Gravitational radiation in $d > 4$ from effective field theory

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Some years ago, a new powerful technique, known as the classical effective field theory, was proposed to describe classical phenomena in gravitational systems. Here we show how this approach can be useful to investigate theoretically important issues, such as gravitational radiation in any spacetime dimension. In particular, we derive for the first time the Einstein-Infeld-Hoffman Lagrangian and we compute Einstein's quadrupole formula for any number of flat spacetime dimensions.

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I. INTRODUCTION

One of the most conceptually simple and interesting problems in any theory of gravitation concerns the dynamics of two structureless particles, moving under their mutual gravitational interaction. This is the two-body problem [1], which has proven to be fairly simple in Newtonian gravity but a formidable task in general relativity. A clear understanding of its solution allows for tests of Einstein's and alternative theories of gravity [2,3], in the weak and strong field regime. With the advent of high-sensitivity gravitational wave experiments, such as LIGO and others [4], the importance of the two-body problem has exponentially increased. Indeed, one of the main targets for these experiments is the inspiral and coalescence of neutron star or black hole binaries. An accurate solution to this two-body problem is necessary not only to compare theory with experiment but also to improve the chances of detection: the signals expected to impinge on the detectors will be completely buried in noise, and the most powerful way to dig them out, matched filtering [5], requires a very accurate knowledge of theoretical templates. This curious interplay between theoretical and experimental needs has boosted the effort to find accurate solutions to the dynamics of gravitationally interacting, but otherwise isolated bodies in general relativity.

Only recently have long-term stable numerical evolutions of black hole binaries been achieved [6], but these still cover only a very restricted part of the evolution, since they are extremely demanding from a computational point of view. Even if the situation improves dramatically in the near future, there is only a limited amount of computational power and therefore numerical waveforms always cover only a limited amount of time in the dynamics of the two-body problem. Analytical solutions are required, and they serve two very important purposes. First of all, analytical solutions are valuable on their own, as solutions which in some regime are accurate enough and describe well enough the problem at hand. They usually provide physical insight that is absent on numerical calculations. A second important purpose relates to the validation of numerical codes, allowing simultaneously a extension of the numerics by some kind of matching procedure. Analytical solutions to the two-body problem have been studied for many decades now, starting from Einstein himself. The traditionally most powerful technique, the post-Newtonian (PN) formalism [7–9], expands the problem as a series in powers of v/c.

In an impressive tour de force, Goldberger and Rothstein [10–12] and Porto and Rothstein [13–18] and later Kol and Smolkin [19] have shown how to extend and simplify considerably the PN calculations, using methods borrowed from the effective field theory (EFT) approach to quantum field theory. As discussed in [10,19], this EFT of gravity is a *classical* theory, justifying the use of the acronym ClEFT (classical effective field theory) [19]. Although the computations are carried out in terms of Feynman diagrams using the quantum field theory language, no quantum effects are considered. The objects whose physical properties one is interested in are macroscopic, and hence the quantum corrections to the classical observables are negligible. This method is especially well suited to treat the early stages of the binary inspiral problem when the two objects are nonrelativistic. In this problem there are three widely separated scales: the typical internal scale of the objects r_0, the orbital distance r between them, and the radiation wavelength $\ell \sim r/v$, where v is the typical velocity of the system. At the computational level, having an exact description of the system does not seem to be practical nor necessary. The reason is that in gravitational wave physics

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one is only interested in the radiation that reaches the detector, and therefore it is sufficient to have an effective description of the system that captures this information. This is precisely what the ClEFT does: by systematically integrating out short distance scales, one is left with an effective description of the system in terms of the relevant degrees of freedom at the scale we are interested in. For the binary problem, these are radiation gravitons coupled to point particles, and the relevant length scale is ℓ. The main advantage of this approach is that these degrees of freedom are described by a Lagrangian and to obtain the observables one has to compute Feynman diagrams, as in standard quantum field theory. Therefore, the procedure is very systematic.

Another advantage of the ClEFT approach is that it has manifest power counting in the small parameter of the theory, namely, v/c. Therefore, one can systematically compute a given observable to any desired accuracy since one can determine which terms in the Lagrangian and which Feynman diagrams will contribute to that order. This is important from a practical point of view because highly accurate theoretical predictions are required to compare to the putative gravitational wave signals to be detected by the interferometers. The ClEFT approach to gravitational radiation also provides a systematic way to deal with the divergences that plague the PN calculations. These divergences can be thought of UV divergences that arise because one is treating extended objects, like black holes or neutron stars, as being pointlike. In EFT these divergences can be simply dealt with dimensional regularization. Moreover, the finite-size effects can be easily taken into account by including in the Lagrangian higher order operators that are consistent with the symmetries of the problem. The coefficients of these operators are then determined by a matching calculation in the full theory, see [10,12] for a more detailed discussion.

In most realistic situations one expects that the two compact objects in the binary system are spinning. Therefore, from both a theoretical and experimental point of view it is important to incorporate rotation to the ClEFT. This has been done in [13–20]. The resulting EFT of gravity has manifest power counting and one can compute observables using Feynman diagrams. Therefore, it retains the computational power of the original ClEFT of gravity of [10].

Finally, we should point out that the ClEFT approach has also been successfully applied to other problems of interest which involve widely separated length scales. In particular, it has been shown that the ClEFT methods are particularly useful to compute, in a perturbative expansion, the physical parameters of caged black holes [19,21]. In this situation, there are two length scales: the size of the black hole r_0 and the length of the compact circle L, and the small parameter is a power of their ratio.

What we want to show here is that the ClEFT approach can and should be used in other contexts as well, in particular, in calculations of gravitational wave emission in higher dimensional scenarios. Gravitational radiation properties (intensity, distribution, etc.) depend on the dimensionality, curvature, etc., of the model under consideration. Thus, the study of gravitational waves in different higher dimensional scenarios may provide a means to constraint or distinguish different models. Wave propagation in higher dimensions is intrinsically different from the four-dimensional case [22–28]: waves in flat even-dimensional spacetimes propagate on the light cone, just as in our usual four-dimensional world, but waves in odd-dimensional spacetime do not. The extension of the notion and properties of gravitational waves to any even-dimensional spacetimes is almost straightforward. By isolating the most important contribution to the Green's function, we recently computed the field in the wave zone and the quadrupole formula in general even-dimensional spacetimes [27]. This formula provides the simplest, lowest-order contribution to gravitational radiation emission in general situations of physical interest. It is relevant, for instance, for string-motivated braneworld scenarios where string-field gravity plays an important role [29].

Unfortunately, wave propagation in general odd-dimensional spacetimes are much harder to deal with [22,30,31]. In particular, some of the rules we are used to are not valid in these spacetimes. For instance, in odd-dimensional flat spacetimes a propagating wave leaves a "tail" behind. This means that a pulse of gravitational waves (or any other massless field) travels not only along the light cone but also spreads out behind it, and slowly dies off in tails. Progress in the study of gravitational waves has been hampered by this fact.

We will show in this work that the ClEFT approach is able to give us the correct quadrupole formula in flat, even-dimensional spacetimes and in the same stroke extend that formula to odd-dimensional spacetimes. The fundamental reason for this, other than the simplicity of the ClEFT approach is that it works in the momentum space, where the Green's function has the same functional behavior. At the same time, we will also derive for the first time the generalization of the Einstein-Infeld-Hoffman [32] Lagrangian to an arbitrary number of spacetime dimensions.

The plan of the paper is the following. In Sec. II, we start by describing the problem and review the ClEFT approach [10] that we use. In Sec. III, we find the higher dimensional Einstein-Infeld-Hoffman Lagrangian. Finally, in Sec. IV we determine the quadrupole formula in a general flat higher dimensional spacetime. In Appendix A, we derive the Feynman rules for the graviton propagators, graviton vertices and point-particle vertices needed in the main body of the text. In Appendix B, we present some useful relations needed to compute Feynman diagram contributions. We try to be self-contained in our presentation.

We take d to be the number of spacetime dimensions. Newton's constant in d dimensions is represented by G_d

and is related to the gravitational coupling κ_g through (3.4) . We take the $(+, -, \cdots, -)$ signature for the Minkowski spacetime. The speed of light is taken to be $c \equiv 1$.

II. DESCRIPTION OF THE PROBLEM: EFFECTIVE FIELD THEORY APPROACH

A. Length scales in a binary system

We are interested on finding the first PN correction to the gravitational interaction between two bodies, and the quadrupole formula expressing the gravitational energy emitted by a system moving at low velocities. One of the simplest backgrounds to address simultaneously, these issues are a binary system. The interacting bodies can be black holes or neutron stars with typical mass m. Such a system has three length scales. To start with, we have the size $r_0 \propto m$ of the gravitational bodies. Then, we have the typical length separation r between them. Finally, we have the wavelength ℓ of the emitted radiation. Alternatively, we can make use of the typical velocity v of the system. Then, its orbital period is $T \sim r/v$, and its evolution sources the emission of gravitational waves with frequency T^{-1} and wavelength $\ell \sim r/v$ (here and henceforth we take unit light velocity, $c \equiv 1$). If the system's constituents move slowly compared with the speed of light, $v \ll 1$, we clearly have three widely separated length scales,

$$r_0 \ll r \ll \ell. \tag{2.1}$$

In this slow motion condition we can study some important properties of the system using a PN expansion in powers of $v \ll 1$. Alternatively, as we will do, we can follow the ClEFT approach, specially tailored for these problems. Both formalisms consist on systematically solving Einstein's equations with nonrelativistic (NR) sources by taking a power series expansion in the small velocity parameter. The three length scales are not independent. Indeed the Virial theorem for a small motion system governed by an almost Newtonian interaction states that

$$v^2 \sim \frac{\kappa_g^2 m}{r^{d-3}}, \tag{2.2}$$

where κ_g^2 is the gravitational coupling proportional to Newton's constant G_d; see (3.4). This relation between the kinetic energy and the Newtonian potential will be important later since it tells us that $\frac{\kappa_g^2 m}{r^{d-3}}$ also contributes at the same order as v^2 in a velocity expansion.

The gravitational field $g^f_{\mu\nu}$ created by this system is a solution of Einstein's equations, which follow from extremization of the Einstein-Hilbert (EH) action

$$S_{\rm EH}[g_f] = \frac{2}{\kappa_g^2}\int d^dx\sqrt{g_f}R[g_f], \tag{2.3}$$

where $\sqrt{g_f}$ is the determinant of the metric $g^f_{\mu\nu}$ and $R[g_f]$ the corresponding Ricci scalar. As is well-known, this yields a nontrivial nonlinear system of differential equations and no exact solution $g^f_{\mu\nu}$ is known for the two-body problem.

B. Integrate out the internal structure or short length scale physics

In the ClEFT framework we first decompose the full, or exact gravitational field $g^f_{\mu\nu}$ into two components. Take $g^s_{\mu\nu}$ to be the metric that describes the gravitational field in the short length scale region of order r_0, and $g_{\mu\nu}$ the metric encoding a similar information but this time in the region with length scales of order r and larger. Then, within a good approximation, we can decompose the full metric as

$$g^f_{\mu\nu} = g^s_{\mu\nu} + g_{\mu\nu}. \tag{2.4}$$

We can now do a first disentanglement of length scales, namely, we separate the short length scale $g^s_{\mu\nu}$ from the long length scale $g_{\mu\nu}$. Within a traditional effective field approach, this is achieved by integrating out $g^s_{\mu\nu}$ through the path integral

$$\exp(iS_{\rm eff}[g_{\mu\nu}]/\hbar) = \int \mathcal{D}g^s_{\mu\nu}\exp(iS_{\rm EH}[g^f_{\mu\nu}, x]/\hbar), \tag{2.5}$$

where $\hbar$ is Planck's constant. In the classical limit, $\hbar \to 0$, this path integral reduces in the saddle point approximation to the computation around the classical solution. This is the regime of interest for us here, since we want to use EFT techniques to compute purely classical (tree-level) results (for a clear discussion of this issue see [19]). In this classical limit we can then write

$$S_{\rm eff}[g, x_a] = S_{\rm EH}[g] + S_M[g, x_a], \tag{2.6}$$

where

$$\begin{aligned} S_{\rm EH}[g] &= \int d^dxL[g], \\ L[g] &= \frac{2}{\kappa_g^2}\sqrt{g}R[g]; \\ S_M[g, x_a] &= S_{pp}[g, x_a] + \cdots, \\ S_{pp}[g, x_a] &= -\sum_a m_a\int d\tau_a. \end{aligned} \tag{2.7}$$

This effective action describes the motion of the point-particle ensemble in the $g_{\mu\nu}$ background. It preserves the symmetries of the original theory, namely, general coordinate invariance, worldline reparametrization invariance and, since we do not consider internal spin degrees of freedom, $SO(d-1)$ invariance. The black hole or neutron star effective action S_M is to leading order given by the point-particle effective action S_{pp}, where $d\tau_a = \sqrt{g_{\mu\nu}(x_a)dx^\mu_a dx^\nu_a}$ is the proper time along the worldline

x_a^μ of the ath particle. The dots in S_M represent higher order terms that describe nonminimal couplings of the point particles to the spacetime metric and are discussed in detail in [12]. They account for finite-size effects. Therefore they are not relevant to compute the Einstein-Hoffmann-Infeld correction to the Newtonian interaction neither to determine the quadrupole formula we are interested in. Indeed, these quantities are clearly independent of the internal or finite-size structure of the gravitational bodies. They would be essential to address dissipative or absorption processes on the black hole horizon or star's surface [12,17].

Resuming, after the first disentanglement, where we integrated out the internal structure of the gravitational bodies, we have a low energy effective theory of point particles coupled to gravity described by (2.6).

C. Potential and radiation gravitons

We can now proceed to further disentangle the two other length scales of the problem. Since we are in the regime (2.1), i.e., r, $\ell \gg r_0$ we can treat the long length scale metric $g_{\mu\nu}$ perturbatively. That is, for distances much larger than r_0 we are in the weak field regime and we can linearize $g_{\mu\nu}$ around flat spacetime $\eta_{\mu\nu}$

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa_g \delta g_{\mu\nu} = \eta_{\mu\nu} + \kappa_g (H_{\mu\nu} + h_{\mu\nu}), \quad |H_{\mu\nu}|, |h_{\mu\nu}| \ll 1. \tag{2.8}$$

We have decomposed the small perturbations $\delta g_{\mu\nu}$ into two components [10]. The so-called potential graviton $H_{\mu\nu}$ describes the gravitational field for length scales of order of the orbital length r, while $h_{\mu\nu}$ is often denoted as the radiation graviton describing the gravitational field at larger distances of the order of $\ell \sim r/v$. Therefore, the potential graviton $H_{\mu\nu}$ is naturally the mediator of the gravitational interaction between the point particles, i.e., it is responsible for the force that binds the two-body system (or n body). On the other hand, the long wavelength radiation gravitons describe the gravitational waves emitted by the system during its time evolution. They are the ones that can propagate to infinity and be detected by an asymptotic observer.

In the regime of (2.8), a small v expansion of the point-particle action S_{pp} yields

$$S_{pp} = \sum_a m_a \int dx_a^0 \bigg[-\frac{\kappa_g}{2} \delta g_{00} - \kappa_g \delta g_{0i} \mathbf{v}_a^i - \frac{\kappa_g}{4} \delta g_{00} \mathbf{v}_a^2 - \frac{\kappa_g}{2} \delta g_{ij} \mathbf{v}_a^i \mathbf{v}_a^j + \frac{\kappa_g^2}{8} \delta g_{00}^2 + \frac{1}{2} \mathbf{v}_a^2 + \frac{1}{8} \mathbf{v}_a^4 + \cdots \bigg], \tag{2.9}$$

where we defined the velocity vector $\mathbf{v}_a^i = \frac{dx_a^i}{dx_a^0}$, and we kept only the low order terms in the expansion. Higher order terms not shown will not be used. This action describes the nonlinear interactions between the point particles and the gravitational field.

