

**Spontaneous symmetry breaking in light front field theory**

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A semiclassical picture of spontaneous symmetry breaking in light front field theory is formulated. It is based on a finite-volume quantization of self-interacting scalar fields obeying antiperiodic boundary conditions. This choice avoids a necessity to solve the zero-mode constraint and enables one to define unitary operators which shift the scalar field by a constant. The operators simultaneously transform the light front Fock vacuum to coherent states with lower energy than the Fock vacuum and with nonzero expectation value of the scalar field. The new vacuum states are noninvariant under the discrete or continuous symmetry of the Hamiltonian. Spontaneous symmetry breaking is described in this way in the two-dimensional  $\lambda\phi^4$  theory and in the three-dimensional  $O(2)$ -symmetric sigma model. A qualitative treatment of topological kink solutions in the first model and a derivation of the Goldstone theorem in the second one are given. Symmetry breaking in the case of periodic boundary conditions is also briefly discussed.

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**I. INTRODUCTION**

Spontaneous symmetry breaking is a fundamental non-perturbative phenomenon of quantum field theory. It occurs when the Hamiltonian of a theory is symmetric under a group of transformations while the ground state is non-invariant. For continuous symmetries it follows that there exists a field operator (elementary or composite) with nonzero expectation value in this vacuum state. As a consequence, the spectrum of such a theory contains a massless state, the Nambu-Goldstone boson [1–3], if the space dimension is greater than one [4,5]. This overall picture of the broken phase is well understood in the conventional field theory which parametrizes the space-time by means of the four-vector  $x^\mu = (t, x, y, z)$ .

On the other hand, spontaneous symmetry breaking (SSB) still remains a bit mysterious in the light front (LF) field theory, which is defined by the choice of the LF variables  $x^\mu = (x^+, x^-, x, y)$ ,  $x^\pm = x^0 \pm x^3$ , and by quantization on a surface of constant LF time  $x^+$ . The main reason for difficulties with obtaining a clear picture of SSB in the LF theory is that, due to positivity of the LF momentum operator  $P^+$ , the vacuum state of the interacting LF theory coincides with the free Fock vacuum if independent Fourier modes carrying  $p^+ = 0$  (dynamical LF zero modes) can be neglected. This simplicity of the vacuum state is very useful in bound-state calculations but it appears to be problematic in other nonperturbative issues because it prohibits any vacuum structure in continuum LF theories where dynamical zero modes seem indeed to be negligible. It is often believed that the vacuum aspects enter into the LF theory via nondynamical constrained zero modes which are in principle obtained as solutions of corresponding constraint equations.

The present work is based on a different concept: the “trivial” LF vacuum, being a simple but rigorously defined nonperturbative state, is viewed as an intermediate construction, not the ultimate physical vacuum state. It can often be systematically transformed into more complex objects by unitary operators that implement a symmetry of a given field theoretic model. These operators are well-defined (at least with a cutoff on number of field modes) in an infrared-regularized formulation—quantization in a finite volume (or on a line of length  $L$  in two dimensions) with fields (anti)periodic in space coordinates. Large gauge transformations and chiral symmetry are two examples of this approach [6,7]. A similar treatment for scalar field theories has not been given so far. The reason was that symmetry generators in a scalar theory always annihilate the LF Fock vacuum because, due to positivity of the momentum  $P^+$ , they cannot contain terms composed of purely creation operators [8,9] if there are no dynamical zero modes in the theory. Without such terms it is not possible to transform the LF Fock vacuum into a more complex object and therefore one cannot construct multiple vacua which are a necessary condition for any SSB. A simple observation that underlies the present work is that for scalar theories with polynomial self-interaction and negative quadratic term the LF Fock vacuum is not the state of minimum LF energy [10]. The energy is lower in a specific coherent state and this state is *not* annihilated by the symmetry generators. Hence, the unitary operators implementing the discrete or continuous symmetry will generate, when applied to this state, a discrete or continuous set of new (semiclassical) vacuum states.

Light front versions of SSB have been studied by a few groups of authors. In the unbroken phase, a zero-mode coherent state vacuum has been derived in  $\lambda\phi^4$  theory in two dimensions and used along with the variational method [11]. Scalar zero mode has been assumed to be an in-

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dependent dynamical variable. If one imposes periodic boundary conditions (a standard choice), this mode is however a dependent variable satisfying an operator constraint. Approximate methods of its solution indicated a development of the broken phase above the critical coupling [12,13]. The value of the critical coupling and the critical exponent  $\eta$  have also been determined using the Haag expansion [14]. In the broken phase, a variational approach with a coherent state  $|\alpha\rangle$  as a trial lowest-energy state was used for small coupling with the zero mode manifestly neglected [15]. Two (approximately) degenerate lowest energy levels were found and the correct value of the mass of the lowest excitation in the broken phase (a kink) was extracted by minimizing the expectation value of the Hamiltonian in the  $|\alpha\rangle$  states subject to the constraint on the dimensionless momentum  $K = \frac{L}{2\pi} P^+$ . The subsequent discrete light-cone quantization (DLCQ) studies confirmed this picture in truly nonperturbative calculations and led to a detailed prediction of kink and antikink mass and a few additional observables [16,17].

In four dimensions, it is usually assumed that the scalar zero mode contains a constant piece. As a consequence, symmetry breaking is found to manifest itself in a rather unusual way by a nonconservation of the current even in the symmetry limit while the physical vacuum is identified with the Fock vacuum [18,19]. The concept of vacuum triviality underlies also an approach to dynamical symmetry breaking [20] based on a derivation of gap equations from the LF constraint equations.

In the broken phase, considered scalar models possess two or more degenerate minima of the classical potential. As already indicated, one might expect that even in the LF theory the Fock vacuum is not the true physical vacuum in this case and that a unitary operator could be constructed which would shift the scalar field  $\phi(x)$  to the true minimum of the LF energy. Unfortunately, for  $\phi(x^+, x^-)$  periodic in the space coordinate  $x^-$ , such a construction is very difficult. This is due to the complicated nonlinear operator zero-mode constraint. On the other hand, choosing antiperiodic boundary conditions in  $x^-$  [21] (which is a consistent choice for polynomial interactions with even powers of fields) allows one to define shift operators which transform the Fock vacuum to new states that correspond to lower LF energy. They are coherent states of a large but finite number of Fourier modes. For simplicity, we will illustrate this mechanism in two well-known low-dimensional scalar models. The first one is the two-dimensional  $\lambda\phi^4$  theory in broken phase, possessing classically two degenerate ground states. The second model is a three-dimensional  $O(2)$ -symmetric linear sigma model. It has a continuum of degenerate vacuum states and one can expect the Goldstone phenomenon to take place. Both models are superrenormalizable. Renormalization can be performed by normal ordering the Hamiltonian or equivalently by adding a mass counterterm (a tadpole) in

the first case and a tadpole together with the second-order self-energy counterterm in the second case [22,23].

