\mathbf{r}-\mathbf{r}'|\cos\theta_1]$, θ is the angle defined by the vectors $\mathbf{p}$ and $(\mathbf{r}-\mathbf{r}')$ $[\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}') = p|\mathbf{r}-\mathbf{r}'|\cos\theta]$. In this case, the angle α is given by $\cos\alpha = \cos\theta\cos\theta_1 + \sin\theta\sin\theta_1\cos(\phi-\phi_1)$. The Fourier transform of $\tilde{G}(\mathbf{p})$ cannot be solved exactly, but an explicit solution can be achieved for the case $\boldsymbol{\kappa} \ll 1$, for which it holds $(1+\boldsymbol{\kappa}^2\sin^2\alpha)^{-1} \simeq (1-\boldsymbol{\kappa}^2\sin^2\alpha)$. The Green function then takes the form

$$G(\mathbf{r}-\mathbf{r}') = -\frac{1}{4\pi}\left\{\left(1-\frac{\kappa^2}{2}\right)\frac{1}{|\mathbf{r}-\mathbf{r}'|} + \frac{(\boldsymbol{\kappa}\cdot(\mathbf{r}-\mathbf{r}'))^2}{2|\mathbf{r}-\mathbf{r}'|^3}\right\}. \tag{21}$$

Using the Green function (21) and Eq. (19), the scalar potential due to general sources (at order $\boldsymbol{\kappa}^2$) is

$$A_0(\mathbf{r}) = \frac{1}{4\pi}\left\{c(\kappa)\int d^3\mathbf{r}'\,\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{2}\times\int d^3\mathbf{r}'\,\frac{[\boldsymbol{\kappa}\cdot(\mathbf{r}-\mathbf{r}')]^2}{|\mathbf{r}-\mathbf{r}'|^3}\rho(\mathbf{r}') - \int d^3\mathbf{r}'\,\frac{\boldsymbol{\kappa}\cdot\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right\}, \tag{22}$$

with $c(\kappa) = (1-\kappa^2/2)$. Such expression reveals that the Lorentz-violating charge corrections to A_0 are proportional to $\boldsymbol{\kappa}^2$, while the current corrections are of first order in $\boldsymbol{\kappa}$. With this expression, we may immediately evaluate the scalar potential for a pointlike charge at rest $[\rho(\mathbf{r}') = e\delta(\mathbf{r}')]$ and a pointlike charge at stationary motion with velocity $\mathbf{u}$, $[\mathbf{j}(\mathbf{r}') = e\mathbf{u}\delta(\mathbf{r}')]$. Direct integration yields

$$A_0(\mathbf{r}) = \frac{e}{4\pi}\left\{\frac{c(\kappa)}{r} - \frac{\boldsymbol{\kappa}\cdot\mathbf{u}}{r} - \frac{1}{2}\frac{(\boldsymbol{\kappa}\cdot\mathbf{r})^2}{r^3}\right\}. \tag{23}$$

This potential leads to the following expressions for the electric field of a static and stationary charge:

$$\mathbf{E}(\mathbf{r}) = \frac{e}{4\pi}\left\{c(\kappa)\frac{\mathbf{r}}{r^3} - \frac{3(\boldsymbol{\kappa}\cdot\mathbf{r})^2}{2r^5}\mathbf{r} + \frac{(\boldsymbol{\kappa}\cdot\mathbf{r})}{r^3}\boldsymbol{\kappa}\right\}, \tag{24}$$

$$\mathbf{E}(\mathbf{r}) = -\frac{e}{4\pi}(\boldsymbol{\kappa}\cdot\mathbf{u})\frac{\mathbf{r}}{r^3}, \tag{25}$$

respectively. Here, both fields present a $1/r^2$ decaying behavior. Although the static field (24) decays as $1/r^2$, its behavior is non-Coulombian once the magnitude of the second term changes with direction and the third term points in the $\boldsymbol{\kappa}$-direction. The presence of the coefficient $c(\kappa)$ in the Coulombian term reveals that the LV background also induces a screening in the magnitude of the electric charge. Such effects may be contrasted with the ones induced by the Carroll-Field-Jackiw background (V^β) on the Maxwell theory. Indeed, it is known that the electric field engendered by a static (or moving stationary) charge remains exactly Coulombian for the case of a timelike background $V^\beta = (v_0, 0)$ [12].

