

Intersecting solitons, amoeba, and tropical geometryToshiaki Fujimori,^{1,*} Muneto Nitta,^{2,+} Kazutoshi Ohta,^{3,‡} Norisuke Sakai,^{4,§} and Masahito Yamazaki^{5,||}¹*Department of Physics, Tokyo Institute of Technology Tokyo 152-8551, Japan*²*Department of Physics, Keio University, Hiyoshi, Yokohama, Kanagawa 223-8521, Japan*³*Department of Physics, Tohoku University, Sendai 980-8578, Japan*⁴*Department of Mathematics, Tokyo Woman's Christian University, Tokyo 167-8585, Japan*⁵*Department of Physics, University of Tokyo, Tokyo 113-0033, Japan*

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We study the generic intersection (or web) of vortices with instantons inside, which is a $1/4$ Bogomol'nyi-Prasad-Sommerfield state in the Higgs phase of five-dimensional $\mathcal{N} = 1$ supersymmetric $U(N_C)$ gauge theory on $\mathbb{R}_t \times (\mathbb{C}^*)^2 \simeq \mathbb{R}^{2,1} \times T^2$ with $N_F = N_C$ Higgs scalars in the fundamental representation. In the case of the Abelian-Higgs model ($N_F = N_C = 1$), the intersecting vortex sheets can be beautifully understood in a mathematical framework of amoeba and tropical geometry, and we propose a dictionary relating solitons and gauge theory to amoeba and tropical geometry. A projective shape of vortex sheets is described by the amoeba. Vortex charge density is uniformly distributed among vortex sheets, and negative contribution to instanton charge density is understood as the complex Monge-Ampère measure with respect to a plurisubharmonic function on $(\mathbb{C}^*)^2$. The Wilson loops in T^2 are related with derivatives of the Ronkin function. The general form of the Kähler potential and the asymptotic metric of the moduli space of a vortex loop are obtained as a by-product. Our discussion works generally in non-Abelian gauge theories, which suggests a non-Abelian generalization of the amoeba and tropical geometry.

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I. INTRODUCTION

The study of topological solitons is intimately connected with the development of mathematics. Originally solitons were found as solutions to nonlinear (field) equations in various physical systems, but many mathematical concepts have been developed at the same time in order to understand properties and integrability of the soliton equations. Indeed, many mathematical tools, such as representation theory of differential operator algebra and infinite dimensional Grassmannian, are required to investigate the solitons. The language of two-dimensional conformal field theory is also useful in studies of a class of the soliton systems, such as Korteweg–de Vries (KdV), Toda, Kadomtsev-Petviashvili (KP) equations. The investigation of solitons in gauge theory, such as instantons and monopoles, gives not only nonperturbative information about the field theory, but also stimulates new developments of differential geometry.

The purpose of the present article is to study topological solitons using novel mathematical objects known as amoeba and tropical geometry. We find a one-to-one correspondence between amoeba/tropical geometry and solitons in Yang-Mills-Higgs theory. Yang-Mills gauge fields coupled to the Higgs fields naturally appear as a bosonic

part in supersymmetric Yang-Mills theory with eight supercharges. When a sufficient number of the Higgs fields get vacuum expectation values (VEVs), the theory is in the Higgs phase with completely broken gauge symmetry. Typical solitons in the Higgs phase known for a long time are vortices in the Abelian $U(1)$ gauge theory coupled to a single complex Higgs field (the Abelian-Higgs model) [1]. These vortices have been recently extended to the non-Abelian case, vortices in completely broken non-Abelian $U(N)$ gauge symmetry [2–12]. The other fundamental solitons found relatively recently in Yang-Mills-Higgs theory (or corresponding nonlinear sigma models) are domain walls or kinks [6,13–16]. Since both vortices and domain walls preserve half of the supercharges when embedded into supersymmetric theories, they are called $1/2$ Bogomol'nyi-Prasad-Sommerfield (BPS) solitons. Composite solitons in the Higgs phase have recently been studied extensively, especially in supersymmetric gauge theories with eight supercharges [17–19]. Since the magnetic field has to vanish in the Higgs phase, magnetic monopoles are confined by vortices (confined monopoles) [20–23]. Although isolated instantons shrink to points in the Higgs phase, they can lie inside a vortex core (trapped instantons) [22,23]. Vortex strings can end on a domain wall [24] or stretch between domain walls [25]. When a (composite) soliton configuration breaks n directions of translational symmetry, it is defined to have n codimensions. The composite soliton with the lowest codimension is a web or a network of domain walls, whose moduli space and dynamics have been worked out recently [26–30]. All of these composite solitons preserve $1/4$ of supercharges if

*fujimori@th.phys.titech.ac.jp

+nitta@phys-h.keio.ac.jp

‡kohta@phys.tohoku.ac.jp

§sakai@lab.twcu.ac.jp

||yamazaki@hep-th.phys.s.u-tokyo.ac.jp

we realize the Yang-Mills-Higgs theory as a supersymmetric Yang-Mills theory, and are called 1/4 BPS states. It has been found that all these 1/4 BPS composite solitons are related by the (Scherk-Schwarz twisted) dimensional reduction, starting from the instanton-vortex system [23]. Therefore understanding the instanton-vortex system is of primary importance for studying BPS solitons in supersymmetric Yang-Mills-Higgs theories with eight supercharges.¹ Unfortunately, generic configurations of instantons and vortex sheets as codimension four solitons have not been worked out, apart from a trapped instanton as a lump on a (uncurved) vortex plane, or an intersection point of two orthogonal (uncurved) vortex planes [23]. It is important to characterize generic configurations of instantons and vortex sheets as codimension four solitons in a precise and transparent manner. We can call these soliton webs of vortices. We show that these soliton webs are nicely described in terms of the amoeba and the tropical geometry.

Interestingly, amoeba and tropical geometry have already appeared in physics literature, in the context of topological strings. Topological string amplitude (or its building block, topological vertex) can be described by means of a melting crystal picture [34,35], which was originally introduced as a statistical model in mathematical physics. Shape of the melting crystal in the thermodynamic limit is well interpreted in terms of an amoeba, which is a logarithmic projection of a smooth Riemann surface. On the other hand, an emaciated body of the amoeba corresponding to zero temperature limit can be captured by tropical geometry, where we can obtain some properties of the Riemann surface from a skeleton of the amoeba. In the context of superstring theory, the amoeba and tropical geometry appear as, respectively, quantum ($g_s \gg 1$) and classical ($g_s \rightarrow 0$) shape of intersecting five-branes (five-brane junctions), which are dual to a suitable Calabi-Yau geometry where the topological string is defined. Other examples appear in an instanton and BPS state counting problem of the supersymmetric (quiver) gauge theories [36–38], since these systems are closely related to the topological string amplitudes and realized in terms of the five-brane web.

The five-brane system is useful for understanding the relationship between the gauge and string theory, and the amoeba and tropical geometry, which is the main subject of this article. Our study is inspired by these successful

¹It is worth pointing out that the other series of 1/4 BPS systems exists. This contains a triple intersection of vortices where a set of two vortices has one common codimension [31], contrary to the instanton-vortex system [23] where vortices have no (or two) common codimensions. The former preserves (1,1) SUSY and the latter (2,0) SUSY in terms of two-dimensional supersymmetry (SUSY)[32]. Different sets of 1/4 BPS equations in these two series are obtained as dimensional reductions of the unique set of the 1/8 BPS equations [32,33].

applications of the amoeba and tropical geometry to physics. We find a similarity between the five-brane web and the 1/4 BPS composite solitons of vortex sheets and instantons. Some of properties of the webs of vortices are still unclear. In particular, the instanton charge appears at the intersection point of the vortices, but we have not obtained a tool to see a distribution of the extra instanton charge on the vortex web. The five-brane web in the superstring theory and vortex web possess common properties. Indeed, toric diagrams and geometry play an important role in both sides, and their moduli spaces are described by similar quotient spaces and the moduli space of the vortex web should be included in that of the five-brane web. So we expect that the amoeba and tropical geometry, which is important for understanding the dynamics of the five-brane web, are also useful for analyzing the web of the solitons in the Yang-Mills-Higgs system.

In this paper, we study the most generic configurations of 1/4 BPS solitons of instantons and vortices in the Higgs phase of the five-dimensional $\mathcal{N} = 1$ supersymmetric $U(N_C)$ gauge theory (with eight supercharges) on $\mathbb{R}_t \times (\mathbb{C}^*)^2 \sim \mathbb{R}^{2,1} \times T^2$ with $N_F = N_C$ Higgs scalars in the fundamental representation, by using the moduli matrix formalism [18]. Torus $T^2 = S^1 \times S^1$ allows us to obtain other 1/4 BPS solitons readily through dimensional reduction. We show that vortex sheets are defined by zeros of a Laurent polynomial of two complex coordinates of $(\mathbb{C}^*)^2$, and instanton positions are given by common zeros with another polynomial. We find the above expectation of the importance of amoeba and tropical geometry to be correct. We describe physical quantities of the intersecting solitons (soliton web) in terms of the mathematical language of amoeba and tropical geometry. We also see the important objects in the amoeba and tropical geometry, such as the logarithmic mapping, the Ronkin function, and the Monge-Ampère measure, which also have essential meanings on the soliton side. We describe properties of the soliton web in terms of these mathematical objects. We find that the moduli matrix approach is very useful in the translation between physical and mathematical languages.