A short description of SSB in the case of periodic boundary conditions will also be given. We will show that some features of the broken phase are similar in both cases. Since our approach is based on the quantization in a finite spatial volume with antiperiodic boundary conditions, we use the correspondingly defined sign function and Dirac's delta function. Their regularized form is displayed in the Appendix.

## II. SPONTANEOUS SYMMETRY BREAKING IN $\lambda\phi^4(1+1)$ THEORY

Let us consider two-dimensional  $\lambda\phi^4$  theory in the broken phase. It is defined by the covariant Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \mu^2 \phi^2 - \frac{\lambda}{4} \phi^4, \quad \mu^2 > 0, \quad (1)$$

which is invariant under the discrete transformation of the real scalar field  $\phi(x) \rightarrow -\phi(x)$ . Classically, the potential energy in (1) has two minima at  $\phi_c = \pm \mu/\sqrt{\lambda}$ . In the tree-level analysis, one usually shifts the field by  $\pm \phi_c$  and obtains two Lagrangians which reveal the particle spectrum of the theory in terms of "small" oscillations above  $\phi_c$ . The original symmetry becomes hidden in the sense that the two Lagrangians are individually not symmetric under  $\phi(x) \rightarrow -\phi(x)$  but the symmetry operation transforms one to the other. Recall that due to the existence of more than one minimum of the potential, the model exhibits in addition to symmetry breaking also nontrivial topological properties [24]. There exist solutions of the classical equations of motion with finite energy which interpolate between the minima. They carry a conserved topological charge, proportional to the difference of the field values at the boundaries, and corresponding to the conserved topological current  $k^\rho = \frac{\sqrt{\lambda}}{\mu} \epsilon^{\rho\nu} \partial_\nu \phi$ .

The Lagrangian (1) is expressed in terms of the LF variables as

$$\mathcal{L}_{\text{lf}} = 2\partial_+ \phi \partial_- \phi + \frac{1}{2} \mu^2 \phi^2 - \frac{\lambda}{4} \phi^4, \quad (2)$$

where  $\partial_\pm = \partial/\partial x^\pm$ . We restrict the spatial coordinate by  $-L \leq x^- \leq L$ . In order to obtain a clear physical picture of SSB it is desirable to avoid the complicated nonlinear operator zero-mode constraint present in the case of periodic boundary conditions. We impose therefore the antiperiodic boundary condition (BC)  $\phi(L) = -\phi(-L)$  which results in discrete Fourier modes

$$p_n^+ = \frac{2\pi}{L} n, \quad n = 1/2, 3/2, \dots \infty. \quad (3)$$

The antiperiodic BC also implies that in the quantum theory we can define the operator of the topological charge  $Q = \frac{\sqrt{\lambda}}{\mu} [\phi(L) - \phi(-L)] = 2 \frac{\sqrt{\lambda}}{\mu} \phi(L)$ .

The standard canonical treatment yields the energy-momentum tensor components  $T^{+-}$  and  $T^{++}$  which define the LF Hamiltonian  $P^-$

$$\begin{aligned} P^- &= \frac{1}{2} \int_{-L}^{+L} dx^- T^{+-}(x^-) \\ &= \frac{1}{2} \int_{-L}^{+L} dx^- \left[ -\mu^2 \phi^2 + \frac{\lambda}{2} \phi^4 \right], \end{aligned} \quad (4)$$

as well as the LF momentum operator

$$P^+ = \frac{1}{2} \int_{-L}^{+L} dx^- T^{++}(x^-) = \frac{1}{2} \int_{-L}^{+L} dx^- 4[\partial_- \phi \partial_- \phi]. \quad (5)$$

The field expansion in terms of the Fourier modes at  $x^+ = 0$  reads

$$\phi(0, x^-) = \frac{1}{\sqrt{2L}} \sum_{n=1/2}^{\infty} \frac{1}{\sqrt{p_n^+}} [a_n e^{-(i/2)p_n^+ x^-} + a_n^\dagger e^{(i/2)p_n^+ x^-}]. \quad (6)$$

The annihilation and creation operators are required to satisfy the quantization condition  $[a_m, a_n^\dagger] = \delta_{mn}$ . As a consequence, one recovers the usual commutator at equal LF times,

$$[\phi(0, x^-), \phi(0, y^-)] = -\frac{i}{8} \epsilon_a(x^- - y^-), \quad (7)$$

where  $\epsilon_a(x^-)$  is the antiperiodic sign function

$$\epsilon_a(x^-) = \frac{4i}{L} \sum_{n=1/2}^{\infty} \frac{1}{p_n^+} [e^{-(i/2)p_n^+ x^-} - e^{(i/2)p_n^+ x^-}], \quad (8)$$

defined in terms of the discrete momenta (3). The conjugate momentum  $\Pi_\phi$  is not equal to the time derivative of the scalar field in the LF theory. It is a dependent variable, determined by  $\phi(x)$  itself,  $\Pi_\phi = 2\partial_- \phi$  [25]. Hence, the alternative form of the basic commutation relation, following from Eq. (7), is

$$[\phi(0, x^-), \Pi_\phi(0, y^-)] = \frac{i}{2} \delta_a(x^- - y^-), \quad (9)$$

where  $\delta_a(x^-)$  is the antiperiodic delta function  $\delta_a(x^-) = 1/2\partial_- \epsilon_a(x^-)$ . The same quantization rules can be obtained more rigorously by the Dirac-Bergmann method [26] for constrained systems.

Consider now a unitary operator

$$U(b) = \exp \left[ -2ib \int_{-L}^{+L} dx^- \Pi_\phi(x^-) \right]. \quad (10)$$

For antiperiodic boundary conditions, it reduces to

$$U(b) = e^{-8ib\phi(L)} \quad (11)$$

and translates the field  $\phi(x^-)$  by a constant  $b$  as can be easily shown by using the operator identity  $\exp(A)B \exp(-A) = B + [A, B] + \dots$ :

$$\begin{aligned} U(b)\phi(x^-)U^{-1}(b) &= \phi(x^-) - 8ib[\phi(L), \phi(x^-)] \\ &= \phi(x^-) - b\epsilon_a(L - x^-). \end{aligned} \quad (12)$$

Thus, the antiperiodic scalar field can be shifted by a constant without violating its antiperiodicity. The reason for that is the simple property of the sign function  $\epsilon_a(L - x^-)$ : it is equal to 1 for all  $x^-$  in the box except for the endpoints where it drops to zero. This is of course a direct consequence of the basic property  $\epsilon_a(0) = \epsilon_a(2L) = 0$ . It is much more difficult to perform a similar shift of the field in the case of periodic boundary condition because of the presence of the *a priori* unknown operator zero mode. Recall for comparison that, since in the conventional spacelike quantization the conjugate momentum is a dynamical quantity, the volume integration in the shift operator analogous to Eq. (10) projects out only its zero-mode component [27].