Let us have a closer look at the properties of the two kind of gravitons. The potential gravitons $H_{\mu\nu}$ have momentum that scales as $k^\mu = (k^0, \mathbf{k}) \sim (\frac{v}{r}, \frac{1}{r})$. Indeed, their energy is given by the source's frequency but, since they have length scale r, their spatial momentum is considerably larger, $|\mathbf{k}| \sim \frac{1}{r}$. Therefore, they have spacelike momentum, $k^\mu k_\mu < 0$. They are off shell and cannot contribute to propagating degrees of freedom to infinity. As an important consequence, derivatives of potential gravitons scale as $\partial_\alpha H^{\mu\nu} = (\partial_t H^{\mu\nu}, \partial_i H^{\mu\nu}) \sim (\frac{v}{r}, \frac{1}{r}) H^{\mu\nu}$. That is, in a small velocity expansion the spatial derivative is one order lower in v than the time derivative and gives the leading order contribution. To emphasize this order distinction between time and spatial derivatives it is useful to work with the Fourier transform of $H_{\mu\nu}$,

$$H_{\mu\nu}(x) = \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} H_{\mathbf{k}_{\mu\nu}}(x^0), \qquad \int_{\mathbf{k}} \equiv \int \frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1}}. \tag{2.10}$$

A time derivative acts as $\partial_t H \to \int \partial_t H_{\mathbf{k}}$ with $\partial_t \sim v/r$, but a spatial derivative gets now replaced by a spatial momentum factor, $\partial_i H \to \int \mathbf{k}_i H_{\mathbf{k}}$ with $|\mathbf{k}_i| \sim 1/r$. The advantage of working with the Fourier transform is that the counting of powers of v is now much more explicit: a term with a time derivative $\int \partial_t H_{\mathbf{k}}$ is immediately identified to be one order higher in v than a spatial term $\int \mathbf{k}_i H_{\mathbf{k}}$. This provides a practical "visual" advantage when power counting a long expression.

We now turn to the radiation gravitons. Their momentum scales as $k^\mu = (k^0, \mathbf{k}) \sim (\frac{v}{r}, \frac{v}{r})$, i.e., since they have length scale r/v their spatial momentum has the same scale as their energy. So, these gravitons are on shell, $k^\mu k_\mu = 0$, and propagate at the speed of light to infinity where they can be detected. Derivatives acting on a radiation graviton introduce a power of v/r, $\partial_\alpha h_{\mu\nu} \sim \frac{v}{r} h_{\mu\nu}$, i.e., time and spatial derivatives contribute equally in the power expansion of v, contrary to what happens with the potential gravitons. Not less important, the ClEFT relies on a small velocity expansion. So, when power counting powers of v we must do a multipole expansion of the radiation graviton $h_{\mu\nu}(x)$,[1]

$$h_{\mu\nu}(x^0, \mathbf{x}) = h_{\mu\nu}(x^0, \mathbf{X}) + \delta\mathbf{x}^i \partial_i h_{\mu\nu}|_{x=(x^0,\mathbf{X})} + \tfrac{1}{2} \delta\mathbf{x}^i \delta\mathbf{x}^j \partial_i \partial_j h_{\mu\nu}|_{x=(x^0,\mathbf{X})} + \mathcal{O}(v^3), \tag{2.11}$$

where $\delta\mathbf{x} = \mathbf{x} - \mathbf{X}$ and $\mathbf{X} = (\sum m_a \mathbf{x}_a)/\sum m_a$ is the center of mass of the system (or any other reference point of the particle ensemble). That is, we must do a Taylor expansion of $h_{\mu\nu}(x)$ around the center of mass to consis-

[1] Indices are lowered with the Minkowski metric $\eta_{\mu\nu}$.

tently identify the leading and subleading contributions in the v expansion. Indeed note that δx^i scales as $\delta x^i \sim r$ and the derivatives scale as $\partial_i h_{\mu\nu} \sim v/r$. Therefore each new term in the multipole expansion contributes with an extra power of v relative to the previous Taylor term.

Summarizing, the ingredients to an EFT with a well-defined power counting scheme in powers of v are the decomposition of the gravitational field into the potential and radiation contributions, the expansion of the particle action, the Fourier transform of the potential graviton and finally the multipole expansion of the radiation graviton.

D. Integrate out the intermediate orbital length scale

To derive an EFT that has manifest velocity power counting rules, we now need to integrate out the orbital scale, i.e., the potential modes $H_{\mu\nu}$, by computing the functional integral

$$\exp iS_{\rm NR}[h, x_a]/\hbar) = \int \mathcal{D}H_{\mathbf{k}_{\mu\nu}} \exp iS_{\rm eff}[g, x_a]/\hbar, \quad \text{with} \quad \hbar \to 0, \tag{2.12}$$

where $S_{\rm eff}[g, x_a]$ is defined in (2.6) and the computation is done in the classical limit $\hbar \to 0$. In the process, the radiation graviton is treated as a slowly varying background field. After this operation we are left with a new effective action $S_{\rm NR}[h, x_a]$ that describes the NR or Post-Newtonian approximation to general relativity, in an expansion in powers of v. It encodes the information concerning the gravitational interaction between the point particles, mediated by the potential gravitons, but also the coupling between the emitting system and radiation gravitons.

The explicit operation of integrating out the potential gravitons will be done in the next two sections using the appropriate Feynman diagrams and techniques. The action $S_{\rm NR}[h, x_a]$, to the desired order in v, is given by a sum over Feynman diagrams that must satisfy some rules.

(i) To start with, note that the worldline particles will be represented by a solid line. The point particle's worldlines represent very short wavelengths and thus "infinitely" heavy fields that were integrated out to get $S_{\rm eff}[g, x_a]$. They thus have no associated propagator (roughly, their propagator would be $\frac{1}{k^2+m^2} \sim \frac{1}{m^2}$ since they are heavy fields) and are treated as background nondynamical fields. So in the Feynman diagrams we have no sum over the momentum of the point particles. This justifies why loops that are closed by the particles are not quantum loops but give instead a contribution to the tree-level result. It also justifies that we should only consider diagrams that keep connected when we remove the particle worldlines.

(ii) Next, in the Feynman diagrams, potential graviton propagators can appear only as internal lines but never as external lines. The reason being that potential gravitons have an interaction range of the order of only the orbital distance r. Thus, they mediate the gravitational interaction between the particles but cannot propagate to the asymptotic region.

(iii) Finally, the Feynman diagrams can only contain external radiation graviton lines that propagate to infinity. These gravitons have a long wavelength and do not contribute to the binding force between particles. So, there are no internal radiation graviton lines.

Summarizing, integrating out the nondynamical potential gravitons generates the gravitational interactions between the NR particles. In the functional integral (2.12), the effective action for these interactions arise from diagrams with no external radiation gravitons. This action will be computed in Sec. III. To order $\mathcal{O}(v^2)$ it is the Einstein-Infeld-Hoffmann (EIH) Lagrangian, $L_{\rm EIH}$. On the other hand, the functional integral (2.12) also provides the coupling of radiation gravitons to the particle ensemble through the diagrams with external radiation gravitons. This action $L_{\rm rad}$ will be computed in Sec. IVA.

To compute the Feynman diagram contributions we need the Feynman rules for the graviton propagators, graviton vertices, and point-particle vertices. In Appendix A we give these rules and the details of their computation. To find these Feynman rules we have to introduce the standard gauge fixing and ghost Lagrangian contributions. We will follow the background field method [33,34] that fixes the gauge in such a way that preserves the invariance under diffeomorphisms of the background metric. This gauge fixing scheme guarantees that the obtained action is gauge invariant. So the gauge fixing action satisfying these properties, given in Eq. (A4), must be added to the action in the right-hand side of Eq. (2.12). There is also a ghost field contribution, Eq. (A4), but this only enters in quantum loop corrections, not considered here. Thus, the ghost contribution will not be considered in the functional integral (2.12).

E. Integrate out the long wavelength radiation scale

Having obtained the Lagrangian $L_{\rm rad}$ describing the interaction between radiation and NR particles from $S_{\rm NR}$, we can finally integrate out the radiation gravitons $h_{\mu\nu}$ through the functional integral

$$\exp iS_{\rm QF}[x_a]/\hbar = \int \mathcal{D}h_{\mu\nu} \exp iS_{\rm rad}[h, x_a]/\hbar, \quad \text{with} \quad \hbar \to 0. \tag{2.13}$$

Integrating out $h_{\mu\nu}$ means that we are left with an effective action $S_{\rm QF}[x_a]$, that will be used to compute the quadrupole formula (QF), that depends only on the particle coordinates. Therefore, this integration arises from Feynman diagrams that have no external radiation graviton lines. It yields the desired quadrupole formula. The explicit computation will be done in Sec. IV B.

F. Power counting scheme

The graviton field was decomposed into the potential and radiation components, (2.8). Furthermore, we did a small velocity expansion of the particle action (2.9), we took the Fourier transform (2.10) of the potential graviton, and we did a multipole expansion (2.11) of the radiation field. These operations are key steps in the ClEFT formalism since they provide the arena for a clear identification of power counting rules in the small velocity expansion. Having it at hand we can uniquely assign powers of the velocity parameter v to any Feynman diagram. We therefore can organize systematically the Feynman diagrams in powers of v and compute their contribution to our observable up to the order we wish. Before computing in the next sections the observables we are interested on, it is useful to list these power counting rules.

Let us start with the potential graviton. Its propagator, here represented in short as $\langle H_{\mathbf{k}_{\alpha\beta}} H_{\mathbf{q}_{\mu\nu}}\rangle$, is given by (A13). Time scales as $x^0 \sim r/v$ and potential graviton momentum goes as $|\mathbf{k}| \sim 1/r$. Since a delta function scales as the inverse of its argument, (A13) tells us that the propagator for $H_{\mathbf{k}_{\mu\nu}}$ goes as $\langle H_{\mathbf{k}_{\alpha\beta}} H_{\mathbf{q}_{\mu\nu}}\rangle \sim [\delta(\mathbf{k})][\mathbf{k}^{-2}] \times [\delta(x^0)] \sim r^{d-1} r^2 v/r \sim r^d v$. Therefore, this fixes the scale of the potential graviton as $H_{\mathbf{k}_{\mu\nu}} \sim v^{1/2} r^{d/2}$.

TABLE I. Power counting rules for time coordinate x^0, spacetime derivatives ∂_μ, potential spatial momentum $\mathbf{k}$, potential graviton $H_{\mathbf{k}_{\mu\nu}}$, radiation graviton $h_{\mu\nu}$, and the coupling $\kappa_g m$. Note that after the Fourier transform treatment (2.10) of the potential gravitons, derivatives of any graviton introduce for sure a factor of v/r. When power counting we should have in mind the useful relation, $[\kappa_g] \sim [\kappa_g m]\frac{vr}{L}$.

x^0	∂_μ	$\mathbf{k}$	$H_{\mathbf{k}_{\mu\nu}}$	$h_{\mu\nu}$	$\kappa_g m$
r/v	v/r	$1/r$	$v^{1/2} r^{d/2}$	$(v/r)^{(d-2)/2}$	$L^{1/2} v^{1/2} r^{(d-4)/2}$

Next, consider the radiation graviton whose propagator $\langle h_{\alpha\beta} h_{\mu\nu}\rangle$ is written in (A8). Since the radiation graviton momentum scales as $|k^\mu| \sim v/r$, this propagator scales as $\langle h_{\alpha\beta} h_{\mu\nu}\rangle \sim [d^d k][k^{-2}] \sim (v/r)^d (v/r)^{-2} \sim r^{2-d} v^{d-2}$ and thus the radiation graviton has scale $h_{\mu\nu} \sim (v/r)^{(d-2/2)}$.

Finally, to find the scale of $\kappa_g m$ (where we recall that $\kappa_g^2 \propto G_d$) we introduce the orbital angular momentum,

$$L = mvr, \tag{2.14}$$

and we use the virial relation (2.2) to get $\kappa_g^2 m^2 \sim m v^2 r^{d-3} = L v r^{d-4}$. Thus, $\kappa_g m \sim L^{1/2} v^{1/2} r^{(d-4/2)}$. These power counting rules are summarized in Table I. Furthermore, in Table II we collect the power counting rules for the graviton correlation functions and point-particle vertices that are derived and presented in Appendix A.

Summarizing, an inspection of these power counting rules reveals that tree-level classical results always follow from Feynman diagrams with a power in the angular momentum of L^1 (for the diagrams contributing to the gravitational interaction between bodies) and $L^{1/2}$ (for diagrams describing the interaction of the system with radiation). Quantum loop corrections would correspond to diagrams with extra powers of $\hbar/L \ll 1$. So, the orbital angular momentum is the parameter that counts loops in this classical EFT. Keeping attached to tree-level diagrams we can get our classical observables up to the desired order in the velocity $v \ll 1$, having always in mind that loops closed by a "heavy" particle worldline are not quantum.

The unique power counting scheme providing a straightforward systematic analysis, and the clear representation of the computation provided by the Feynman diagrams are probably the strongest qualities of the gravitational classical EFT formalism. On the top of this, the concepts and techniques of regularization and renormalization are naturally incorporated and can be used to fix divergencies that appear at the classical level. For example we will later used dimensional regularization.

TABLE II. Power counting rules for the radiation graviton propagator $\langle h_{\alpha\beta} h_{\mu\nu}\rangle$, potential graviton propagator $\langle H_{\mathbf{k}_{\alpha\beta}} H_{\mathbf{q}_{\mu\nu}}\rangle$, $\mathcal{O}(v^2)$ correction to the potential graviton propagator $\langle H_{\mathbf{k}_{\alpha\beta}} H_{\mathbf{q}_{\mu\nu}}\rangle_\otimes$, three-potential graviton correlation function $\langle H_{\mathbf{k}_{\alpha\beta}} H_{\mathbf{q}_{\mu\nu}} H_{\mathbf{p}_{\gamma\sigma}}\rangle$, three-radiation-potential graviton correlation function $\langle h H_{\mathbf{k}_{\alpha\beta}} H_{\mathbf{q}_{\mu\nu}}\rangle$, and for the several point-particle vertices $V^{(i)}_{\mu\nu}$ displayed in (A36). These graviton propagators or correlation functions and point-particle vertices are derived and presented in Appendix A.

$\langle h_{\alpha\beta} h_{\mu\nu}\rangle$	$\langle H_{\mathbf{k}_{\alpha\beta}} H_{\mathbf{q}_{\mu\nu}}\rangle$	$\langle H_{\mathbf{k}_{\alpha\beta}} H_{\mathbf{q}_{\mu\nu}}\rangle_\otimes$	$\langle H_{\mathbf{k}_{\alpha\beta}} H_{\mathbf{q}_{\mu\nu}} H_{\mathbf{p}_{\gamma\sigma}}\rangle$	$\langle h H_{\mathbf{k}_{\alpha\beta}} H_{\mathbf{q}_{\mu\nu}}\rangle$
$r^{2-d} v^{d-2}$	$r^d v$	$r^d v^3$	$r^{3d/2} L^{-1/2} v^{7/2}$	$r^d L^{-1/2} v^{(d+3)/2}$
$V^{(1)}_{\mu\nu} H^{\mu\nu}_{\mathbf{k}}$	$V^{(2)}_{\mu\nu} H^{\mu\nu}_{\mathbf{k}}$	$V^{(3)}_{\mu\nu} H^{\mu\nu}_{\mathbf{k}}$	$V^{(4)}_{\mu\nu} H^{\mu\nu}_{\mathbf{k}}$	$V^{(5)}_{\alpha\beta\mu\nu} H^{\alpha\beta}_{\mathbf{k}} H^{\mu\nu}_{\mathbf{k}}$
$L^{1/2} v^0$	$L^{1/2} v$	$L^{1/2} v^2$	$L^{1/2} v^2$	v^2

III. EINSTEIN-INFELD-HOFFMANN LAGRANGIAN IN HIGHER DIMENSIONS

In Newton's gravity theory, perturbations propagate at infinite velocity and only the mass sources the force between gravitational bodies. Einstein realized that not only the rest energy but also kinetic and graviton energies contribute to the gravitational interaction of a system, and that gravitons propagate not instantaneously but at the speed of light. Newton's gravitational force has therefore general relativity corrections that can be organized systematically in a power expansion in the velocity of the interacting bodies. The leading v-dependent correction of order $\mathcal{O}(v^2)$ was first computed by Einstein, Infeld, and Hoffmann in 1938 [32] (in four dimensions) using a PN formalism and reproduced in [10] and later in [20] using the classical EFT approach. In this section we compute, for the first time, the Einstein-Infeld-Hoffmann correction to the gravitational interaction between two bodies in any dimension using the EFT approach introduced in [10].