The organization of the present paper is as follows. In Sec. II, we review the 1/4 BPS equations for vortices and instantons in the Higgs phase of five-dimensional $\mathcal{N} = 1$ supersymmetric $U(N_C)$ gauge theory on $\mathbb{R}_t \times (\mathbb{C}^*)^2 \sim \mathbb{R}^{2,1} \times T^2$ with $N_F = N_C$ Higgs scalars in the fundamental representation, using the moduli matrix formalism. We mainly consider vortices in the overall $U(1)$ gauge theory except in Sec. V. In Sec. III, we consider a simpler case of vortex on cylinder $\mathbb{R} \times S^1 \simeq \mathbb{C}^*$. In Sec. IV, we study the most general situation of vortex sheets on $(\mathbb{C}^*)^2$. Section IVA gives amoeba corresponding to vortex sheets. Section IV B relates it to the tropical geometry and gives an example. Section IV C describes a general formula to compute the topological charges. Section IV D gives the metric of the moduli space of a vortex loop. We also

obtained a new general formula for the Kähler metric which is valid for arbitrary values of N_C and N_F . In Sec. V, we examine the instanton number in non-Abelian gauge theory. Section VA reviews the instanton number for instantons trapped inside a non-Abelian vortex plane [23]. Section VB discusses more general configurations of the instantons trapped inside a non-Abelian vortex web. Section VI is devoted to conclusion and discussion.

II. VORTICES AND INSTANTONS

In this section, we first review the construction of vortex solutions in $U(N_C)$ gauge theory in $(4+1)$ -dimensional spacetime $\mathbb{R}_t \times (\mathbb{C}^*)^2 \sim \mathbb{R}^{2,1} \times T^2$ with N_F Higgs fields in the fundamental representation. By introducing additional N_F Higgs fields in the fundamental representation, this theory can also be regarded as the bosonic part of a five-dimensional $\mathcal{N} = 1$ supersymmetric $U(N_C)$ gauge theory with N_F hypermultiplets in the fundamental representation, but the fermionic part (and another set of N_F Higgs scalars) is irrelevant and is omitted in the following discussion. We introduce (x_1, y_1, x_2, y_2) and $z_1 \equiv x_1 + iy_1$, $z_2 \equiv x_2 + iy_2$ as real and complex coordinates of $(\mathbb{C}^*)^2$, respectively.

The Lagrangian of the theory takes the form

$$\mathcal{L} = \text{Tr} \left[-\frac{1}{2g^2} F_{\mu\nu} F^{\mu\nu} + \mathcal{D}_\mu H (\mathcal{D}^\mu H)^\dagger - \frac{g^2}{4} (HH^\dagger - c \mathbf{1}_{N_C})^2 \right], \quad (2.1)$$

where the Higgs fields are expressed as an $N_C \times N_F$ matrix H^{rA} ($r = 1, \dots, N_C$, $A = 1, \dots, N_F$). The covariant derivative is defined by $\mathcal{D}_\mu H = \partial_\mu H + iW_\mu H$ and the field strength by $F_{\mu\nu} = -i[\mathcal{D}_\mu, \mathcal{D}_\nu] = \partial_\mu W_\nu - \partial_\nu W_\mu + i[W_\mu, W_\nu]$. The constants g and c are the gauge coupling constant and the Fayet-Iliopoulos (FI) parameter, respectively. At the vacua (the minima of the potential) of this theory, the Higgs fields H get VEV and $U(N)$ gauge symmetry is completely broken. Namely the theory has only the Higgs branch due to the nonzero FI term. The moduli space of the vacua is given by a complex Grassmannian

$$G(N_F, N_C) = \frac{SU(N_F)}{SU(N_C) \times SU(N_F - N_C) \times U(1)}. \quad (2.2)$$

Considering a static gauge configuration, we find that there is a lower bound of the energy [22,23]

$$\begin{aligned} E &\geq -\frac{1}{g^2} \int \text{Tr}(F \wedge F) - c \int \text{Tr} F \wedge \omega \\ &= \frac{8\pi^2}{g^2} I + 2\pi c V, \end{aligned} \quad (2.3)$$

where the two-form $\omega \equiv \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2)$ is the Kähler form on $(\mathbb{C}^*)^2$, and we have defined the total instanton charge I as an integral of the instanton charge

density I , and the vortex charge V as a divergent integral of the vortex charge density \mathcal{V}

$$I \equiv \int I \equiv -\frac{1}{8\pi^2} \int \text{Tr}(F \wedge F) = \int ch_2, \quad (2.4)$$

$$V \equiv \int \mathcal{V} \equiv -\frac{1}{2\pi} \int \text{Tr} F \wedge \omega = \int c_1 \wedge \omega. \quad (2.5)$$

The lower bound Eq. (2.3) is saturated if the following BPS equations [22,23]

$$\begin{aligned} F_{\bar{z}_1 \bar{z}_2} &= 0, & \mathcal{D}_{\bar{z}_i} H &= 0, \\ -2i(F_{z_1 \bar{z}_1} + F_{z_2 \bar{z}_2}) &= \frac{g^2}{2} (HH^\dagger - c \mathbf{1}_{N_C}), \end{aligned} \quad (2.6)$$

are satisfied. When FI parameter c is sent to zero, the Higgs field H vanishes and these equations reduce to the anti-self-dual equations for Yang-Mills instantons, whereas when we neglect the z_2 -dependence (or the z_1 -dependence) of the fields they reduce to simple vortex equations for vortices on the z_1 -plane (or the z_2 -plane). These vortices are two-codimensional surfaces in the four-dimensional space $(\mathbb{C}^*)^2$. Therefore the BPS equations (2.6) contain at least instantons and intersecting vortex sheets. As we will see below, these equations describe webs of vortex sheets in general. The equations (2.6) were earlier found for those on arbitrary Kähler manifold [39] and were simply called ‘‘vortex equations’’ although they contain instantons also.² It has been shown in [23] that solutions to the BPS equations (2.6) on $(\mathbb{C}^*)^2$ (or \mathbb{C}^2) preserve a quarter of supercharges in the supersymmetric gauge theory with eight supercharges. So the configuration of the solution is called a 1/4 BPS state in this sense. The energy of the BPS configuration is determined by the topological charges (2.4) and (2.5).

The vortex charge V can be evaluated from the sum of the area of each vortex sheet, as we will see in Sec. IV. The total instanton charge I can be decomposed into the intersection charge $I_{\text{intersection}}$ and the instanton number $I_{\text{instanton}}$ as

$$I = -I_{\text{intersection}} + I_{\text{instanton}}, \quad (2.7)$$

$$\begin{aligned} I_{\text{intersection}} &\equiv \int J_{\text{intersection}} \equiv \frac{1}{8\pi^2} \int \text{Tr} F \wedge \text{Tr} F \\ &= \frac{1}{2} \int c_1 \wedge c_1, \end{aligned} \quad (2.8)$$

²In Ref. [39] the vortex equations are defined on arbitrary Kähler manifold M of complex dimension n (with $n = 2$ not necessary). There the bound is given by $\int_M \text{Tr}(F \wedge F) \wedge \omega^{n-2}$ and $\int_M \text{Tr} F \wedge \omega^{n-1}$ with the Kähler two-form ω , instead of the charges (2.4) and (2.5). Furthermore at least in the case of $N_C = N_F = 1$ these generalized equations can be obtained as equivariant dimensional reduction of the Donaldson-Uhlenbeck-Yau equations on $M \times S^2$ with a monopole configuration on S^2 [40].

$$I_{\text{instanton}} \equiv \int I_{\text{instanton}} \equiv \int c_2. \quad (2.9)$$

The intersection charge has negative contribution to the energy of the BPS configuration, which can be regarded as binding energy of intersecting vortex sheets. On the other hand the instanton number (not to be confused with total instanton charge) counts to the number of usual (particle-like) instantons and has positive contribution to the energy.

Let us solve the BPS equations (2.6). The first BPS equation $F_{\bar{z}_1 \bar{z}_2} \equiv -i[\mathcal{D}_{\bar{z}_1}, \mathcal{D}_{\bar{z}_2}] = 0$ in (2.6) is equivalent to an integrability condition³ for the differential operators $\mathcal{D}_{\bar{z}_i}$, which states that there exists an $N_C \times N_C$ matrix-valued function⁴ $S(z_i, \bar{z}_i) \in U(N_C)^{\mathbb{C}} = GL(N_C, \mathbb{C})$ such that

$$W_{\bar{z}_i} = -iS^{-1} \partial_{\bar{z}_i} S. \quad (2.10)$$

Defining an $N_C \times N_F$ matrix

$$H_0 \equiv SH, \quad (2.11)$$

the second equation in (2.6) reduces to

$$\partial_{\bar{z}_i} H_0 = 0. \quad (2.12)$$

This means that the elements of the matrix H_0 should be holomorphic⁵ with respect to the complex coordinates z_i . The matrix-valued quantity S is determined from the last equation in Eq. (2.6), which can be rewritten in terms of an $N_C \times N_C$ positive definite Hermitian matrix⁶

$$\Omega \equiv SS^\dagger \quad (2.13)$$

into

$$\begin{aligned} & \partial_{\bar{z}_1}(\Omega \partial_{z_1} \Omega^{-1}) + \partial_{\bar{z}_2}(\Omega \partial_{z_2} \Omega^{-1}) \\ &= -\frac{g^2 c}{4}(\mathbf{1}_{N_C} - \Omega_0 \Omega^{-1}), \end{aligned} \quad (2.14)$$

where we have defined

$$\Omega_0 \equiv \frac{1}{c} H_0 H_0^\dagger. \quad (2.15)$$

³If we identify the Higgs field H^{rA} as a set of N_F sections of a rank N_C vector bundle E on the base Kähler manifold, then the first BPS equation $F_{\bar{z}_1 \bar{z}_2} = 0$ is equivalent to the condition of the existence of the holomorphic frame $\{\tilde{e}_i\}$ ($i = 1, \dots, N_C$), which satisfy $\mathcal{D}_{\bar{z}_i} \tilde{e}_i = 0$.

