PHYSICAL REVIEW D **78,** 105007 (2008)

Fermionic vacuum polarization by a cylindrical boundary in the cosmic string spacetime

E. R. Bezerra de Mello,[1,*] V. B. Bezerra,[1,+] A. A. Saharian,[1,2,‡] and A. S. Tarloyan[2,3]

[1]*Departamento de Física, Universidade Federal da Paraíba, 58.059-970, Caixa Postal 5.008, João Pessoa, PB, Brazil*

[2]*Department of Physics, Yerevan State University, 0025 Yerevan, Armenia*

[3]*Yerevan Physics Institute, 0036 Yerevan, Armenia*

(Received 8 September 2008; published 26 November 2008)

The vacuum expectation values of the energy-momentum tensor and the fermionic condensate are analyzed for a massive spinor field obeying the MIT bag boundary condition on a cylindrical shell in the cosmic string spacetime. Both regions inside and outside the shell are considered. By applying to the corresponding mode sums a variant of the generalized Abel-Plana formula, we explicitly extract the parts in the expectation values corresponding to the cosmic string geometry without boundaries. In this way the renormalization procedure is reduced to that for the boundary-free cosmic string spacetime. The parts induced by the cylindrical shell are presented in terms of integrals rapidly convergent for points away from the boundary. The behavior of the vacuum densities is investigated in various asymptotic regions of the parameters. In the limit of large values of the planar angle deficit, the boundary-induced expectation values are exponentially suppressed. As a special case, we discuss the fermionic vacuum densities for the cylindrical shell on the background of the Minkowski spacetime.

DOI: 10.1103/PhysRevD.78.105007 PACS numbers: 03.70.+k, 11.27.+d, 98.80.Cq

I. INTRODUCTION

It is well known that different types of topological objects may have been formed in the early universe after Planck time by the vacuum phase transition [1]. Depending on the topology of the vacuum manifold these are domain walls, strings, monopoles, and textures. Among them the cosmic strings are of special interest. Although the recent observational data on the cosmic microwave background radiation have ruled out cosmic strings as the primary source for primordial density perturbations, they are still candidates for the generation of a number of interesting physical effects such as the generation of gravitational waves, gamma ray bursts, and high-energy cosmic rays (see, for instance, [2]). Recently, cosmic strings attract a renewed interest partly because a variant of their formation mechanism is proposed in the framework of brane inflation [3].

In the simplest theoretical model describing the infinite straight cosmic string the spacetime is locally flat except on the string where it has a delta shaped Riemann curvature tensor. In quantum field theory the corresponding nontrivial topology induces nonzero vacuum expectation values (VEVs) for physical observables. Explicit calculations for the geometry of a single cosmic string have been done for different fields [4–29]. Vacuum polarization effects by higher-dimensional composite topological defects constituted by a cosmic string and global monopole are investigated in Refs. [30] for scalar and fermionic fields. Another type of vacuum polarization arises when boundaries are present. The imposed boundary conditions on quantum fields alter the zero-point fluctuations spectrum and result in additional shifts in the vacuum expectation values of physical quantities. This is the well-known Casimir effect (for a review see [31]). In Ref. [32], we have studied both types of sources for the polarization of the scalar vacuum, namely, a cylindrical boundary and a cosmic string, assuming that the boundary is coaxial with the string and that on this surface the scalar field obeys Robin boundary condition. For a massive scalar field with an arbitrary curvature coupling parameter we evaluated the Wightman function and the vacuum expectation values of the field squared and the energy-momentum tensor. The polarization of the electromagnetic vacuum by a conducting cylindrical shell in the cosmic string spacetime is investigated in [33] (for a combination of topological and boundary-induced quantum effects in the gravitational field of a global monopole see Refs. [34–36].)

Continuing in this line of investigation, in the present paper we analyze the polarization of the fermionic vacuum by a cylindrical shell coaxial with the cosmic string on which the field obeys the MIT bag boundary condition. We evaluate the fermionic condensate and vacuum expectation values of the energy-momentum tensor in both interior and exterior regions of the shell. The renormalized vacuum expectation value of the energy-momentum tensor for a fermionic field in the geometry of a cosmic string without boundaries is investigated in [7,9,21,22,28]. In addition to describing the physical structure of the quantum field at a given point, the energy-momentum tensor acts as a source of gravity in the Einstein equations and plays an important role in modelling a self-consistent dynamics involving the gravitational field. In the problem under consideration all calculations can be performed in a closed form and it constitutes an example in which the topological and

*emello@fisica.ufpb.br
+valdir@fisica.ufpb.br
‡saharian@ictp.it

boundary-induced polarizations of the vacuum can be separated in different contributions.

From the point of view of the physics in the region outside the string, the geometry considered in the present paper can be viewed as a simplified model for the nontrivial core. This model presents a framework in which the influence of the finite core effects on physical processes in the vicinity of the cosmic string can be investigated. In particular, it enables to specify conditions under which the idealized model with the core of zero thickness can be used. The corresponding results may shed light upon features of finite core effects in more realistic models, including those used for stringlike defects in crystals and superfluid helium. In addition, the problem considered here is of interest as an example with combined topological and boundary-induced quantum effects in which the vacuum characteristics such as energy density and stresses can be found in closed analytic form. From the results of the present paper, as a special case, we obtain the fermionic Casimir densities for a cylindrical shell with MIT boundary conditions in the Minkowski background (for the combined effects of a magnetic fluxon and MIT boundary conditions on the vacuum energy of a Dirac field see Refs. [37]). Note that, in addition to traditional problems of quantum field theory under the presence of material boundaries, the Casimir effect for cylindrical geometries can also be important to the flux tube models of confinement [38,39] and for determining the structure of the vacuum state in interacting field theories [40].

We have organized the paper as follows. In the next section, the eigenspinors for the region inside the cylindrical boundary are constructed and the eigenvalues of the corresponding quantum numbers are specified. These eigenspinors are the basis for the analysis of the Casimir densities in the following sections. In Sec. III, by using a variant of the generalized Abel-Plana formula, we extract from the mode sum of the fermionic condensate the part corresponding to the geometry of a cosmic string without the shell. The part induced by the shell is investigated in various asymptotic regions for the parameters. Section IV is devoted to the investigation of the boundary-induced parts in the vacuum expectation value of the energy-momentum tensor inside the cylindrical shell. The vacuum densities in the region outside the cylindrical shell are discussed in Sec. V. The main results of the paper are summarized in Sec. VI. In the appendix we give an alternative representation for the fermionic condensate and the expectation value of the energy-momentum tensor for a massive fermionic field in the geometry of a cosmic string without boundaries.

II. EIGENSPINORS INSIDE A CYLINDRICAL SHELL

We consider the background spacetime corresponding to an infinitely long straight cosmic string with the conical line element

$$ds^2 = dt^2 - dr^2 - r^2 d\phi^2 - dz^2, \tag{2.1}$$

where $0 \leq \phi \leq \phi_0 \leq 2\pi$ and the spatial points (r, ϕ, z) and $(r, \phi + \phi_0, z)$ are to be identified. The planar angle deficit is related to the mass per unit length of the string, μ_0, by $2\pi - \phi_0 = 8\pi G\mu_0$, where G is the Newton gravitational constant. In the discussion below, in addition to the parameter ϕ_0, we will use the combination

$$q = 2\pi/\phi_0. \tag{2.2}$$

The dynamics of a massive fermionic field is governed by the Dirac equation

$$i\gamma^\mu \nabla_\mu \psi - m\psi = 0, \tag{2.3}$$

with the covariant derivative operator defined as

$$\nabla_\mu = \partial_\mu + \Gamma_\mu. \tag{2.4}$$

Here $\gamma^\mu = e^\mu_{(a)}\gamma^{(a)}$ are the Dirac matrices in curved spacetime and Γ_μ is the spin connection given in terms of the flat space Dirac matrices $\gamma^{(a)}$ by the relation

$$\Gamma_\mu = \tfrac{1}{4}\gamma^{(a)}\gamma^{(b)}e^\nu_{(a)}e_{(b)\nu;\mu}. \tag{2.5}$$

Note that in this formula ";" means the standard covariant derivative for vector fields. In the relations above $e^\mu_{(a)}$ is the tetrad basis satisfying $e^\mu_{(a)}e^\nu_{(b)}\eta^{ab} = g^{\mu\nu}$, where η^{ab} is the Minkowski spacetime metric tensor.

In this paper we are interested in the change of the VEVs of the fermionic condensate and the energy-momentum tensor for a fermionic field, induced by a cylindrical shell coaxial with the string on which the field obeys the MIT bag boundary condition:

$$(1 + i\gamma^\mu n_\mu)\psi = 0, \qquad r = a, \tag{2.6}$$

where a is the cylinder radius and $n_\mu = (0, 1, 0, 0)$ is the outward-pointing normal to the boundary. For the evaluation of the VEVs we will use the direct mode summation procedure. In this approach we need to have the complete set of the eigenfunctions satisfying boundary condition (2.6).

In order to find these eigenfunctions, we will use the standard representation of the flat space Dirac matrices:

$$\gamma^{(0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^{(a)} = \begin{pmatrix} 0 & \sigma_a \\ -\sigma_a & 0 \end{pmatrix}, \qquad a = 1, 2, 3, \tag{2.7}$$

with $\sigma_1, \sigma_2, \sigma_3$ being the Pauli matrices. We take the tetrad fields in the form used before in [41] (see also [28,42]):

$$e^\mu_{(a)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(q\phi) & -\sin(q\phi)/r & 0 \\ 0 & \sin(q\phi) & \cos(q\phi)/r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.8}$$

where the index a identifies the rows of the matrix. With this choice, the gamma matrices are given by

$$\gamma^0 = \gamma^{(0)}, \qquad \gamma^l = \begin{pmatrix} 0 & \beta^l \\ -\beta^l & 0 \end{pmatrix}, \qquad l = 1, 2, 3, \quad (2.9)$$

where we have introduced the 2×2 matrices

$$\beta^1 = \begin{pmatrix} 0 & e^{-iq\phi} \\ e^{iq\phi} & 0 \end{pmatrix}, \qquad \beta^2 = -\frac{i}{r}\begin{pmatrix} 0 & e^{-iq\phi} \\ -e^{iq\phi} & 0 \end{pmatrix}, \qquad \beta^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.10)$$

For the spin connection and the combination appearing in the Dirac equation we find

$$\Gamma_\mu = \frac{1-q}{2}\gamma^{(1)}\gamma^{(2)}\delta^2_\mu, \qquad \gamma^\mu\Gamma_\mu = \frac{1-q}{2r}\gamma^1, \quad (2.11)$$

and the Dirac equation takes the form

$$\left(\gamma^\mu\partial_\mu + \frac{1-q}{2r}\gamma^1 + im\right)\psi = 0. \quad (2.12)$$

For positive frequency solutions, assuming the time-dependence of the eigenfunctions in the form $e^{-i\omega t}$ and decomposing the bispinor ψ into the upper and lower components, denoted by φ and χ, respectively, we find the equations

$$\begin{aligned} &\left(\beta^l\partial_l + \frac{1-q}{2r}\beta^1\right)\varphi - i(\omega+m)\chi = 0, \\ &\left(\beta^l\partial_l + \frac{1-q}{2r}\beta^1\right)\chi - i(\omega-m)\varphi = 0. \end{aligned} \quad (2.13)$$

Substituting the function χ from the first equation into the second one, we obtain the second order differential equation for the function φ:

$$\left[-g^{nl}\partial_n\partial_l + \frac{1}{r}\partial_1 + \frac{q-1}{r}\beta^1\beta^2\partial_2 - \frac{(q-1)^2}{4r^2} + \omega^2 - m^2\right]\varphi = 0, \quad (2.14)$$

where $n, l = 1, 2, 3$. The same equation is obtained for the function χ.