As discussed previously, the potential gravitons govern the intermediate length scale physics at distances of the order of the orbital length r. Therefore they mediate the gravitational interaction between the NR bodies. The radiation gravitons, being long wavelength fields when compared to the typical distance between the particles, do not mediate this interaction. We thus have to compute the functional integral (2.12) keeping only Feynman diagram contributions with (internal) potential gravitons and no radiation gravitons.

In Subsec. III A, we first compute the $\mathcal{O}(v^0)$ Newton interaction and fix our choice for Newton's constant. Then, in Subsec. III B, we find the Einstein-Infeld-Hoffmann correction.

A. Newtonian Lagrangian

The Feynman diagrams that contribute to leading $\mathcal{O}(Lv^0)$ order are those of Fig. 1. The first diagram has a nonvanishing contribution given by

$$\begin{aligned}\text{Fig.1a} &= V^{\alpha\beta}_{(1)}\langle H_{\mathbf{k}_{\alpha\beta}}(x_2^0)H_{\mathbf{q}_{\mu\nu}}(x_1^0)\rangle V^{\mu\nu}_{(1)}\\ &= \frac{i\kappa_g^2 m_1 m_2}{4}\frac{d-3}{d-2}\int dx_1^0 dx_2^0\,\delta(x_1^0-x_2^0)\\ &\quad\times\int_{\mathbf{k}}\frac{1}{\mathbf{k}^2}e^{-i\mathbf{k}\cdot(\mathrm{x}_1-\mathrm{x}_2)}\\ &= i\int dt\frac{d-3}{d-2}\frac{\Gamma(\frac{d-3}{2})}{16\pi^{(d-1)/2}}\frac{\kappa_g^2 m_1 m_2}{|\mathrm{x}_1(t)-\mathrm{x}_2(t)|^{d-3}},\end{aligned} \tag{3.1}$$

where we used the Feynman rule (A36) for the point-particle vertex $V^{\mu\nu}_{(1)}$ and (A13) for the potential graviton propagator $\langle H_{\mathbf{k}_{\alpha\beta}}H_{\mathbf{q}_{\mu\nu}}\rangle$. We also used relations (B1)–(B3). In the end we did the identification $t \equiv x_1^0$. Note that this diagram scales as Lv^0 since $V_{(1)}\langle H_{\mathbf{k}}H_{\mathbf{q}}\rangle V_{(1)} \sim (L^{1/2}v^0)^2 \sim Lv^0$ (see Table II).

The two last self-energy diagrams in Fig. 1 are pure counterterm and have no physical effect. They renormalize the particle masses and formally vanish after doing dimensional regularization, as pointed out in [10]. For example, diagram 1(b) gives

$$\text{Fig.1b} = (-1)^d\left(\frac{d-3}{d-2}\right)\kappa_g^2 m_1^2\int dx_1^0 dx_2^0\int_{\mathbf{k}}\frac{1}{\mathbf{k}^2}, \tag{3.2}$$

where $\mathbf{k}$ is the momentum circulating in the loop. The last integral in the expression above is divergent and it gives rise to a renormalization of the particle's mass. However, in dimensional regularization it can be formally set to zero by noting that it arises as the $\mathbf{x}\to 0$ limit of the integral

$$\int\frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1}}e^{-i\mathbf{k}\cdot\mathbf{x}}\frac{1}{(\mathbf{k}^2)^\alpha} = \frac{1}{(4\pi)^{(d-1)/2}}\frac{\Gamma(\frac{d-1}{2}-\alpha)}{\Gamma(\alpha)}\times\left(\frac{\mathbf{x}^2}{4}\right)^{\alpha-(d-1)/2}. \tag{3.3}$$

Sending $\mathbf{x}\to 0$ before fixing d shows that this integral vanishes. An obvious equivalent argument also makes Fig.1c = 0.

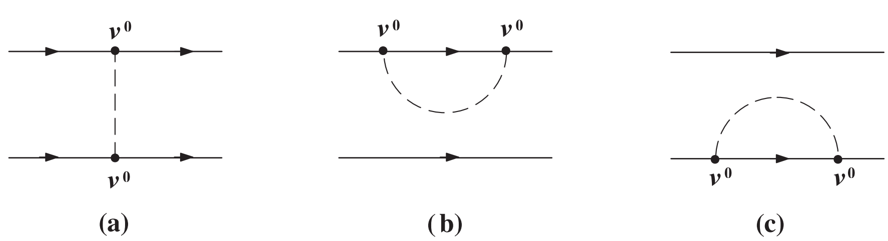


FIG. 1. Feynman diagrams that give the leading Newtonian interaction. They are of order Lv^0. Actually, the self-energy diagrams (b) and (c) give a vanishing contribution after doing dimensional regularization. Solid lines represent particle worldlines and dashed lines the potential graviton propagator (A13). Dots are point-particle vertices. In these diagrams they are $V^{(1)}_{\mu\nu}$ in (A36). Note that loops closed by a heavy particle worldline are not quantum and contribute to tree-level results (i.e., we have no integration over the particle momentum). Quantum loop corrections would correspond to diagrams with extra powers of $\hbar/L \ll 1$, i.e., with graviton loops. (The reader interested in quantum corrections to the gravitational interaction between two bodies can see [38–40] and references therein).

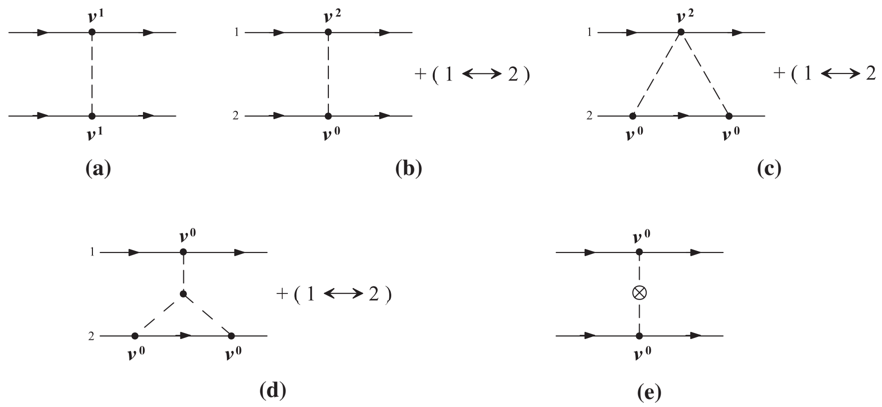


FIG. 2. Feynman diagrams for the Einstein-Infeld-Hoffmann correction to the Newton interaction. They are of order Lv^2. Only (internal) potential gravitons contribute as mediators of the interaction.

At this point we specify our normalization for the gravitational coupling κ_g in terms of the d-dimensional Newton's constant G_d. We choose, as in $d = 4$, to work with the normalization

$$\kappa_g^2 \equiv 32\pi G_d. \tag{3.4}$$

Defining the gravitational coupling in this way (i.e., in a d-independent form) has the advantage that the black hole entropy in d dimensions is always the horizon area divided by $4G_d$. For this reason this is the normalization most commonly chosen (see, e.g.,, [27,35,36]).[2] With this choice, the $\mathcal{O}(v^0)$ Newtonian potential energy reads

$$U_N = \frac{d-3}{d-2}\frac{2\Gamma(\frac{d-3}{2})}{\pi^{(d-3)/2}}\frac{G_d m_1 m_2}{|\mathbf{x}_1(t) - \mathbf{x}_2(t)|^{d-3}}. \tag{3.5}$$

B. Einstein-Infeld-Hoffmann Lagrangian

The Feynman diagrams contributing to the next-to-leading order $\mathcal{O}(Lv^2)$ are those of Fig. 2. Their sum gives the Einstein-Infeld-Hoffmann correction to the Newton interaction.

Using Feynman rules (A13) and (A36), respectively, for the point-particle vertices $V^{\mu\nu}_{(j)}$ $(j = 1, \cdots, 5)$ and potential graviton propagator $\langle H_{\mathbf{k}_{\alpha\beta}} H_{\mathbf{q}_{\mu\nu}}\rangle$, together with relations (B1) and (B3) one gets the contributions from each of the diagrams of Fig. 2.

Starting with the first diagram, one has

$$\begin{aligned}\text{Fig .2a} &= V^{\alpha\beta}_{(2)}\langle H_{\mathbf{k}_{\alpha\beta}}(x_2^0) H_{\mathbf{q}_{\mu\nu}}(x_1^0)\rangle V^{\mu\nu}_{(2)} \\ &= i\int dt \frac{-4\Gamma(\frac{d-3}{2})}{\pi^{(d-3)/2}}\frac{G_d m_1 m_2}{|\mathbf{x_1}(t) - \mathbf{x_2}(t)|^{d-3}}(\mathbf{v}_1\cdot\mathbf{v}_2).\end{aligned} \tag{3.6}$$

[2] Instead, if we prefer to have a Newtonian potential with a d-independent prefactor, $G_d m/r^{d-3}$, we should have chosen the normalization $\kappa_g^2 \equiv \frac{d-2}{d-3}\frac{16\pi^{(d-1/2)}}{\Gamma(\frac{d-3}{2})}G_d$.

For the second diagram, there are two point-particle vertices, namely $V^{\mu\nu}_{(3)}$ and $V^{\mu\nu}_{(4)}$ in (A36), scaling as $L^{1/2}v^2$ (see Table II). Also, there are two mirror contributions resulting from interchanging the two particles, $(1 \leftrightarrow 2)$. Taking this into account we get,

$$\begin{aligned}\text{Fig.2b} &= V^{\alpha\beta}_{(3+4)}\langle H_{\mathbf{k}_{\alpha\beta}}(x_2^0) H_{\mathbf{q}_{\mu\nu}}(x_1^0)\rangle V^{\mu\nu}_{(1)} + (1\leftrightarrow 2) \\ &= i\int dt \frac{d-1}{d-2}\frac{\Gamma(\frac{d-3}{2})}{\pi^{(d-3)/2}}\frac{G_d m_1 m_2}{|\mathbf{x}_1(t) - \mathbf{x}_2(t)|^{d-3}} \\ &\quad\times(\mathbf{v}_1^2 + \mathbf{v}_2^2).\end{aligned} \tag{3.7}$$

To compute the third diagram we introduce a symmetrization factor of $\frac{1}{2}$ to account for the symmetry under the exchange of the two graviton lines, yielding

$$\begin{aligned}\text{Fig.2c} &= \frac{1}{2}V^{\alpha\beta}_{(1)}\langle H_{\mathbf{k}_{\alpha\beta}}(x_2^0) H_{\mathbf{q}_{\gamma\sigma}}(x_1^0)\rangle V^{\gamma\sigma\lambda\eta}_{(5)} \\ &\quad\times\langle H_{\mathbf{q}'_{\lambda\eta}}(x_1^0) H_{\mathbf{p}_{\mu\nu}}(x_2'^0)\rangle V^{\mu\nu}_{(1)} + (1\leftrightarrow 2) \\ &= i\int dt\, 2\left(\frac{d-3}{d-2}\frac{\Gamma(\frac{d-3}{2})}{\pi^{(d-3)/2}}\right)^2\frac{G_d^2 m_1 m_2(m_1+m_2)}{|\mathbf{x}_1(t) - \mathbf{x}_2(t)|^{2(d-3)}}.\end{aligned} \tag{3.8}$$

The fourth diagram requires the Feynman rule for the three-point correlation function for the interaction between three-potential gravitons, $\langle H_{\mathbf{k}_{00}} H_{\mathbf{q}_{00}} H_{\mathbf{p}_{00}}\rangle$. This is computed in Appendix A 5 and given by (A34). The symmetrization factor of $\frac{1}{2}$ is also needed. Using Eqs. (B1) and (B3) and the integrals (B5) we get

$$\begin{aligned}\text{Fig.2d} &= \frac{1}{2}V^{\alpha\beta}_{(1)}\langle H_{\mathbf{k}_{\alpha\beta}}(x_1^0) H_{\mathbf{q}_{\gamma\sigma}}(x_2^0) H_{\mathbf{p}_{\mu\nu}}(x_2'^0)\rangle V^{\gamma\sigma}_{(1)} V^{\mu\nu}_{(1)} \\ &\quad+ (1\leftrightarrow 2) \\ &= -i\int dt\left(\frac{d-3}{d-2}\frac{2\Gamma(\frac{d-3}{2})}{\pi^{(d-3)/2}}\right)^2\frac{G_d^2 m_1 m_2(m_1+m_2)}{|\mathbf{x}_1 - \mathbf{x}_2|^{2(d-3)}}.\end{aligned} \tag{3.9}$$

To get the contribution from the diagram of Fig. 2(e) we need the v^3 correction to the potential graviton propagator, $\langle H_{\mathbf{k}_{\alpha\beta}} H_{\mathbf{q}_{\mu\nu}} \rangle_\otimes$, given in (A15). We get

$$\mathrm{Fig.2e} = V^{\alpha\beta}_{(1)} \langle H_{\mathbf{k}_{\alpha\beta}}(x_2^0) H_{\mathbf{q}_{\mu\nu}}(x_1^0) \rangle_\otimes V^{\mu\nu}_{(1)} = i \int dt \frac{d-3}{d-2} \frac{\Gamma(\frac{d-3}{2})}{\pi^{(d-3)/2}} \frac{G_d m_1 m_2}{|\mathbf{x}_1(t) - \mathbf{x}_2(t)|^{d-3}} \times \left(\mathbf{v}_1 \cdot \mathbf{v}_{12} - (d-3) \frac{(\mathbf{v}_1 \cdot \mathbf{x}_{12})(\mathbf{v}_2 \cdot \mathbf{x}_{12})}{|\mathbf{x}_1 - \mathbf{x}_2|^2} \right), \tag{3.10}$$

where $\mathbf{x}_{12} \equiv \mathbf{x}_1 - \mathbf{x}_2$. To evaluate the result of acting with the ∂_0 operator (appearing in $\langle H_{\mathbf{k}_{\alpha\beta}} H_{\mathbf{q}_{\mu\nu}} \rangle_\otimes$) we used the second relation of (B2) together with $\partial_{t_b} \mathbf{x}_a = \delta^a_b \mathbf{v}_a$.