⁴The complexified gauge transformation S^i_r can be interpreted as the change of basis from the unitary frame (orthonormal frame) $\{e_r\}$ ($r = 1, \dots, N_C$) to the holomorphic frame $\{\tilde{e}_i\}$.

⁵In other words, H_0^{iA} is a set of N_F holomorphic sections of the holomorphic vector bundle E .

⁶This matrix Ω can be interpreted as the inverse of the Hermitian metric in terms of the holomorphic frame $\{\tilde{e}_i\}$ which is the identity matrix in terms of the unitary frame $\{e_r\}$.

We call Eq. (2.14) the ‘‘master equation’’ of the instanton-vortex system.⁷

Using these redefined fields, we can solve the BPS equations by the following procedure. Taking an arbitrary holomorphic matrix $H_0(z)$ and solving Eq. (2.14) in terms of Ω , then we can determine S up to $U(N_C)$ gauge transformation $S \rightarrow SU^\dagger$ and physical fields can be obtained via the relations

$$W_{\bar{z}_i} = -iS^{-1} \partial_{\bar{z}_i} S, \quad H = S^{-1} H_0. \quad (2.16)$$

Equations (2.12) and (2.14) have a ‘‘gauge symmetry’’, which we call ‘‘V-transformation,’’ defined by

$$(H_0, S) \rightarrow (VH_0, VS), \quad V(z) \in GL(N_C, \mathbb{C}). \quad (2.17)$$

Note that the physical fields $W_{\bar{z}}$ and H are invariant under the V-transformation, so this defines an equivalence relation called the ‘‘V-equivalence’’.⁸ Assuming that there exists a unique solution of Eq. (2.14) for a given $H_0(z)$,⁹ we find that there exists a one-to-one correspondence between the equivalence class $H_0 \sim VH_0$ and a point on the moduli space of the BPS configurations. In this sense, we call $H_0(z)$ a ‘‘moduli matrix’’ and the parameters contained in H_0 are identified with the moduli parameters of the BPS configurations.

Now let us consider the case $N_C = N_F = N$ which is often called a local theory. If the determinant of the Higgs fields H vanishes in some regions, the broken gauge symmetry is partially restored in those regions. As dictated by the Meissner effect in the Higgs phase, this gauge symmetry restoration occurs where the magnetic flux penetrates and a vorticity arises around the zero of the Higgs field. Therefore the vanishing determinant $\det H = 0$ defines a two-dimensional surface of vortex positions in the four-dimensional space: it is equivalently given by¹⁰

$$\det H_0(z_1, z_2) = 0. \quad (2.18)$$

In closing this section, we comment on the vortex solutions of the $(2+1)$ -dimensional $U(N_C)$ gauge theory on $\mathbb{R}_t \times \mathbb{C}^*$ with N_F massless Higgs fields in the fundamental representation, since we study this case in the next section. Historically, the non-Abelian vortices were first found [2,3] on \mathbb{C} in the color-flavor locked phase of $U(N_C)$ gauge

⁷When the FI parameter c goes to zero, the Higgs phase no longer exists. In this case, the right-hand side of Eq. (2.14) vanishes, and Eq. (2.14) becomes the so-called Yang’s equation [41] for usual instantons not accompanied by vortices.

⁸This equivalence relation originates from the redundancy of the holomorphic frame $\{\tilde{e}_i\}$.

⁹This assumption is correct at least when the base space is compact and Kähler, since the uniqueness and existence of solutions to the BPS equations (2.6) were rigorously proved in terms of the Hitchin-Kobayashi correspondence [39].

¹⁰The determinant $\det H_0$ can be regarded as the holomorphic section of the determinant line bundle $\wedge^{N_C} E$, and the vortex sheet corresponds to the effective divisor associated with the holomorphic section $\det H_0$.

theory with N_F massless Higgs fields in the fundamental representation. After their discovery the non-Abelian vortices have been extensively studied by many authors [4]. In particular, the non-Abelian vortices on a cylinder \mathbb{C}^* have been studied in [7,11]. In the moduli matrix formalism, the discussion of the \mathbb{C}^* case is completely parallel to the $(\mathbb{C}^*)^2$ case and the necessary formulae are simply obtained by neglecting the z_2, \bar{z}_2 dependence. For example, the BPS bound (2.3) is reduced to

$$E \geq -c \int d^2x \text{Tr} F_{xy}, \quad (2.19)$$

and the master equation (2.14) for $\Omega(z_1)$ is given by [5]

$$\partial_{\bar{z}_1}(\Omega \partial_{z_1} \Omega^{-1}) = -\frac{g^2 c}{4} (\mathbf{1}_{N_C} - \Omega_0 \Omega^{-1}). \quad (2.20)$$

For the Abelian-Higgs model, $N_C = N_F = 1$, this equation reduces to the so-called Taubes's equation [42] after some redefinition.

III. VORTICES ON A CYLINDER \mathbb{C}^*

In this section, we first consider the simpler case of vortices on a cylinder $\mathbb{R} \times S^1 \simeq \mathbb{C}^*$ before discussing the intersecting vortices in four dimensions. The vortices are BPS solutions of the $(2+1)$ -dimensional $U(N_C)$ gauge theory with N_F massless Higgs fields in the fundamental representation, with one spatial direction compactified [7,11]. In the supersymmetric system, the vortices preserve half of the supercharges. Although our real interest is in the complex two-dimensional case ($\mathbb{R}^2 \times T^2 \simeq (\mathbb{C}^*)^2$), the one-dimensional case is simpler and useful to understand the discussion in the next section.

Let (x, y) and $z \equiv x + iy$ be real and complex coordinates of $\mathbb{R} \times S^1 \simeq \mathbb{C}^*$, respectively. The coordinate of S^1 has a period $2\pi R$, namely $y \sim y + 2\pi R$. We here concentrate on the case of $N_C = N_F = N$. The moduli matrix formalism works as well in this $(2+1)$ -dimensional case, and the BPS solutions are parametrized by the moduli matrix $H_0(z)$. Since the moduli matrix should satisfy the periodic boundary condition $H_0(z + 2\pi i R) = H_0(z)$, the determinant of the moduli matrix can be expanded as a Fourier series

$$\det H_0 = \sum_{n \in \mathbb{Z}} a_n e^{nz/R}. \quad (3.1)$$

Introducing a new coordinate $u \equiv e^{z/R}$, this can be rewritten as

$$\det H_0 = P(u) \equiv \sum_{n \in \mathbb{Z}} a_n u^n. \quad (3.2)$$

The positions of the vortices are determined by zeros of this Laurent polynomial. By performing an appropriate V -transformation $H_0(z) \rightarrow V(z)H_0(z)$, $\det H_0$ reduces to

$$\det H_0 = \prod_{i=1}^k (e^{z_i/R} - e^{z_i/R}) = \prod_{i=1}^k (u - u_i), \quad u_i \equiv e^{z_i/R}, \quad (3.3)$$

with k denoting the number of the vortices. The solution Ω of the master equation (2.20) for the vortices approaches to $\Omega_0 = \frac{1}{c} H_0 H_0^\dagger$ in the strong gauge coupling limit $g \rightarrow \infty$. In this limit the configuration of the magnetic flux of the overall $U(1)$ becomes singular such as

$$\begin{aligned} \text{Tr} F &= i \bar{\partial} \partial \log \det \Omega \rightarrow i \bar{\partial} \partial \log |\det H_0|^2 \\ &= -2\pi \sum_{i=1}^k \delta^2(z - z_i) dx \wedge dy, \end{aligned} \quad (3.4)$$

with $\partial = dz \partial_z, \bar{\partial} = d\bar{z} \partial_{\bar{z}}$. This reflects the fact that the size of the vortices is proportional to $l \equiv 1/g\sqrt{c}$ and becomes zero in the strong coupling limit¹¹. Using the configuration in the infinite coupling limit, we find that the topological charge is given by the number of zeros of the polynomial $P(u)$

$$- \frac{1}{2\pi} \int d^2x \text{Tr} F_{xy} = k. \quad (3.5)$$

If we dimensionally reduce the theory on S^1 , then the vortex can be viewed as a domain wall in $(1+1)$ -dimensional theory [14]. These field theoretical BPS solitons are realized by kinky D-brane configurations in superstring theory [15,16], and the relation between vortices and domain walls is understood via T-duality [7]. In field theory language, the profile of the kink solution of the domain wall is described by a logarithm of a Wilson line along S^1

$$\hat{\Sigma}(x) \equiv -\frac{1}{2\pi i R} \log \left[\mathbf{P} \exp \left(i \int_{S^1} dy W_y \right) \right]. \quad (3.6)$$

$\hat{\Sigma}(x)$ can also be viewed as the adjoint scalar in the T-dual (dimensionally reduced) theory. See Fig. 1(b) for an example of a $\text{Tr} \hat{\Sigma}(x)$ plot.