Because the above equation is in diagonal matrix form, we decompose the spinor φ into the upper, φ_1, and lower, φ_2, components. Taking the eigenfunctions corresponding to these components in the form $\varphi_l = R_l(r)e^{i(qn_l\phi+kz-\omega t)}$, $l = 1, 2$, we can see that the solutions of the equations for the radial functions, regular on the string, are expressed in terms of the Bessel function of the first kind: $R_l(r) = C_lJ_{\beta_l}(\lambda r)$, where the order is defined by the relations

$$\beta_1 = |qn_1 + (q-1)/2|, \qquad \beta_2 = |qn_2 - (q-1)/2|, \quad (2.15)$$

with $n_{1,2} = 0, \pm1, \pm2, \ldots$, and

$$\lambda = \sqrt{\omega^2 - k^2 - m^2}. \quad (2.16)$$

Hence, the components of the upper spinor are given by the formula

$$\varphi_l = C_lJ_{\beta_l}(\lambda r)\exp[i(qn_l\phi + kz - \omega t)], \qquad l = 1, 2. \quad (2.17)$$

Having the upper spinor, we can find the components χ_l of the lower one by using the first equation in (2.13). From this equation we find the following relations:

$$n_2 = n_1 + 1, \qquad \beta_2 = \beta_1 + \epsilon_{n_1}, \qquad \epsilon_{n_1} \equiv \mathrm{sgn}(n_1), \quad (2.18)$$

and

$$\chi_l = B_lJ_{\beta_l}(\lambda r)\exp[i(qn_l\phi + kz - \omega t)], \qquad l = 1, 2, \quad (2.19)$$

with the coefficients

$$B_1 = \frac{kC_1 - i\epsilon_{n_1}\lambda C_2}{\omega + m}, \qquad B_2 = -\frac{kC_2 - i\epsilon_{n_1}\lambda C_1}{\omega + m}.$$

We can see that the bispinor with the components defined by relations (2.17) and (2.19) is an eigenfunction of the projection of the total momentum along the cosmic string:

$$\hat{J}_3\psi = \left(-i\partial_\phi + i\frac{q}{2}\gamma^{(1)}\gamma^{(2)}\right)\psi = qj\psi, \quad (2.20)$$

where

$$j = n_1 + 1/2, \qquad j = \pm1/2, \qquad \pm3/2, \ldots \quad (2.21)$$

For the further specification of the eigenfunctions we can impose an additional condition relating the constants C_1 and C_2. As such a condition, following [43], we will require the following relations between the upper and lower components:

$$\chi_1 = \kappa\varphi_1, \qquad \chi_2 = -\frac{\varphi_2}{\kappa}. \quad (2.22)$$

From the expressions for the spinor components we find the eigenvalues of the parameter κ,

$$\kappa = \kappa_s \equiv \frac{\omega + s\sqrt{\omega^2 - k^2}}{k}, \qquad s = \pm1, \quad (2.23)$$

and the relation

$$C_2 = i\epsilon_{n_1}\frac{\kappa_s}{\lambda}(m + s\sqrt{\omega^2 - k^2})C_1, \quad (2.24)$$

for the coefficients in (2.17).

Hence, the positive frequency solutions to the Dirac equation, specified by the set of quantum numbers $\sigma = (\lambda, j, k, s)$, has the form

$$\psi_\sigma^{(+)} = C_\sigma^{(+)} \begin{pmatrix} J_{\beta_1}(\lambda r) \\ i\epsilon_j \kappa_s b_s^{(+)} J_{\beta_2}(\lambda r) e^{iq\phi} \\ \kappa_s J_{\beta_1}(\lambda r) \\ -i\epsilon_j b_s^{(+)} J_{\beta_2}(\lambda r) e^{iq\phi} \end{pmatrix} \times \exp[i(q(j-1/2)\phi + kz - \omega t)], \tag{2.25}$$

where the orders of the Bessel functions are defined in terms of j as

$$\beta_1 = |qj - 1/2| = q|j| - \epsilon_j/2, \quad \beta_2 = |qj + 1/2| = q|j| + \epsilon_j/2. \tag{2.26}$$

In (2.25) and in the consideration below we use the notations

$$b_s^{(\pm)} = \frac{\pm m + s\sqrt{\lambda^2 + m^2}}{\lambda}. \tag{2.27}$$

Note that one has the relation $b_s^{(\mp)} = 1/b_s^{(\pm)}$.

The eigenvalues of the radial quantum number λ are determined from the boundary condition (2.6) imposed on the eigenspinor (2.25). For fixed values of j and s, this leads to the single equation

$$J_{\beta_1}(\lambda a) + \epsilon_j b_s^{(+)} J_{\beta_2}(\lambda a) = 0. \tag{2.28}$$

Using the recurrence relations for the Bessel functions, this equation may also be written in the form

$$\tilde{J}_{\beta_1}(\lambda a) = 0. \tag{2.29}$$

Here and in what follows we use the notation

$$\tilde{J}_{\beta_1}(x) = xJ'_{\beta_1}(x) + (\mu - s\sqrt{x^2 + \mu^2} - \epsilon_j\beta_1)J_{\beta_1}(x) = -x\epsilon_j[J_{\beta_2}(x) + \epsilon_j b_s^{(-)} J_{\beta_1}(\lambda a)], \tag{2.30}$$

with $\mu = ma$. We will denote the solutions of Eq. (2.29) by $\lambda a = \lambda_{\beta_1,l}$, $l = 1, 2, \ldots$, assuming that they are arranged in ascending order. Now the set of quantum numbers is specified by $\sigma = (l, j, k, s)$.

The coefficient $C_\sigma^{(+)}$ in (2.25) is determined from the normalization condition

$$\int d^3x \sqrt{\gamma} \psi_\sigma^{(+)+} \psi_{\sigma'}^{(+)} = \delta_{\sigma\sigma'}, \tag{2.31}$$

where γ is the determinant of the spatial metric and the integration goes over the region inside the cylindrical shell. The delta symbol on the right-hand side of Eq. (2.31) is understood as the Dirac delta function for continuous quantum numbers (k) and the Kronecker delta for discrete ones (l, j, s). Substituting the eigenspinors (2.25) into Eq. (2.31) and using the value of the standard integral involving the square of the Bessel function [44], we find

$$(C_\sigma^{(+)})^{-2} = 2\pi\phi_0 a^2 J_{\beta_1}^2(x) \frac{\kappa_s^2 + 1}{x^2}[2(x^2 + \mu^2) + s(2\beta_1\epsilon_j + 1)\sqrt{x^2 + \mu^2} + \mu], \tag{2.32}$$

with the notation $x = \lambda a$.

The negative frequency eigenspinors are found in a way similar to that used above for the positive frequency ones and have the form

$$\psi_\sigma^{(-)} = C_\sigma^{(-)} \begin{pmatrix} J_{\beta_2}(\lambda r) \\ i\epsilon_j \kappa_s b_s^{(-)} J_{\beta_1}(\lambda r) e^{iq\phi} \\ \kappa_s J_{\beta_2}(\lambda r) \\ -i\epsilon_j b_s^{(-)} J_{\beta_1}(\lambda r) e^{iq\phi} \end{pmatrix} \times \exp[-i(q(j+1/2)\phi + kz - \omega t)], \tag{2.33}$$

with the same notations as in (2.25). The boundary condition imposed on this eigenspinor leads to the same Eq. (2.28) for the eigenvalues of λ. The coefficient $C_\sigma^{(-)}$ is found from the orthonormalization condition which is similar to (2.31) and has the form $C_\sigma^{(-)} = C_\sigma^{(+)}/b_s^{(-)}$.

III. FERMIONIC CONDENSATE

Fermionic condensate is evaluated by using the mode-sum formula

$$\langle 0|\bar{\psi}\psi|0\rangle = \sum_\sigma \bar{\psi}_\sigma^{(-)} \psi_\sigma^{(-)}, \tag{3.1}$$

where $\bar{\psi}_\sigma^{(-)} = \psi_\sigma^{(-)+}$ is the Dirac adjoint and

$$\sum_\sigma = \sum_{j=\pm 1/2, \pm 3/2, \cdots} \int_{-\infty}^{+\infty} dk \sum_{s=\pm 1} \sum_{l=1}^{\infty}. \tag{3.2}$$

Substituting the eigenspinor (2.33) into (3.1), we find

$$\langle 0|\bar{\psi}\psi|0\rangle = \sum_\sigma (\kappa_s^2 - 1) C_\sigma^{(-)2} [b_s^{(-)2} J_\beta^2(\lambda r) - J_{\beta+\epsilon_j}^2(\lambda r)]_{\lambda = \lambda_{\beta,l}/a}, \tag{3.3}$$

where κ_s is defined by the expression (2.23) and

$$\beta = \beta_1 = q|j| - \epsilon_j/2. \tag{3.4}$$

The fermionic condensate given by formula (3.3) is divergent and some regularization procedure is necessary. We will assume that a cutoff function is introduced in formula (3.3) without explicitly writing it.

As the explicit form for $\lambda_{\beta,l}$ is not known, formula (3.3) is not convenient for the direct evaluation of the condensate. In addition, the separate terms in the mode sum are highly oscillatory for large values of the quantum numbers. A convenient form can be obtained by applying to the series over l the summation formula, previously derived in Ref. [35] (see, also, [45]). In [35] by using the generalized Abel-Plana formula, it has been shown that for a function $h(z)$ analytic in the half-plane $\mathrm{Re}z > 0$ and satisfying the condition

$$|h(z)| < \varepsilon(x)e^{c|y|}, \qquad z = x + iy, \qquad |z| \to \infty, \quad (3.5)$$

with $c < 2$ and $\varepsilon(x) \to 0$ for $x \to \infty$, the following formula takes place:

$$\sum_{l=1}^{\infty} T_\beta(\lambda_{\beta,l})h(\lambda_{\beta,l}) = \int_0^\infty h(x)dx + \frac{\pi}{2}\mathop{\mathrm{Res}}_{z=0}\left[h(z)\frac{\tilde{Y}_\beta(z)}{\tilde{J}_\beta(z)}\right] - \frac{1}{\pi}\int_0^\infty dx\left[e^{-\beta\pi i}h(xe^{\pi i/2})\frac{K_\beta^{(+)}(x)}{I_\beta^{(+)}(x)} + e^{\beta\pi i}h(xe^{-\pi i/2})\frac{K_\beta^{(-)}(x)}{I_\beta^{(-)}(x)}\right], \quad (3.6)$$

where $I_\beta(x)$, $K_\beta(x)$ are the modified Bessel functions, and $T_\beta(z)$ is defined by the relation

$$zT_\beta^{-1}(z) = J_\beta^2(z)\left[z^2 + (\mu - \epsilon_j\beta)(\mu^2 - s\sqrt{z^2 + \mu^2}) + \frac{sz^2}{2\sqrt{z^2 + \mu^2}}\right]. \quad (3.7)$$

In formula (3.6) we used the notations

$$F^{(\pm)}(z) = \begin{cases} zF'(z) + (\mu - s\sqrt{\mu^2 - z^2} - \epsilon_j\beta)F(z), & |z| < \mu, \\ zF'(z) + (\mu \mp si\sqrt{z^2 - \mu^2} - \epsilon_j\beta)F(z), & |z| > \mu, \end{cases} \quad (3.8)$$

for a given function $F(z)$.