Summing all contributions (3.6), (3.7), (3.8), (3.9), and (3.10) and adding also the first relativistic correction to the kinetic energy [which is also of order Lv^2, as can be seen from the last term in expansion (2.9)], we finally get

$$L_{\mathrm{EIH}} = \frac{1}{8} \sum_a m_a \mathbf{v}_a^4 - 2 \left(\frac{d-3}{d-2} \frac{\Gamma(\frac{d-3}{2})}{\pi^{(d-3)/2}} \right)^2 \times \frac{G_d^2 m_1 m_2 (m_1 + m_2)}{|\mathbf{x}_1(t) - \mathbf{x}_2(t)|^{2(d-3)}} + \frac{d-3}{d-2} \frac{\Gamma(\frac{d-3}{2})}{\pi^{(d-3)/2}} \times \frac{G_d m_1 m_2}{|\mathbf{x}_1 - \mathbf{x}_2|^{d-3}} \left[\frac{d-1}{d-3} (\mathbf{v}_1^2 + \mathbf{v}_2^2) - \frac{3d-5}{d-3} \mathbf{v}_1 \cdot \mathbf{v}_2 - (d-3) \frac{(\mathbf{v}_1 \cdot \mathbf{x}_{12})(\mathbf{v}_2 \cdot \mathbf{x}_{12})}{|\mathbf{x}_1 - \mathbf{x}_2|^2} \right]. \tag{3.11}$$

L_{EIH} is the Einstein-Infeld-Hoffmann correction to the Newtonian potential in a d-dimensional background. For $d = 4$, (3.11) reduces to the result originally derived in [32] and latter reproduced with the ClEFT in [10,20].

A final comment is in order. An improved version of the ClEFT introduced in [10] was recently proposed [19]. This improved version [19] is probably the most economic computational way to get the desired classical observables when the geometry is stationary, by optimizing the original EFT construction. Its key novelty is the observation that for stationary geometries one can do a Kaluza-Klein dimensional reduction along the time direction. Generically and in short, this introduces scalar and vector fields that replace the original tensorial graviton and whose propagators are simpler. This technique also emphasizes the classical nature of the ClEFT and eliminates unnecessary quantum features (after all we only want tree-level results). This method has been used [20] to recover the EIH correction in four dimensions. Most of our computations were done prior to publication of [19], which is the reason why we did not follow this approach here.

IV. QUADRUPOLE FORMULA IN HIGHER DIMENSIONS

Our next step is to find the Lagrangian L_{rad} describing the coupling between the NR source and the radiation gravitons, up to leading order in v. As we will soon show, this occurs at order $Lv^{(d+1)/2}$ and is done in Sec. IV A. The quadrupole formula is then computed in Sec. IV B.

A. Lagrangian for the interaction between particles and radiation

To find L_{rad} we take the small v expansion (2.9) of the point-particle NR ensemble in the long wavelength regime, i.e., with $\delta g_{\mu\nu} = h_{\mu\nu}$. In this expansion we just consider terms that couple to the graviton $h_{\mu\nu}$. These are the terms that can potentially describe the emission of waves out to infinity. Finally, we take the multipole expansion (2.11) of the radiation gravitons to organize systematically the Feynman diagrams in a ladder of powers of v.

Consider the first such term in (2.9), describing the coupling of h_{00} to the mass monopole $\sum m_a$,

$$- \sum m_a \int dx_a^0 \frac{\kappa_g}{2} h_{00}(x) = - \sum m_a \int dx_a^0 \frac{\kappa_g}{2} \left(h_{00}(x^0, \mathbf{X}) + \delta \mathbf{x}_a^i \partial_i h_{00}|_{(x^0, \mathbf{X})} + \frac{1}{2} \delta \mathbf{x}_a^i \delta \mathbf{x}_a^j \partial_i \partial_j h_{00}|_{(x^0, \mathbf{X})} + \mathcal{O}(v^3) \right). \tag{4.1}$$

Using the power counting rules of Table I, this expression scales as $[dx^0][\kappa_g m][h_{00}] \sim \sqrt{L} v^{(d-3)/2}(1 + v + v^2 + \cdots)$. We are free to work in the center of mass (CM) frame where one has $\mathbf{X} \equiv 0$. Then, the second term in (4.2) vanishes, and in the last term one has $\delta \mathbf{x}^i = \mathbf{x}^i - \mathbf{X}^i = \mathbf{x}^i$. The first term in (4.2) gives the leading order contribution to the coupling between particles and radiation,

$$L^{(a)}_{\mathrm{rad}} = - \frac{\kappa_g}{2} \sum m_a h_{00}(x^0, \mathbf{X}) \to \mathrm{Fig.3a}, \tag{4.2}$$

and corresponds to the contribution coming from the Feynman diagram of Fig. 3(a), that is $\mathrm{Fig.3a} = i \int dt L^{(a)}_{\mathrm{rad}}$. The third term in (4.2) is stored into the $\sqrt{L} v^{(d+1)/2}$ contribution.

The second coupling term in (2.9),

$$- \sum m_a \int dx_a^0 \kappa_g \mathbf{v}_a^i h_{0i}(x) = - \sum m_a \int dx_a^0 \kappa_g \mathbf{v}_a^i (h_{0i}(x^0, \mathbf{X}) + \delta \mathbf{x}_a^j \partial_j h_{0i}|_{(x^0, \mathbf{X})} + \mathcal{O}(v^2)), \tag{4.3}$$

scales as $\sqrt{L} v^{(d-1)/2}(1 + v + \cdots)$. By definition, the first contribution is proportional to the CM momentum,

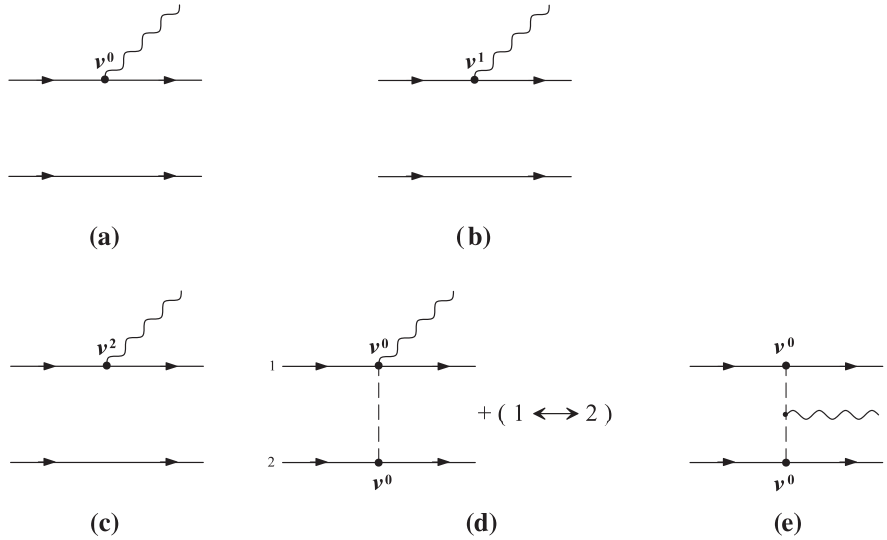


FIG. 3. Feynman diagrams for the coupling between the NR source and the radiation gravitons, up to leading order in v where radiation emission occurs. Diagram (a) is of order $\sqrt{L}v^{(d-3)/2}$, diagram (b) is $\mathcal{O}(\sqrt{L}v^{(d-1)/2})$, and (c)–(e) are order $\sqrt{L}v^{(d+1)/2}$. Only external radiation gravitons, represented by wavy lines, describe the radiated waves.

$\sum m_a \mathbf{v}_a^i \equiv \mathbf{P}_{\rm cm}^i$ and vanishes in the CM frame, that we choose to work on. Thus,

$$L_{\rm rad}^{(b)} = 0 \rightarrow \text{Fig.3b}, \tag{4.4}$$

and the last term in (4.4) is stored into the $\sqrt{L}v^{(d+1)/2}$ contribution. The vanishing of the $\sqrt{L}v^{(d-1)/2}$ contribution reflects the absence of radiation emission of dipole nature in gravity.

The third and fourth terms in (2.9) both scale to leading order as $\sqrt{L}v^{(d+1)/2}$. This corresponds to taking only the leading order term $h_{\mu\nu}(x^0, \mathbf{X})$ in the multipole expansion of $h_{\mu\nu}(x)$. These two terms together with the two terms that have been previously stored yield the following $\sqrt{L}v^{(d+1)/2}$ contribution to the interaction Lagrangian between particles and radiation,

$$L_{\rm rad}^{(c)} = -\frac{\kappa_g}{2} \sum m_a \bigg(\frac{1}{2} \mathbf{v}_a^2 h_{00}(x^0, \mathbf{X}) + \mathbf{v}_a^i \mathbf{v}_a^j h_{ij}(t, \mathbf{X}) + \frac{1}{2} \mathbf{x}_a^i \mathbf{x}_a^j \partial_i \partial_j h_{00}|_{(x^0, \mathbf{X})} + 2 \mathbf{x}_a^i \mathbf{v}_a^j \partial_i h_{0j}|_{(x^0, \mathbf{X})} \bigg) \rightarrow \text{Fig.3c}. \tag{4.5}$$

As we shall soon realize, the leading terms in $L_{\rm rad}$ contributing to radiation emission are of order $\sqrt{L}v^{(d+1)/2}$. Therefore we disregard the other terms in (2.9), all of order higher than $\sqrt{L}v^{(d+1)/2}$. However, other Feynman diagrams contribute at order $\sqrt{L}v^{(d+1)/2}$, namely, the diagrams drawn in Figs. 3(d) and 3(e). They contain both internal potential gravitons and external radiation gravitons.

The Feynman diagram 3(d) is formally the tensorial product of diagram 1(a) with diagram 3(a) and yields the contribution[3]

$$\text{Fig.3d} = [V_{(1)}^{\alpha\beta} \langle H_{\mathbf{k}_{\alpha\beta}}(x_2^0) H_{\mathbf{q}_{\mu\nu}}(x_1^0) \rangle V_{(1)}^{\mu\nu}] \bigg[-\frac{\kappa_g}{2} h_{00}(x^0, \mathbf{X}) \bigg] + (1 \leftrightarrow 2) = i \int dt L_{\rm rad}^{(d)} \rightarrow L_{\rm rad}^{(d)} = -\frac{d-3}{d-2} \frac{\Gamma(\frac{d-3}{2})}{16\pi^{(d-1)/2}} \frac{\kappa_g^3 m_1 m_2}{|\mathbf{x}_1(t) - \mathbf{x}_2(t)|^{d-3}} h_{00}(x^0, \mathbf{X}). \tag{4.6}$$

To find the contribution of the diagram shown in Fig. 3(e) we need the three-point correlation function for the interaction vertex between two potential gravitons and a radiation graviton, $\langle h(x^0, X) H_{\mathbf{k}_{00}}(x_1^0) H_{\mathbf{q}_{00}}(x_2^0) \rangle$. This is computed in Appendix A 4 and given by (A30). Using also the vertex rule for the coupling $V_{(1)}^{\alpha\beta}$ between a particle and a potential graviton we find[4]

[3]Using the power counting rules of Tables I and II, diagram 3(d) indeed contributes as $V_{(1)} \langle H_{\mathbf{k}} H_{\mathbf{q}} \rangle V_{(1)} \kappa_g h_{00} \sim (L^{1/2} v^0)^2 (L^{1/2} v^{1/2} r^{(d-4)/2} \frac{vr}{L})(\frac{v}{r})^{(d-2)/2} \sim \sqrt{L} v^{(d+1)/2}$.

[4]Diagram 3(e) indeed scales as $V_{(1)} \langle h H_{\mathbf{k}} H_{\mathbf{q}} \rangle V_{(1)} \sim (L^{1/2} v^0 / (r^{d/2} v^{1/2}))^2 (r^d L^{-1/2} v^{(d+3)/2}) \sim \sqrt{L} v^{(d+1)/2}$, where we used the rules of Table II.

$$\begin{aligned}\text{Fig.3e} &= V_{(1)}^{\alpha\beta}\langle h(x^0,X)H_{\mathbf{k}_{\alpha\beta}}(x_2^0)H_{\mathbf{q}_{\mu\nu}}(x_1^0)\rangle V_{(1)}^{\mu\nu} \\ &= i\int dt\frac{\kappa_g^3 m_1 m_2}{4}\frac{d-3}{d-2}\int_{\mathbf{k}} e^{-i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}\frac{1}{\mathbf{k}^4}\left[\frac{3}{2}\mathbf{k}^2 h^{00} - \left(\frac{1}{2}\mathbf{k}^2\eta_{ij} + \mathbf{k}_i\mathbf{k}_j\right)h^{ij}\right] \\ &= i\int dt\frac{d-3}{d-2}\frac{\Gamma(\frac{d-3}{2})}{16\pi^{(d-1)/2}}\frac{\kappa_g^3 m_1 m_2}{|\mathbf{x}_{12}(t)|^{d-3}}\left(\frac{3}{2}h^{00} + \frac{d-3}{2}\frac{\mathbf{x}_{12}^i\mathbf{x}_{12}^j}{|\mathbf{x}_{12}(t)|^2}h_{ij}\right) = i\int dt L_{\text{rad}}^{(e)} \to L_{\text{rad}}^{(e)} \\ &= \frac{d-3}{d-2}\frac{3\Gamma(\frac{d-3}{2})}{32\pi^{(d-1)/2}}\frac{\kappa_g^3 m_1 m_2}{|\mathbf{x}_1(t)-\mathbf{x}_2(t)|^{d-3}}h^{00}(x^0,\mathbf{X}) - \frac{\kappa_g}{2}\sum_{a=1,2}m_a\mathbf{x}_a^i\ddot{\mathbf{x}}_a^j h_{ij}(x^0,\mathbf{X}),\end{aligned} \tag{4.7}$$

where $\ddot{\mathbf{x}} \equiv \frac{d^2\mathbf{x}}{dt^2}$. We used the momentum integrals (B3) and (B4) and Newton's laws, $\mathbf{F}_a = m_a\ddot{\mathbf{x}}_a$ and $\mathbf{F} = -\nabla U_N$, with U_N being Newton's potential energy (3.5). Indeed, they allow us to write

$$\begin{aligned}&\frac{(d-3)^2}{d-2}\frac{\Gamma(\frac{d-3}{2})}{32\pi^{(d-1)/2}}\frac{\kappa_g^3 m_1 m_2}{|\mathbf{x}_{12}(t)|^{d-3}}\frac{\mathbf{x}_{12}^i\mathbf{x}_{12}^j}{|\mathbf{x}_{12}(t)|^2}h_{ij} \\ &\quad = -\frac{\kappa_g}{2}(\mathbf{x}_1^i - \mathbf{x}_2^i)(\nabla^j U_N)h_{ij} = -\frac{\kappa_g}{2}\sum_{a=1,2}m_a\mathbf{x}_a^i\ddot{\mathbf{x}}_a^j h_{ij}.\end{aligned} \tag{4.8}$$

Summing the contributions from all the Feynman diagrams of Fig. 3, i.e., adding (4.2), (4.3), (4.4), (4.5), (4.6), and (4.7) we get the interaction Lagrangian between particles and radiation up to order $\sqrt{L}v^{(d+1)/2}$,

$$\begin{aligned}S_{\text{rad}} &= \int dt L_{\text{rad}}, \\ L_{\text{rad}} &= L_{\text{rad}}^{(a)} + L_{\text{rad}}^{(b)} + L_{\text{rad}}^{(c)} + L_{\text{rad}}^{(d)} + L_{\text{rad}}^{(e)} + \cdots.\end{aligned} \tag{4.9}$$