It is convenient to define a function $N_P(x)$ associated with the Laurent polynomial $P(u) = \det H_0(z)$ by

$$\begin{aligned} N_P(x) &\equiv \lim_{g \rightarrow \infty} \int_0^{2\pi R} \frac{dy}{2\pi R} \frac{1}{2} \log \det \Omega \\ &= \int_0^{2\pi R} \frac{dy}{2\pi R} \log |\det H_0| \\ &= \frac{1}{R} \sum_{i=1}^k (x \theta(x - x_i) + x_i \theta(x_i - x)). \end{aligned} \quad (3.7)$$

In the final line, we have used the Jensen's formula [43].¹²

¹¹The appearance of length scale $l \equiv 1/g\sqrt{c}$ is understood from the master equation (2.20). In (2.20), the parameters g and c appear only in the combined form $g^2 c$, and thus l is the only length scale made from $g^2 c$.

¹²The classical Jensen formula states that for arbitrary holomorphic function $f(x)$ with zeros at a_i ($i = 1, 2, \dots, k$), we have

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta = \log |f(0)| + \sum_{i=1}^{N_r} \log \left| \frac{r}{a_i} \right|, \quad (3.8)$$

where we have chosen indices i such that $a_i < r$ for $i = 1, 2, \dots, N_r$ and $a_i > r$ otherwise.

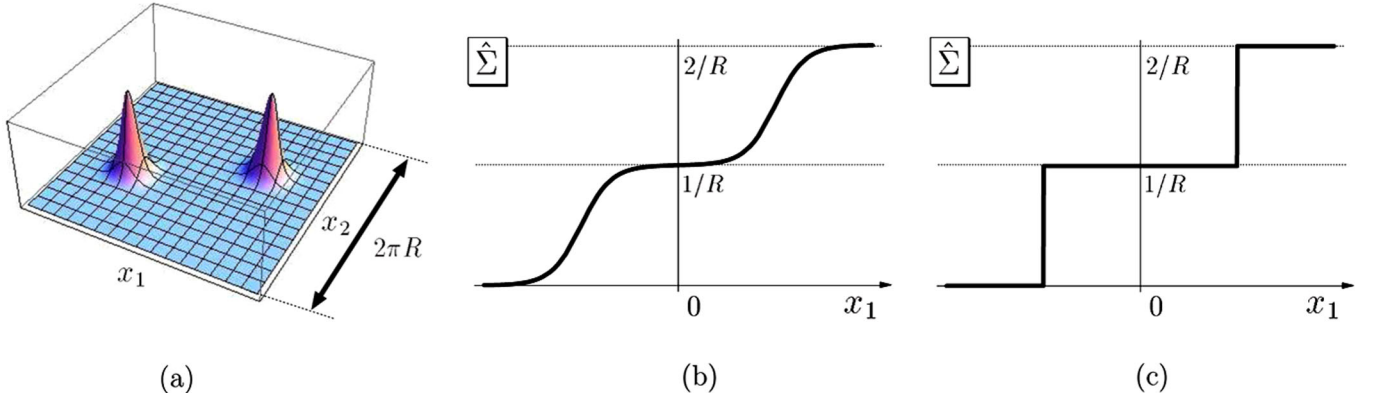


FIG. 1 (color online). (a) Represents the energy density of two vortices. The energy is localized around the center of the vortices. (b) Shows the profile of a kink solution, or equivalently $\text{Tr}\hat{\Sigma}$ as defined in (3.6). In the strong gauge coupling limit, the profile $\lim_{g \rightarrow \infty} \text{Tr}\hat{\Sigma} = \partial_x N_p(x)$ reduces to the stepwise shape as shown in (c).

As we will see in the next section, this piecewise linear function is the ‘‘Ronkin function’’ in one dimension. By using this function N_p , the trace of the adjoint scalar in the infinite gauge coupling limit can be written by a step function

$$\lim_{g \rightarrow \infty} \text{Tr}\hat{\Sigma} = \partial_x N_p(x) = \frac{1}{R} \sum_{i=1}^k \theta(x - x_i), \quad (3.9)$$

$$x_i \equiv R \log|u_i| = \text{Re}z_i.$$

Note that in the strong gauge coupling limit, the smooth kinky profile reduces to a stepwise profile¹³ as shown in Fig. 1(c).

We can also express topological charges in terms of N_p and its derivative $\hat{\Sigma}$. The BPS bound (2.19) is now rewritten as

$$E \geq -c \int d^2x \text{Tr}F_{xy} = \hat{c} \int dx \partial_x \text{Tr}\hat{\Sigma}, \quad \hat{c} \equiv 2\pi R c, \quad (3.10)$$

where we have used $\int dy \partial_y \text{Tr}W_x = 0$ and $\text{Tr}\hat{\Sigma} = -\frac{1}{2\pi R} \int dy \text{Tr}W_y$. In the context of the domain wall, the quantity $\hat{c} \int dx \partial_x \text{Tr}\hat{\Sigma}$ provides the sum of the charges (masses) of the domain walls [14]. Interestingly, the energy of the BPS configuration can be determined only from $\text{Tr}\hat{\Sigma}$, namely, the zero mode of the overall $U(1)$ gauge field $\text{Tr}W_y$. This is because the topological charge is determined only from the boundary condition and all the massive Kaluza-Klein (KK) modes vanish at spatial infinities $x \rightarrow \pm\infty$ in the BPS configurations.

We finally discuss the Kähler metric on the moduli space of BPS vortices. The moduli space of the vortices is a Kähler manifold parametrized by the moduli parameters

$\phi_i, \bar{\phi}_i$. The Kähler metric of the moduli space is directly calculated from the solution of the master equation as follows [6] (see also [18,23])

$$K_{i\bar{j}} = c \int d^2x \mathcal{K}_{i\bar{j}}(z, \bar{z}, \phi, \bar{\phi}, R, l), \quad (3.11)$$

$$\mathcal{K}_{i\bar{j}} \equiv \text{Tr}[\partial_i \partial_{\bar{j}} \log \Omega + 4l^2 (\partial_{\bar{z}}(\Omega \partial_i \Omega^{-1}) \partial_{\bar{j}}(\Omega \partial_z \Omega^{-1}) - \partial_{\bar{z}}(\Omega \partial_z \Omega^{-1}) \partial_{\bar{j}}(\Omega \partial_i \Omega^{-1}))], \quad (3.12)$$

where $\partial_i \equiv \partial/\partial \phi_i$, $\partial_{\bar{j}} \equiv \partial/\partial \bar{\phi}_j$ are derivatives with respect to the moduli parameters and $l \equiv 1/g\sqrt{c}$ is the length scale of the vortex core. Note that the matrix-valued function Ω depends on the parameters g and c only through l .

For concreteness, let us consider k -vortex configurations in the Abelian-Higgs model ($N_C = N_F = 1$). In this case the $2k$ -dimensional moduli space is parametrized by the positions of vortices $z_i (i = 1, \dots, k)$. For well-separated vortices $|z_i - z_j| \gg R, l$, the asymptotic metric is obtained by taking the limit $l \rightarrow 0, R \rightarrow 0$. In the small vortex limit $l \rightarrow 0$ the Kähler metric becomes

$$K_{i\bar{j}} \approx c \int d^2x \frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log|H_0|^2. \quad (3.13)$$

Therefore the Kähler potential K , which determines the metric by a relation $K_{i\bar{j}} = \frac{\partial^2 K}{\partial z_i \partial \bar{z}_j}$, can be written by

$$K \approx 4\pi c \int dx (F_p(x, z_i, \bar{z}_i) - f(x, z_i) - \overline{f(x, z_i)}), \quad (3.14)$$

where $F_p(x) \equiv \lim_{R \rightarrow 0} R N_p(x)$ and $f(z_i)$ is a holomorphic function which is required to make the Kähler potential finite. This F_p is a one-dimensional ‘‘tropical polynomial’’ that we will extend to the two-dimensional case in the next section. Since the asymptotic forms of the function $F_p(x)$ are given by

$$F_p(x) = \begin{cases} x_1 + \dots + x_k, & x \rightarrow -\infty \\ kx, & x \rightarrow \infty \end{cases}, \quad (3.15)$$

¹³This limit is different from the one taken in [7] where the profile is not a step function but has a constant slope in the interval of a vortex size.