By taking into account the relation

$$C_\sigma^{(-)2} = \frac{k^2 x}{8\pi\phi_0 a\omega}\frac{\sqrt{x^2 + \mu^2} + s\mu}{a\omega + s\sqrt{x^2 + \mu^2}}\frac{T_\beta(x)}{\sqrt{x^2 + \mu^2}}, \quad (3.9)$$

we can write the mode sum for the fermionic condensate in the form

$$\langle 0|\bar{\psi}\psi|0\rangle = -\frac{1}{4\pi\phi_0 a^2}\sum_\sigma \frac{xT_\beta(x)}{a\omega}f_\beta(x, xr/a)|_{x=\lambda_{\beta,l}}, \quad (3.10)$$

with the notation

$$f_\beta(x, y) = (\mu - s\sqrt{x^2 + \mu^2})J_\beta^2(y) + (\mu + s\sqrt{x^2 + \mu^2})J_{\beta+\epsilon_j}^2(y). \quad (3.11)$$

Note that in (3.10)

$$a\omega = \sqrt{x^2 + k^2a^2 + \mu^2}, \quad (3.12)$$

and we have the property

$$e^{-\beta\pi i}f_\beta(xe^{\pi i/2}, ye^{\pi i/2}) = e^{\beta\pi i}f_\beta(xe^{-\pi i/2}, ye^{-\pi i/2}), \quad (3.13)$$

for $x < \mu$. Now we apply to the sum over l in (3.10) formula (3.6) taking $h(x) = xf(x, xr/a)/(a\omega)$. For this function the residue term in (3.6) vanishes. The part in the fermionic condensate with the last integral on the right-hand side of (3.6) vanishes in the limit $a \to \infty$, whereas the part with the first integral does not depend on a. From here it follows that the latter presents the fermionic condensate in the geometry of a cosmic string without boundaries. This can also be seen by direct evaluation.

Indeed, when the cylindrical boundary is absent, the positive and negative frequency eigenspinors are still given by formulae (2.25) and (2.33), where now the spectrum for λ is continuous, $0 \le \lambda < \infty$. The corresponding normalization coefficients are found from the condition (2.31) and have the form

$$(C_\sigma^{(\pm)})^{-2} = 2\pi\phi_0\frac{\kappa_s^2 + 1}{\lambda}(1 + b_s^{(\pm)2}). \quad (3.14)$$

Substituting the eigenspinors into the mode-sum formula (3.1), for the fermionic condensate in the geometry of a cosmic string without boundaries we find

$$\langle 0|\bar{\psi}\psi|0\rangle_s = -\frac{q}{8\pi^2}\sum_{j=\pm 1/2,\pm 3/2,\cdots}\int_{-\infty}^{+\infty}dk\int_0^\infty d\lambda\sum_{s=\pm 1}\frac{\lambda}{\omega} \times [(m - s\sqrt{\lambda^2 + m^2})J_\beta^2(\lambda r) + (m + s\sqrt{\lambda^2 + m^2})J_{\beta+\epsilon_j}^2(\lambda r)]. \quad (3.15)$$

This coincides with the result obtained from the first term on the right of formula (3.6) applied to mode sum (3.10). Formula (3.15) is further simplified by taking into account the expression for β and after the summation over s:

$$\langle 0|\bar{\psi}\psi|0\rangle_s = -\frac{qm}{\pi^2}\sum_j\int_0^\infty dk\int_0^\infty d\lambda\frac{\lambda}{\omega}[J_{qj-1/2}^2(\lambda r) + J_{qj+1/2}^2(\lambda r)]. \quad (3.16)$$

Here and in what follows

$$\sum_j = \sum_{j=1/2,3/2,\cdots}. \quad (3.17)$$

As it is seen from formula (3.16), for a massless field the fermionic condensate vanishes in the boundary-free cosmic string spacetime. Since the geometry outside of the string is flat, the renormalization of the fermionic condensate given by (3.16) is done by subtracting the corresponding quantity for the boundary-free Minkowski spacetime. The latter is obtained from (3.16) taking in this formula $q = 1$. Note that for the Minkowski case the summation over j is explicitly done by using the formula

$$\sum_j[J_{j-1/2}^2(\lambda r) + J_{j+1/2}^2(\lambda r)] = 2{\sum_{n=0}^{\infty}}' J_n^2(\lambda r) = 1, \quad (3.18)$$

where the prime on the sign of summation means that the

$n = 0$ term should be halved. The renormalized fermionic condensate in the geometry of a cosmic string without boundaries is further investigated in the appendix.

From the discussion above it follows that, after the application of the summation formula (3.6), the part in the fermionic condensate with the second integral on the right-hand side of this formula corresponds to the VEV induced by the presence of the cylindrical shell. By using property (3.13) and noting that under the change $s \to -s$ we have $F^{(+)}(x) \to F^{(-)}(x)$, $F^{(-)}(x) \to F^{(+)}(x)$, the fermionic condensate in the geometry with the cylindrical shell is presented in the decomposed form

$$\langle 0|\bar{\psi}\psi|0\rangle = \langle 0|\bar{\psi}\psi|0\rangle_{\rm s} + \langle\bar{\psi}\psi\rangle_{\rm cyl}. \tag{3.19}$$

Here the second term on the right-hand side,

$$\langle\bar{\psi}\psi\rangle_{\rm cyl} = \frac{q}{4\pi^3 a^2}\sum_{j=\pm 1/2,\pm 3/2,\cdots}\sum_{s}\int_{-\infty}^{+\infty}dk \times \int_{\sqrt{\mu^2+a^2k^2}}^{\infty}\frac{dxx}{\sqrt{x^2-\mu^2-a^2k^2}}\frac{K_\beta^{(+)}(x)}{I_\beta^{(+)}(x)} \times [(\mu - is\sqrt{x^2-\mu^2})I_\beta^2(xr/a) - (\mu + is\sqrt{x^2-\mu^2})I_{\beta+\epsilon_j}^2(xr/a)], \tag{3.20}$$

is the part in the fermionic condensate induced by the cylindrical boundary. The number of the integrations in this formula is reduced by using the relation

$$\int_0^\infty dk\int_{\sqrt{\mu^2+a^2k^2}}^\infty \frac{dx\,xG(x)}{\sqrt{x^2-\mu^2-a^2k^2}} = \frac{\pi}{2a}\int_\mu^\infty duuG(u). \tag{3.21}$$

Further, redefining $s \to -s$ in the part of the summation over the negative values j, it can be seen that the negative and positive values of j give the same contributions to the fermionic condensate. Finally, we arrive at the following formula:

$$\langle\bar{\psi}\psi\rangle_{\rm cyl} = \frac{q}{\pi^2 a^3}\sum_j\int_\mu^\infty dx\,x\,{\rm Re}\Big[\frac{\bar{K}_{\beta_j}(x)}{\bar{I}_{\beta_j}(x)}F_{\beta_j}(x, xr/a)\Big], \tag{3.22}$$

where

$$\beta_j = qj - 1/2, \tag{3.23}$$

and the function $F_{\beta_j}(x, y)$ is given by the expression

$$F_{\beta_j}(x, y) = (\mu - i\sqrt{x^2-\mu^2})I_{\beta_j}^2(y) - (\mu + i\sqrt{x^2-\mu^2})I_{\beta_j}^2(y). \tag{3.24}$$

In (3.22), for the modified Bessel functions we use the barred notations

$$\bar{F}_\beta(x) = xF'_\beta(x) + (\mu - i\sqrt{x^2-\mu^2} - \beta)F_\beta(x) = \eta_F xF_{\beta+1}(x) + (\mu - i\sqrt{x^2-\mu^2})F_\beta(x), \tag{3.25}$$

with $F = I, K$, and $\eta_I = 1$, $\eta_K = -1$.

The boundary-induced part (3.22) is finite for points away from the cylindrical shell and the renormalization is necessary for the boundary-free part, $\langle 0|\bar{\psi}\psi|0\rangle_{\rm s}$, only. Note that the ratio in the integrand of Eq. (3.22) may also be presented in the form

$$\frac{\bar{K}_\beta(x)}{\bar{I}_\beta(x)} = -\frac{W_\beta(x) - \mu + i\sqrt{x^2-\mu^2}}{x^2[I_{\beta+1}^2(x) + I_\beta^2(x)] + 2\mu xI_\beta(x)I_{\beta+1}(x)}, \tag{3.26}$$

where

$$W_\beta(x) = x^2[I_{\beta+1}(x)K_{\beta+1}(x) - K_\beta(x)I_\beta(x)] + 2\mu xI_\beta(x)K_{\beta+1}(x). \tag{3.27}$$

For a massless fermionic field, from (3.22) and (3.26) we find

$$\langle\bar{\psi}\psi\rangle_{\rm cyl} = -\frac{q}{\pi^2 a^3}\sum_j\int_0^\infty dx\,x\frac{I_{\beta_j}^2(xr/a) + I_{\beta_j+1}^2(xr/a)}{I_{\beta_j}^2(x) + I_{\beta_j+1}^2(x)}. \tag{3.28}$$

In this case the boundary-free part vanishes and the fermionic condensate is always negative.

Now we turn to the investigation of the fermionic condensate given by Eq. (3.22) in the asymptotic regions of the parameters. For large values of the cylinder radius we use the asymptotic formulae for the modified Bessel functions when the argument is large. By taking into account that the main contribution into the integral in (3.22) comes from the lower limit of the integral, to the leading order we find

$$\langle\bar{\psi}\psi\rangle_{\rm cyl} \approx -\frac{qm}{4\pi a^2}e^{-2am}\sum_j\sum_{\delta=\pm 1}(1 + \delta qj)I_{qj-\delta/2}^2(mr), \tag{3.29}$$

for $am \gg 1$. For a massless field, expanding the integrand in (3.28) we see that the main contribution is due to the term with $j = 1/2$ and one has

$$\langle\bar{\psi}\psi\rangle_{\rm cyl} \approx -\frac{2^{1-q}q}{\pi^2 a^3}\frac{(r/a)^{q-1}}{\Gamma^2((q+1)/2)} \times \int_0^\infty dx\frac{x^q}{I_{(q-1)/2}^2(x) + I_{(q+1)/2}^2(x)}, \tag{3.30}$$

for $r \ll a$.