We should note however that the above considerations were a bit formal and the actual situation is slightly more complicated. The point is that the operator  $U(b)$  (11) exists (is nonzero) only if we impose a cutoff on the number of modes [see Eq. (17) and the discussion after Eq. (23)]. Consequently, the sign function in (12) is replaced by a truncated series  $\epsilon_\Lambda(x^-)$  defined by Eq. (8) with  $n \leq \Lambda$ .

We may use  $U(b)$  to generate a family of shifted vacuum states  $|b\rangle = U(b)|0\rangle$ , where  $|0\rangle$  is the Fock vacuum  $a_n|0\rangle = 0$ . Can one of these states be a better candidate for the true physical vacuum? To determine this, let us minimize the expectation value of the LF Hamiltonian,

$$\begin{aligned} \langle b|P^-|b\rangle &= \langle 0|U^{-1}(b)P^-U(b)|0\rangle \\ &= \langle 0|\frac{1}{2} \int_{-L}^{+L} dx^- T_b^{+-}(x^-)|0\rangle, \end{aligned} \quad (13)$$

where

$$\begin{aligned} T_b^{+-}(x^-) &= \left[ -\mu^2(\phi + b\epsilon_\Lambda(L - x^-))^2 \right. \\ &\quad \left. + \frac{\lambda}{2}(\phi + b\epsilon_\Lambda(L - x^-))^4 \right]. \end{aligned} \quad (14)$$

As shown in the Appendix, for sufficiently large value of  $\Lambda$  the function  $\epsilon_\Lambda(L - x^-)$  differs only negligibly from unity on the interval  $-L \leq x^- \leq L$ . The same is true also for its powers. We will therefore suppress henceforth the symbol of the sign function in the formulae similar to (14). Also, due to the finite number of Fourier modes, the function  $\epsilon_\Lambda(L - x^-)$  does not have an exactly rectangular shape but it is smooth in the neighborhood of the points  $x^- = \pm L$  (see the Appendix).

Now, for the expectation value of the energy we find  $\langle b|P^-|b\rangle = Lb^2(\frac{\lambda}{2}b^2 - \mu^2)$  which has a nontrivial minimum for  $b^2 = \frac{\mu^2}{\lambda} \equiv v^2$ . The LF energy density is lower in the new vacuum  $|v\rangle$ :

$$\langle v|P^-|v\rangle/2L = -\frac{\mu^4}{4\lambda} < \langle 0|P^-|0\rangle/2L = 0. \quad (15)$$

The vacuum expectation value (VEV) of the scalar field in this state coincides with the position of the minimum of the classical potential

$$\begin{aligned} \langle v|\phi(x)|v\rangle &= \langle 0|U^{-1}(v)\phi(x)U(v)|0\rangle \\ &= \frac{\mu}{\sqrt{\lambda}}\epsilon_\Lambda(x^- - L) = \frac{\mu}{\sqrt{\lambda}}. \end{aligned} \quad (16)$$

The last equality holds in the sense discussed after Eq. (14). The fact that the field expectation value is not a perfect constant is irrelevant here. The crucial point is that the scalar field can be shifted by a nonoperator piece which is a  $c$  number multiplied by a function approaching unity for the number of field modes tending to infinity [28].

Inserting the field expansion (6) into the definition of  $U(v)$ , we get a coherent state representing the physical vacuum of the model in the semiquantum approximation:

$$\begin{aligned} |v\rangle &= \exp\left\{v \sum_{n=1/2}^{\Lambda} \tilde{c}_n(a_n^\dagger - a_n)\right\}|0\rangle \\ &= \mathcal{N} \exp\left\{v \sum_{n=1/2}^{\Lambda} \tilde{c}_n a_n^\dagger\right\}|0\rangle, \end{aligned} \quad (17)$$

where

$$\begin{aligned} \tilde{c}_n &= 4(-1)^{n-1/2}/\sqrt{\pi n}, \\ \mathcal{N} &= \exp\left\{-\frac{v^2}{2} \sum_{n=1/2}^{\Lambda} \tilde{c}_n^2\right\} \approx \exp\left\{-\frac{8v^2}{\pi} \ln \Lambda\right\}. \end{aligned} \quad (18)$$

Notice that the coherent states (17) are  $L$ -independent and also correctly normalized,  $\langle v|v\rangle = 1$ . Further, the scalar product  $\langle -v|v\rangle = \mathcal{N}^4 = \Lambda^{-32v^2/\pi}$  and thus the overlap between the two vacua is very small for large value of  $\Lambda$  (and approaches zero for  $\Lambda$  tending to infinity). This means that, in contrast to the spacelike theory, the two vacua become orthogonal even in the finite volume in the (formal) limit of infinite number of degrees of freedom.

The corresponding multiparticle spaces cannot be generated by applying creation operators  $a_n^\dagger$  on  $|v\rangle$  because these states do not form an orthogonal basis. Instead, one can transform the original Fock states, built on  $|0\rangle$ , by means of  $U(v)$  [29,32]. The Hamiltonian matrix elements will be (up to normalization) of the form

$$\langle 0|a_{m_1} a_{m_2} \dots a_{m_i} U^{-1}(v)P^-U(v)a_{n_1}^\dagger \dots a_{n_2}^\dagger a_{n_i}^\dagger |0\rangle. \quad (19)$$

Thus, the physically relevant Hamiltonian is the transformed (“effective”) one, equal to  $P_{(v)}^- = U^{-1}(v)P^-U(v)$ :

$$P_{(v)}^- = \frac{1}{2} \int_{-L}^{+L} dx^- : \left[ 2\mu^2 \phi^2 + \frac{\lambda}{2} \phi^4 + 2\lambda v \phi^3 - \frac{\mu^4}{2\lambda} \right] :. \quad (20)$$

It has a correct sign of the term quadratic in  $\phi$  and hence it describes a massive scalar field with mass equal to  $\sqrt{2}\mu$ . However, it has lost the symmetry of the original Hamiltonian under  $\phi(x) \rightarrow -\phi(x)$ —this symmetry has been broken by choosing  $|v\rangle$  as the vacuum state. Actually, the theory originally had also the second ground state. This one can demonstrate by considering a unitary operator that implements the original discrete symmetry

$$V(\pi) = \exp\left[-i\pi \sum_{n=1/2}^{\Lambda} a_n^\dagger a_n\right]. \quad (21)$$

It acts correctly on the creation and annihilation operators

$$V(\pi)a_n V^{-1}(\pi) = -a_n, \quad V(\pi)a_n^\dagger V^{-1}(\pi) = -a_n^\dagger \quad (22)$$

and hence leaves  $P^-$  invariant,  $V(\pi)P^-V^{-1}(\pi) = P^-$ . The operator  $V(\pi)$  generates the second vacuum

$$V(\pi)|v\rangle = | -v\rangle, \quad (23)$$

since  $V(\pi)U(v) = U(-v)V(\pi)$ . The latter relation follows from the operator identity  $\exp(A)\exp(B) = \exp(e^\rho B) \times \exp(A)$ , valid if  $[A, B] = \rho B$  ( $\rho$  = real parameter.) We easily find  $\langle -v|\phi(x^-)| -v\rangle = -v$ . The corresponding effective Hamiltonian  $P_{(-v)}^-$  in the space sector built on  $| -v\rangle$  coincides with the expression (20) up to the opposite sign of the cubic term. Although both Hamiltonians are individually not invariant, they are connected by the “parity” transformation  $P_{(-v)}^- = V(\pi)P_{(v)}^-V^{-1}(\pi)$  and vice versa. We can choose either of the two vacua and their corresponding effective Hamiltonian to describe the physical system under study.