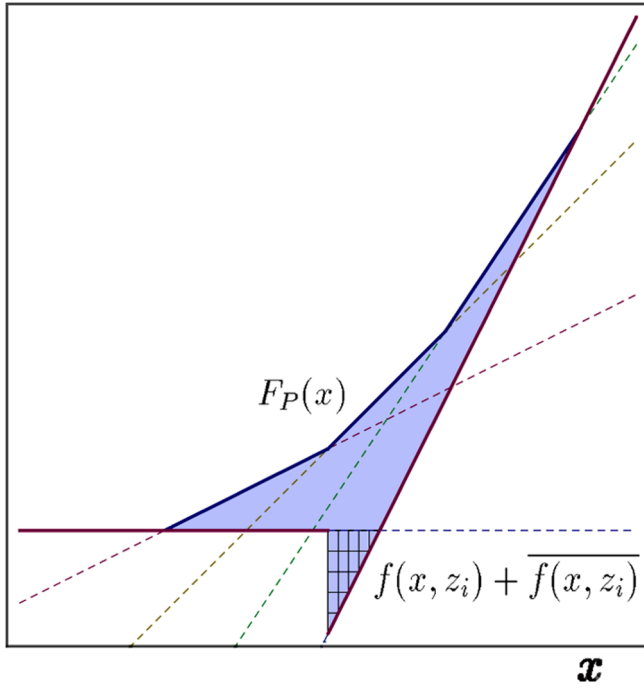


FIG. 2 (color online). The asymptotic Kähler potential can be evaluated as the area of the region surrounded by $F_P(x)$ and $f(x, z_i) + \overline{f(x, z_i)}$ (shaded regions). The area of the meshed region gives the contribution from the center of mass modulus $\frac{\pi ck}{2}(z_c + \bar{z}_c)^2$, $z_c \equiv (z_1 + z_2 + \dots + z_k)/k$.

a possible choice of the function $f(x, z_i)$ is

$$f(x, z_i) = \frac{z_1 + \dots + z_k}{2} \theta(-x) + \frac{kx}{2} \theta(x). \quad (3.16)$$

Then the asymptotic Kähler potential can be evaluated as the area of the shaded region in Fig. 2, which is given by

$$K \approx 2\pi c \sum_{i=1}^k x_i^2 = \frac{\pi c}{2} \sum_{i=1}^k (z_i + \bar{z}_i)^2. \quad (3.17)$$

Using this Kähler potential, the Kähler metric is given by $K_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K = \pi c \delta_{i\bar{j}}$. So the effective Lagrangian which describes the dynamics of well-separated vortices becomes

$$L_{\text{eff}} = K_{i\bar{j}} \dot{z}_i \dot{\bar{z}}_j = \pi c \sum_{i=1}^k |\dot{z}_i|^2. \quad (3.18)$$

This shows that the well-separated vortices behave as undistinguished free particles with the mass $2\pi c$.

In the next section, we move on to the case of vortex on $(\mathbb{C}^*)^2$. Although the story is more complicated and general, we will encounter similar structures to those described in this section.

IV. WEBS OF VORTEX SHEETS ON $(\mathbb{C}^*)^2$

A. Vortex sheets and amoeba

Let us consider the vortex-instanton system on $(\mathbb{C}^*)^2 \simeq \mathbb{R}^2 \times T^2$. As before, we will use (x_1, y_1, x_2, y_2) and $z_1 \equiv x_1 + iy_1, z_2 \equiv x_2 + iy_2$ as real and complex coordinates of $(\mathbb{C}^*)^2$, respectively. The coordinates of T^2 are identified with periods $(2\pi R_1, 2\pi R_2)$, namely $y_i \sim y_i + 2\pi R_i$. In this case, the determinant of the moduli matrix $\det H_0$, which defines the vortex sheets, is written in the form of the Fourier series

$$\det H_0(z_1, z_2) = \sum_{(n_1, n_2) \in \mathbb{Z}^2} a_{n_1, n_2} e^{(n_1/R_1)z_1 + (n_2/R_2)z_2}. \quad (4.1)$$

If we define new cylindrical coordinates (u_1, u_2) on $(\mathbb{C}^*)^2$ by $u_i \equiv e^{(z_i/R_i)}$, $\det H_0$ is now written by a Laurent polynomial

$$P(u_1, u_2) \equiv \det H_0 = \sum_{(n_1, n_2) \in \mathbb{Z}^2} a_{n_1, n_2} u_1^{n_1} u_2^{n_2}. \quad (4.2)$$

The positions of the vortices are described by zeros of $P(u_1, u_2)$ similarly to those on \mathbb{C}^* in the previous section, but the vortices form a two-dimensional sheet (surface) in $(\mathbb{C}^*)^2$ in the present case.

We define the ‘‘Newton polytope’’ $\Delta(P) \subset \mathbb{R}^2$ of a Laurent polynomial $P(u_1, u_2)$ by

$$\Delta(P) = \text{conv.hull}\{(n_1, n_2) \in \mathbb{Z}^2 | a_{n_1, n_2} \neq 0\}. \quad (4.3)$$

Conversely, $P(u_1, u_2)$ is called the Newton polynomial of Δ , when its Newton polytope Δ is convex. In the discussion of domain wall webs, the Newton polytope $\Delta(P)$ was called the ‘‘grid diagram’’ [26–30]. When we say ‘‘Newton polynomial,’’ the coefficients a_{n_1, n_2} in (4.2) are arbitrary parameters. Namely, a_{n_1, n_2} are regarded as moduli parameters of the vortices.

Analogous to the case of the vortices on the cylinder discussed in the previous section, a web of vortices on $(\mathbb{C}^*)^2$ is now dimensionally reduced to a web of domain walls on \mathbb{R}^2 [26–30]. In order to see the connection better, we define ‘‘amoeba’’ of P by¹⁴

$$\mathcal{A}_P = \{(R_1 \log|u_1|, R_2 \log|u_2|) \in \mathbb{R}^2 | P(u_1, u_2) = 0\}. \quad (4.4)$$

Note here that $R_1 \log|u_1| = x_1$ and $R_2 \log|u_2| = x_2$. This is a projection of the shape of the vortex sheet onto two noncompact directions. See Fig. 3 for an example of amoeba.

From this example, we can see that the amoeba has several asymptotic regions extending to infinity, which are called ‘‘tentacles’’ (pseudopods) in the literature. In physics terms, each of these tentacles represents a semi-

¹⁴Amoeba can be defined for $(\mathbb{C}^*)^n$ with arbitrary integer n , but we only use the case of $n = 2$.

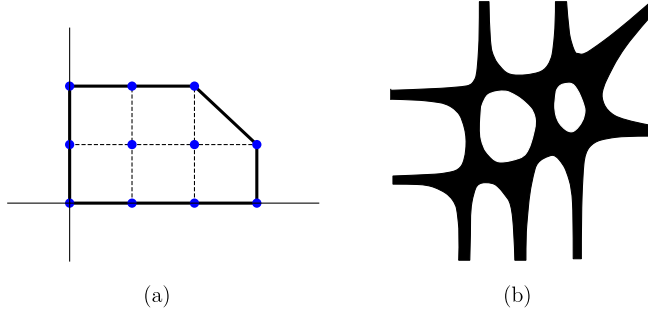


FIG. 3 (color online). An example of amoeba for $P(u_1, u_2) = a_{0,0} + a_{1,0}u_1 + a_{2,0}u_1^2 + a_{3,0}u_1^3 + a_{0,1}u_2 + a_{1,1}u_1u_2 + a_{2,1}u_1^2u_2 + a_{3,1}u_1^3u_2 + a_{0,2}u_2^2 + a_{1,2}u_1u_2^2 + a_{2,2}u_1^2u_2^2$; (a) Newton polytope and (b) amoeba.

infinite cylinder of the vortex and, for generic coefficients, corresponds to the normals to the Newton polytope (see Fig. 3).¹⁵ We also have (again, for generic values of moduli) holes for each internal lattice point of the Newton polytope. We learn from these facts that the amoeba is a projection of generic webs of the vortices.

The notion of amoeba dates back to [44]. It was originally studied in the context of monodromy of the so-called Gel'fand-Kapranov-Zelevinsky or GKZ-hypergeometric (originally called \mathcal{A} -hypergeometric) functions. It is also intimately connected with real algebraic geometry (Hilbert's 16th problem). Furthermore, it plays an important role in the discussion of the tropical geometry, as we will see. What we have found is that amoeba also appears quite naturally in the discussion of the webs of the vortices. See also the work [38], which discusses amoeba in the context of instanton counting.