Integrating by parts $L_{\text{rad}}^{(d)}$ and the last term of $L_{\text{rad}}^{(e)}$, we can write (4.9) as

$$\begin{aligned}L_{\text{rad}} = -\frac{\kappa_g}{2}\Bigg[&\left(\sum_a m_a\left(1+\frac{1}{2}\mathbf{v}_a^2\right) - \frac{d-3}{d-2}\frac{2\Gamma(\frac{d-3}{2})}{\pi^{(d-3)/2}}\right. \\ &\left.\times\frac{G_d m_1 m_2}{|\mathbf{x}_1(t)-\mathbf{x}_2(t)|^{d-3}}\right)h^{00} \\ &+ \sum_a m_a(\mathbf{x}_a^i\mathbf{v}_a^j - \mathbf{x}_a^j\mathbf{v}_a^i)\partial_i h_{0j} \\ &- \frac{1}{2}\sum_a m_a\mathbf{x}_a^i\mathbf{x}_a^j(\partial_0\partial_i h_{0j} + \partial_0\partial_j h_{0i} - \partial_0^2 h_{ij} \\ &- \partial_i\partial_j h_{00})\Bigg].\end{aligned} \tag{4.10}$$

This Lagrangian can be written in a more suggestive form. Introducing the orbital angular momentum,

$$\mathbf{L}_k = \epsilon_{ijk}\mathbf{x}_a^i\mathbf{v}_a^j, \tag{4.11}$$

one has $\sum m_a(\mathbf{x}_a^i\mathbf{v}_a^j - \mathbf{x}_a^j\mathbf{v}_a^i)\partial_i h_{0j} = \epsilon^{ijk}\mathbf{L}_k\partial_j h_{0i}$, where we used $\epsilon^{ijk}\epsilon_{nmk} = \delta_m^i\delta_n^j - \delta_n^i\delta_m^j$. To interpret the last term in (4.11), use the definition of the quadrupole moment of a gravitational source,

$$I^{ij} = \sum_a m_a\mathbf{x}_a^i\mathbf{x}_a^j, \tag{4.12}$$

and the Riemann tensor $R_{0i0j}^{(1)}$ of the radiated gravitons [this follows from], taking Minkowski as the background metric),

$$R_{0i0j}^{(1)} = \frac{\kappa_g}{2}(\partial_0\partial_i h_{0j} + \partial_0\partial_j h_{0i} - \partial_0^2 h_{ij} - \partial_i\partial_j h_{00}). \tag{4.13}$$

The Lagrangian describing the coupling of radiation to NR sources can then be written as,

$$\begin{aligned}L_{\text{rad}} = &-\frac{\kappa_g}{2}\left[\sum_a m_a\left(1+\frac{1}{2}\mathbf{v}_a^2\right) - U_N\right]h^{00} \\ &- \frac{\kappa_g}{2}\epsilon^{ijk}\mathbf{L}_k\partial_j h_{0i} - \frac{1}{2}I^{ij}R_{0i0j}^{(1)} + \cdots,\end{aligned} \tag{4.14}$$

where U_N is defined in (3.5). The first term in L_{rad} represents the coupling of h^{00} to the rest mass plus Newtonian energy of the system. It results from the fact that in general relativity, not only the rest mass but also the kinetic and gravitational potential energy contribute to the effective mass of the system. This mass correction or monopole term is conserved at order $\sqrt{L}v^{(d-3)/2}$. Thus, it does not contribute to radiation emission, which is a dissipative effect. The dipole contribution also vanishes as observed in (4.4).

The second term in L_{rad}, represents the coupling of the orbital angular momentum of the system to the radiation gravitons h_{0i}, and is of order $\sqrt{L}v^{(d-1)/2}$. At this order, the orbital angular momentum is conserved and thus this term does not contribute also to radiation emission.

The leading order term describing radiation emission is the third term in L_{rad}, $-\frac{1}{2}I^{ij}R_{0i0j}^{(1)}$, which is of order $\sqrt{L}v^{(d+1)/2}$. Note that this term relevant for the emission process arises from diagram 3(e), which has three-point graviton self-interaction. This clearly shows that the quadrupole gravitational emission is sourced by the nonlinearities of general relativity. Ultimately, this is a consequence of the fact that graviton energies also source the gravitational field in general relativity. In the next subsection we use this radiation term to find the emitted radiation power. As in the $d = 4$ case [10], the Lagrangian (4.14) is manifestly gauge invariant under infinitesimal gauge transformations, showing the power counting scheme in its full glory.

B. Quadrupole formula

We now want to compute the quadrupole formula that gives the total energy per unit time radiated through gravitational waves by an NR source. The total number of gravitons emitted by the slow motion system over the period T is given by

$$\int dE d\Omega \frac{d^2\Gamma}{dE d\Omega} = \frac{2}{T}\,\mathrm{Im} S_{\rm QF}[x_a]. \tag{4.15}$$

Here, $d\Gamma$ is the differential graviton emission rate, and $S_{\rm QF}[x_a]$ is the effective action that follows from integrating out the radiation gravitons through the functional integral (2.13). The imaginary part of this action, $\mathrm{Im}S_{\rm QF}$, gives the quantity we are interested in and is given by the imaginary part of the self-energy Feynman diagram represented in Fig. 4. Only the last term in (4.14) contributes to radiation emission since this is the only term with an imaginary part, to leading order. The self-energy diagram then gives

$$\mathrm{Fig.4} = -\frac{\kappa_g^2}{8}\int dx_1^0 dx_2^0 I^{ij}(x_1^0) I^{km}(x_2^0) \times \langle R^{(1)}_{0i0j}(x_1^0, \mathbf{X}) R^{(1)}_{0k0m}(x_2^0, \mathbf{X})\rangle. \tag{4.16}$$

To find the two-point function of the linearized Riemann tensor we take the definition (4.13) of $R^{(1)}_{0i0j}$ and the radiation graviton propagator (A8), yielding

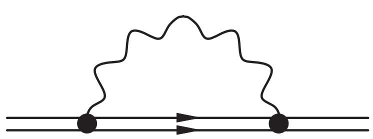


FIG. 4. Self-energy diagram whose imaginary contribution gives the quadrupole formula for the radiated power in the form of gravitational waves by an NR system. The double solid lines represent the NR particles. In the low energy ClEFT describing the coupling of the particles to the radiated gravitons, length scales smaller than the orbital distance cannot be resolved. This is pictorially emphasized in this diagram by drawing the particle lines close to each other. This diagram is to be understood as a standard self-energy diagram: at a certain point in the past the system radiates a graviton [described by (4.14)] which is then absorbed back in the future. The amplitude for this process is complex. Its imaginary part encodes the information on the radiated energy through gravitational waves that propagates to the asymptotic region. Again, this is a tree-level diagram since the loop is closed by heavy particle worldlines whose momenta are not integrated.

$$\langle R^{(1)}_{0i0j}(x_1^0, \mathbf{X}) R^{(1)}_{0k0m}(x_2^0, \mathbf{X})\rangle = \frac{ik_0^4}{k^2 + i\epsilon}\frac{d(d-3)}{4(d+1)(d-2)} \times \left[\frac{1}{2}(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk}) - \frac{1}{d-1}\delta_{ij}\delta_{km}\right], \tag{4.17}$$

where we used that from the on shell condition for the radiation gravitons, $k_0^2 - \mathbf{k}^2 = 0$, follows that $\mathbf{k}_i\mathbf{k}_j = \frac{k_0^2}{d-1}\delta_{ij}$ and $\mathbf{k}_i\mathbf{k}_j\mathbf{k}_k\mathbf{k}_m = \frac{k_0^4}{d^2-1}(\delta_{ij}\delta_{km} + \delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})$. Note that the two-point function in (4.17) is proportional to the projection operator onto symmetric and traceless two-index spatial tensors. So when it acts on the quadrupole moment in (4.16), only the traceless part of the quadrupole moment I^{ij} will effectively contribute. This traceless component is given by

$$Q^{ij} = \sum_a m_a\left(\mathbf{x}_a^i\mathbf{x}_a^j - \frac{1}{d-1}\mathbf{x}_a^2\delta^{ij}\right). \tag{4.18}$$

Introducing its Fourier transform,

$$Q^{ij}(k_0) = \int dx^0 Q^{ij}(x^0) e^{-ik_0x^0}, \tag{4.19}$$

and using (4.16) and (4.17), becomes

$$\mathrm{Fig.4} = -i\frac{\kappa_g^2}{32}\frac{d(d-3)}{(d+1)(d-2)}\int\frac{d^dk}{(2\pi)^d} \times \frac{(k^0)^4}{k^2 + i\epsilon}|Q_{ij}(k_0)|^2, \tag{4.20}$$

with $|Q_{ij}|^2 = Q^{ij}Q^*_{ij}$. To find the imaginary part of (4.20), that gives $\mathrm{Im}S_{\rm QF}$, one uses the well-known relation for the principal value of a function,

$$\int dk^0 \frac{f(k^0)}{k_0^2 - \mathbf{k}^2 + i\epsilon} = P\int dk^0\frac{f(k^0)}{k_0^2 - \mathbf{k}^2} \mp i\pi\int dk^0 f(k^0)\delta(k_0^2 - \mathbf{k}^2),$$
$$\delta(k_0^2 - \mathbf{k}^2) = \frac{1}{2|k^0|}\delta(k^0 - |\mathbf{k}|)\delta(k^0 + |\mathbf{k}|), \tag{4.21}$$

where P stands for the principle value. This relation extracts the imaginary contribution of diagram 4,

$$\mathrm{Im}\, S_{\rm QF} = -\frac{\kappa_g^2}{64}\frac{d(d-3)}{(d+1)(d-2)} \times \int\frac{d^{d-1}k}{(2\pi)^{d-1}}|\mathbf{k}|^3|Q_{ij}(|\mathbf{k}|)|^2. \tag{4.22}$$

Here, the integrand gives the differential graviton emission rate along the evolution of the system. The differential power radiated is then this emission rate times the energy $|\mathbf{k}|$ of the graviton. And the total power radiated during the period $T \to \infty$ is then

$$P = \int dE d\Omega E \frac{d^2\Gamma}{dE d\Omega} = \frac{2E}{T} \text{Im} S_{\text{QF}}[x_a], \qquad (4.23)$$

that is

$$P = -\frac{\kappa_g^2}{32T} \frac{d(d-3)}{(d+1)(d-2)} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} |\mathbf{k}|^4 |Q_{ij}(|\mathbf{k}|)|^2. \qquad (4.24)$$

Exploring the spherical symmetry of the problem one has $\int d^{d-1}\mathbf{k} = \Omega_{d-2} \int dk |\mathbf{k}|^{d-2}$, with $\Omega_{d-2} = \frac{2\pi^{(d-1)/2}}{\Gamma(\frac{d-1}{2})}$ being the area of the unit $(d-2)$ sphere. Moreover, the on shell condition for the graviton momentum allows one to make the replacement $|\mathbf{k}| = -k^0$. Using $k^0 \equiv \omega$, and the normalization (3.4) for the gravitational coupling, the energy radiated per unit of frequency finally reads

$$\frac{dE}{d\omega} = G_d \frac{2^{2-d}\pi^{-(d-5)/2} d(d-3)}{(d-2)(d+1)\Gamma[\frac{d-1}{2}]} \omega^{d+2} |Q_{ij}(\omega)|^2. \qquad (4.25)$$

This is the celebrated quadrupole formula, valid for any spacetime dimension $d > 3$. For $d = 4$ it was originally computed by Einstein and reproduced using the EFT formalism in [10,20]. For even d, our formula agrees with the expression first computed in [27] using a standard Green's function formalism. Using the ClEFT approach we have shown, for the first time, that the formula is also valid for odd d-dimensional spacetimes. This was an open problem up to now [27] and this is thus one of our main results.

V. RELATION WITH PREVIOUS WORK

The reason for the difficulties to derive the quadrupole formula in the traditional approach is the following [27,28]. Wave propagation of massless fields is considerably different depending on whether the number of spacetime dimensions d is even or odd. This follows from the fact that Green's function has support only on the light cone in the former case, but also inside of it in the latter background. Indeed, in even d the Green's function is roughly [27] $G^{\text{ret}} \sim \partial_r^{(d-4)/2}[r^{-1}\delta(t-r)]$ (where r stands for radial distance and t for time), while in odd d one has $G^{\text{ret}} \sim \partial_r^{(d-3)/2}[\frac{\Theta(t-r)}{\sqrt{t^2-r^2}}]$ [where $\Theta(x)$ is the step function]. As a consequence, wave tails or wakes develop in odd dimensions [28]. Now, traditional methods to compute the quadrupole formula require solving the linearized Einstein field equations in the harmonic gauge, $\Box h_{\mu\nu} = 16\pi G_d S_{\mu\nu}$, where $\Box$ is the d-dimensional Laplacian, and $S_{\mu\nu} = T_{\mu\nu} - \frac{1}{d-2}\eta_{\mu\nu}T$ encodes the information on the energy-momentum tensor of the NR source. The general solution of this inhomogeneous differential equation can be expressed with the help of the retarded Green's function $G^{\text{ret}}(t-t', \mathbf{x}-\mathbf{x}')$, as

$$h_{\mu\nu}(t, \mathbf{x}) = 16\pi G_d \int dt' \int d^{d-1}\mathbf{x}' S_{\mu\nu}(t', \mathbf{x}') \times G^{\text{ret}}(t-t', \mathbf{x}-\mathbf{x}'), \qquad (4.26)$$

plus any solution to the homogeneous equation. For even d dimensions, we have a trivial integral over a delta function. However, in odd d, the structure of the Green's function displayed above makes the integral in (4.26) highly nontrivial, and so far this integration has not performed.

In a forthcoming publication [37] we will show that the quadrupole formula (4.25) can also be reproduced using a more traditional Green's function formalism. The key observation will be that the Green's function in *momentum* space has the same functional form for both even and odd spacetimes. So, carrying the calculations in momentum space instead of in the position space (as is standard) allows to rederive (4.25). We will further discuss in detail the differences in even and odd d spacetimes, and explore (4.25).

APPENDIX A: FEYNMAN RULES FOR PROPAGATORS AND VERTICES

To compute the Feynman diagram contributions of Figs. 1–3 we need Feynman rules for the graviton propagators, graviton vertices, and point-particle vertices. In this appendix we give these rules and details of their computation. The derivation is self-contained. We adopt the background field method introduced by Dewitt [33] in 1967 and fully developed by t' Hooft and Veltman [34].

We start by considering metric perturbations $\delta g_{\mu\nu}$ around the unperturbed background $g^{(0)}_{\mu\nu}$, up to the order we will be interested,[5]

$$\begin{aligned} g_{\mu\nu} &= g^{(0)}_{\mu\nu} + \kappa_g \delta g_{\mu\nu}, \\ g^{\mu\nu} &= g^{\mu\nu}_{(0)} - \kappa_g \delta g^{\mu\nu} + \kappa_g^2 \delta g^{\mu}{}_{\alpha} \delta g^{\alpha\nu} + \mathcal{O}(\delta g^3), \\ \sqrt{g} &= \sqrt{g_{(0)}} e^{1/2\,\text{Tr}[\ln(\delta^\mu{}_\nu + \kappa_g \delta g^\mu{}_\nu)]} \\ &= \sqrt{g_{(0)}} \Big[1 + \frac{\kappa_g}{2}\delta g - \frac{\kappa_g^2}{4}\Big(\delta g_{\alpha\beta}\delta g^{\alpha\beta} - \frac{1}{2}\delta g^2\Big)\Big] \\ &\quad + \mathcal{O}(\delta g^3), \end{aligned} \qquad (A1)$$

where we used $g^{\mu\gamma}g_{\gamma\nu} = \delta^\mu{}_\nu$, $\delta g = g^{\mu\nu}_{(0)}\delta g_{\mu\nu}$, $\det M = e^{\text{Tr}[\ln M]}$, and the Taylor expansion for $\ln(1+x)$ and e^x.