An alternative way of obtaining the coherent state vacuum (17) is to minimize the expectation value of the Hamiltonian in the coherent states  $|\alpha\rangle$ ,  $|\alpha\rangle \sim \exp(\sum \alpha_n a_n^\dagger)|0\rangle$ , imposing the condition that the expectation value of the antiperiodic field is constant (which drops to zero at the interval boundaries to satisfy antiperiodicity). If one requires instead of a constant value for  $\langle \alpha|\phi(x^-)|\alpha\rangle$  the value  $-v$  for  $-L \leq x^- \leq 0$  and  $v$  for  $0 \leq x^- \leq L$ , i.e. a steplike shape, one obtains a configuration that also minimizes the LF energy and qualitatively approximates a kink [16]:

$$|\alpha\rangle = \exp\left[v \sum_{n=1/2}^{\Lambda} \alpha_n (a_n^\dagger - a_n)\right]|0\rangle, \quad \alpha_n = \frac{4i}{\sqrt{\pi n}}. \quad (24)$$

In  $x$  representation, the state  $|\alpha\rangle$  can be expressed in terms of the unitary operator  $W(v)$  as

$$|\alpha\rangle = W(v)|0\rangle, \quad W(v) = e^{i8v\phi(0)} \quad (25)$$

and one easily obtains

$$\langle \alpha|\phi(x)|\alpha\rangle = \langle 0|W^{-1}(v)\phi(x)W(v)|0\rangle = v\epsilon_\Lambda(x^-), \quad (26)$$

which is the result indicated above. Note also that the kink state  $|\alpha\rangle$  has for large  $\Lambda$  a negligibly small overlap with the vacuum state [33]  $\langle v|\alpha\rangle \sim \exp(-\ln\Lambda)$ . These states are orthogonal in the formal limit  $\Lambda \rightarrow \infty$ . They belong to the sectors with different topological charges (superselection sectors):

$$\begin{aligned}\langle\alpha|Q|\alpha\rangle &= v^{-1}\langle 0|W^{-1}(v)\phi(L)W(v)|0\rangle = 8i[\phi(L), \phi(0)] \\ &= \epsilon_\Lambda(L) = 1. \\ \langle v|Q|v\rangle &= v^{-1}\langle 0|U^{-1}(v)\phi(L)U(v)|0\rangle \\ &= v^{-1}\langle 0|\phi(L)|0\rangle = 0.\end{aligned}\quad (27)$$

Quantitative predictions of the properties of kink and anti-kink in quantum theory were obtained by LF Hamiltonian matrix diagonalizations using discretized light-cone quantization [15,16].

Finally, let us discuss the LF momentum of the coherent-state vacuum  $U(v)|0\rangle$  and of the transformed Fock states  $U(v)a_{m_1}^\dagger a_{m_2}^\dagger \dots |0\rangle$ . Our vacua are not momentum eigenstates since they are by definition only eigenstates of the annihilation operator. They represent an approximation to the true physical vacuum. One can calculate expectation values of physical quantities in these states. The VEV of an unordered  $P^+$  would be

$$\begin{aligned}\langle v|P^+|v\rangle &= 2 \int_{-L}^{+L} dx^- \langle 0|[\partial_-(\phi(x) + v\epsilon_\Lambda(L-x^-))]^2|0\rangle \\ &= \frac{\pi}{L} \sum_{n=1/2}^{\Lambda} \left( n + \frac{32}{\pi} v^2 \right).\end{aligned}\quad (28)$$

The first term on the right-hand side is removed by normal ordering. The second term, equal to  $16v^2\delta_\Lambda(0)$ , is a consequence of the fact that  $\partial_-\epsilon_\Lambda(L-x^-) = -2\delta_\Lambda(L-x^-)$  which for  $\Lambda \rightarrow \infty$  is singular just at the endpoints  $x^- = \pm L$ . For finite  $\Lambda$  the second term is a finite constant  $C$ . It is also present in the expectation values of the LF momentum of particle states

$$\begin{aligned}\langle 0|a_l U^{-1}(v)P^+ U(v)a_l^\dagger|0\rangle &= p_l^+ + C, \\ \langle 0|a_k a_l U^{-1}(v)P^+ U(v)a_k^\dagger a_l^\dagger|0\rangle &= p_k^+ + p_l^+ + C,\end{aligned}\quad (29)$$

and similarly for higher many-particle states. Thus the LF momentum of the transformed states is shifted by the same constant value which is physically irrelevant since it cancels in the differences between any two levels. We shall therefore subtract this unphysical constant. Let us remark that the necessity to perform the (trivial) renormalization of the  $P^+$  operator may seem a little unusual but actually it is natural and physically transparent: the shift of the scalar field due to  $U(v)$  is almost precisely equal to a constant in the whole box except for the small neighborhood of the endpoints. The expectation values of the momentum operator receive large but common contributions from the neighborhood of the endpoints due to an  $x^-$  integral over  $[\delta_\Lambda(L-x^-)]^2$ .

Since the approximative vacuum states  $|v\rangle$  are not eigenstates of  $P^+$ , the translational invariance of the theory can only be formulated in a weaker form. The Heisenberg equation  $-2i\partial_-\phi(x) = [P^+, \phi(x)]$  is satisfied on the vacuum state in the sense of matrix elements. The usual condition  $\exp(i\vec{a} \cdot \vec{P})|\text{vac}\rangle = |\text{vac}\rangle$  implying  $\langle \text{vac}|\exp(i\vec{a} \cdot \vec{P})|\text{vac}\rangle = 1$  is replaced by  $\langle v|\exp(\frac{i}{2}a^- P^+)|v\rangle = \exp(\frac{i}{2}a^- C)$  here.

As is well-known, the studied two-dimensional scalar model exhibits a second-order phase transition from symmetric to broken phase [22]. This has been demonstrated at the level of effective Hamiltonians by varying the coupling constant and keeping the mass parameter fixed.