The electric field generated by the sources $(\rho, \mathbf{j})$, read off from Eq. (22)

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi}\left\{c(\kappa)\int\frac{\rho(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}' + \frac{1}{2}\nabla\int d^3\mathbf{r}'\,\frac{[\boldsymbol{\kappa}\cdot(\mathbf{r}-\mathbf{r}')]^2}{|\mathbf{r}-\mathbf{r}'|^3}\rho(\mathbf{r}') - \int\frac{\boldsymbol{\kappa}\cdot\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')d^3\mathbf{r}'\right\}. \tag{26}$$

The magnetic field stemming from Eq. (18) is then given by (at $\boldsymbol{\kappa}^2$ order)

$$\mathbf{B}(\mathbf{r}) = \frac{1}{(4\pi)}\left\{\int \frac{\rho(\mathbf{r}')\boldsymbol{\kappa}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}' - \int \frac{\boldsymbol{\kappa}\times(\mathbf{r}-\mathbf{r}')(\boldsymbol{\kappa}\cdot\mathbf{j}(\mathbf{r}'))}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r}' + \nabla\times\int d^3\mathbf{r}'\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right\}. \quad (27)$$

This expression shows that the charges yield a first order LV contribution to the magnetic field while the currents provide only a second order contribution. The last term of the expression above is the usual contribution of the Maxwell theory. For the case of pointlike sources $[\rho(\mathbf{r}') = q\delta(\mathbf{r}'), \mathbf{j} = q\mathbf{u}\delta(\mathbf{r}')]$, the resulting magnetic field (at $\boldsymbol{\kappa}^2$ order) is

$$\mathbf{B}(\mathbf{r}) = \frac{q}{4\pi}\left\{[1-(\boldsymbol{\kappa}\cdot\mathbf{u})]\frac{\boldsymbol{\kappa}\times\mathbf{r}}{r^3} + \frac{\mathbf{u}\times\mathbf{r}}{r^3}\right\}. \quad (28)$$

Such a solution has two components: one pointing in the direction $\boldsymbol{\kappa}\times\mathbf{r}$, other in the direction $\mathbf{u}\times\mathbf{r}$, revealing that the magnetic field is always orthogonal to the position vector $\mathbf{r}$. For a static pointlike charge, the associated magnetic field is

$$\mathbf{B}(\mathbf{r}) = \frac{q}{4\pi}\frac{\boldsymbol{\kappa}\times\mathbf{r}}{r^3}. \quad (29)$$

For a consistency issue, it should be mentioned that the result (28) can be obtained directly from Eq. (17) for pointlike sources. In fact, proposing a Fourier transform expression $B(\mathbf{r}) = (2\pi)^{-3}\int \tilde{B}(\mathbf{p})\exp(-i\mathbf{p}\cdot\mathbf{r})d^3\mathbf{p}$ and replacing it in the full expression (17), we achieve

$$\tilde{B}(\mathbf{p}) = -q\left[i\frac{(1-\boldsymbol{\kappa}\cdot\mathbf{u})\boldsymbol{\kappa}\times\mathbf{p}}{\mathbf{p}^2(1+\boldsymbol{\kappa}^2)-(\boldsymbol{\kappa}\cdot\mathbf{p})^2} + i\frac{\mathbf{u}\times\mathbf{p}}{\mathbf{p}^2}\right], \quad (30)$$

whose Fourier transform, at order κ^2, provides exactly the outcome of Eq. (28).

A. The dipole approximation

In the case of spatially distributed sources, we can work in the dipole approximation $|\mathbf{r}-\mathbf{r}'|^{-1} = r^{-1} + (\mathbf{r}\cdot\mathbf{r}')/r^3$. With it, Eq. (22) provides

$$A_0(\mathbf{r}) = \frac{1}{4\pi}\left\{c(\kappa)\frac{q}{r} - \frac{(\boldsymbol{\kappa}\cdot\mathbf{r})^2}{2r^3}q + \left[\frac{c(\kappa)}{r^3} - \frac{3(\boldsymbol{\kappa}\cdot\mathbf{r})^2}{2r^5}\right](\mathbf{r}\cdot\mathbf{p}) + \frac{(\boldsymbol{\kappa}\cdot\mathbf{r})}{r^3}(\boldsymbol{\kappa}\cdot\mathbf{p}) - \frac{1}{r^3}\mathbf{r}\cdot(\boldsymbol{\kappa}\times\mathbf{m})\right\}. \quad (31)$$

Here, we have used $q = \int\rho(\mathbf{r}')d^3\mathbf{r}'$ as the electric charge, $\mathbf{p} = \int\mathbf{r}'\rho(\mathbf{r}')d^3\mathbf{r}'$ as the electric dipole moment, and $\mathbf{m} = \frac{1}{2}\int\mathbf{r}'\times\mathbf{j}(\mathbf{r}')d^3\mathbf{r}'$ as the magnetic dipole moment associated with the current $\mathbf{j}$ (considering a localized and divergenceless current density).