The metric perturbation (A1) naturally induces perturbations on the affine connections, Riemann, and Ricci tensors. The perturbations introduced in these tensors can be found in [34] and a nice reference to find the details of their computation is [38]. Here we present the first order perturbation in the Riemann tensor, because we need this quantity to get (4.13),

[5]We use the $(+, -, \cdots, -)$ signature so $\sqrt{g} = \sqrt{\pm \det g_{\mu\nu}}$ for odd and even d, respectively.

$$ {}^{(1)}R^{\mu}_{\nu\alpha\beta} = \frac{\kappa_g}{2}(\nabla_\alpha\nabla_\nu\delta g^{\mu}{}_{\beta} - \nabla_\beta\nabla_\nu\delta g^{\mu}{}_{\alpha} - \nabla_\alpha\nabla^\mu\delta g_{\nu\beta} + \nabla_\beta\nabla^\mu\delta g_{\nu\alpha} + {}^{(0)}R^{\mu}_{\gamma\alpha\beta}\delta g^{\gamma}_{\nu} + {}^{(0)}R^{\gamma}_{\nu\beta\alpha}\delta g^{\mu}{}_{\gamma}). \tag{A2} $$

It also induces perturbations on the Ricci scalar. Expanding the Einstein-Hilbert Lagrangian (2.7) in powers of the gravitational field $\delta g_{\mu\nu}$ one finds that $L[g] = L^{(0)} + L^{(1)} + L^{(2)} + \mathcal{O}(\delta g^3)$ with: $L^{(0)} = 2R^{(0)}/\kappa_g^2$, $L^{(1)} = 0$ (due to the unperturbed Einstein's equations), and

$$ L^{(2)} = \sqrt{g_{(0)}}\bigg[\tfrac{1}{2}\Big(\tfrac{1}{2}\delta g^2 - \delta g^{\mu}{}_{\nu}\delta g^{\nu}{}_{\mu}\Big)R^{(0)} + (2\delta g^{\mu}{}_{\alpha}\delta g^{\alpha\nu} - \delta g\delta g^{\mu\nu})R^{(0)}_{\mu\nu} - \tfrac{1}{2}(\nabla_\alpha\delta g)(\nabla^\alpha\delta g) + (\nabla_\mu\delta g)(\nabla_\nu\delta g^{\mu\nu}) + \tfrac{1}{2}(\nabla_\alpha\delta g_{\mu\nu})(\nabla^\alpha\delta g^{\mu\nu}) - (\nabla_\alpha\delta g^{\mu\nu})(\nabla_\mu\delta g^{\alpha}{}_{\nu})\bigg], \tag{A3} $$

after partial integrations (PI) and moduli total divergencies (MTD) in the action integral.

At this point we have to choose the gauge. The background field method adopts the viewpoint in which the background unperturbed metric $g^{\mu\nu}_{(0)}$ is left invariant under diffeomorphism transformations, $x^\mu \to x^\mu - \xi^\mu$ (where ξ^μ infinitesimal vector field), while the metric perturbations transform as $\delta g_{\mu\nu} \to \delta g_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu$ (where ∇_μ is the covariant derivative with respect to $g^{(0)}_{\mu\nu}$), hence its name. Choosing to work in the harmonic gauge, $\nabla_\alpha\delta g^{\alpha}{}_{\mu} - \frac{1}{2}\nabla_\mu\delta g = 0$, one then has the gauge fixing Lagrangian and associated ghost Lagrangian [34],

$$ L_{\rm GF} = \sqrt{g_{(0)}}\Big(\nabla_\alpha\delta g^{\alpha}{}_{\mu} - \tfrac{1}{2}\nabla_\mu\delta g\Big)\Big(\nabla_\alpha\delta g^{\alpha\mu} - \tfrac{1}{2}\nabla^\mu\delta g\Big), $$
$$ L_{\rm ghost} = \sqrt{\eta}\,\eta^{*\mu}(\nabla_\alpha\nabla^\alpha\eta_{\mu\nu} - R^{(0)}_{\mu\nu})\eta^\nu, \tag{A4} $$

where η^μ is the fermionic ghost field (GF). The ghost contribution would be important only if we were interested in computing quantum loop corrections. Since we only want tree-level results we will make no further reference to it.

Defining $L_{\delta g^2} \equiv L^{(2)} + L_{\rm GF}$ one then has

$$ L_{\delta g^2} = \sqrt{g_{(0)}}[\tfrac{1}{2}(\tfrac{1}{2}\delta g^2 - \delta g^{\mu}{}_{\nu}\delta g^{\nu}{}_{\mu})R^{(0)} + (2\delta g^{\mu}{}_{\alpha}\delta g^{\alpha\nu} - \delta g\delta g^{\mu\nu})R^{(0)}_{\mu\nu} + \tfrac{1}{2}(\nabla_\alpha\delta g_{\mu\nu})(\nabla^\alpha\delta g^{\mu\nu}) - (\nabla_\alpha\delta g^{\mu\nu})(\nabla_\mu\delta g^{\alpha}{}_{\nu}) - \tfrac{1}{4}(\nabla_\alpha\delta g)(\nabla^\alpha\delta g) + (\nabla_\alpha\delta g^{\alpha}{}_{\mu})(\nabla_\nu\delta g^{\nu\mu})], \tag{A5} $$

where the covariant derivatives ∇_μ are taken with respect to the background unperturbed metric $g^{\mu\nu}_{(0)}$.

At this point, as motivated in the discussion before (2.8), we take the unperturbed background to be Minkowski spacetime, $g^{(0)}_{\mu\nu} \equiv \eta_{\mu\nu}$. After PI and MTD we then have

$$ L_{\delta g^2} = \tfrac{1}{2}(\partial_\alpha\delta g_{\mu\nu})\partial^\alpha\delta g^{\mu\nu} - \tfrac{1}{4}(\partial_\alpha\delta g)\partial^\alpha\delta g. \tag{A6} $$

1. Radiation graviton propagator

So far, our discussion is independent on the length scale of the gravitational perturbation. We now take the graviton perturbation to be a long wavelength radiation graviton, i.e., we make the replacement $\delta g_{\mu\nu} \to h_{\mu\nu}$. Then, again after PI and MTD, (A6) reads

$$ L_{h^2} = -\tfrac{1}{2}h_{\alpha\beta}\mathcal{D}_h^{\alpha\beta\mu\nu}h_{\mu\nu}, \quad \text{with} \quad \mathcal{D}_h^{\alpha\beta\mu\nu} = \tfrac{1}{2}(\eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\beta}\eta^{\mu\nu})\partial_\lambda\partial^\lambda, \tag{A7} $$

and the associated action is $S_{h^2} = \int d^dx L_{h^2}$.

To find the graviton propagator we have to get the symmetric inverse of the bilinear operator $\mathcal{D}_h^{\alpha\beta\mu\nu}$. This amounts to find the inverse tensor $P_{\gamma\sigma\alpha\beta}$ that multiplied by the tensorial part of $\mathcal{D}_h^{\alpha\beta\mu\nu}$ gives the symmetric identity $I_{\gamma\sigma}{}^{\mu\nu}$ defined in (A20), and to find the inverse of the differential operator $\partial_\lambda\partial^\lambda$. This is simply the Feynman propagator for a massless boson, $D_F(x-y)$. The Feynman rule for the radiation graviton propagator is then

αβ μν

$$ = \langle h_{\alpha\beta}(x)h_{\mu\nu}(y)\rangle = D_F(x-y)\,P_{\alpha\beta\mu\nu}\,, \tag{A8} $$

with

$$ D_F(x-y) = \int\frac{d^dk}{(2\pi)^d}\frac{i}{k^2+i\epsilon}e^{-ik\cdot(x-y)}, \quad P_{\alpha\beta\mu\nu} = \frac{1}{2}\Big(\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\alpha\nu}\eta_{\beta\mu} - \frac{2}{d-2}\eta_{\alpha\beta}\eta_{\mu\nu}\Big). \tag{A9} $$

This propagator has the power counting scale $\langle h_{\alpha\beta}h_{\mu\nu}\rangle \sim r^{2-d}v^{d-2}$, as shown in Sec. II F.

2. Potential graviton propagator: Correction to the potential propagator

To find the potential graviton propagator we go back to (A6). This time we take the graviton perturbation to be a potential graviton, i.e., we do the replacement $\delta g_{\mu\nu} \to H_{\mu\nu}$,

$$ L_{H^2} = \tfrac{1}{2}(\partial_\alpha H_{\mu\nu})\partial^\alpha H^{\mu\nu} - \tfrac{1}{4}(\partial_\alpha H)\partial^\alpha H. \tag{A10} $$

Introducing the Fourier transform (2.10) of $H_{\mu\nu}$, and using the integral representation of the delta function (B1) one gets

$$ L_{H^2} = -\frac{1}{2}\int_{\mathbf{k}}\Big[\mathbf{k}^2 H_{\mathbf{k}_{\mu\nu}}H^{\mu\nu}_{-\mathbf{k}} - \frac{\mathbf{k}^2}{2}H_{\mathbf{k}}H_{-\mathbf{k}}\Big] - \frac{1}{2}\times\int_{\mathbf{k}}\Big(-\partial_0 H_{\mathbf{k}_{\mu\nu}}\partial^0 H^{\mu\nu}_{-\mathbf{k}} + \frac{1}{2}\partial_0 H_{\mathbf{k}}\partial^0 H_{-\mathbf{k}}\Big), \tag{A11} $$

where $H_{\mathbf{k}} \equiv H^{\mu}_{\mathbf{k}\mu}$, and the associated action is $S_{H^2} = \int dx^0 L_{H^2}$. The terms in the last curved brackets are suppressed relative to the square brackets terms by a power of v^2. They are treated perturbatively, as operator insertions, in correlation functions (see discussion below). So, taking by now only the leading contribution of (A11), it can be written as

$$L_{H^2} = -\frac{1}{2}\int_{\mathbf{k}} H_{-\mathbf{k}_{\alpha\beta}} \mathcal{D}_H^{\alpha\beta\mu\nu} H_{\mathbf{k}_{\mu\nu}}, \quad \text{with} \quad \mathcal{D}_H^{\alpha\beta\mu\nu} = \frac{1}{2}(\eta^{\alpha\mu}\eta^{\beta\nu} + \eta^{\alpha\nu}\eta^{\beta\mu} - \eta^{\alpha\beta}\eta^{\mu\nu})\mathbf{k}^2. \tag{A12}$$

To find the potential graviton propagator we have to get the symmetric inverse of the bilinear operator $\mathcal{D}_H^{\alpha\beta\mu\nu}$. The inverse of its tensorial part is $P_{\alpha\beta\mu\nu}$, and the inverse of its momentum part is simply $\mathbf{k}^{-2}$. The Feynman rule for the potential graviton propagator is then

$$\alpha\beta - - - - - \mu\nu = \langle H_{\mathbf{k}\alpha\beta}(x^0) H_{\mathbf{q}\mu\nu}(x'^0)\rangle = -(2\pi)^{d-1}\delta(\mathbf{k}+\mathbf{q})\frac{i}{\mathbf{k}^2} \times \delta(x^0 - x'^0) P_{\alpha\beta\mu\nu}. \tag{A13}$$

with $P_{\alpha\beta\mu\nu}$ defined in (A9). As shown in Sec. II F, the power counting scale for this propagator is $\langle H_{\mathbf{k}_{\alpha\beta}} H_{\mathbf{q}_{\mu\nu}}\rangle \sim r^d v$.

To compute diagram 2(e) in the main body of the text we need the next-to-leading order correction to the potential graviton propagator. This accounts for the contribution of the curved brackets terms in (A11) that we neglected to get the leading propagator (A13). The quickest way to get this propagator correction is to note that the graviton propagator is proportional to $\frac{1}{k^2} = \frac{1}{k_0^2 - \mathbf{k}^2}$. For a potential graviton, and in the small v limit, one has $\frac{k_0^2}{\mathbf{k}^2} = v^2 \ll 1$ which allows the Taylor expansion,

$$\frac{1}{k_0^2 - \mathbf{k}^2} = -\frac{1}{\mathbf{k}^2}\left(1 + \frac{k_0^2}{\mathbf{k}^2} + \mathcal{O}(v^4)\right). \tag{A14}$$

Since $k^0 \leftrightarrow \partial_0$ the potential graviton propagator correction contributing as $r^d v^3$ corresponds to insert the operator ∂_0^2 (which we denote with the subscript $\otimes$) in the correlation function (A13), yielding

$$\alpha\beta - - \otimes - - \mu\nu = \langle H_{\mathbf{k}\alpha\beta}(x^0) H_{\mathbf{q}\mu\nu}(x'^0)\rangle_{\otimes} = -(2\pi)^{d-1}\delta(\mathbf{k}+\mathbf{q})\frac{i\partial_0^2}{\mathbf{k}^2} \times \delta(x^0 - x'^0) P_{\alpha\beta\mu\nu}. \tag{A15}$$

3. Three-radiation graviton vertex

To get the three-radiation graviton vertex using the background field method we have to go back to (A5) and expand the background field $g^{\mu\nu}_{(0)}$ up to first order,

$$g^{(0)}_{\mu\nu} = \eta_{\mu\nu} + \kappa_g \bar{h}_{\mu\nu}, \tag{A16}$$

where the background metric $\eta_{\mu\nu}$ is taken to be Minkowski spacetime and $\bar{h}_{\mu\nu}$ represents the new small perturbations around it. Of course, (A16) also induces perturbations on the metric determinant $\sqrt{g_{(0)}}$, affine connections, Ricci tensor $R^{(0)}_{\mu\nu}$ and Ricci scalar $R^{(0)}$ that appear in (A16).

The result of this expansion is the Lagrangian for the three-graviton interaction,

$$S_{\bar{h}h^2} = \int d^dx L_{\bar{h}h^2}, \qquad L_{\bar{h}h^2}(x) = -\frac{\kappa_g}{2}\bar{h}^{\mu\nu}(x) T^{h^2}_{\mu\nu}(x), \tag{A17}$$

with

$$\begin{aligned} T^{h^2}_{\mu\nu}(x) = & -h^{\alpha\beta}\partial_\mu\partial_\nu h_{\alpha\beta} + \tfrac{1}{2}h\partial_\mu\partial_\nu h + [\tfrac{1}{4}\partial_\mu\partial_\nu - \tfrac{3}{8}\eta_{\mu\nu}\partial^2](h^2 - 2h^{\alpha\beta}h_{\alpha\beta}) - \partial^2(h_{\alpha\mu}h^{\alpha}{}_{\nu} - hh_{\mu\nu}) \\ & - [\partial_\alpha\partial_\mu(hh_\nu{}^\alpha) + \partial_\alpha\partial_\nu(hh_\mu{}^\alpha)] + 2\partial_\alpha\partial_\beta(h^\alpha{}_\mu h^\beta{}_\nu - h^{\alpha\beta}h_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}h^{\alpha\gamma}h^\beta{}_\gamma + \tfrac{1}{2}\eta_{\mu\nu}hh^{\alpha\beta}) \\ & + 2\partial_\alpha[h^{\alpha\beta}(\partial_\mu h_{\beta\nu} + \partial_\nu h_{\beta\mu})] - h^\alpha{}_\mu\partial^2 h_{\alpha\nu} - h^\alpha{}_\nu\partial^2 h_{\alpha\mu} + h_{\mu\nu}\partial^2 h + \tfrac{1}{2}\eta_{\mu\nu}(h^{\alpha\beta}\partial^2 h_{\alpha\beta} - \tfrac{1}{2}h\partial^2 h). \end{aligned} \tag{A18}$$

The Feynman rule for the three-radiation graviton vertex follows from its definition in momentum space,

$$(V_{\bar{h}h^2})^{\mu\nu}_{\alpha\beta\gamma\sigma} = i(2\pi)^d \int d^dx \int d^dx_1 d^dx_2 d^dx_3 e^{i(kx_1 + px_2 + qx_3)} \frac{\delta}{\delta h^{\alpha\beta}(x_1)} \frac{\delta}{\delta h^{\gamma\sigma}(x_2)} \frac{\delta L_{\bar{h}h^2}(x)}{\delta \bar{h}_{\mu\nu}(x_3)},$$
$$\text{with} \quad \frac{\delta h_{\alpha\beta}(x)}{\delta h^{\mu\nu}(x_i)} = \delta^d(x - x_i) I_{\alpha\beta\mu\nu}, \qquad k + p + q = 0, \tag{A19}$$