On the other hand, the DLCQ method with antiperiodic fields has been used recently to study the mass eigenvalue spectrum and other observables in the broken phase for strong coupling [34]. An interesting level crossing, a change in the expectation value of the integral of the  $\phi^2$  operator, and an evidence of kink condensation has been obtained suggesting an onset of symmetry restoring phase transition.

Our formulation of the broken phase in the two-dimensional  $\lambda\phi^4$  theory may also be used in the Hamiltonian matrix diagonalizations for strong coupling [35]. One can test a possibility that the new matrix elements (19) generated by working with the vacuum  $|v\rangle$  and the Hamiltonian  $P_v^-$  may be important for an improved description of the energy eigenstates and of other observables for small coupling constant and especially for strong coupling and the symmetry restoration region.

Recall that in the DLCQ method one diagonalizes Hamiltonian matrices for a fixed value of the dimensionless momentum  $K = \frac{L}{2\pi}P^+$ . The value of  $K$  determines simultaneously the maximum momentum mode in the Fock expansion of the field and hence the summation in the coherent-state vacuum (17) will be truncated by  $K$ . The corresponding transformed DLCQ Hamiltonian  $\tilde{H} = \frac{2\pi}{L}P^-$  takes the form

$$\begin{aligned}\tilde{H} &= -\mu^2 \sum_{n=1/2}^{\Lambda} \frac{1}{n} A_n^\dagger A_n + \frac{\lambda}{8\pi} \sum_{klmn} \frac{1}{\sqrt{klmn}} \\ &\quad \times [2A_k^\dagger A_l A_m A_n \delta_{k,l+m+n} + 3A_k^\dagger A_l^\dagger A_m A_n \delta_{k+l,m+n} \\ &\quad + 2A_k^\dagger A_l^\dagger A_m^\dagger A_n \delta_{k+l+m,n}], \\ A_n &= U^{-1}(v)a_n U(v) = a_n + v\tilde{c}_n.\end{aligned}\quad (30)$$

For large  $\Lambda$  this Hamiltonian approaches the limiting form (20) as can be shown by evaluating explicitly the powers of the operators  $A_n$ , regrouping terms, and using the definition of the sign function in terms of discrete modes. In real DLCQ computations one should diagonalize the above Hamiltonian calculated in the Fock basis for given value of  $K \approx 40-60$  and then extrapolate results to  $K \rightarrow \infty$ .

### A. SSB with periodic boundary conditions

As discussed in the introduction, there have been attempts to understand SSB in the LF theory either without imposing boundary conditions explicitly or by employing periodic ones, typically starting from the symmetric phase of the theory. Some aspects have also been studied with the antiperiodic fields [18,36]. Can one give a direct formulation of the broken phase using periodic boundary conditions (PBC)? The problem is complicated because one has to solve the operator constraint for the dependent zero-mode  $\phi_0$ . At present, this appears possible only for small coupling, where one can use perturbation theory. Perturbative solution is however quite interesting because it corresponds to the semiclassical regime of the broken phase and one can compare the results with the results of the previous section. The physical picture obtained by imposing antiperiodic boundary condition should be quite accurate far from the critical region, i.e. for small value of the coupling constant. Since a derivation of a semiclassical vacuum state similar to the case of antiperiodic boundary conditions seems not to be possible for PBC, one may expect that the physical vacuum state will coincide with the Fock vacuum and SSB will manifest itself by the presence of two Hamiltonians [12].

The field equation for the scalar field following from the Lagrangian (2) is

$$4\partial_+\partial_-\phi = \mu^2\phi + \lambda\phi^3. \quad (31)$$

The scalar field can be decomposed as  $\phi(x) = \phi_0(x^+) + \varphi(x^+, x^-)$ , with  $\phi_0$  being the  $x^-$ -independent part carrying  $p^+ = 0$ . Projection of the field equation (31) on the zero-mode sector

$$\mu^2\phi_0 = -\lambda \int_{-L}^{+L} dx^- (\phi_0 + \varphi)^3 \quad (32)$$

shows that  $\phi_0$  is a dependent variable which has to be expressed in terms of all other (normal) modes [37]. The perturbative solution of the classical zero-mode constraint was given by Robertson [38] and has two physical branches:

$$\begin{aligned} \phi_0^{(1)} &= \frac{\mu}{\sqrt{\lambda}} - \frac{3}{2} \frac{\sqrt{\lambda}}{\mu} \int_{-L}^{+L} \frac{dx^-}{2L} \varphi^2 - \frac{1}{2} \frac{\lambda}{\mu^2} \int_{-L}^{+L} \frac{dx^-}{2L} \varphi^3 \\ \phi_0^{(2)} &= -\frac{\mu}{\sqrt{\lambda}} + \frac{3}{2} \frac{\sqrt{\lambda}}{\mu} \int_{-L}^{+L} \frac{dx^-}{2L} \varphi^2 - \frac{1}{2} \frac{\lambda}{\mu^2} \int_{-L}^{+L} \frac{dx^-}{2L} \varphi^3. \end{aligned} \quad (33)$$

To the given order it can be taken over to the quantum theory since there is no ordering ambiguity. Note that the solutions contain a constant piece and their structure differs completely from the perturbative solution in the symmetric phase [39] because of the opposite sign of the  $\mu^2$  term in the field equation. Under  $\varphi \rightarrow -\varphi$ , we have  $\phi_0^{(1)} \rightarrow -\phi_0^{(2)}$  and vice versa. When these two solutions are inserted into the PBC Hamiltonian, analogous to (4),

$$P^- = \frac{1}{2} \int_{-L}^{+L} dx^- \left[ -\mu^2(\phi_0 + \varphi)^2 + \frac{\lambda}{2}(\phi_0 + \varphi)^4 \right], \quad (34)$$

one indeed gets through  $O(\lambda)$  two Hamiltonians

$$\begin{aligned} P_{(\pm v)}^- &= \frac{1}{2} \int_{-L}^{+L} dx^- \left[ 2\mu^2\varphi^2 + \frac{\lambda}{2}\varphi^4 \pm 2\mu\sqrt{\lambda}\varphi^3 - \frac{\mu^4}{2\lambda} \right. \\ &\quad \left. - \frac{9}{2}\lambda\varphi^2 \int_{-L}^{+L} \frac{dx^-}{2L} \varphi^2 \right]. \end{aligned} \quad (35)$$

Their structure is similar to the Hamiltonians  $P_v^-$  from the case of antiperiodic boundary conditions. Each Hamiltonian separately violates the symmetry under  $\varphi \rightarrow -\varphi$  but the transformation connects them. Any of them can be chosen for calculating physical properties of the system. Their eigenstates will also be connected by the parity transformation. It is an interesting problem for DLCQ to find the lowest energy levels of the Hamiltonians (35).