For points near the string, $r/a \ll 1$, the main contribution into the boundary-induced VEV (3.22) comes from the lowest mode $j = 1/2$ and to the leading order we find

$$\langle\bar{\psi}\psi\rangle_{\rm cyl} \approx \frac{2^{1-q}q(r/a)^{q-1}}{\pi^2 a^3\Gamma^2((q+1)/2)}\int_\mu^\infty dx\,x^q\,{\rm Re}\left[\frac{\bar{K}_{(q-1)/2}(x)}{\bar{I}_{(q-1)/2}(x)}\times(\mu - i\sqrt{x^2-\mu^2})\right]. \tag{3.31}$$

This quantity is nonzero in the case of the cylindrical boundary in the Minkowski bulk and vanishes for the cosmic string geometry with $q>1$. For a massless field formula (3.31) is reduced to Eq. (3.30). For points near the boundary the main contribution comes from large values of j and in this case we can use the uniform asymptotic expansions for the modified Bessel functions for large values of the order (see, for instance, [46]). In this way, to the leading order we have

$$\langle\bar{\psi}\psi\rangle_{\rm cyl} \approx -\frac{1}{4\pi^2(a-r)^3}, \tag{3.32}$$

for $(1-r/a)\ll 1$. As we see, the leading term does not depend on the planar angle deficit and corresponds to the same one obtained for a cylindrical boundary in the Minkowski bulk.

Now we consider the limit when the parameter q is large which corresponds to small values of ϕ_0 and, hence, to a large planar angle deficit. In this limit the order of the modified Bessel functions in (3.22) is large and we replace these functions by their uniform asymptotic expansions. Assuming fixed value of the ratio r/a, the integral is estimated by the Laplace method and to the leading order we have

$$\langle\bar{\psi}\psi\rangle_{\rm cyl} \approx -\frac{q^2\exp[-(1-(r/a)^2)\mu/2]}{2\pi^2 r^3[1-(r/a)^2]}\left(\frac{r}{a}\right)^q, \qquad q\gg 1. \tag{3.33}$$

Hence, for large values of the angle deficit the fermionic condensate is exponentially suppressed.

IV. ENERGY-MOMENTUM TENSOR

In this section we consider the VEV of the energy-momentum tensor of the fermionic field inside a cylindrical shell on which the field satisfies the boundary condition (2.6). This VEV can be evaluated by making use of the mode-sum formula

$$\langle 0|T_{\mu\nu}|0\rangle = \frac{i}{2}\sum_\sigma[\bar{\psi}_\sigma^{(-)}(x)\gamma_{(\mu}\nabla_{\nu)}\psi_\sigma^{(-)}(x) - (\nabla_{(\mu}\bar{\psi}_\sigma^{(-)}(x))\gamma_{\nu)}\psi_\sigma^{(-)}(x)], \tag{4.1}$$

with the negative frequency eigenspinors given by (2.33). We can see that the vacuum energy-momentum tensor is diagonal and the separate components are given by the formulae (no summation over ν)

$$\langle 0|T_\nu^\nu|0\rangle = \frac{q}{8\pi^2 a^3}\sum_\sigma \frac{x^3 T_\beta(x)}{a\omega} f_\beta^{(\nu)}(x, xr/a)|_{x=\lambda_{\beta,l}}, \tag{4.2}$$

where the following notations were introduced

$$f_\beta^{(0)}(x,y) = -\frac{a^2\omega^2}{x^2}\left[\left(1-\frac{s\mu}{\sqrt{x^2+\mu^2}}\right)J_\beta^2(y) + \left(1+\frac{s\mu}{\sqrt{x^2+\mu^2}}\right)J_{\beta+\epsilon_j}^2(y)\right],$$

$$f_\beta^{(1)}(x,y) = J_\beta^2(y) + J_{\beta+\epsilon_j}^2(y) - \frac{2\beta+\epsilon_j}{y}J_\beta(y)J_{\beta+\epsilon_j}(y),$$

$$f_\beta^{(2)}(x,y) = \frac{2\beta+\epsilon_j}{y}J_\beta(y)J_{\beta+\epsilon_j}(y),$$

$$f_\beta^{(3)}(x,y) = (k^2/\omega^2)f_\beta^{(0)}(x,y). \tag{4.3}$$

The other notations in (4.2) are the same as in (3.10). The VEV given by (4.2) is divergent and, as in the case of the fermionic condensate, it is assumed that a cutoff function is present. Now, we can explicitly verify that the VEVs (4.2) satisfy the trace relation

$$\langle 0|T_\nu^\nu|0\rangle = m\langle 0|\bar{\psi}\psi|0\rangle.$$

In order to extract from the VEV of the energy-momentum tensor the part corresponding to the geometry of a string without boundaries, we apply to the series over l in (4.2) the summation formula (3.6) with

$$h(x) = \frac{x^3}{a\omega}f_\beta^{(\nu)}(x, xr/a). \tag{4.4}$$

In a way similar to that used in the case of the fermionic condensate, the VEV can be written in the decomposed form:

$$\langle 0|T_\mu^\nu|0\rangle = \langle 0|T_\mu^\nu|0\rangle_{\rm s} + \langle T_\mu^\nu\rangle_{\rm cyl}, \tag{4.5}$$

where $\langle 0|T_\mu^\nu|0\rangle_{\rm s}$ is the fermionic energy-momentum tensor in the geometry of a cosmic string when the cylindrical shell is absent. The second term on the right-hand side of formula (4.5) is induced by the cylindrical shell and is given by the formula (no summation over ν)

$$\langle T_\nu^\nu\rangle_{\rm cyl} = \frac{q}{\pi^2 a^4}\sum_j\int_\mu^\infty dx\,x^3\,{\rm Re}\left[\frac{\bar{K}_{\beta_j}(x)}{\bar{I}_{\beta_j}(x)}F_{\beta_j}^{(\nu)}(x, xr/a)\right], \tag{4.6}$$

where β_j is defined by Eq. (3.23). In this formula we have introduced the notations

$$F_{\beta_j}^{(0)}(x,y) = \frac{\mu^2/x^2-1}{2}\sum_{\delta=\pm1}\delta\left(1+\frac{\delta i\mu}{\sqrt{x^2-\mu^2}}\right)I_{qj-\delta/2}^2(y),$$

$$F_{\beta_j}^{(1)}(x,y) = I_{\beta_j}^2(y) - I_{\beta_j+1}^2(y) - (2qj/y)I_{\beta_j}(y)I_{\beta_j+1}(y),$$

$$F_{\beta_j}^{(2)}(x,y) = (2qj/y)I_{\beta_j}(y)I_{\beta_j+1}(y), \tag{4.7}$$

and $F_\beta^{(3)}(x,y) = F_\beta^{(0)}(x,y)$. Note that the function for the radial component is also presented in the form

$$F^{(1)}_{\beta_j}(x, y) = I_{\beta_j}(y)I'_{\beta_j+1}(y) - I_{\beta_j+1}(y)I'_{\beta_j}(y). \tag{4.8}$$

As we see, the vacuum stress along the axis of the string is equal to the energy density. Of course, this property is a direct consequence of the boost invariance along this axis.

By using Eq. (3.26), we may write the expressions for the components of the energy-momentum tensor in a more explicit form given by

$$\langle T^0_0\rangle_{\rm cyl} = \frac{q}{2\pi^2 a^4}\sum_j \int_\mu^\infty dx\, x(1-\mu^2/x^2)\frac{W_{\beta_j}(x)[I^2_{\beta_j}(xr/a) - I^2_{\beta_j+1}(xr/a)] - 2\mu I^2_{\beta_j}(xr/a)}{I^2_{\beta_j+1}(x) + I^2_{\beta_j}(x) + 2(\mu/x)I_{\beta_j}(x)I_{\beta_j+1}(x)}, \tag{4.9}$$

for the energy density and by (no summation over ν)

$$\langle T^\nu_\nu\rangle_{\rm cyl} = -\frac{q}{\pi^2 a^4}\sum_j \int_\mu^\infty dx \frac{x[W_{\beta_j}(x) - \mu]F^{(\nu)}_{\beta_j}(x, xr/a)}{I^2_{\beta_j+1}(x) + I^2_{\beta_j}(x) + 2(\mu/x)I_{\beta_j}(x)I_{\beta_j+1}(x)}, \tag{4.10}$$

for the radial and azimuthal stresses, $\nu = 1, 2$.

The VEV of the fermionic energy-momentum tensor in the geometry of a cosmic string when the cylindrical shell is absent, corresponds to the first term on the right-hand side of the summation formula (3.6). For this VEV we have the mode sum (no summation over ν)

$$\langle 0|T^\nu_\nu|0\rangle_{\rm s} = \frac{q}{8\pi^2}\sum_{j=\pm 1/2,\pm 3/2,\cdots}\int_{-\infty}^{+\infty} dk \times \int_0^\infty d\lambda \sum_{s=\pm 1}\frac{\lambda^3}{\omega} f^{(\nu)}_\beta(\lambda a, \lambda r). \tag{4.11}$$

The summation over s in this formula is done explicitly and noting that the negative and positive values of j give the same contributions, we find

$$\langle 0|T^\nu_\nu|0\rangle_{\rm s} = \frac{q}{\pi^2}\sum_j \int_0^\infty dk \int_0^\infty d\lambda \frac{\lambda^3}{\omega} g^{(\nu)}_{\beta_j}(\lambda, \lambda r), \tag{4.12}$$

where now

$$\begin{aligned}
g^{(0)}_{\beta_j}(\lambda, y) &= -\frac{\omega^2}{\lambda^2}[J^2_{\beta_j}(y) + J^2_{\beta_j+1}(y)],\\
g^{(1)}_{\beta_j}(\lambda, y) &= J^2_{\beta_j}(y) + J^2_{\beta_j+1}(y) - \frac{2qj}{y}J_{\beta_j}(y)J_{\beta_j+1}(y),\\
g^{(2)}_{\beta_j}(\lambda, y) &= \frac{2qj}{y}J_{\beta_j}(y)J_{\beta_j+1}(y),\\
g^{(3)}_{\beta_j}(\lambda, y) &= (k^2/\omega^2)g^{(0)}_{\beta_j}(\lambda, y).
\end{aligned} \tag{4.13}$$

As in the case of the fermionic condensate, the VEV (4.12) is renormalized by subtracting the corresponding VEV in the Minkowski spacetime without boundaries. The boundary-free renormalized VEV of the energy-momentum tensor for a massive fermionic field is further investigated in the appendix.

It can be explicitly verified that the boundary-induced parts satisfy the trace relation $\langle T^\nu_\nu\rangle_{\rm cyl} = m\langle\bar\psi\psi\rangle_{\rm cyl}$ and the covariant conservation equation $\nabla_\nu\langle T^\nu_\mu\rangle_{\rm cyl} = 0$. For the geometry under consideration the latter is reduced to the single equation

$$\partial_r[r\langle T^1_1\rangle_{\rm cyl}] = \langle T^2_2\rangle_{\rm cyl}. \tag{4.14}$$

This equation is easily checked by using the relation $\partial_y(yF^{(1)}_\beta(x, y)) = F^{(2)}_\beta(x, y)$ between the radial and azimuthal functions in (4.7). In the case of a massless fermionic field, for the VEV induced by the cylindrical boundary from (4.9) and (4.10) we have (no summation over ν)

$$\langle T^\nu_\nu\rangle_{\rm cyl} = \frac{q}{\pi^2 a^4}\sum_j \int_0^\infty dx\, x^3 \times \frac{I_{\beta_j}(x)K_{\beta_j}(x) - I_{\beta_j+1}(x)K_{\beta_j+1}(x)}{I^2_{\beta_j}(x) + I^2_{\beta_j+1}(x)}F^{(0,\nu)}_{\beta_j}(xr/a), \tag{4.15}$$

where $F^{(0,\nu)}_\beta(y) = F^{(\nu)}_\beta(x, y)$ for $\nu = 1, 2$, and the corresponding function for the energy density is defined as

$$F^{(0,0)}_{\beta_j}(y) = -\tfrac{1}{2}[I^2_{\beta_j}(y) - I^2_{\beta_j+1}(y)]. \tag{4.16}$$

In the special case with $q = 1$, from the formulae given above we obtain the fermionic Casimir densities for a cylindrical boundary in the Minkowski spacetime. On the left panel of Fig. 1 we have plotted the corresponding VEVs for the energy density and radial stress as functions of the radial coordinate for a massless fermionic field. On the right panel the boundary-induced parts are presented in the geometry of a cosmic string with the parameter $q = 2$.