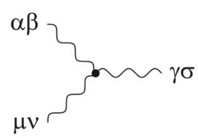
and $I_{\alpha\beta\mu\nu}$ being the symmetric identity tensor,

$$I_{\alpha\beta\mu\nu} \equiv \tfrac{1}{2}(\eta_{\alpha\mu}\eta_{\beta\nu} + \eta_{\alpha\nu}\eta_{\beta\mu}). \tag{A20}$$

We assumed that all momenta k, p, q are incoming and thus conservation of momentum in this convention reads as displayed in the end of (A19). After a long but straightforward computation that also makes use of the integral representation (B1) for the delta function we arrive at the desired Feynman rule for the three-radiation graviton vertex,

$$
\begin{aligned}
(V_{\bar{h}h^2})^{\mu\nu}_{\alpha\beta\gamma\sigma} = -\frac{i\kappa_g}{2}\bigg\{&\bigg[k^\mu k^\nu + (k+q)^\mu (k+q)^\nu + q^\mu q^\nu - \frac{3}{2}\eta^{\mu\nu}q^2\bigg]\bigg(I_{\alpha\beta\gamma\sigma} - \frac{1}{2}\eta_{\alpha\beta}\eta_{\gamma\sigma}\bigg)\\
&+ 2q_\lambda q_\delta[I^{\lambda\delta}{}_{\alpha\beta}I^{\mu\nu}{}_{\gamma\sigma} + I^{\lambda\delta}{}_{\gamma\sigma}I^{\mu\nu}{}_{\alpha\beta} - I^{\mu\delta}{}_{\alpha\beta}I^{\nu\lambda}{}_{\gamma\sigma} - I^{\nu\lambda}{}_{\alpha\beta}I^{\mu\delta}{}_{\gamma\sigma}] + q_\lambda q^\mu(\eta_{\alpha\beta}I^{\lambda\nu}{}_{\gamma\sigma} + \eta_{\gamma\sigma}I^{\lambda\nu}{}_{\alpha\beta})\\
&+ q_\lambda q^\nu(\eta_{\alpha\beta}I^{\lambda\mu}{}_{\gamma\sigma} + \eta_{\gamma\sigma}I^{\lambda\mu}{}_{\alpha\beta}) - q^2(\eta_{\alpha\beta}I^{\mu\nu}{}_{\gamma\sigma} - \eta_{\gamma\sigma}I^{\mu\nu}{}_{\alpha\beta}) - \eta^{\mu\nu}q^\lambda q^\delta(\eta_{\alpha\beta}I_{\gamma\sigma\delta\lambda} + \eta_{\gamma\sigma}I_{\alpha\beta\delta\lambda})\\
&- 2q^\lambda[I_{\alpha\beta\lambda\delta}(I^{\delta\nu}{}_{\gamma\sigma}(k+q)^\mu + I^{\delta\mu}{}_{\gamma\sigma}(k+q)^\nu) - I_{\gamma\sigma\lambda\delta}(I^{\delta\nu}{}_{\alpha\beta}k^\mu + I^{\delta\mu}{}_{\alpha\beta}k^\nu)]\\
&+ q^2(I^{\delta\mu}{}_{\alpha\beta}I_{\gamma\sigma\delta}{}^\nu + I_{\alpha\beta\delta}{}^\nu I^{\delta\mu}{}_{\gamma\sigma}) + \eta^{\mu\nu}q^\lambda q_\delta(I_{\alpha\beta\lambda\rho}I^{\rho\delta}{}_{\gamma\sigma} + I_{\gamma\sigma\lambda\rho}I^{\rho\delta}{}_{\alpha\beta})\\
&+ (k^2 + (k+q)^2)\bigg[I^{\delta\mu}{}_{\alpha\beta}I_{\gamma\sigma\delta}{}^\nu + I^{\delta\nu}{}_{\alpha\beta}I_{\gamma\sigma\delta}{}^\mu - \frac{1}{2}\eta^{\mu\nu}\bigg(I_{\alpha\beta\gamma\sigma} - \frac{1}{2}\eta_{\alpha\beta}\eta_{\gamma\sigma}\bigg)\bigg]\\
&- (k^2\eta_{\alpha\beta}I^{\mu\nu}{}_{\gamma\sigma} + (k+q)^2\eta_{\gamma\sigma}I^{\mu\nu}{}_{\alpha\beta})\bigg\}.
\end{aligned}
\tag{A21}
$$

This Feynman rule is independent of the dimensionality of the spacetime (contrary to the rule for the graviton propagator and potential graviton vertices). In particular, (A21) is the same as the Feynman rule first obtained in four dimensions by [39,40] (for the details see [38]). The three-point correlation function for the three-radiation graviton interaction, $\langle h_{\mu\nu}h_{\alpha\beta}h_{\gamma\sigma}\rangle$, can be obtained trivially from the tensorial product of this vertex rule with the radiation graviton propagators (A8),



$$
\begin{aligned}
&= \langle h_{\alpha\beta}(x_1)h_{\mu\nu}(x_2)h_{\gamma\sigma}(x_3)\rangle\\
&= \langle h_{\alpha\beta}(x_1)h_{\hat{\alpha}\hat{\beta}}(x_1')\rangle\,(V_{\bar{h}H^2})^{\hat{\alpha}\hat{\beta}\hat{\mu}\hat{\nu}\hat{\gamma}\hat{\sigma}}\,\langle h_{\mu\nu}(x_2)h_{\hat{\mu}\hat{\nu}}(x_2')\rangle\langle h_{\gamma\sigma}(x_3)h_{\hat{\gamma}\hat{\sigma}}(x_3')\rangle\,.
\end{aligned}
\tag{A22}
$$

We do not need (A21) or (A22) (we just quote them for completeness), but we will make use of (A18) in the next subsections.

4. Three-radiation-potential graviton vertex and correlation function

The action describing the three-radiation-potential graviton interaction is

$$
S_{\bar{h}H^2} = \int d^dx L_{\bar{h}H^2}, \qquad L_{\bar{h}H^2}(x) = -\frac{\kappa_g}{2}\bar{h}^{\mu\nu}(x)T^{H^2}_{\mu\nu}(x),
\tag{A23}
$$

where $T^{H^2}_{\mu\nu}(x)$ is obtained from $T^{h^2}_{\mu\nu}(x)$ by doing the replacement $h \to H$ in (A18).

We proceed using the Fourier transform (2.10) and keeping only the leading order terms in the velocity. This yields

$$
\begin{aligned}
T^{H^2}_{\mu\nu}(x) &= \int_{\mathbf{k}}\int_{\mathbf{q}} e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{x}}\mathcal{T}^{H^2}_{\mu\nu}(x^0),\\
\mathcal{T}^{H^2}_{\mu\nu}(x^0) &= \bigg(\mathbf{q}_u\mathbf{q}_v + \frac{1}{2}\eta_{\mu\nu}\mathbf{q}^2\bigg)\bigg(H^{\alpha\beta}_{\mathbf{k}}H_{\mathbf{q}_{\alpha\beta}} - \frac{1}{2}H_{\mathbf{k}}H_{\mathbf{q}}\bigg) - \mathbf{q}^2(H^\alpha_{\mathbf{k}_\mu}H_{\mathbf{q}_{\alpha\nu}} + H^\alpha_{\mathbf{k}_\nu}H_{\mathbf{q}_{\alpha\mu}} - H_{\mathbf{k}_{\mu\nu}}H_{\mathbf{q}})\\
&\quad + \frac{1}{2}\bigg(\mathbf{k}_u\mathbf{k}_v + \mathbf{q}_u\mathbf{q}_v + 2\mathbf{k}_{(u}\mathbf{q}_{v)} + \frac{3}{2}\eta_{\mu\nu}(\mathbf{k}^2 + \mathbf{q}^2 + 2\mathbf{k}\cdot\mathbf{q})\bigg)\bigg(H^{\alpha\beta}_{\mathbf{k}}H_{\mathbf{q}_{\alpha\beta}} - \frac{1}{2}H_{\mathbf{k}}H_{\mathbf{q}}\bigg)\\
&\quad + (\mathbf{k}_i\mathbf{k}_u + \mathbf{q}_i\mathbf{q}_u + 2\mathbf{k}_{(i}\mathbf{q}_{u)})H_{\mathbf{k}}H^i_{\mathbf{q}_\nu} + (\mathbf{k}_i\mathbf{k}_v + \mathbf{q}_i\mathbf{q}_v + 2\mathbf{k}_{(i}\mathbf{q}_{v)})H_{\mathbf{k}}H^i_{\mathbf{q}_\mu} - 2(\mathbf{k}_i\mathbf{k}_j + \mathbf{q}_i\mathbf{q}_j + 2\mathbf{k}_{(i}\mathbf{q}_{j)})\\
&\quad \times\bigg(H^i_{\mathbf{k}_\mu}H^j_{\mathbf{q}_\nu} - H^{ij}_{\mathbf{k}}H_{\mathbf{q}_{\mu\nu}} - \frac{1}{2}\eta_{\mu\nu}(H^{i\alpha}_{\mathbf{k}}H^j_{\mathbf{q}_\alpha} - H_{\mathbf{k}}H^{ij}_{\mathbf{q}})\bigg) - 2(\mathbf{q}_i\mathbf{q}_u + \mathbf{k}_i\mathbf{q}_u)H^{i\alpha}_{\mathbf{k}}H_{\mathbf{q}_{\alpha\nu}} - 2(\mathbf{q}_i\mathbf{q}_v + \mathbf{k}_i\mathbf{q}_v)H^{i\alpha}_{\mathbf{k}}H_{\mathbf{q}_{\alpha\mu}}\\
&\quad - (\mathbf{k}^2 + \mathbf{q}^2 + 2\mathbf{k}\cdot\mathbf{q})(H_{\mathbf{k}_{\alpha\mu}}H^\alpha_{\mathbf{q}_\nu} - H_{\mathbf{k}}H_{\mathbf{q}_{\mu\nu}}) + \text{higher order } v \text{ terms.}
\end{aligned}
\tag{A24}
$$

Notice that in this expression we use the Latin letters u, v when only spatial μ, ν make a leading order contribution, and we use parenthesis () around the indices to denote symmetrization, $\mathbf{k}_{(i}\mathbf{q}_{j)} = (\mathbf{k}_i\mathbf{q}_j + \mathbf{q}_i\mathbf{k}_j)/2$.

Proceeding, we do the multipole expansion (2.11) of $\bar{h}_{\mu\nu}(x)$ around the system's center of mass $\mathbf{X}$, and keep only the leading order term in the velocity v, $\bar{h}_{\mu\nu}(x^0, \mathbf{X})$. So, to leading order $\bar{h}_{\mu\nu}(x)$ only depends on x^0 but not on $\mathbf{x}$. This allows to write (A23) in lowest order as

$$S_{\bar{h}H^2} = \int dx^0 \bar{h}_{\mu\nu}(x^0, \mathbf{X}) \int_{\mathbf{k}} \int_{\mathbf{q}} \int d^{d-1}\mathbf{x} e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{x}} \left(-\frac{\kappa_g}{2} \mathcal{T}^{H^2}_{\mu\nu}(x^0) \right). \tag{A25}$$

The spatial integral can be done using the integral representation of the delta function $\int_{\mathbf{x}} e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{x}} = \delta(\mathbf{k}+\mathbf{q})$. This operation results in $\mathbf{q} \to -\mathbf{k}$ in (A24) and only the first line survives to the procedure. We can finally write the leading order action for the three-graviton interaction between potential and radiation gravitons as

$$S_{\bar{h}H^2} = \int dx^0 L_{\bar{h}H^2}(x^0, \mathbf{X}), \quad \text{with}$$

$$\begin{aligned} L_{\bar{h}H^2}(x^0, \mathbf{X}) = {} & \kappa_g \bar{h}^{00} \int_{\mathbf{k}} \mathbf{k}^2 \left[H^{\mu}_{\mathbf{k}_0} H_{-\mathbf{k}_{\mu 0}} - \frac{1}{2} H_{\mathbf{k}_{00}} H_{-\mathbf{k}} - \frac{1}{4} H^{\mu\nu}_{\mathbf{k}} H_{-\mathbf{k}_{\mu\nu}} + \frac{1}{8} H_{\mathbf{k}} H_{-\mathbf{k}} \right] \\ & + \kappa_g \bar{h}^{0i} \int_{\mathbf{k}} \mathbf{k}^2 [2 H^{\mu}_{\mathbf{k}_0} H_{-\mathbf{k}_{\mu i}} - H_{\mathbf{k}_{0i}} H_{-\mathbf{k}}] + \kappa_g \bar{h}^{ij} \int_{\mathbf{k}} \left[-\frac{1}{2} \mathbf{k}_i \mathbf{k}_j \left(H^{\mu\nu}_{\mathbf{k}} H_{-\mathbf{k}_{\mu\nu}} - \frac{1}{2} H_{\mathbf{k}} H_{-\mathbf{k}} \right) + \mathbf{k}^2 H_{\mathbf{k}_{i\mu}} H^{\mu}_{-\mathbf{k}_j} \right. \\ & \left. - \frac{1}{2} \mathbf{k}^2 H_{\mathbf{k}_{ij}} H_{-\mathbf{k}} - \frac{\mathbf{k}^2}{4} \eta_{ij} \left(H^{\mu\nu}_{\mathbf{k}} H_{-\mathbf{k}_{\mu\nu}} - \frac{1}{2} H_{\mathbf{k}} H_{-\mathbf{k}} \right) \right], \end{aligned} \tag{A26}$$

where $\bar{h}^{00}$, $\bar{h}^{0i}$, $\bar{h}^{ij}$ are functions of $(x^0, \mathbf{X})$ and $H^{\mu\nu}_{\mathbf{k}} = H^{\mu\nu}_{\mathbf{k}}(x^0)$.