To see the relation between the webs of the vortex sheets and the webs of the domain walls, it is convenient (just as in the previous section) to define $\hat{\Sigma}_1(x_1, x_2)$ by

$$\hat{\Sigma}_1(x_1, x_2) \equiv -\frac{1}{2\pi R_1} \oint \frac{dy_2}{2\pi i R_2} \times \log \left[\mathbf{P} \exp \left(i \oint dy_1 W_{y_1} \right) \right], \quad (4.5)$$

and similarly for $\hat{\Sigma}_2(x_1, x_2)$ by interchanging the subscript 1 with 2. These adjoint scalar fields $\hat{\Sigma}_i(x_1, x_2)$ on \mathbb{R}^2 are interpreted as the zero modes of the gauge fields in the Kaluza-Klein decomposition and exhibit the two-dimensional kink profiles. The trace of these adjoint scalar fields can be simply written as

$$\begin{aligned} \text{Tr}[\hat{\Sigma}_i(x_1, x_2)] &= -\frac{1}{2\pi R_1} \frac{1}{2\pi R_2} \int d^2y \text{Tr} W_{y_i} \\ &= \frac{1}{8\pi^2 R_1 R_2} \frac{\partial}{\partial x_i} \int_{T^2} d^2y \log \det \Omega, \end{aligned} \quad (4.6)$$

where we have used Eq. (2.10) and $\Omega = SS^\dagger$. The matrix-valued function Ω is simplified in the strong gauge coupling limit to [recall master equation (2.14)]

$$\lim_{g \rightarrow \infty} \Omega = \Omega_0 = \frac{1}{c} H_0 H_0^\dagger. \quad (4.7)$$

Although the vortex sheet becomes thin and singular in the strong coupling limit, the matrix Ω_0 still has important physical information on vortices and instantons. The traces of the adjoint scalar fields Eq. (4.6) in the strong coupling limit are given by

$$\lim_{g \rightarrow \infty} \text{Tr}[\hat{\Sigma}_i(x_1, x_2)] = \frac{\partial}{\partial x_i} N_P(x_1, x_2), \quad (4.8)$$

where $N_P(x_1, x_2)$ is nothing but the ‘‘Ronkin function’’ [45] in two dimensions

$$\begin{aligned} N_P(x_1, x_2) &= \frac{1}{2\pi R_1} \frac{1}{2\pi R_2} \int_{T^2} d^2y \log |\det H_0(z_1, z_2)| \\ &= \frac{1}{(2\pi i)^2} \int_{|u_i|=e^{x_i/R_i}} \frac{du_1}{u_1} \wedge \frac{du_2}{u_2} \log |P(u_1, u_2)|, \end{aligned} \quad (4.9)$$

defined from the Laurent polynomial $P(u_1, u_2) = \det H_0(z_1, z_2)$.

The Ronkin function has several interesting properties. First of all, it is convex [45]. Second, the derivatives of the Ronkin function, $\text{Tr} \hat{\Sigma}_1$ and $\text{Tr} \hat{\Sigma}_2$, take constant values in each complement of the amoeba,¹⁶ and those constant values (multiplied by $R_1 R_2$) are given by the lattice points in the Newton polytope of P [46]. More generally, $R_1 R_2 (\text{Tr} \hat{\Sigma}_1, \text{Tr} \hat{\Sigma}_2)$ as a function defined on \mathbb{R}^2 (including points on the amoeba) take values within the Newton polytope $\Delta(P)$ of P .

B. Relation with tropical geometry

Now one difference arises from the previous section. In the complex one-dimensional case discussed in the previous section, the Ronkin function N_P is piecewise linear when we take the thin wall limit $l = 1/g\sqrt{c} \rightarrow 0$. In the two-dimensional case, however, the Ronkin function and its derivative are smooth even when the gauge coupling goes to infinity.

We can still consider another limit in which the derivative of the Ronkin function becomes discontinuous. The limit is $R_1 = R_2 = R \rightarrow 0$ with fixed

¹⁵When moduli parameters are chosen to be special value, several tentacles of amoeba can merge into one. In this case, the multiplicity of the spires is considered to be greater than one.

¹⁶In [46] and other literature, these constant values are called the orders of the complement of the amoeba.

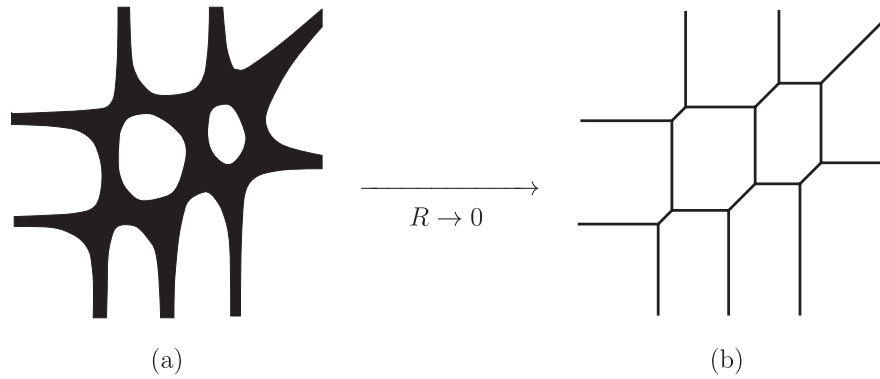


FIG. 4. An example of (a) the amoeba and (b) corresponding tropical variety.

$$r_{n_1, n_2} \equiv R \log |a_{n_1, n_2}|. \tag{4.10}$$

This limit corresponds to dimensionally reducing the theory to $(2 + 1)$ dimensions, neglecting all KK modes. In this limit, the amoeba degenerates into a set of lines (“spines”), which is called “tropical variety” in the tropical geometry literature. Physically speaking, the vortices reduce to the domain walls by dimensional reduction, and the tropical variety signifies the location of the domain walls. At the same time, the Ronkin function becomes a piecewise linear function $F_P(x_1, x_2)$ defined by

$$F_P(x_1, x_2) = \lim_{R \rightarrow 0} R \log |P(u_1, u_2)| \\ = \max_{(n_1, n_2) \in V(Q)} (n_1 x_1 + n_2 x_2 + r_{n_1, n_2}), \tag{4.11}$$

where $V(Q)$ is a set of the vertices associated with the Newton polytope Q , and r_{n_1, n_2} in Eq. (4.10) are constants determined from constants a_{n_1, n_2} in (4.2). This function F_P coincides with the Ronkin function N_P on the complement of the amoeba (recall N_P is linear on each complement).

If we compare (4.11) with (4.2), we notice that the sum and products in the polynomial $\sum a_{n_1, n_2} u_1^{n_1} u_2^{n_2}$ are replaced by a maximum function $\max_{(n_1, n_2) \in V(Q)} (n_1 x_1 + n_2 x_2 + r_{n_1, n_2})$ of the linearized functions. The formal reasoning is given as follows. If we define $\tilde{x}_1 = \exp(x_1/R)$, $\tilde{x}_2 = \exp(x_2/R)$, $\tilde{x}_1 + \tilde{x}_2 = \exp(x_3/R)$, and $\tilde{x}_1 \tilde{x}_2 = \exp(x_4/R)$, then we find

$$x_3 = \max(x_1, x_2), \quad x_4 = x_1 + x_2 \tag{4.12}$$

in the $R \rightarrow \infty$ limit. Hence in the tropical limit, the ring $(\mathbb{R}, +, \times)$ is replaced by an idempotent semiring¹⁷ $(\mathbb{R}, \oplus, \otimes)$, with a tropical addition \oplus and a tropical multiplication \otimes given by

$$x_1 \oplus x_2 = \max(x_1, x_2), \quad x_1 \otimes x_2 = x_1 + x_2, \tag{4.13}$$

¹⁷A semiring is an algebraic structure similar to a ring, but without the requirement that each element must have an additive inverse.

respectively. The semiring $(\mathbb{R}, \oplus, \otimes)$ is sometimes called the tropical semiring or the max-plus algebra.

The operation replacing the addition and multiplication with the tropical addition and tropical multiplication is also called dequantization or ultradiscretization. It appears in a discretization of integrable soliton equations such as KdV, Toda, and KP hierarchies and also in cellular automata. These integrable soliton systems seem to be completely different from the vortex-instanton system we are considering, but it is interesting that the same structure plays important roles in many integrable systems.

We have mentioned the tropical limit and tropical semiring, but then what is the corresponding geometry? In usual algebraic geometry, we consider geometry corresponding to a commutative ring. In contrast, the geometry corresponding to tropical semiring $(\mathbb{R}, \oplus, \otimes)$ is called *tropical (algebraic) geometry*.¹⁸ We can formulate and prove “tropical analogue” of many theorems in usual algebraic geometry, such as the Riemann-Roch theorem and the Bezout’s theorem. Although the study of idempotent semirings in applied mathematics (such as control theory and optimization) has a long history [50], the study of corresponding geometry is relatively new and it is still an active area of research (see [47,51,52]). The tropical geometry now has diverse applications, ranging from string networks [53], enumeration of curves [54], mirror symmetry [55], and even computational biology [56].

We can consider a tropical version of algebraic variety, namely, tropical variety. In the literature, it is often defined as a non-Archimedean amoeba, but for our applications, it suffices to define it as the set of points where the piecewise linear function $F_P(x_1, x_2)$ (“tropical polynomial”) is not differentiable. This is nothing but the skeleton (spine) of the amoeba in the limit $R \rightarrow 0$, and its physical meaning is the position of the domain walls, namely, the position of the stepwise kinks appearing in the profiles of $\text{Tr} \hat{\Sigma}_i(x_1, x_2)$

¹⁸According to [47], the name “tropical” was coined by a French mathematician Jean-Eric Pin [48], in honor of his Brazilian colleague Imere Simon [49].

($i = 1, 2$). An example of the tropical varieties is shown in Fig. 4. As shown there, tropical varieties (in the situation we want to consider) are obtained from triangulation of the Newton polytope ([51], proposition 3.5). In this sense, this is similar to the so-called (p, q) web or web diagram in [57]. We will make more comments on this analogy in the last section devoted to the discussion.

Example

Let us here give a simple and concrete example for later discussions. An example of the Newton polytope is given

in Fig. 5(a) and we set $R_1 = R_2 = 1$ for simplicity in the following. Then the corresponding Newton polynomial is given by

$$P(u_1, u_2) = u_1 + u_2 + 1 = e^{z_1} + e^{z_2} + 1, \quad (4.14)$$

and its amoeba and corresponding variety are shown in Fig. 5.