For large values of the cylinder radius the main contribution into the integral in (4.6) comes from the lower limit and in the leading order we have (no summation over ν)

$$\begin{aligned}
\langle T^0_0\rangle_{\rm cyl} &\approx -\frac{qm^2}{8\pi a^2}e^{-2am}\sum_j\sum_{\delta=\pm 1} I^2_{qj-\delta/2}(mr),\\
\langle T^\nu_\nu\rangle_{\rm cyl} &\approx -\frac{q^2m^2}{4\pi a^2}e^{-2am}\sum_j qjF^{(\nu)}_{\beta_j}(ma, mr),
\end{aligned} \tag{4.17}$$

where $\nu = 1, 2$ and $am \gg 1$. For a massless fermionic field the main contribution comes from the $j = 1/2$ term and from (4.15) one finds

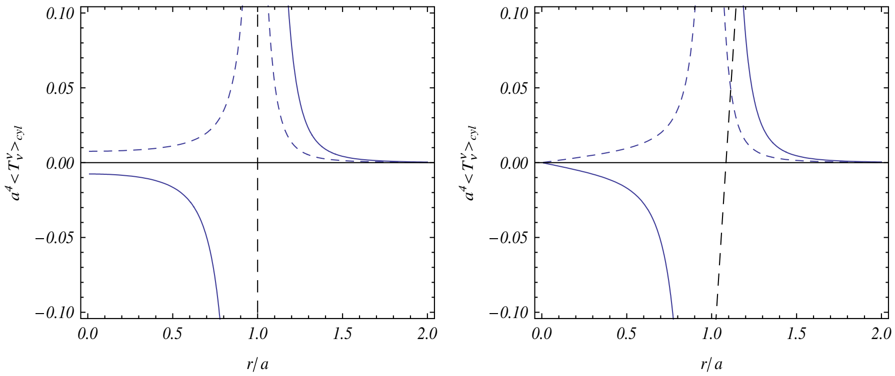

FIG. 1 (color online). Boundary-induced parts in the VEVs of the energy density, $a^4\langle T_0^0\rangle_{\rm cyl}$ (full curves), and radial stress, $a^4\langle T_1^1\rangle_{\rm cyl}$ (dashed curves), for a massless fermionic field as functions of the radial coordinate. The left panel is plotted for $q = 1$ (Minkowski spacetime) and for the right panel $q = 2$.

$$\langle T_0^0\rangle_{\rm cyl} \approx -\frac{2^{-q}q(r/a)^{q-1}}{\pi^2 a^4\Gamma^2((q+1)/2)} \times \int_0^\infty dx\, x^{q+2} \times \frac{I_{(q-1)/2}(x)K_{(q-1)/2}(x) - I_{(q+1)/2}(x)K_{(q+1)/2}(x)}{I^2_{(q-1)/2}(x) + I^2_{(q+1)/2}(x)}, \tag{4.18}$$

for the energy density and

$$\langle T_1^1\rangle_{\rm cyl} \approx \frac{-2}{q+1}\langle T_0^0\rangle_{\rm cyl}, \qquad \langle T_2^2\rangle_{\rm cyl} \approx \frac{-2q}{q+1}\langle T_0^0\rangle_{\rm cyl}, \tag{4.19}$$

for the radial and azimuthal stresses.

Now we consider the behavior of the boundary-induced VEV of the energy-momentum tensor near the string and for points near the boundary. In the first case one has $r/a \ll 1$ and the main contribution in (4.6) comes from the lowest mode $j = 1/2$. By using the formulae for the modified Bessel functions for small values of the argument, to the leading order one finds (no summation over ν)

$$\langle T_\nu^\nu\rangle_{\rm cyl} \approx \frac{q(r/2a)^{q-1}}{\pi^2 a^4\Gamma^2((q+1)/2)} \times \int_\mu^\infty dx\, x^{q+2}\, {\rm Re}\left[\frac{\bar K_{(q-1)/2}(x)}{\bar I_{(q-1)/2}(x)} F^{(\nu)}(x)\right], \tag{4.20}$$

where

$$F^{(0)}(x) = \frac{\mu^2/x^2 - 1}{2}(1 + i\mu/\sqrt{x^2-\mu^2}), \quad F^{(2)}(x) = qF^{(1)}(x) = q/(q+1). \tag{4.21}$$

For a massless fermionic field this formula is reduced to the results given by Eqs. (4.18) and (4.19).

For points near the cylindrical boundary, we replace the modified Bessel functions by the corresponding uniform asymptotic expansions for large values of the order. Unlike to the case of the fermionic condensate here the leading terms in the VEVs of the energy-momentum tensor vanish and it is necessary to take the next terms in the asymptotic expansions. In particular, for the function appearing in the integrands of the vacuum stresses we have

$${\rm Re}\left[\frac{\bar K_\beta(\beta x)}{\bar I_\beta(\beta x)}\right] \approx \frac{1 - t^2 - 2\beta}{1-t}\frac{\pi t^2}{2\beta} e^{-2\beta\eta(x)}, \tag{4.22}$$

where

$$t = \frac{1}{\sqrt{1+x^2}}, \qquad \eta(x) = \sqrt{1+x^2} + \ln\frac{x}{1+\sqrt{1+x^2}}. \tag{4.23}$$

Substituting this into the expression for the azimuthal stress, to the leading order we find

$$\langle T_2^2\rangle_{\rm cyl} \approx \frac{1-5\mu}{60\pi^2 a(a-r)^3}, \qquad (1 - r/a) \ll 1. \tag{4.24}$$

The corresponding expressions for the radial stress and the energy density are found from the conservation equation and the trace relation by using the result (3.32) for the fermionic condensate. In this way we obtain the following formulae:

$$\langle T_0^0\rangle_{\rm cyl} \approx -\frac{1+10\mu}{120\pi^2 a(a-r)^3}, \quad \langle T_1^1\rangle_{\rm cyl} \approx \frac{1-5\mu}{120\pi^2 a^2(a-r)^2}. \tag{4.25}$$

As in the case of the fermionic condensate, the leading terms in the VEVs of the energy-momentum tensor com-

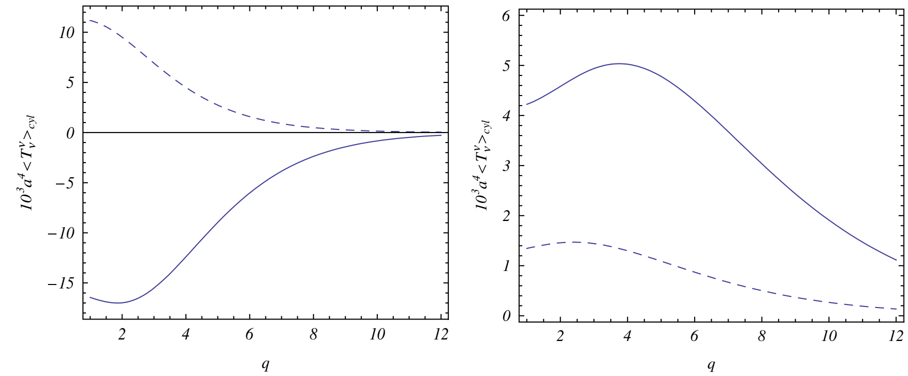

FIG. 2 (color online). Boundary-induced parts in the VEVs of the energy density, $a^4\langle T_0^0\rangle_{\rm cyl}$ (full curves), and radial stress, $a^4\langle T_1^1\rangle_{\rm cyl}$ (dashed curves), for a massless fermionic field as functions of the parameter q. The left panel corresponds to the interior region with $r/a = 0.5$ and the right panel corresponds to the exterior region with $r/a = 1.5$.

ponents do not depend on the planar angle deficit and are the same as for a cylindrical boundary in the Minkowski bulk. Note that, in dependence of the parameter μ, the vacuum stresses near the boundary can be either positive or negative, whereas the energy density remains always negative. As the boundary-free part is finite everywhere outside the string axis, for points near the boundary the total VEV is dominated by the boundary-induced part.

Similar to the case of the fermionic condensate, it can be seen that for large values of the parameter q the boundary-induced part in the VEV of the energy-momentum tensor is suppressed by the factor $(r/a)^q$. The dependence of the boundary-induced VEVs on the parameter q is presented in Fig. 2 for a massless fermionic field. On the left (right) panel the energy density and radial stresses are plotted for the value of the ratio $r/a = 0.5$ ($r/a = 1.5$).

In Fig. 3 we give the boundary-induced parts in the energy density and the radial stress in the geometry of a cosmic string with $q = 2$ as functions of the mass. The graphs on the left panel are for the interior quantities at $r/a = 0.5$ and the graphs on the right panel are for the exterior ones evaluated at $r/a = 1.5$. As it is seen, we have a nontrivial dependence on the mass and in the case of a massive field the polarization effects induced by the boundary can be stronger than for a massless field.

V. VACUUM DENSITIES IN THE EXTERIOR REGION

A. Eigenspinors

In this section we consider the fermionic condensate and the VEV of the energy-momentum tensor in the region outside the cylindrical shell. As in the interior case this can be done by the direct mode summation. The corresponding eigenspinors have the form (2.25) and (2.33) with the difference that now, instead of the Bessel functions $J_{\beta_{1,2}}(\lambda r)$, the linear combinations of the functions $J_{\beta_{1,2}}(\lambda r)$ and $Y_{\beta_{1,2}}(\lambda r)$ should be taken, with $Y_\nu(x)$ being the Neumann function. The ratio of the coefficients in this combination is determined from the boundary condition (2.6) imposed on the cylindrical surface. In this way for the positive and negative frequency eigenspinors we have the expressions

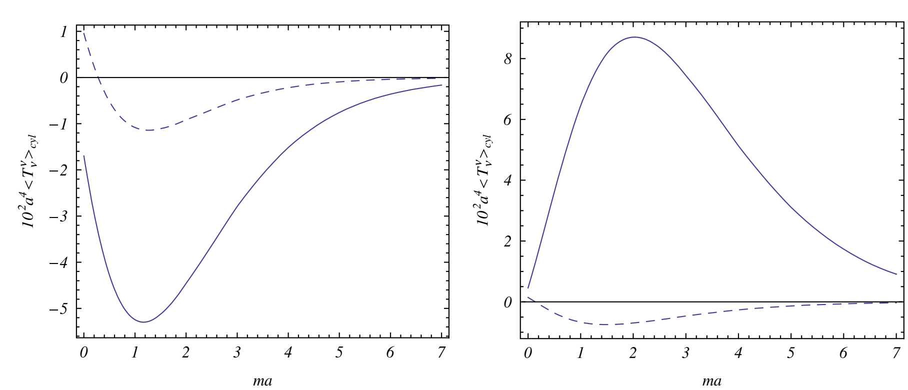

FIG. 3 (color online). Boundary-induced parts in the VEVs of the energy density, $a^4\langle T_0^0\rangle_{\rm cyl}$ (full curves), and radial stress, $a^4\langle T_1^1\rangle_{\rm cyl}$ (dashed curves), as functions of the parameter ma for $q = 2$. The left panel corresponds to the interior region with $r/a = 0.5$ and the right panel corresponds to the exterior region with $r/a = 1.5$.