The corresponding electric field is obtained from (22) via $\mathbf{E} = -\nabla A_0$, or introducing the dipolar approximation directly in (26)

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi}\left\{c(\kappa)\left[\frac{q}{r^3}\mathbf{r} - \frac{\mathbf{p}}{r^3} + \frac{3(\mathbf{r}\cdot\mathbf{p})}{r^5}\mathbf{r}\right] - \frac{3q(\boldsymbol{\kappa}\cdot\mathbf{r})^2}{2r^5}\mathbf{r} - \frac{15(\boldsymbol{\kappa}\cdot\mathbf{r})^2(\mathbf{r}\cdot\mathbf{p})}{2r^7}\mathbf{r} + \frac{3(\boldsymbol{\kappa}\cdot\mathbf{r})(\boldsymbol{\kappa}\cdot\mathbf{p})}{r^5}\mathbf{r} + \frac{3(\boldsymbol{\kappa}\cdot\mathbf{r})^2}{2r^5}\mathbf{p} + \frac{q(\boldsymbol{\kappa}\cdot\mathbf{r})}{r^3}\boldsymbol{\kappa} + \frac{3(\boldsymbol{\kappa}\cdot\mathbf{r})(\mathbf{r}\cdot\mathbf{p})}{r^5}\boldsymbol{\kappa} - \frac{(\boldsymbol{\kappa}\cdot\mathbf{p})}{r^3}\boldsymbol{\kappa} - \frac{3[\mathbf{r}\cdot(\boldsymbol{\kappa}\times\mathbf{m})]}{r^5}\mathbf{r} + \frac{\boldsymbol{\kappa}\times\mathbf{m}}{r^3}\right\}. \quad (32)$$

It exhibits a $1/r^2$ behavior. The first three terms (in brackets) at zeroth order represent the usual Coulombian behavior (the ones of the usual Maxwell theory), whereas the following nine terms represent the non-Coulombian electric character. The last two terms are the corrections stemming from the magnetic moment of the current. All these terms could originate new interesting phenomena potentially observable both at microscopic (atomic) and macroscopic levels.

In the dipole approximation, the general expression (18) reads as

$$\mathbf{B} = \boldsymbol{\kappa}\times\mathbf{E} + \frac{1}{4\pi}\left[\frac{3(\mathbf{m}\cdot\mathbf{r})}{r^5}\mathbf{r} - \frac{\mathbf{m}}{r^3}\right], \quad (33)$$

with the last two terms coming from the usual Maxwell theory.

The magnetic field in the dipole approximation can be obtained directly from Eq. (27) or by using the general expression (33) with (32); thus it amounts to

$$\mathbf{B}(\mathbf{r}) = \frac{1}{4\pi}\left\{\left[\frac{q}{r^3} + \frac{3(\mathbf{r}\cdot\mathbf{p})}{r^5} - \frac{3\mathbf{r}\cdot(\boldsymbol{\kappa}\times\mathbf{m})}{r^5}\right]\boldsymbol{\kappa}\times\mathbf{r} - \frac{\boldsymbol{\kappa}\times\mathbf{p}}{r^3} + \frac{\boldsymbol{\kappa}\times(\boldsymbol{\kappa}\times\mathbf{m})}{r^3} + \left[\frac{3(\mathbf{m}\cdot\mathbf{r})}{r^5}\mathbf{r} - \frac{\mathbf{m}}{r^3}\right]\right\}. \quad (34)$$

The non-Maxwellian terms are induced by the LV background. As already noted, the LV first order effects are induced by the charge distribution.

B. Some applications

Now, we can make some illustrative applications. We begin evaluating the LV (magnetic) contribution to the scalar potential due to a circular ring of current (I_0) of radius R, confined in the x-y plane, described by the following current density $j(\mathbf{r}') = I_0[\delta(\cos\theta')\delta(r'-R)/R]\hat{\mathbf{e}}_{\phi'}$, with $\hat{\mathbf{e}}_{\phi'} = -\sin\phi'\hat{i} + \cos\phi'\hat{j}$. Since the geometry is cylindrically symmetric, we may choose the observation point in the x-z plane ($\phi = 0$) for purposes of calculation. Replacing such current density (with $\rho = 0$) in Eq. (22), we achieve

$$A_0(\mathbf{r}) = -\frac{I_0 R}{(4\pi)}\boldsymbol{\kappa}\cdot\Big[-\int_0^{2\pi}\frac{\sin\phi' d\phi'}{\sqrt{a-b\cos\phi'}}\hat{\imath} + \int_0^{2\pi}\frac{\cos\phi' d\phi'}{\sqrt{a-b\cos\phi'}}\hat{\jmath}\Big], \tag{35}$$