### III. SYMMETRY BREAKING IN $\lambda(\phi^*\phi)^2(2+1)$ THEORY

As the next step, we could consider a two-dimensional theory of a self-interacting complex scalar field. The corresponding Hamiltonian has a continuous symmetry instead of the discrete one. The full treatment requires a discussion of the LF version of the Coleman theorem which prohibits SSB in one space dimension [5]. Since this topic deserves a separate analysis, here we will study the  $O(2)$ -symmetric sigma model in three dimensions. It is defined by the classical Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi^\dagger \partial^\mu \phi + \frac{1}{2} \mu^2 \phi^\dagger \phi - \frac{1}{4} \lambda (\phi^\dagger \phi)^2. \quad (36)$$

The system will be studied in a finite volume  $V = 4LL_\perp$ ,  $-L \leq x^- \leq L$ ,  $-L_\perp \leq x_\perp \leq L_\perp$ . Scalar fields are taken antiperiodic in both  $x^-$  and the transverse coordinate  $x_\perp$ . In terms of two real scalar fields introduced by  $\phi(x) = \sigma(x) + i\pi(x)$ , the corresponding LF Lagrangian density

$$\begin{aligned} \mathcal{L}_{\text{lf}} &= 2\partial_+\sigma\partial_-\sigma + 2\partial_+\pi\partial_-\pi - \frac{1}{2}(\partial_\perp\sigma)^2 - \frac{1}{2}(\partial_\perp\pi)^2 \\ &\quad + \frac{\mu^2}{2}(\sigma^2 + \pi^2) - \frac{\lambda}{4}(\sigma^2 + \pi^2)^2 \end{aligned} \quad (37)$$

is invariant under  $O(2)$  rotations

$$\begin{aligned} \sigma(x) &\rightarrow \sigma(x) \cos\alpha - \pi(x) \sin\alpha, \\ \pi(x) &\rightarrow \sigma(x) \sin\alpha + \pi(x) \cos\alpha. \end{aligned} \quad (38)$$

The associated conserved current is  $j^\mu = \sigma\partial^\mu\pi - \partial^\mu\sigma\pi$ . The field expansions at  $x^+ = 0$  are

$$\sigma(\underline{x}) = \frac{1}{\sqrt{V}} \sum_{\underline{n}} \frac{1}{\sqrt{p_n^+}} [a(p_{\underline{n}}) e^{-ip_{\underline{n}} \cdot \underline{x}} + a^\dagger(p_{\underline{n}}) e^{ip_{\underline{n}} \cdot \underline{x}}], \quad (39)$$

$$\pi(\underline{x}) = \frac{1}{\sqrt{V}} \sum_{\underline{n}} \frac{1}{\sqrt{p_{\underline{n}}^+}} [c(p_{\underline{n}}) e^{-ip_{\underline{n}} \cdot \underline{x}} + c^\dagger(p_{\underline{n}}) e^{ip_{\underline{n}} \cdot \underline{x}}]. \quad (40)$$

We use the notation  $\underline{x} = (x^-, x_\perp)$ ,  $\underline{n} \equiv (n, n_\perp)$ ,  $p_{\underline{n}} = (p_{\underline{n}}^+, p_{\underline{n}\perp}) = (\frac{2\pi}{L} n, \frac{\pi}{L_\perp} n_\perp)$  with  $n, n_\perp = 1/2, 3/2, \dots, \infty$ . The conjugate momenta are  $\Pi_\sigma = 2\partial_- \sigma$ ,  $\Pi_\pi = 2\partial_- \pi$ . The  $\sigma$  field operators satisfy the commutation relation

$$[\sigma(0, \underline{x}), \sigma(0, \underline{y})] = -\frac{i}{8} \epsilon_a(x^- - y^-) \delta_a(x_\perp - y_\perp). \quad (41)$$

The commutator of the  $\pi$  fields has the same form. The Hamiltonian is

$$P^- = \int_V d^2 \underline{x} [(\partial_\perp \sigma)^2 + (\partial_\perp \pi)^2 + 2V(\sigma^2 + \pi^2)],$$

$$V(\sigma^2 + \pi^2) = -\frac{\mu^2}{2}(\sigma^2 + \pi^2) + \frac{\lambda}{4}(\sigma^2 + \pi^2)^2, \quad (42)$$

where  $d^2 \underline{x} = \frac{1}{2} dx^- dx_\perp$ . In principle, both  $\sigma(x)$  and  $\pi(x)$  can be transformed by the unitary operators  $U_\sigma(b)$  and  $U_\pi(b)$  in analogy with Eq. (12). It is simpler however to start by shifting only one field which we choose in accord with the standard treatment to be  $\sigma$ :

$$U_\sigma(b) \sigma(\underline{x}) U_\sigma^\dagger(b) = \sigma(\underline{x}) - b \epsilon_\Lambda(L - x^-) \epsilon_\Lambda(x_\perp - L_\perp), \quad (43)$$

with

$$U_\sigma(b) = \exp\left[-4ib \int_V d^2 \underline{x} \Pi_\sigma(\underline{x})\right]$$

$$= \exp\left[-8ib \int_{-L_\perp}^{+L_\perp} dx_\perp \sigma(L, x_\perp)\right]. \quad (44)$$

By minimization of  $\langle b; 0 | P^- | b; 0 \rangle$ , where  $|b; 0\rangle = U_\sigma(b) |0\rangle$ , we find that the (approximate) physical vacuum  $|v; 0\rangle = U_\sigma(v) |0\rangle$  corresponds to the value  $b = \frac{\mu}{\sqrt{\lambda}} \equiv v$  and

$$|v; 0\rangle = \exp\left\{-v \sum_{\underline{n}} \tilde{c}(p_{\underline{n}}) [a^\dagger(p_{\underline{n}}) - a(p_{\underline{n}})]\right\} |0\rangle, \quad (45)$$

$$\tilde{c}(p_{\underline{n}}) = \frac{8}{\pi} \sqrt{\frac{L_\perp}{2\pi}} \frac{(-1)^{n+n_\perp}}{\sqrt{nn_\perp}}. \quad (46)$$

The rotations (38) are implemented by the unitary operators  $V(\alpha) = e^{i\alpha Q}$ , where  $Q = \int_V d^2 \underline{x} j^+(\underline{x})$ :

$$\sigma(x) \rightarrow V(\alpha) \sigma(x) V^\dagger(\alpha), \quad \pi(x) \rightarrow V(\alpha) \pi(x) V^\dagger(\alpha), \quad (47)$$

$$V(\alpha) = \exp\left[\alpha \sum_{\underline{n}} (a^\dagger(p_{\underline{n}}) c(p_{\underline{n}}) - c^\dagger(p_{\underline{n}}) a(p_{\underline{n}}))\right]. \quad (48)$$

The operators  $V(\alpha)$  extend the ‘‘primary’’ vacuum  $|v; 0\rangle$  to the infinite family  $|v; \alpha\rangle = V(\alpha) |v; 0\rangle$ . Explicitly, we get

$$|\alpha; v\rangle = \exp\left\{-v \sum_{\underline{n}} \tilde{c}(p_{\underline{n}}) [(a^\dagger(p_{\underline{n}}) - a(p_{\underline{n}})) \cos\alpha + (c^\dagger(p_{\underline{n}}) - c(p_{\underline{n}})) \sin\alpha]\right\} |0\rangle. \quad (49)$$

In spite of the presence of the box length  $L_\perp$  in the coherent state (45), the orthogonality  $\langle v; \alpha | v; \alpha' \rangle = \delta_{\alpha\alpha'}$  holds in the limit of infinite number of longitudinal modes  $n$ .