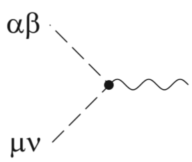
The Feynman rule for the three-radiation-potential graviton vertex follows from the definition,

$$(V_{\bar{h}H^2})_{\alpha\beta\mu\nu} = i \int dx^0 \int dx^0_1 dx^0_2 \frac{\delta}{\delta H^{\alpha\beta}_{\mathbf{k}_1}(x^0_1)} \frac{\delta L_{\bar{h}H^2}(x^0, \mathbf{X})}{\delta H^{\mu\nu}_{\mathbf{k}_2}(x^0_2)}, \tag{A27}$$

and use of the functional derivative,

$$\frac{\delta H_{\mathbf{k}_{\alpha\beta}}(x^0)}{\delta H^{\mu\nu}_{\mathbf{q}}(x^0_i)} = \delta(\mathbf{k}+\mathbf{q}) \delta(x^0 - x^0_i) I_{\alpha\beta\mu\nu}, \tag{A28}$$

with $I_{\alpha\beta\mu\nu}$ defined in (A20). Finally, the three-point correlation function for the interaction between two potential gravitons and radiation graviton is (dropping the bars over the graviton)



$$= \langle h(x^0, X) H_{\mathbf{k}_{\alpha\beta}}(x^0_1) H_{\mathbf{q}_{\mu\nu}}(x^0_2) \rangle$$

$$\begin{aligned} & = \langle H_{\mathbf{k}_{\alpha\beta}}(x^0_1) H_{\mathbf{k}'_{\gamma\sigma}}(x'^0_1) \rangle \, (V_{\bar{h}H^2})^{\gamma\sigma\lambda\eta} \, \langle H_{\mathbf{q}_{\mu\nu}}(x^0_2) H_{\mathbf{q}'_{\lambda\eta}}(x'^0_2) \rangle \\ & = -i\kappa_g (2\pi)^{d-1} \delta(x^0_1 - x^0_2) \delta(\mathbf{k}+\mathbf{q}) \frac{1}{\mathbf{k}^2 \mathbf{q}^2} \\ & \quad \times \left\{ h^{00} \mathbf{k}^2 \left[-\frac{1}{2} P_{\alpha\beta\mu\nu} + P_{\alpha\beta\mu 0} \eta_{0\nu} + P_{\alpha\beta 0\nu} \eta_{0\mu} - \frac{2}{d-2} (P_{\alpha\beta 00} \eta_{\mu\nu} + \eta_{\alpha\beta} P_{00\mu\nu}) \right] \right. \\ & \quad + h^{0i} \mathbf{k}^2 \left[2 (P_{\alpha\beta\mu i} \eta_{0\nu} + P_{\alpha\beta i\nu} \eta_{0\mu}) + \frac{4}{d-2} (\eta_{\alpha\beta} P_{0i\mu\nu} - P_{\alpha\beta 0i} \eta_{\mu\nu}) \right] \\ & \quad \left. + h^{ij} \left[-\left(\mathbf{k}_i \mathbf{k}_j + \frac{1}{2} \eta_{ij} \mathbf{k}^2 \right) P_{\alpha\beta\mu\nu} + \mathbf{k}^2 \left(I_{\alpha\beta\mu j} \eta_{i\nu} + I_{\alpha\beta j\nu} \eta_{i\mu} - \frac{2}{d-2} I_{\alpha\beta ij} \eta_{\mu\nu} \right) \right] \right\}. \end{aligned} \tag{A29}$$

In the main text we need only the $\alpha, \beta, \mu, \nu = 0$ components which reads

$$\langle h(x^0, X) H_{\mathbf{k}_{00}}(x^0_1) H_{\mathbf{q}_{00}}(x^0_2) \rangle = -i\kappa_g (2\pi)^{d-1} \delta(x^0_1 - x^0_2) \delta(\mathbf{k}+\mathbf{q}) \frac{1}{\mathbf{k}^2 \mathbf{q}^2} \frac{d-3}{d-2} \left[\frac{3}{2} \mathbf{k}^2 h^{00} - \left(\frac{1}{2} \mathbf{k}^2 \eta_{ij} + \mathbf{k}_i \mathbf{k}_j \right) h^{ij} \right]. \tag{A30}$$

Using the power counting rules of Table I we find that $\langle h H_{\mathbf{k}_{\alpha\beta}} H_{\mathbf{q}_{\mu\nu}} \rangle \sim L^{-1/2} v^{d+3/2} r^d$.

5. Three-potential graviton vertex and correlation function

To obtain the Lagrangian for the three-potential graviton interaction we begin by making the replacement $\bar{h}^{\mu\nu} \to \bar{H}^{\mu\nu}$ in (A23). We then take the Fourier transform (2.10) of the potential graviton $\bar{H}^{\mu\nu}$ and use the integral representation of the delta function (B1) to get

$$S_{\bar{H}H^2} = \int dx^0 L_{\bar{H}H^2},$$

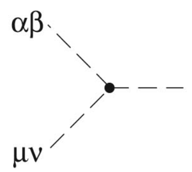
$$L_{\bar{H}H^2}(x^0) = -\frac{\kappa_g}{2}\int_{\mathbf{k}}\int_{\mathbf{q}}\int_{\mathbf{p}}(2\pi)^{d-1}\delta(\mathbf{k}+\mathbf{q}+\mathbf{p})\bar{H}^{\mu\nu}_{\mathbf{p}}(x^0) \times \mathcal{T}^{H^2}_{\mu\nu}(x^0), \tag{A31}$$

where $\mathcal{T}^{H^2}_{\mu\nu}(x^0)$ is defined in (A24).

The Feynman rule for the three-potential graviton vertex follows from the definition,

$$(V_{\bar{H}H^2})_{\alpha\beta\mu\nu\gamma\sigma} = i\int dx^0 \int dx^0_1 dx^0_2 dx^0_3 \frac{\delta}{\delta H^{\alpha\beta}_{\mathbf{k}_1}(x^0_1)} \times \frac{\delta}{\delta H^{\mu\nu}_{\mathbf{k}_2}(x^0_2)}\frac{\delta L_{\bar{H}H^2}(x^0)}{\delta H^{\gamma\sigma}_{\mathbf{k}_3}(x^0_3)}, \tag{A32}$$

and the use of (A28). The three-point correlation function for the interaction between three potential gravitons is finally



$$\gamma\sigma = \langle H_{\mathbf{k}\,\alpha\beta}(x^0_1)H_{\mathbf{q}\,\mu\nu}(x^0_2)H_{\mathbf{p}\,\gamma\sigma}(x^0_3)\rangle$$

$$\begin{aligned}
&= \langle H_{\mathbf{k}\,\alpha\beta}(x^0_1)H_{\mathbf{k}'\,\hat{\alpha}\hat{\beta}}(x'^0_1)\rangle\,(V_{\bar{H}H^2})^{\hat{\alpha}\hat{\beta}\hat{\mu}\hat{\nu}\hat{\gamma}\hat{\sigma}}\,\langle H_{\mathbf{q}\,\mu\nu}(x^0_2)H_{\mathbf{q}'\,\hat{\mu}\hat{\nu}}(x'^0_2)\rangle\langle H_{\mathbf{p}\,\gamma\sigma}(x^0_3)H_{\mathbf{p}'\,\hat{\gamma}\hat{\sigma}}(x'^0_3)\rangle\\
&= -\frac{\kappa_g}{2}(2\pi)^{d-1}\delta(x^0_1-x^0_2)\delta(x^0_1-x^0_3)\delta\left(\mathbf{k}+\mathbf{q}+\mathbf{p}\right)\frac{1}{\mathbf{k}^2\,\mathbf{q}^2\,\mathbf{p}^2}\\
&\quad\times\Big\{2P_{\alpha\beta\mu\nu}P^{ij}{}_{\gamma\sigma}\left(\mathbf{k}_i\mathbf{k}_j+\mathbf{q}_i\mathbf{q}_j+\mathbf{k}_{(i}\mathbf{q}_{j)}\right)\\
&\quad-\left[2P^{\lambda\eta}{}_{\gamma\sigma}\left(P_{\alpha\beta\eta(\mu}\,\eta_{\nu)\lambda}+\frac{1}{d-2}\,P_{\lambda\eta\alpha\beta}\,\eta_{\mu\nu}\right)+\frac{3}{d-2}\,P_{\alpha\beta\mu\nu}\,\eta_{\gamma\sigma}\right]\left(\mathbf{k}^2+\mathbf{q}^2+2\mathbf{k}\cdot\mathbf{q}\right)\\
&\quad+\left[P_{\gamma\sigma\alpha(\mu}\eta_{\nu)\beta}+P_{\gamma\sigma\beta(\mu}\eta_{\nu)\alpha}+\frac{1}{d-2}\,\left(2P_{\alpha\beta\gamma\sigma}\,\eta_{\mu\nu}-P_{\alpha\beta\mu\nu}\,\eta_{\gamma\sigma}\right)\right]\left(\mathbf{k}^2+\mathbf{q}^2\right)\\
&\quad-4\left(\mathbf{k}_i\mathbf{k}_j+\mathbf{q}_i\mathbf{q}_j+2\mathbf{k}_{(i}\mathbf{q}_{j)}\right)\\
&\quad\times\left[P^{i\lambda}{}_{\gamma\sigma}\left(\frac{2}{d-2}\,P^{j}{}_{\lambda\alpha\beta}\,\eta_{\mu\nu}+P_{\lambda\eta\alpha\beta}P^{j\eta}{}_{\mu\nu}\right)+P^{\lambda\eta}{}_{\gamma\sigma}\left(P^{i}{}_{\lambda\mu\nu}P^{j}{}_{\eta\alpha\beta}-P_{\lambda\eta\alpha\beta}P^{ij}{}_{\mu\nu}\right)\right.\\
&\quad\left.+\frac{1}{d-2}\left(\frac{1}{d-2}\,P^{ij}{}_{\alpha\beta}\,\eta_{\mu\nu}\,\eta_{\gamma\sigma}+I_{\alpha\beta(\mu}{}^{j}\,\eta^{i}{}_{\nu)}\eta_{\gamma\sigma}-\frac{1}{d-2}\eta_{\alpha\beta}\,\eta_{\gamma\sigma}\,I^{ij}{}_{\mu\nu}\right)\right]\Big\},
\end{aligned} \tag{A33}$$

where we used (again) the rule that a sum over Latin indices means that only spatial indices contribute (to leading order), and the parenthesis () around the indices denote symmetrization. In the main text we need only the α, β, μ, ν, γ, $\sigma = 0$ components which reads

$$\langle H_{\mathbf{k}_{00}}(x^0_1)H_{\mathbf{q}_{00}}(x^0_2)H_{\mathbf{p}_{00}}(x^0_3)\rangle = \kappa_g(2\pi)^{d-1}\delta(x^0_1-x^0_2)\delta(x^0_1-x^0_3)\delta(\mathbf{k}+\mathbf{q}+\mathbf{p}) \times\frac{(d-3)^2}{(d-2)^2}\frac{\mathbf{k}^2+\mathbf{q}^2+\mathbf{p}^2}{\mathbf{k}^2\mathbf{q}^2\mathbf{p}^2}. \tag{A34}$$

Using the power counting rules of Table I we find that $\langle H_{\mathbf{k}_{\alpha\beta}}H_{\mathbf{q}_{\mu\nu}}H_{\mathbf{p}_{\gamma\sigma}}\rangle \sim L^{-1/2}v^{7/2}r^{3d/2}$.

6. Point-particle vertices

The Feynman rules for the nonlinear vertex interactions of particles with potential gravitons are obtained by taking the small velocity expansion (2.9) of the point-particle action S_{pp}, with $\delta g_{\mu\nu} \to H_{\mu\nu}$. In the expansion (2.9) we only display the lowest-order terms that contribute to the leading order observables of interest. We also take the Fourier transform (2.10) of the potential graviton to keep a well-defined track of the velocity power counting.

The Feynman rules for the interaction vertex between the particle and a potential graviton vertex then follows from the definition of vertex interaction factor,

$$V_{\mu\nu} = i\frac{\delta S_{pp}}{\delta H^{\mu\nu}_{\mathbf{k}}}, \qquad V_{\mu\nu\alpha\beta} = i\frac{\delta^2 S_{pp}}{\delta H^{\mu\nu}_{\mathbf{k}}\delta H^{\alpha\beta}_{\mathbf{q}}}, \tag{A35}$$

and the use of (A28). The point-particle vertex Feynman rules are then [following the order of appearance in (2.9)]

$$V^{(1)}_{\mu\nu} = -\frac{i\kappa_g m_a}{2}\int dx^0_a \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\eta_{0\mu}\eta_{0\nu},$$
$$V^{(2)}_{\mu\nu} = -\frac{i\kappa_g m_a}{2}\int dx^0_a \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} v^i_a(\eta_{0\mu}\eta_{i\nu} + \eta_{0\nu}\eta_{i\mu}),$$
$$V^{(3)}_{\mu\nu} = -\frac{i\kappa_g m_a}{2}\int dx^0_a \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} v^i_a v^j_a(\eta_{i\mu}\eta_{j\nu} + \eta_{i\nu}\eta_{j\mu}),$$
$$V^{(4)}_{\mu\nu} = -\frac{i\kappa_g m_a}{4}\int dx^0_a \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\mathbf{v}^2_a\eta_{0\mu}\eta_{0\nu},$$
$$V^{(5)}_{\mu\nu\alpha\beta} = \frac{i\kappa^2_g m_a}{4}\int dx^0_a \int_{\mathbf{k}}\int_{\mathbf{q}} e^{i(\mathbf{k}+\mathbf{q})\cdot\mathbf{x}}\eta_{0\mu}\eta_{0\nu}\eta_{0\alpha}\eta_{0\beta}. \tag{A36}$$

The power counting rules for the vertices are shown in Table II [the kinetic contributions in (2.9), i.e., the two last terms, have power counting rules of $L^{1/2}v^0$ and $L^{1/2}v^2$, respectively].

APPENDIX B: USEFUL RELATIONS FOR FEYNMAN DIAGRAM COMPUTATIONS

In this appendix we present some standard formulas that are useful to compute Feynman diagram contributions.

We start with the integral representation of the delta function

$$\delta(\mathbf{k}) \equiv \delta^{d-1}(\mathbf{k}) = \int \frac{d^{d-1}\mathbf{x}}{(2\pi)^{d-1}} e^{i\mathbf{k}\cdot\mathbf{x}} \equiv \int_{\mathbf{x}} e^{i\mathbf{k}\cdot\mathbf{x}},$$
$$\delta(\mathbf{x}) = \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{B1}$$

and the well-known integrals,

$$\int f(z)\delta(z-a) = f(a), \qquad \int f'(z)\delta(z-a) = -f'(a), \tag{B2}$$

where $f'(z) = \frac{df}{dz}$.

The integrals over the momentum are computed using the following relations,

$$\int \frac{d^n\mathbf{k}}{(2\pi)^n}\frac{1}{(\mathbf{k}^2)^\alpha}e^{-i\mathbf{k}\cdot\mathbf{x}} = \frac{1}{(4\pi)^{n/2}}\frac{\Gamma(n/2-\alpha)}{\Gamma(\alpha)}\left(\frac{\mathbf{x}^2}{4}\right)^{\alpha-n/2}, \tag{B3}$$

$$\int \frac{d^{d-1}\mathbf{k}}{(2\pi)^{d-1}}\frac{\mathbf{k}_i\mathbf{k}_j}{\mathbf{k}^4}e^{-i\mathbf{k}\cdot\mathbf{x}} = \frac{\Gamma(\frac{d-3}{2})}{8\pi^{(d-1)/2}}|\mathbf{x}|^{-(d-3)} \times\left[\delta_{ij} - (d-3)\frac{\mathbf{x}_i\mathbf{x}_j}{|\mathbf{x}|^2}\right], \tag{B4}$$

and

$$\int \frac{d^{2\eta}\mathbf{k}}{(2\pi)^{2\eta}}\frac{1}{\mathbf{k}^2(\mathbf{k}-\mathbf{p})^2} = I_0(\mathbf{p}^2)^{\eta-2},$$
$$I_0 \equiv \frac{\Gamma(2-\eta)[\Gamma(\eta-1)]^2}{(4\pi)^\eta\Gamma(2\eta-2)},$$
$$\int \frac{d^{2\eta}\mathbf{k}}{(2\pi)^{2\eta}}\frac{\mathbf{k}_i}{\mathbf{k}^2(\mathbf{k}-\mathbf{p})^2} = \frac{1}{2}I_0\mathbf{p}_i(\mathbf{p}^2)^{\eta-2}, \tag{B5}$$
$$\int \frac{d^{2\eta}\mathbf{k}}{(2\pi)^{2\eta}}\frac{\mathbf{k}_i\mathbf{k}_j}{\mathbf{k}^2(\mathbf{k}-\mathbf{p})^2} = -\frac{\delta_{ij}}{4(2\eta-1)}I_0(\mathbf{p}^2)^{\eta-1} + \mathbf{p}_i\mathbf{p}_j\frac{\eta}{2(2\eta-1)}I_0(\mathbf{p}^2)^{\eta-2}.$$

The integral (B3) is needed to evaluate all Feynman diagrams. Relation (B4) is needed to compute (4.7), and relations (B5) are needed to compute (3.9).

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