with

$$a=(r^2+R^2), \qquad b=2rR\sin\theta. \tag{36}$$

While the first integral is null, the second integral yields a non-null result

$$A_0(\mathbf{r}) = -\frac{I_0 R}{(4\pi)}\frac{4}{\sqrt{a+b}}\Big[\frac{a}{b}K(\alpha)-\frac{a+b}{b}E(\alpha)\Big]\boldsymbol{\kappa}\cdot\hat{\jmath}, \tag{37}$$

where K and E represent the complete elliptic functions of first and second kind, respectively, with $\alpha=\sqrt{2b/(a+b)}$. The result of Eq. (37) can be expressed as an expansion of the ratio $(b/2a)=[rR\sin\theta/(r^2+R^2)]$

$$A_0(\mathbf{r}) = -\frac{mr\sin\theta}{4\pi(r^2+R^2)^{3/2}}\Big[1+15\frac{R^2r^2\sin^2\theta}{8(r^2+R^2)^2}+\cdots\Big]\boldsymbol{\kappa}\cdot\hat{\jmath}, \tag{38}$$

where $m=\pi R^2 I_0$ is the magnitude of the dipolar moment associated with the current, given as $\mathbf{m}=m\hat{k}$. Now, it is important to show that this result can be reconciled with the dipolar expansion of Eq. (31). Using the identification $mr\sin\theta\hat{\jmath}=\mathbf{m}\times\mathbf{r}$ (valid for the configuration of evaluation), we rewrite the scalar potential as

$$A_0(\mathbf{r}) = \frac{1}{4\pi}\frac{1}{(r^2+R^2)^{3/2}}\Big[1+15\frac{R^2r^2\sin^2\theta}{8(r^2+R^2)^2}+\cdots\Big]\mathbf{r}\cdot(\mathbf{m}\times\boldsymbol{\kappa}), \tag{39}$$

which in the limit $r\gg R$ recovers the behavior predicted in Eq. (31), namely

$$A_0(\mathbf{r}) = \frac{1}{4\pi}\Big[\frac{\mathbf{r}\cdot(\mathbf{m}\times\boldsymbol{\kappa})}{r^3}+\cdots\Big]. \tag{40}$$

In this limit the associated electric field reveals a typical dipolar behavior as well, as can be verified by simple inspection.

Another example is the magnetic field generated by a ring of radius R containing charge (Q), located at plane x-y, whose charge density is read as $\rho(r')=(Q/2\pi R^2)\delta(r'-R)\delta(\cos\theta')$. The magnetic field generated by such charge is given by the first term of Eq. (27)

$$\mathbf{B}(\mathbf{r}) = \frac{1}{(4\pi)}\frac{Q}{2\pi R^2}\times\Big[\int\frac{\boldsymbol{\kappa}\times(\mathbf{r}-\mathbf{r}')\delta(r'-R)\delta(\cos\theta')}{|\mathbf{r}-\mathbf{r}'|^3}d^3r'\Big], \tag{41}$$

which implies

$$\mathbf{B}(\mathbf{r}) = \frac{1}{(4\pi)}\frac{Q}{2\pi}\Big[(\boldsymbol{\kappa}\times\mathbf{r})\int_0^{2\pi}\frac{1}{[a-b\cos\phi']^{3/2}}d\phi' - (\boldsymbol{\kappa}\times\hat{\imath})R\int_0^{2\pi}\frac{\cos\phi'}{[a-b\cos\phi']^{3/2}}d\phi'\Big], \tag{42}$$

with the parameters a, b given by Eq. (36); and it was used in $\mathbf{R}=R(\cos\phi'\hat{\imath}+\sin\phi'\hat{\jmath})$. The corresponding solution is

$$\mathbf{B}(\mathbf{r}) = \frac{1}{(4\pi)}\frac{Q}{2\pi}\frac{4}{\sqrt{a+b}(a-b)}\Big\{(\boldsymbol{\kappa}\times\mathbf{r})E(\alpha)-R(\boldsymbol{\kappa}\times\hat{\imath})\times\Big[\frac{(a-b)}{b}K(\alpha)-\frac{a}{b}E(\alpha)\Big]\Big\}. \tag{43}$$