$$\psi_\sigma^{(\pm)} = C_\sigma^{(\pm)} \begin{pmatrix} Z_{\beta_\pm}(\lambda a, \lambda r) \\ i\kappa_s \epsilon_{n_\pm} b_s^{(\pm)} Z_{\beta_\mp}(\lambda a, \lambda r) e^{iq\phi} \\ \kappa_s Z_{\beta_\pm}(\lambda a, \lambda r) \\ -i\epsilon_{n_\pm} b_s^{(\pm)} Z_{\beta_\mp}(\lambda a, \lambda r) e^{iq\phi} \end{pmatrix} \times \exp[\pm i(q(j \mp 1/2)\phi + kz - \omega t)], \quad (5.1)$$

where the function $Z_\nu(x, y)$ is defined by the formula

$$Z_{\beta_\pm}(x, y) = \tilde{Y}_\beta(x) J_{\beta_\pm}(y) - \tilde{J}_\beta(x) Y_{\beta_\pm}(y), \quad (5.2)$$

and

$$\beta_+ = \beta_1 = \beta, \qquad \beta_- = \beta_2, \quad (5.3)$$

with $\beta_{1,2}$ given by relations (2.26). Here the notation $\tilde{Y}_\beta(x)$ is defined by (2.30) with the replacement $J \to Y$.

The eigenspinors are orthonormalized by condition (2.31), where now the radial integration goes over the exterior region. The eigenvalues for λ are continuous and on the right-hand side of the normalization condition we have $\delta(\lambda - \lambda')$. Since the radial integral diverges for $\lambda' = \lambda$, the main contribution to this integral comes from large values r and we can replace the Bessel and Neumann functions with the arguments λr, by the corresponding asymptotic expressions. In this way, for the normalization coefficients in (5.1), we find

$$(C_\sigma^{(\pm)})^{-2} = 2\pi\phi_0(\kappa_s^2 + 1)(1 + b_s^{(\pm)2}) \frac{1}{\lambda} [\tilde{J}_\beta^2(\lambda a) + \tilde{Y}_\beta^2(\lambda a)], \quad (5.4)$$

where κ_s and $b_s^{(\pm)}$ are defined by Eqs. (2.23) and (2.27).

B. Fermionic condensate

Substituting the eigenspinors (5.1) into the mode-sum formula (3.1), for the fermionic condensate in the region outside the cylindrical shell we obtain the formula

$$\langle 0|\bar{\psi}\psi|0\rangle = \frac{q}{8\pi^2} \sum_\sigma \frac{\lambda}{\omega[\bar{J}_\beta^2(\lambda a) + \bar{Y}_\beta^2(\lambda a)]} \times [(-m + s\sqrt{\lambda^2 + m^2}) Z_\beta^2(\lambda a, \lambda r) - (m + s\sqrt{\lambda^2 + m^2}) Z_{\beta+\sigma_j}^2(\lambda a, \lambda r)], \quad (5.5)$$

where

$$\sum_\sigma = \sum_{j=\pm 1/2, \pm 3/2, \cdots} \int_{-\infty}^{+\infty} dk \int_0^\infty d\lambda \sum_{s=\pm 1}. \quad (5.6)$$

In order to extract from the VEV (5.5) the part induced by the cylindrical shell, we subtract the fermionic condensate for the geometry of a string without the shell. As it has been shown before, the latter is given by formula (3.15). For further evaluation of the difference, we use the identities

$$\frac{Z_{\beta+\sigma_j}^2(\lambda a, \lambda r)}{\bar{J}_\beta^2(\lambda a) + \bar{Y}_\beta^2(\lambda a)} - J_{\beta+\sigma_j}^2(\lambda r) = -\frac{1}{2} \sum_{l=1,2} \frac{\tilde{J}_\beta(\lambda a)}{\tilde{H}_\beta^{(l)}(\lambda a)} H_{\beta+\sigma_j}^{(l)2}(\lambda r), \quad (5.7)$$

where $\sigma_j = 0$ or ϵ_j, and $H_\nu^{(1,2)}(x)$ are the Hankel functions. As a result for the fermionic condensate we obtain

$$\langle 0|\bar{\psi}\psi|0\rangle = \langle 0|\bar{\psi}\psi|0\rangle_{\rm s} + \frac{q}{16\pi^2} \sum_\sigma \sum_{l=1,2} \frac{\lambda}{\omega} \frac{\tilde{J}_\beta(\lambda a)}{\tilde{H}_\beta^{(l)}(\lambda a)} \times [(m - s\sqrt{\lambda^2 + m^2}) H_\beta^{(l)2}(\lambda r) + (m + s\sqrt{\lambda^2 + m^2}) H_{\beta+\sigma_j}^{(l)2}(\lambda r)]. \quad (5.8)$$

Now, in the complex plane λ we rotate the integration contour in the integral over λ on the right-hand side of the formula (5.8) by the angle $\pi/2$ for the $l = 1$ term and by the angle $-\pi/2$ for the $l = 2$ term. By using the symmetry properties of the integrands, it can be seen that the parts of the integrals over $(0, i\sqrt{k^2 + m^2})$ and $(0, -i\sqrt{k^2 + m^2})$ are cancelled. The number of the remaining integrations is reduced by using the formula (3.21). In this way, introducing the modified Bessel functions, we present the fermionic condensate in the decomposed form (3.19), where the part induced by the cylindrical boundary in the region $r > a$ is given by the expression

$$\langle \bar{\psi}\psi \rangle_{\rm cyl} = \frac{q}{\pi^2 a^3} \sum_j \int_\mu^\infty dx\, x\, {\rm Re}\left[\frac{\bar{I}_{\beta_j}(x)}{\bar{K}_{\beta_j}(x)} F_{\beta_j}^{(\rm ex)}(x, xr/a)\right], \quad (5.9)$$

where β_j is defined by Eq. (3.23) and we are using the notation

$$F_{\beta_j}^{(\rm ex)}(x, y) = (\mu - i\sqrt{x^2 - \mu^2}) K_{\beta_j}^2(y) - (\mu + i\sqrt{x^2 - \mu^2}) K_{\beta_j}^2(y). \quad (5.10)$$

Note that the ratio of the combinations of the modified Bessel functions in (5.9) can be written in the form

$$\frac{\bar{I}_\beta(x)}{\bar{K}_\beta(x)} = -\frac{W_\beta(x) - \mu - i\sqrt{x^2 - \mu^2}}{x^2[K_{\beta+1}^2(x) + K_\beta^2(x)] - 2\mu x K_\beta(x) K_{\beta+1}(x)}, \quad (5.11)$$

where the function $W_\beta(x)$ is defined by Eq. (3.27). For a massless fermionic field one has the formula

$$\langle \bar{\psi}\psi \rangle_{\rm cyl} = \frac{q}{\pi^2 a^3} \sum_j \int_0^\infty dx\, x \frac{K_{\beta_j}^2(xr/a) + K_{\beta_j+1}^2(xr/a)}{K_{\beta_j}^2(x) + K_{\beta_j+1}^2(x)}, \quad (5.12)$$

and the fermionic condensate is positive.

Let us consider the behavior of the fermionic condensate in asymptotic regions of the parameter. In the limit $a \to 0$ with fixed values r, we introduce in (5.9) a new integration variable $y = x/a$ and expand the integrand in powers of a. The main contribution comes from the mode $j = 1/2$ and we have the leading term given below

$$\langle \bar{\psi}\psi \rangle_{\rm cyl} \approx \frac{2^{1-q} q (a/r)^q}{\pi^2 \Gamma^2((q+1)/2) r^3} \int_{mr}^{\infty} dx\, x^q [(x^2 - 2m^2 r^2) \times K^2_{(q-1)/2}(x) + x^2 K^2_{(q+1)/2}(x)]. \tag{5.13}$$

For a massless fermionic field, by using the result from [44] for the integral involving the square of the MacDonald function, from here we find

$$\langle \bar{\psi}\psi \rangle_{\rm cyl} \approx \frac{q(q+1)}{2\pi^2 r^3} \left(\frac{a}{r}\right)^q, \qquad a/r \ll 1. \tag{5.14}$$

At large distances from the cylindrical boundary, for a massive field under the condition $mr \gg 1$ the main contribution into the integral in (5.9) comes from the lower limit of the integration and to the leading order we find

$$\langle \bar{\psi}\psi \rangle_{\rm cyl} \approx \frac{q\sqrt{mr} e^{-2mr}}{4\sqrt{\pi} r^3} \sum_j {\rm Im}\left[\frac{\bar{I}_{qj-1/2}(ma)}{\bar{K}_{qj-1/2}(ma)}\right]. \tag{5.15}$$

Here the imaginary part is easily taken by using Eq. (5.11). As we could expect, in this limit the VEV is exponentially suppressed. At large distances and for a massless field the behavior of the fermionic condensate is described by Eq. (5.14). Note that the decreasing of the fermionic condensate at large distances is stronger than in the case when the string is absent. For points near the boundary, by using the uniform asymptotic expansions for the modified Bessel functions, we can see that the leading term in the asymptotic expansion of the fermionic condensate over the distance from the boundary is given by the same expression (3.32) as in the interior region.

C. VEV of the energy-momentum tensor

The VEV for the energy-momentum tensor in the exterior region is found in the way similar to that for the fermionic condensate. Here we omit the details of the calculations and give the final result. The VEV is decomposed into the sum of boundary-free and boundary-induced parts in the form given by (4.5). In the region outside the cylindrical shell the boundary-induced part is (no summation over ν)

$$\langle T_{\nu}^{\nu} \rangle_{\rm cyl} = \frac{q}{\pi^2 a^4} \sum_j \int_{\mu}^{\infty} dx\, x^3 \, {\rm Re}\left[\frac{\bar{I}_{\beta_j}(x)}{\bar{K}_{\beta_j}(x)} F^{({\rm ex})(\nu)}_{\beta_j}(x, xr/a)\right]. \tag{5.16}$$

In this formula we introduced the notations

$$\begin{aligned} F^{({\rm ex})(0)}_{\beta_j}(x,y) &= \frac{\mu^2/x^2 - 1}{2} \sum_{\delta=\pm 1} \delta\left(1 + \frac{i\delta\mu}{\sqrt{x^2-\mu^2}}\right) K^2_{qj-\delta/2}(xr/a), \\ F^{({\rm ex})(1)}_{\beta_j}(x,y) &= K^2_{\beta_j}(y) - K^2_{\beta_j+1}(y) + (2qj/y) K_{\beta_j}(y) K_{\beta_j+1}(y), \\ F^{({\rm ex})(2)}_{\beta_j}(x,y) &= -(2qj/y) K_{\beta_j}(y) K_{\beta_j+1}(y), \end{aligned} \tag{5.17}$$

and $F^{({\rm ex})(3)}_{\beta}(x,y) = F^{({\rm ex})(0)}_{\beta}(x,y)$. As an additional check we can see that these VEVs satisfy the trace relation and the covariant conservation equation. By using formula (5.11), we can write the vacuum densities in the form

$$\langle T_0^0 \rangle_{\rm cyl} = \frac{q}{2\pi^2 a^4} \sum_j \int_{\mu}^{\infty} dx\, x(1 - \mu^2/x^2) \frac{W_{\beta_j}(x)[K^2_{\beta_j}(xr/a) - K^2_{\beta_j+1}(xr/a)] + 2\mu K^2_{\beta_j+1}(xr/a)}{K^2_{\beta_j+1}(x) + K^2_{\beta_j}(x) - 2(\mu/x) K_{\beta_j}(x) K_{\beta_j+1}(x)}, \tag{5.18}$$

for the energy density and in the form (no summation over ν)

$$\langle T_{\nu}^{\nu} \rangle_{\rm cyl} = -\frac{q}{\pi^2 a^4} \sum_j \int_{\mu}^{\infty} dx \times \frac{x[W_{\beta_j}(x) - \mu] F^{({\rm ex})(\nu)}_{\beta_j}(x, xr/a)}{K^2_{\beta_j+1}(x) + K^2_{\beta_j}(x) - 2(\mu/x) K_{\beta_j}(x) K_{\beta_j+1}(x)}, \tag{5.19}$$

for the radial and azimuthal stresses, $\nu = 1, 2$. We recall that the summation over j in these formulae goes in accordance with Eq. (3.17).