The Ronkin function $N_p(x_1, x_2)$ associated with P is shown in Fig. 6(a) and its derivatives $\text{Tr}\hat{\Sigma}_i(x_1, x_2)$ are computed to be

$$\lim_{g \rightarrow \infty} \text{Tr}[\hat{\Sigma}_1(x_1, x_2)] = \begin{cases} 0 & \text{for } x_1 < \log|e^{x_2} - 1| \\ 1 - \frac{1}{\pi} \cos^{-1}\left(\frac{e^{2x_1} - e^{2x_2} - 1}{2e^{x_2}}\right) & \text{for } \log|e^{x_2} - 1| \leq x_1 \leq \log|e^{x_2} + 1|, \\ 1 & \text{for } x_1 > \log|e^{x_2} + 1| \end{cases}$$

$$\lim_{g \rightarrow \infty} \text{Tr}[\hat{\Sigma}_2(x_1, x_2)] = \begin{cases} 0 & \text{for } x_2 < \log|e^{x_1} - 1| \\ 1 - \frac{1}{\pi} \cos^{-1}\left(\frac{e^{2x_2} - e^{2x_1} - 1}{2e^{x_1}}\right) & \text{for } \log|e^{x_1} - 1| \leq x_2 \leq \log|e^{x_1} + 1|, \\ 1 & \text{for } x_2 > \log|e^{x_1} + 1| \end{cases}$$

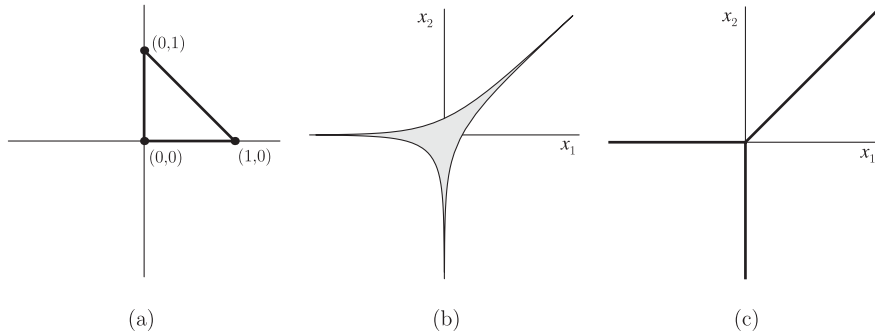


FIG. 5. (a) the Newton polytope, (b) amoeba, and (c) tropical variety, for the Laurent polynomial $P(u_1, u_2) = u_1 + u_2 + 1$.

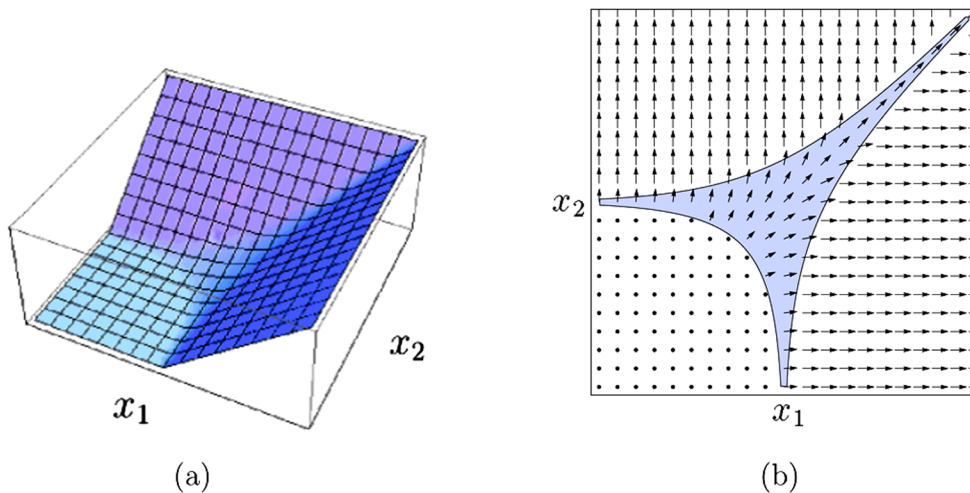


FIG. 6 (color online). (a) Ronkin function and (b) $\text{Tr}\hat{\Sigma}_1$ and $\text{Tr}\hat{\Sigma}_2$ as the gradient of the Ronkin function.

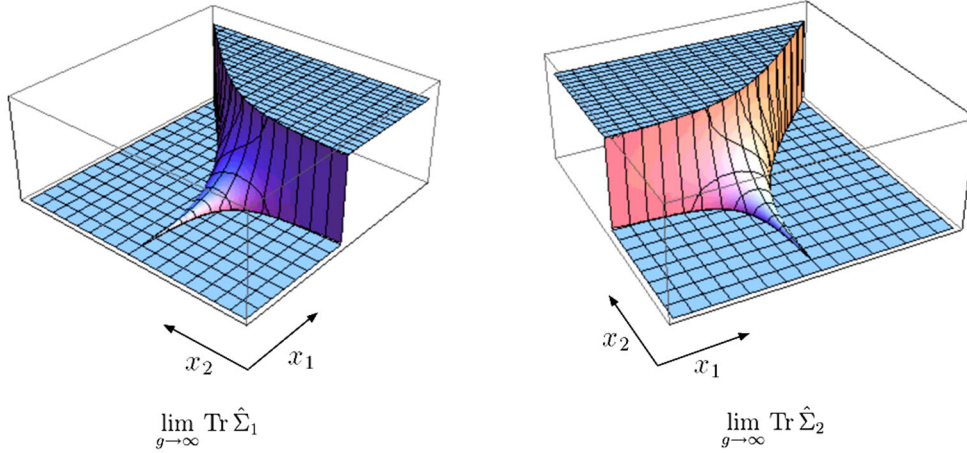


FIG. 7 (color online). The plots of $\text{Tr}\hat{\Sigma}_1$ and $\text{Tr}\hat{\Sigma}_2$.

and are displayed as gradient flows of the Ronkin function in Fig. 6(b), and their plots are given in Fig. 7. Note that $\hat{\Sigma}_i$ takes a constant value at each complement of amoeba, as expected.

C. Topological charges

We now move on to discussion of the topological charges given in (2.4) and (2.5). For a given Laurent polynomial P , these topological charges are evaluated as follows. First let us consider the vortex charge. Since the topological charges are independent of the gauge coupling constant g , we can take the strong gauge coupling limit $g \rightarrow \infty$. In the strong gauge coupling limit, the magnetic flux of the overall $U(1)$ can be written as

$$-\frac{1}{2\pi} \text{Tr}F = \frac{1}{4\pi} dd^c \log \det \Omega \rightarrow \frac{1}{2\pi} dd^c \log |P|, \quad (4.15)$$

where $d^c \equiv -i(\partial - \bar{\partial})$. By using the Poincaré-Lelong formula¹⁹

$$\int_{(\mathbb{C}^*)^2} \frac{1}{2\pi} dd^c \log |P| \wedge \alpha = \int_X \alpha, \quad (4.16)$$

$$X = \{(u_1, u_2) \in (\mathbb{C}^*)^2 | P(u_1, u_2) = 0\},$$

we can show that the vortex charge can be evaluated as

$$V = -c \int_{(\mathbb{C}^*)^2} \text{Tr}F \wedge \omega = 2\pi c \int_X \omega = 2\pi c \text{Area}(X). \quad (4.17)$$

¹⁹This formula is the generalization of the formula $dd^c \log |z| = 2\pi \delta^2(z) dx \wedge dy$ with $\int dx dy \delta^2(z) = 1$.

From this computation, it is clear that the vortex charge is distributed on the surface of the vortex sheets X . We also see that the vortex charge is uniformly distributed along all X , and the total vortex charge is given by the area of the vortex sheets multiplied by the tension $2\pi c$. It is interesting to note that the same formula has appeared in mathematics literature ([58], theorem 6 and [59]).