An expansion in terms of the ratio $[rR\sin\theta/(r^2+R^2)]$ can be performed, implying

$$\mathbf{B}(\mathbf{r}) = \frac{Q}{2\pi^2}\frac{1}{(r^2+R^2)^{3/2}}\Big\{(\boldsymbol{\kappa}\times\mathbf{r})\frac{\pi}{2}\Big[1+\frac{15}{4}\frac{(rR\sin\theta)^2}{(r^2+R^2)^2}\Big] - R(\boldsymbol{\kappa}\times\hat{\imath})\frac{\pi}{2}\Big[-\frac{3}{2}\frac{(rR\sin\theta)}{(r^2+R^2)}-\frac{105}{16}\frac{(rR\sin\theta)^3}{(r^2+R^2)^3}\Big]\Big\}. \tag{44}$$

In the limit $r\gg R$, we have

$$\mathbf{B}(\mathbf{r}) = \frac{Q}{4\pi}\frac{(\boldsymbol{\kappa}\times\mathbf{r})}{r^3}. \tag{45}$$

This outcome coincides with the dipolar expansion (34), once $\mathbf{p}=\int\mathbf{r}'\rho(\mathbf{r}')d^3\mathbf{r}'=0$ for the charge distribution here considered.

Finally, we shall employ the expression for the magnetic field generated by a static charge [see Eq. (29)] to obtain an upper bound on the Lorentz-violating vector. First, we consider the magnetic field created by the electric charge confined in an atomic nucleus (Z). For the sodium element, $Z=11e$ with $e=\sqrt{1/137}$. Evaluating the magnitude of the magnetic field $|\mathbf{B}(\mathbf{r})|=(4\pi)^{-1}Z\kappa/r^2$ at a typical atomic orbital distance ($r=0.75\times10^{-10}$ m), we obtain $|\mathbf{B}(\mathbf{r})|=10^5\kappa$ (eV)^2. Such a field obviously couples with the electron spin, amounting to an energy contribution ($\Delta E=\mu_s\cdot\mathbf{B}$) that may modify the spectral lines, where $\mu_s=g_s(\mu_B/\hbar)\mathbf{S}$ is the spin magnetic momentum and μ_B is the Bohr magneton. Taking $g_s=2$, $S=1/2$, $\mu_B=1.3\times10^{-10}$ (eV)$^{-1}$, we have $\Delta E=1.3\times10^{-5}\kappa$. Regarding that such correction may not be larger than 10^{-10} eV, the following limit is attained: $\kappa<10^{-5}$. For heavier atoms, this limit can be improved to $\kappa<10^{-6}$.

Another case that can provide a better bound consists of a conducting sphere of radius equal to R and endowed with a large electric charge Q. Once the magnetic field from a pointlike charge goes as r^{-2}, according to Eq. (29), a charged sphere should engender a magnetic field proportional to $Q\kappa/r^2$. A $R=0.9$ m sphere charged with $1C$ (maintained in vacuum) generates a magnetic field at $r=1$ m equal to $|\mathbf{B}(\mathbf{r})|=2\times10^4\kappa$ (eV)^2. Remembering that superconducting quantum interference devices are able to detect magnetic field variations as small as 10^{-10} G, an upper limit as stringent as $\kappa < 10^{-16}$ can be, in principle, set up.

III. CONCLUSION

In this work, we have investigated the classical solutions of the Lorentz-violating electrodynamics associated with the *CPT*-even term of the gauge sector of SME. Among the 19 independent components of the tensor $W^{\alpha\nu\rho\varphi}$, we have focused on three components of the parity-odd sector of this tensor, represented by the $\boldsymbol{\kappa}$-vector . The Maxwell equations and wave equations for the potentials and magnetic field were written. The Green method was applied to yield the classical solutions for static and stationary sources. In this way, general solutions for the scalar potential, electric and magnetic fields were constructed for pointlike charge and spatially extended sources. A dipolar expansion was written for the field strengths. It is explicitly shown that charges generate first order effects for the magnetic field while currents imply first order effects for electric fields. Hence, a suitable experiment conceived to constrain the LV parameter should involve one of these two situations. Considering the magnetic field engendered by a macroscopic charged sphere in vacuum, a stringent bound ($\kappa < 10^{-16}$) for the LV coefficient can be stated, the best result for a laboratory based experiment to date.

In an extension of this work, the effects of the six nonbirefringent terms (here taken as null) belonging to the parity-even sector of tensor $W_{\alpha\nu\rho\varphi}$, and comprised by the matrices $\tilde{\kappa}_{e+}$, $\tilde{\kappa}_{e-}$, will be investigated. These terms are contained in the matrix κ_{HB} and in the coefficient $\tilde{\kappa}_{\rm tr}$. We expect that the classical solutions associated with such terms may lead to new effects and upper bounds on these parameters. This work is now in progress.

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