In the case of a massless fermionic field from (5.16) we find the following expressions (no summation over ν):

$$\langle T_{\nu}^{\nu} \rangle_{\rm cyl} = \frac{q}{\pi^2 a^4} \sum_j \int_0^{\infty} dx\, x^3 F^{({\rm ex})(0,\nu)}_{\beta_j}(xr/a) \times \frac{I_{\beta_j}(x) K_{\beta_j}(x) - I_{\beta_j+1}(x) K_{\beta_j+1}(x)}{K^2_{\beta_j}(x) + K^2_{\beta_j+1}(x)}, \tag{5.20}$$

where

$$F^{({\rm ex})(0,0)}_{\beta_j}(y) = \tfrac{1}{2}[K^2_{\beta_j+1}(xr/a) - K^2_{\beta_j}(xr/a)], \tag{5.21}$$

and for the corresponding functions for the radial and azimuthal stresses we have $F^{({\rm ex})(0,\nu)}_{\beta}(y) = F^{({\rm ex})(\nu)}_{\beta}(x,y)$, $\nu = 1, 2$.

Now we turn to the investigation of the VEV in the energy-momentum tensor induced by the cylindrical shell in the exterior region in limiting cases. First let us consider the limit $a \to 0$ for fixed values r. Expanding the integrands in powers of a, we can see that the main contribution comes from the terms with $j = 1/2$, and the leading terms are given by the expressions (no summation over ν)

$$\langle T_0^0\rangle_{\rm cyl} \approx \frac{2^{1-q} qm(a/r)^q}{\pi^2\Gamma^2((q+1)/2)r^3} \times \int_{mr}^{\infty} dx\, x^q(x^2 - m^2r^2)K^2_{(q-1)/2}(x),$$
$$\langle T_\nu^\nu\rangle_{\rm cyl} \approx -\frac{2^{1-q} qm(a/r)^q}{\pi^2\Gamma^2((q+1)/2)r^3}\int_{mr}^{\infty} dx\, x^{q+2} F^{({\rm ex})(\nu)}_{(q-1)/2}(x,x), \tag{5.22}$$

with $\nu = 1, 2$. For a massless field these terms vanish. In this case from Eq. (5.20) we find the following leading behavior (no summation over ν):

$$\langle T_\nu^\nu\rangle_{\rm cyl} \approx \frac{q^2(q+1)A_\nu}{\pi^2(q-1)(q+2)r^4}\left(\frac{a}{r}\right)^{q+1}, \tag{5.23}$$

with the coefficients

$$A_0 = \frac{q+3}{2(q+4)}, \qquad A_1 = \frac{1}{q+4}, \qquad A_2 = -1. \tag{5.24}$$

For $q = 1$ the VEVs behave like $(a/r)^2 r^{-4}\ln(a/r)$.

At large distances from the cylinder and for a massive field the main contribution comes from the lower limit of the integral in (5.16). By using the asymptotic formulae for the MacDonald function for large values of the argument, we find

$$\langle T_0^0\rangle_{\rm cyl} \approx \frac{m}{2}\langle\bar\psi\psi\rangle_{\rm cyl}, \qquad \langle T_1^1\rangle_{\rm cyl} \approx -\frac{1}{2mr}\langle T_2^2\rangle_{\rm cyl},$$
$$\langle T_2^2\rangle_{\rm cyl} \approx -\frac{qme^{-2mr}}{2\pi r^3}\sum_j qj\,{\rm Re}\left[\frac{\bar I_{qj-1/2}(ma)}{\bar K_{qj-1/2}(ma)}\right], \tag{5.25}$$

for $mr \gg 1$. As we see, in this limit $\langle T_1^1\rangle_{\rm cyl} \ll \langle T_2^2\rangle_{\rm cyl} \ll \langle T_0^0\rangle_{\rm cyl}$. At large distances from the cylinder and for a massless field the asymptotic behavior of the boundary-induced parts is given by formula (5.23).

The asymptotic behavior of the VEV for the energy-momentum tensor near the cylindrical shell is found in a way similar to that for the interior region and the leading terms are given by the formulae (4.24) and (4.25). Hence, near the boundary the energy density and the azimuthal stress in the interior and exterior regions have opposite signs, whereas the radial stresses have the same sign. The boundary-induced parts in the VEVs of the energy density and the radial stress for the exterior region are plotted in Figs. 1–3 as functions of the radial coordinate and the parameters q and ma.

VI. CONCLUSION

In this paper the vacuum polarization effects are investigated for a fermionic field in the geometry of a cosmic string with a coaxial cylindrical shell. We have assumed that on the shell the field obeys the MIT bag boundary condition. In order to evaluate the fermionic condensate and the VEV of the energy-momentum tensor one needs the complete set of normalized eigenspinors satisfying the boundary condition. This set for the region inside the cylindrical shell is considered in Sec. II. The corresponding mode sums for both fermionic condensate and the energy-momentum tensor contain a series over the zeros of the combination (2.30) of the Bessel function of the first kind and its derivative. For the summation of these series we used a variant of the generalized Abel-Plana formula previously derived in Ref. [35]. This formula allows us to extract from the respective VEVs the parts corresponding to the cosmic string geometry without a cylindrical shell and to present the part induced by the shell in terms of exponentially convergent integrals for points away from the boundary. In this way the renormalization procedure for the fermionic condensate and the energy-momentum tensor is reduced to the renormalization of the corresponding quantities in the geometry of the boundary-free cosmic string. The renormalized VEV of the energy-momentum tensor for a fermionic field in the boundary-free geometry is well investigated in literature. In the appendix, by using the Abel-Plana summation formula, we give alternative integral representations for both fermionic condensate and the energy-momentum tensor in the case of a massive field.

In the region inside the shell, the parts in the VEVs induced by the presence of the cylindrical boundary are given by formula (3.22) for the fermionic condensate and by Eqs. (4.9) and (4.10), for the vacuum energy densities and stresses. These formulae are further simplified for a massless fermionic field with the vacuum densities given by Eqs. (3.28) and (4.15). For points near the cylindrical shell the boundary-induced parts in the VEVs dominate over the boundary-free parts and diverge on the cylindrical shell. These types of divergences are well known in quantum field theory with boundaries and are investigated for various bulk and boundary geometries. In the problem under consideration, the leading terms in the asymptotic expansions in powers of the distance from the boundary are given by Eq. (3.32) for the fermionic condensate and by Eqs. (4.9) and (4.10) for the components of the energy-momentum tensor. These leading terms do not depend on the planar angle deficit and are the same as for a cylindrical boundary in the Minkowski bulk. The boundary-induced parts in the VEVs vanish on the string axis for $q > 1$ and are nonzero in the case of a cylindrical boundary in the Minkowski bulk. Since the boundary-free part diverges on the string axis, for points near the string it dominates. For large values of the parameter q, which corresponds to a large planar angle deficit, the boundary-induced VEVs are suppressed by the factor $(r/a)^q$. The boundary-induced VEVs have nontrivial dependence on the mass of the field and, as it is illustrated by Fig. 3, for a massive field the polarization effects can be stronger than for a massless one.

Fermionic vacuum densities in the region outside a cylindrical shell with the MIT bag boundary condition are investigated in Sec. V. Subtracting from the mode sums the parts corresponding to the geometry of a string without boundaries and by making use of a complex rotation, we have derived explicit expressions for the boundary-induced VEVs. The corresponding parts in the fermionic condensate and the energy-momentum tensor are given by Eqs. (5.9), (5.18), and (5.19). When the cylinder radius goes to zero, for a fixed value of the radial distance, the boundary-induced part in the VEV of the energy-momentum tensor vanishes as a^q for a massive field and as a^{q+1} for a massless one. At large distances from the cylindrical shell this part is exponentially suppressed for a massive field and decay as $r^{-4}(r/a)^{q+1}$ in the case of a massless field. Note that in the latter case the boundary-free part behaves as r^{-4} and it dominates at large distances. For points near the cylindrical shell the leading terms in the asymptotic expansions in powers of the distance from the boundary are given by the same formulae as for the interior region. In this limit the total VEV is dominated by the boundary-induced part. In dependence of the mass, the vacuum stresses can be either positive or negative, whereas the energy density is positive. In the special case $q = 1$, from the formulae derived in the present paper we obtain the fermionic Casimir densities for a cylindrical boundary in the Minkowski spacetime.

We have considered the idealized geometry of a cosmic string with zero thickness. A realistic model for cosmic string has a nontrivial structure on a length scale defined by the phase transition at which it is formed. As it has been shown in Refs. [13,14,23], the internal structure of the string may have non-negligible effects even at large distances. Here we note that when the cylindrical boundary is present, the VEVs of the physical quantities in the exterior region are uniquely defined by the boundary conditions and the bulk geometry. This means that if we consider a nontrivial core model with finite thickness $b < a$ and with the line element (2.1) in the region $r > b$, the results in the region outside the cylindrical shell will not be changed. In regards to the interior region, the formulae given in this paper are the first stage of the evaluation of the VEVs and other effects could be present in a realistic cosmic string. Note that from the point of view of the physics in the exterior region the cylindrical surface with the MIT bag boundary condition can be considered as a simple model of nontrivial string core.

ACKNOWLEDGMENTS

E. R. B. M. and V. B. B. thank Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), FAPESQ-PB/CNPq (PRONEX), and FAPES-ES/CNPq (PRONEX) for partial financial support. A. A. S. was supported by the Armenian Ministry of Education and Science Grant No. 119 and by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq). A. S. T. was supported in part by Grant No. NFSAT-CRDF UC-06/07.