We can interpret the relation for the vacuum and particle matrix elements of  $P^-$  [cf. Eq. (19)] as defining an effective Hamiltonian  $P_v^- = U_\sigma^\dagger(v) P^- U_\sigma(v)$ :

$$P_v^- = \int_V d^2 \underline{x} \left[ (\partial_\perp \sigma)^2 + (\partial_\perp \pi)^2 + 2\mu^2 \sigma^2 + 2\sqrt{\lambda} \mu \sigma(\sigma^2 + \pi^2) + \frac{\lambda}{2} (\sigma^2 + \pi^2)^2 \right] \quad (50)$$

[see the remark after Eq. (14)]. The form of the above Hamiltonian suggests that  $\sigma(x)$  corresponds to a massive field because its mass term has a correct sign while the mass term is missing for  $\pi(x)$  which became a Goldstone boson field. This tree-level result is more rigorously expressed by the Goldstone theorem.

In the usual proof of the Goldstone theorem [3], one inserts a complete set of energy and momentum operator eigenstates into the VEV of the commutator

$$[Q, \pi(x)] = \sigma(x) \quad (51)$$

and then invokes translational invariance to show a singularity in the spectral function for  $p^2 = 0$  [3,40]. This means that there exists a massless state in the spectrum. We can proceed analogously because we have all the necessary components for the proof. A difference with respect to the usual theory is that here we have an explicit realization of the vacuum in the Fock representation, not just an abstract state with postulated properties. The states  $|\alpha; v\rangle$  represent however only an approximative variational estimate of true degenerate family of ground states. But its existence tells us that there must exist exact eigenstates of the LF Hamiltonian with energy lower than the energy of the Fock vacuum  $|0\rangle$ . This is sufficient for the usual proof of the Goldstone theorem. Some ingredients of the proof are actually valid also if we used the approximative  $|\alpha; v\rangle$  states. Namely, the above commutator (51) is a rigorous consequence of Eqs. (38) and (47). To show that, one only has to use the infinitesimal form of both transformation laws and compare the leading terms in the expansion. The vacuum expectation value of the commutator is

$$\langle v; 0 | [Q, \pi(0)] | v; 0 \rangle = \langle 0 | U_\sigma^{-1}(v) \sigma(0) U_\sigma(v) | 0 \rangle = v. \quad (52)$$

If we denote the set of exact vacuum states by  $|\Omega_\alpha\rangle$ , then we should also have

$$\langle\Omega_0|[Q, \pi(0)]|\Omega_0\rangle = \langle\Omega_0|\sigma(0)|\Omega_0\rangle = f_v, \quad (53)$$

where  $f_v$  is the expectation value (not known precisely) of the  $\sigma$  field in the true physical vacuum  $|\Omega_\alpha\rangle$ . Let  $|n\rangle$  be the set of simultaneous eigenstates of the LF energy and momentum operators  $P^\mu|n\rangle = p^\mu|n\rangle$  where  $p^\mu = (E_n^-, P_n^+, P_n^\perp)$ . Inserting such a complete set in the form of  $\hat{1} = \sum_n |n\rangle\langle n|$  into the relation (53), using the definition of the charge operator as a volume integral of  $j^+(x)$  as well as the translational invariance of the theory,

$$\begin{aligned} j^+(x) &= \exp(ix_\mu P^\mu) j^+(0) \exp(-ix_\mu P^\mu), \\ \exp(ix_\mu P^\mu) |\Omega_\alpha\rangle &= |\Omega_\alpha\rangle, \end{aligned} \quad (54)$$

we find

$$\begin{aligned} &\frac{2}{V} \sum_n \delta^2(\underline{p}_n) \exp\left(-\frac{i}{2} E_n^- x^+\right) \langle\Omega_0|j^+(0)|n\rangle\langle n|\pi(0)|\Omega_0\rangle \\ &\quad - \frac{2}{V} \sum_n \delta^2(\underline{p}_n) \exp\left(\frac{i}{2} E_n^- x^+\right) \langle\Omega_0|\pi(0)|n\rangle\langle n|j^+(0)|\Omega_0\rangle \\ &= f_v. \end{aligned} \quad (55)$$

It follows from the VEV of the volume integral of the commutator  $[\partial_\mu j^\mu, \pi(0)] = 0$  that  $f_v$  has indeed to be  $x^+$ -independent:

$$\begin{aligned} &\left[ \left( \partial_+ \int_V d^2\underline{x} j^+(x) + \int_V d^2\underline{x} \partial_- j^-(x) \right. \right. \\ &\quad \left. \left. + \int_V d^2\underline{x} \partial_\perp j^\perp(x) \right), \pi(0) \right] = 0, \end{aligned} \quad (56)$$

where the second and the third term in the commutator vanishes due to the fact that the current obeys periodic BC in  $x^-$  and  $x_\perp$ . In order that the left-hand side of Eq. (55) is also  $x^+$ -independent, there must exist an eigenstate  $|G\rangle$  of  $P^\mu$  which for  $p^+ = 0$ ,  $p^\perp = 0$  has  $E^- = 0$  (so that the  $x^+$  dependence vanishes), while  $\langle\Omega_0|\pi(0)|G\rangle \neq 0$ ,  $\langle\Omega_0|j^+(0)|G\rangle \neq 0$ . Since  $M^2 = E^- p^+ - p_\perp^2$ , this state is massless. Note that the Nambu-Goldstone state is not simply  $c^\dagger(\underline{k})|\Omega_0\rangle$  since the latter is not an eigenstate of  $P^-$ . The correct linear combination of Fock states can be (at least in principle) obtained by a Hamilton matrix diagonalization.

#### IV. CONCLUSIONS

To summarize, in this work a novel strategy to the spontaneous symmetry breaking phenomenon in the light front description was formulated. The approach is based on quantization in a finite volume and on a unitary transformation of the Fock LF vacuum to the ground states with lower value of the LF energy. These semiclassical

vacua are degenerate and have a form of boson coherent states. The general properties of a spontaneously broken phase of the theory including existence of the massless Goldstone boson have been derived. We believe that the present picture of spontaneous symmetry breaking in light front field theory in terms of semiclassical vacuum states adds a further evidence that there is no conflict between the ‘‘triviality’’ of the LF vacuum of interacting models and a rich nonperturbative content of quantum field theory.