APPENDIX: VACUUM DENSITIES IN THE GEOMETRY OF A COSMIC STRING WITHOUT BOUNDARIES

In this appendix we consider the renormalized fermionic condensate and the VEV of the energy-momentum tensor in the cosmic string geometry when the cylindrical shell is absent. For a massless field the vacuum energy-momentum tensor was found in [7,9]. Fermionic propagators for a massive field are considered in Refs. [21,22]. In the case of a massive field, a representation of the VEVs for the energy-momentum tensor in terms of contour integrals is given in [28]. Here alternative integral formulae are given by applying to the corresponding mode sums the Abel-Plana formula. We will do these calculations by using the mode-sum formulae (3.16) and (4.12) for the boundary-free VEVs.

First let us consider the fermionic condensate. The renormalization is done by subtracting the corresponding quantity for the Minkowski background. The latter is obtained from (3.16) putting $q = 1$. Substituting in the corresponding formulae

$$\frac{1}{\omega} = \frac{2}{\sqrt{\pi}} \int_0^\infty ds e^{-\omega^2 s^2},$$

integrating over k and λ, and introducing a new integration variable $y = r^2/2s^2$, we find the following representation of the renormalized fermionic condensate:

$$\begin{aligned}\langle \bar{\psi} \psi \rangle_{\mathrm{s,ren}} &= \langle 0|\bar{\psi}\psi|0\rangle_{\mathrm{s}} - \langle 0|\bar{\psi}\psi|0\rangle_{\mathrm{M}} \\ &= -\frac{m}{2\pi^2 r^2} \int_0^\infty dy e^{-m^2 r^2/y - y} \\ &\quad \times \sum_{\delta=\pm 1} \sum_j [q I_{qj-\delta/2}(y) - I_{j-\delta/2}(y)]. \end{aligned} \tag{A1}$$

Next, we apply to the series over j the Abel-Plana summation formula in the form (see, for example, [31,45])

$$\sum_{n=0}^\infty f(n + 1/2) = \int_0^\infty dx f(x) - i \int_0^\infty dx \frac{f(ix) - f(-ix)}{e^{2\pi x} + 1}. \tag{A2}$$

It is easily seen that for the summand in (A1) the first integral on the right-hand side of (A2) vanishes and, hence, the divergent parts are explicitly cancelled. Introducing the MacDonald function, we arrive at the following expression:

$$\langle \bar{\psi}\psi \rangle_{\rm ren} = \frac{2m}{\pi^3 r^2} \int_0^\infty dy\, e^{-m^2r^2/2y-y} \times \int_0^\infty dx\, g(q,x)\,{\rm Im}[K_{ix+1/2}(y)], \quad \text{(A3)}$$

where the notation

$$g(q,x) = \cosh(\pi x)\left[\frac{1}{e^{2\pi x/q}+1} - \frac{1}{e^{2\pi x}+1}\right] \quad \text{(A4)}$$

is introduced.

In a similar way we can find the formula for the renormalized VEV of the energy-momentum tensor. For the renormalized energy density and the azimuthal stress we have the representations given below:

$$\langle T_0^0 \rangle_{\rm s,ren} = \frac{r^{-4}}{2\pi^2} \int_0^\infty dy\, y e^{-m^2r^2/2y-y} \times \sum_{\delta=\pm1} \sum_j [q I_{qj-\delta/2}(y) - I_{j-\delta/2}(y)],$$

$$\langle T_2^2 \rangle_{\rm s,ren} = \frac{r^{-4}}{\pi^2} \int_0^\infty dy\, y e^{-m^2r^2/2y-y} \times \sum_{\delta=\pm1} \delta \sum_j j[q^2 I_{qj-\delta/2}(y) - I_{j-\delta/2}(y)]. \quad \text{(A5)}$$

By making use of summation formula (A2) to the series over j, we find the following formulae for the renormalized VEVs:

$$\langle T_0^0 \rangle_{\rm s,ren} = -\frac{2}{\pi^3 r^4} \int_0^\infty dy (y + m^2r^2) e^{-m^2r^2/2y-y} \times \int_0^\infty dx g(q,x)\,{\rm Im}[K_{ix+1/2}(y)],$$

$$\langle T_2^2 \rangle_{\rm s,ren} = \frac{4}{\pi^3 r^4} \int_0^\infty dy\, y e^{-m^2r^2/2y-y} \times \int_0^\infty dx\, x g(q,x)\,{\rm Re}[K_{ix+1/2}(y)]. \quad \text{(A6)}$$

The radial stress is found from (A3) and (A6) by using the trace relation.

Formulae (A6) are further simplified for a massless fermionic field. The integration over y is done by using the formula

$$\int_0^\infty dy\, y e^{-y} K_{ix+1/2}(y) = \frac{\pi(4x^2+1)}{24\cosh(\pi x)}(2ix+3). \quad \text{(A7)}$$

Substituting this into Eq. (A6) and integrating over x, we find

$$\langle T_0^0 \rangle_{\rm s,ren} = -\frac{1}{3}\langle T_2^2 \rangle_{\rm s,ren} = -\frac{(q^2-1)(7q^2+17)}{2880\pi^2 r^4}. \quad \text{(A8)}$$

In the massless case the radial stress is equal to the energy density.

[1] A. Vilenkin, Phys. Rep. **121**, 263 (1985); A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and Other Topological Defects* (Cambridge University Press, Cambridge, England, 1994).
[2] T. Damour and A. Vilenkin, Phys. Rev. Lett. **85**, 3761 (2000); P. Battacharjee and G. Sigl, Phys. Rep. **327**, 109 (2000); V. Berezinski, B. Hnatyk, and A. Vilenkin, Phys. Rev. D **64**, 043004 (2001).
[3] S. Sarangi and S. H. H. Tye, Phys. Lett. B **536**, 185 (2002); E. J. Copeland, R. C. Myers, and J. Polchinski, J. High Energy Phys. 06 (2004) 013; G. Dvali and A. Vilenkin, J. Cosmol. Astropart. Phys. 03 (2004) 010.
[4] T. M. Helliwell and D. A. Konkowski, Phys. Rev. D **34**, 1918 (1986).
[5] W. A. Hiscock, Phys. Lett. B **188**, 317 (1987).
[6] B. Linet, Phys. Rev. D **35**, 536 (1987).
[7] V. P. Frolov and E. M. Serebriany, Phys. Rev. D **35**, 3779 (1987).
[8] J. S. Dowker, Phys. Rev. D **36**, 3095 (1987).
[9] J. S. Dowker, Phys. Rev. D **36**, 3742 (1987).
[10] P. C. W. Davies and V. Sahni, Classical Quantum Gravity **5**, 1 (1988).
[11] A. G. Smith, in *Proceedings of the Cambridge Workshop on The Formation and Evolution of Cosmic Strings, Cambridge, England, 1989*, edited by G. W. Gibbons, S. W. Hawking, and T. Vachaspati (Cambridge University Press, Cambridge, England, 1990).
[12] G. E. A. Matsas, Phys. Rev. D **41**, 3846 (1990).
[13] B. Allen and A. C. Ottewill, Phys. Rev. D **42**, 2669 (1990).
[14] B. Allen, J. G. Mc Laughlin, and A. C. Ottewill, Phys. Rev. D **45**, 4486 (1992).
[15] T. Souradeep and V. Sahni, Phys. Rev. D **46**, 1616 (1992).
[16] K. Shiraishi and S. Hirenzaki, Classical Quantum Gravity **9**, 2277 (1992).
[17] V. B. Bezerra and E. R. Bezerra de Mello, Classical Quantum Gravity **11**, 457 (1994); E. R. Bezerra de Mello, Classical Quantum Gravity **11**, 1415 (1994).
[18] D. Fursaev, Classical Quantum Gravity **11**, 1431 (1994).
[19] G. Cognola, K. Kirsten, and L. Vanzo, Phys. Rev. D **49**, 1029 (1994).
[20] M. E. X. Guimarães and B. Linet, Commun. Math. Phys. **165**, 297 (1994).
[21] B. Linet, J. Math. Phys. (N.Y.) **36**, 3694 (1995).
[22] E. S. Moreira Jnr, Nucl. Phys. **B451**, 365 (1995).
[23] B. Allen, B. S. Kay, and A. C. Ottewill, Phys. Rev. D **53**, 6829 (1996).
[24] M. Bordag, K. Kirsten, and S. Dowker, Commun. Math. Phys. **182**, 371 (1996).
[25] D. Iellici, Classical Quantum Gravity **14**, 3287 (1997).
[26] N. R. Khusnutdinov and M. Bordag, Phys. Rev. D **59**,

064017 (1999).
[27] J. Spinelly and E. R. Bezerra de Mello, Classical Quantum Gravity **20**, 873 (2003).
[28] V. B. Bezerra and N. R. Khusnutdinov, Classical Quantum Gravity **23**, 3449 (2006).
[29] J. Spinelly and E. R. Bezerra de Mello, J. High Energy Phys. 09 (2008) 005.
[30] E. R. Bezerra de Mello and A. A. Saharian, Phys. Lett. B **642**, 129 (2006); Phys. Rev. D **78**, 045021 (2008).
[31] V. M. Mostepanenko and N. N. Trunov, *The Casimir Effect and Its Applications* (Oxford University, New York, 1997); M. Bordag, U. Mohidden, and V. M. Mostepanenko, Phys. Rep. **353**, 1 (2001); K. A. Milton, *The Casimir Effect: Physical Manifestation of Zero-Point Energy* (World Scientific, Singapore, 2002).
[32] E. R. Bezerra de Mello, V. B. Bezerra, A. A. Saharian, and A. S. Tarloyan, Phys. Rev. D **74**, 025017 (2006).
[33] E. R. Bezerra de Mello, V. B. Bezerra, and A. A. Saharian, Phys. Lett. B **645**, 245 (2007).
[34] A. A. Saharian and M. R. Setare, Classical Quantum Gravity **20**, 3765 (2003); Int. J. Mod. Phys. A **19**, 4301 (2004).
[35] A. A. Saharian and E. R. Bezerra de Mello, J. Phys. A **37**, 3543 (2004); Int. J. Mod. Phys. A **20**, 2380 (2005).
[36] E. R. Bezerra de Mello and A. A. Saharian, Classical Quantum Gravity **23**, 4673 (2006).
[37] S. Leseduarte and A. Romeo, Commun. Math. Phys. **193**, 317 (1998); C. G. Beneventano, M. De Francia, K. Kirsten, and E. M. Santangelo, Phys. Rev. D **61**, 085019 (2000); M. De Francia and K. Kirsten, Phys. Rev. D **64**, 065021 (2001).
[38] P. Fishbane, S. Gasiorowicz, and P. Kaus, Phys. Rev. D **36**, 251 (1987); **37**, 2623 (1988).
[39] B. M. Barbashov and V. V. Nesterenko, *Introduction to the Relativistic String Theory* (World Scientific, Singapore, 1990).
[40] J. Ambjørn and S. Wolfram, Ann. Phys. (N.Y.) **147**, 1 (1983).
[41] P. S. Gerbert and R. Jackiw, Commun. Math. Phys. **124**, 229 (1989).
[42] V. D. Skarzhinsky and D. D. Harari, and U. Jasper, Phys. Rev. D **49**, 755 (1994).
[43] M. Bordag and N. Khusnutdinov, Classical Quantum Gravity **13**, L41 (1996).
[44] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series* (Gordon and Breach, New York, 1986), Vol. 2.
[45] A. A. Saharian, International Centre for Theoretical Physics Report No. ICTP/2007/082.
[46] *Handbook of Mathematical Functions*, edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1972).