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#### APPENDIX

We present a few details of the regularized Dirac delta function and the sign function in this Appendix for completeness. Regularization is performed in two steps: a cut-off on number of modes (as discussed in the main text) and a convergence factor governed by a small parameter  $\epsilon$ . The corresponding formulae read

$$\begin{aligned} \delta_\Lambda(x^- - y^-) &= \frac{1}{2L} \sum_{n=1/2}^\Lambda (e^{-(i/2)p_n^+(x^- - y^- - i\epsilon)} \\ &\quad + e^{(i/2)p_n^+(x^- - y^- + i\epsilon)}), \\ \epsilon_\Lambda(x^- - y^-) &= \frac{4i}{L} \sum_{n=1/2}^\Lambda \frac{1}{p_n^+} (e^{-(i/2)p_n^+(x^- - y^- - i\epsilon)} \\ &\quad - e^{(i/2)p_n^+(x^- - y^- + i\epsilon)}). \end{aligned} \quad (A1)$$

The  $\pm i\epsilon$  terms in the exponents ensure a smooth behavior in the neighborhood of the points where these functions diverge (for  $\Lambda \rightarrow \infty$ ) or drop to zero. This is quite analogous to the continuum theory where the same convergence factors guarantee existence of corresponding integrals that replace the discrete series (A1) [41]. Figures 1–4 display differences between the functions with and without the convergence factors for typical values of the box length and of the number of field modes. The shifted function  $\epsilon_\Lambda(L - x^-)$  is equal to unity to a very high precision over the whole interval  $|x^-| \leq L$  except for the endpoints  $x^- = \pm L$  where it behaves in the same manner as  $\epsilon_\Lambda(x^-)$  in the neighborhood of  $x^- = 0$ .

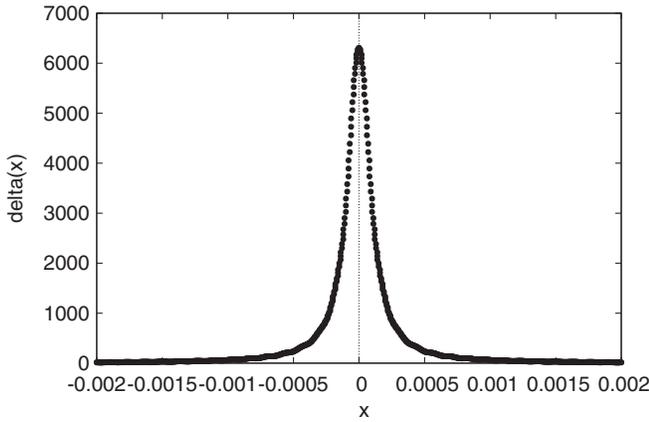


FIG. 1. Regularized delta function  $\delta_\Lambda(x^-)$  for  $L = 20$ ,  $\Lambda = 3 \cdot 10^5$ , and  $\epsilon = 10^{-4}$ .

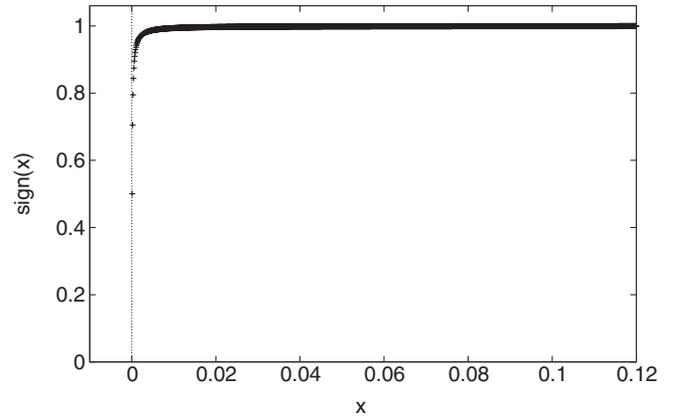


FIG. 3. Regularized sign function  $\epsilon_\Lambda(x^-)$  for  $L = 20$ ,  $\Lambda = 3 \cdot 10^5$ , and  $\epsilon = 10^{-4}$  in the neighborhood of  $x^- = 0$ . The function is a constant to a very high precision over the whole interval except for a tiny neighborhood of the point  $x^- = 0$  (or of the endpoints  $x^- = \pm L$  in the case of the shifted function  $\epsilon_\Lambda(L - x^-)$ ).

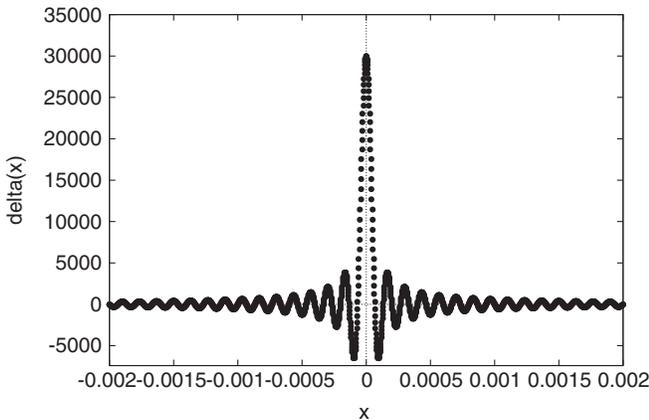


FIG. 2. Unregularized delta function  $\delta_\Lambda(x^-)$  (the parameter  $\epsilon = 0$ ) for  $L = 20$ , and  $\Lambda = 3 \cdot 10^5$ . The oscillations are rather strong and do not vanish for increased values of  $\Lambda$ .

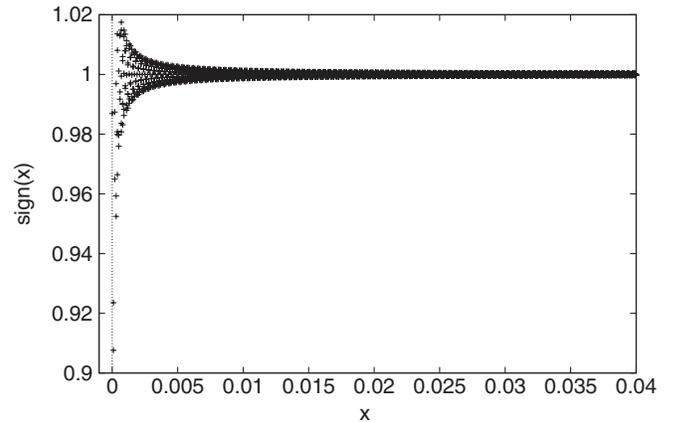


FIG. 4. Detailed behavior of the unregularized ( $\epsilon = 0$ ) sign function  $\epsilon_\Lambda(x^-)$  for  $L = 20$ ,  $\Lambda = 3 \cdot 10^5$  around  $x^- = 0$ . Although the function is again indistinguishable from a constant inside the finite interval, it oscillates around the point  $x^- = 0$  in contrast to the case with a nonzero regulator  $\epsilon$ .

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