We can also give another expression for the vortex charge, using the Ronkin function N_p . The vortex charge is given by

$$V = \hat{c} \int_{\mathbb{R}^2} d^2x (\partial_1 \text{Tr}\hat{\Sigma}_1 + \partial_2 \text{Tr}\hat{\Sigma}_2)$$

$$= \hat{c} \int_{\mathbb{R}^2} d^2x (\partial_1^2 + \partial_2^2) N_p(x_1, x_2), \quad (4.18)$$

where $\hat{c} \equiv 4\pi^2 R_1 R_2 c$. Namely, vortex charge is given by an integration of a Laplacian of the Ronkin function. If we take the limit $R_1 = R_2 = R \rightarrow 0$, the amoeba becomes the tropical variety which can be interpreted as the web diagram of the domain walls. The tension of each wall can be calculated as follows. Since the integrand of Eq. (4.18) becomes $(\partial_1^2 + \partial_2^2) F_p(x_1, x_2)$ in the small radius limit $R \rightarrow 0$, the tension of the domain wall is computed by integrating the Laplacian of the piecewise linear function $F_p(x_1, x_2)$ along the line perpendicular to the wall. If the wall is located along the line $n_1 x_1 + n_2 x_2 + r = (n_1 + p)x_1 + (n_2 + q)x_2 + r'$, then the tension is given by

$$T_{(p,q)} = \frac{\hat{c}}{R} \sqrt{p^2 + q^2}. \quad (4.19)$$

Next let us consider the intersection charge (2.8). By taking the strong gauge coupling limit, the intersection charge density $I_{\text{intersection}}$ becomes a complex Monge-

Ampère measure $(dd^c \log|P|)^2$ on $(\mathbb{C}^*)^2$ associated with a plurisubharmonic²⁰ function $\log|P|$, which is a higher dimensional generalization of the Laplace operator (see [60,61] for discussion on the complex Monge-Ampère measure):

$$\begin{aligned} I_{\text{intersection}} &= \frac{1}{8\pi^2} \text{Tr}F \wedge \text{Tr}F \\ &\rightarrow -\frac{1}{\pi^2} \det\left(\frac{\partial^2 \log|P|}{\partial u_i \partial \bar{u}_j}\right) du_1 \wedge d\bar{u}_1 \wedge du_2 \wedge d\bar{u}_2 \\ &= \frac{1}{8\pi^2} dd^c \log|P| \wedge dd^c \log|P|. \end{aligned} \quad (4.20)$$

Then the intersection charge is evaluated again by using Poincaré-Lelong formula,

$$\begin{aligned} I_{\text{intersection}} &= \frac{1}{8\pi^2} \int dd^c \log|P| \wedge dd^c \log|P| \\ &= \frac{1}{4\pi} \int_X dd^c \log|P|, \end{aligned} \quad (4.21)$$

but this naive evaluation is unfortunately divergent. The divergence comes from the fact that the strong gauge coupling limit in the master equation (2.14) is ill-defined when $\Omega = 0$, since there appears Ω^{-1} in the master equation. In principle, if we can solve the master equation for finite gauge coupling, we could safely obtain a correct value of the intersection charge, but that would be difficult in practice. Instead, we propose to regularize the divergence as follows.

Let P_1 and P_2 be distinguished Laurent polynomials associated with the same Newton polytope $\Delta(P)$ of P , and replace two P 's in (4.22) by P_1 and P_2 , respectively. For generic Laurent polynomials P_1 and P_2 , the intersection points of the zero sets of P_1 and P_2 are discrete points. Then we obtain

$$\begin{aligned} \frac{1}{8\pi^2} \int dd^c \log|P_1| \wedge dd^c \log|P_2| &= \frac{1}{4\pi} \int_{X_1} dd^c \log|P_2| \\ &= \frac{1}{2} \#(X_1 \cdot X_2), \end{aligned} \quad (4.22)$$

where the surfaces X_i ($i = 1, 2$) are defined by $P_i(z_1, z_2) = 0$ and the number of intersection points are denoted as $\#(X_1 \cdot X_2)$.

Thanks to Bernstein's theorem [62],²¹ $\#(X_1 \cdot X_2)$ is independent of the choice of the Laurent polynomials P_1, P_2

²⁰The Monge-Ampère measure is defined for an arbitrary plurisubharmonic function. Here it suffices to know that $\log|P|$ is plurisubharmonic for arbitrary holomorphic function P .

²¹This theorem is a generalization of the well-known Bezout's theorem. See [63] for a leisurely introduction to Bernstein's theorem.

as long as P_1 and P_2 are generic, and is given by $2 \text{Area}(\Delta)$. We thus find that the intersection charge $I_{\text{intersection}}$ is evaluated to be equal to the area of the Newton polygon

$$I_{\text{intersection}} = \text{Area}(\Delta). \quad (4.23)$$

The meaning of this regularization is now clear. The original expression (4.21) is divergent essentially because it is a self-intersection number. We propose to regularize this by infinitesimally changing P , but with fixed boundary conditions at infinity.²²

Instead of invoking Bernstein's theorem, we can take more down-to-earth approach and the calculation goes as follows. This derivation is not independent of the previous argument and moreover not rigorous, but it has an advantage of clarifying the relation with the Ronkin function and real Monge-Ampère measure.

First, it is reasonable to expect²³ that the intersection number does not change under replacements $P_1(u_1, u_2) \rightarrow P_1(|u_1|e^{i\theta_1}, |u_2|e^{i\theta_2})$ and $P_2(u_1, u_2) \rightarrow P_2(|u_1|e^{i\phi_1}, |u_2|e^{i\phi_2})$, as far as $\theta_1, \theta_2, \phi_1$, and ϕ_2 are sufficiently generic:

$$\#(X_1 \cdot X_2) = \#(X_1(\theta_1, \theta_2) \cdot X_2(\phi_1, \phi_2)), \quad (4.24)$$

where

$$X_1(\theta_1, \theta_2) = \{(z_1, z_2) \in (\mathbb{C}^*)^2 | P_1(|u_1|e^{i\theta_1}, |u_2|e^{i\theta_2}) = 0\}, \quad (4.25)$$

$$X_2(\phi_1, \phi_2) = \{(z_1, z_2) \in (\mathbb{C}^*)^2 | P_2(|u_1|e^{i\phi_1}, |u_2|e^{i\phi_2}) = 0\}. \quad (4.26)$$

Certainly the intersection number (4.24) might change if θ_1 and θ_2 are nongeneric, but those special values do not contribute when we integrate over all θ_1 and θ_2 . The same applies to ϕ_1 and ϕ_2 . Therefore the intersection charge can be written as

²²The condition that P_1 and P_2 are Newton polynomials of the convex polytope $\Delta(P)$ is important. Otherwise the answer depends on the choice of P_1 and P_2 . For example, if we take $P_{1,j} = z_1, P_{2,j} = z_1 + 1/j$, then $dd^c \log|P_{1,j}| \wedge dd^c \log|P_{2,j}| = 0$ for all j . If we take instead $P_{3,j} = z_1 + z_2/j$, then $dd^c \log|P_{1,j}| \wedge dd^c \log|P_{3,j}| = \delta_0 \equiv \delta^2(z_1)\delta^2(z_2)dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2$. And for $P_{4,j} = z_1 + z_2^j$ (in a neighborhood of 0), $dd^c \log|P_{1,j}| \wedge dd^c \log|P_{4,j}| = j\delta_0$. All these functions converge to the same $P = z_1$ in the $j \rightarrow \infty$ limit, but give a different answer. We thank Alexander Rashkovski for providing us with this example.

²³Essentially, we are again using Bernstein's theorem here for the rigorous argument.

$$\begin{aligned}
 & \frac{1}{8\pi^2} \int_{(\mathbb{C}^*)^2} dd^c \log|P_1| \wedge dd^c \log|P_2| \\
 &= \frac{1}{8\pi^2} \int \frac{d\theta_1}{2\pi} \frac{d\theta_2}{2\pi} \frac{d\phi_1}{2\pi} \frac{d\phi_2}{2\pi} \\
 & \quad \times \int_{(\mathbb{C}^*)^2} dd^c \log|P_1(|u_1|e^{i\theta_1}, |u_2|e^{i\theta_2})| \\
 & \quad \wedge dd^c \log|P_2(|u_1|e^{i\phi_1}, |u_2|e^{i\phi_2})| \\
 &= \frac{1}{8\pi^2} \int_{\mathbb{R}^2 \times T^2} dd^c N_{P_1}(x_1, x_2) \wedge dd^c N_{P_2}(x_1, x_2) \\
 &= \frac{1}{8\pi^2} \left(\frac{i}{2}\right)^2 \int_{\mathbb{R}^2 \times T^2} dz_i \wedge d\bar{z}_j \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} N_{P_1}(x_1, x_2) \\
 & \quad \wedge dz_k \wedge d\bar{z}_l \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_l} N_{P_2}(x_1, x_2) \\
 &= \frac{1}{8\pi^2} \int_{\mathbb{R}^2 \times T^2} dx_1 \wedge dy_1 \wedge dx_2 \wedge dy_2 \epsilon_{ik} \epsilon_{jl} \\
 & \quad \times \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} N_{P_1}(x_1, x_2) \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_l} N_{P_2}(x_1, x_2) \\
 &= R_1 R_2 \int_{\mathbb{R}^2} dx_1 \wedge dx_2 \mu_{\text{MA}}(P_1, P_2), \tag{4.27}
 \end{aligned}$$

where

$$\begin{aligned}
 \mu_{\text{MA}}(P_1, P_2) &= \frac{1}{2!} \epsilon_{ik} \epsilon_{jl} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} N_{P_1}(x_1, x_2) \\
 & \quad \times \frac{\partial}{\partial x_k} \frac{\partial}{\partial x_l} N_{P_2}(x_1, x_2) \tag{4.28}
 \end{aligned}$$

is known as a real Monge-Ampère measure,²⁴ which is a real analogue of the complex Monge-Ampère measure we explained previously. Note that this is well-defined since N_P is convex, as discussed previously. If we set $P_1, P_2 \rightarrow P$, then

$$I_{\text{intersection}} = R_1 R_2 \int_{\mathbb{R}^2} dx_1 \wedge dx_2 \text{Hessian}(N_P) \tag{4.29}$$

$$= \int d(R_1 \text{Tr} \hat{\Sigma}_1) \wedge d(R_2 \text{Tr} \hat{\Sigma}_2) \tag{4.30}$$

$$= \text{Area}(\Delta(P)). \tag{4.31}$$

In the final line we used the fact that $R_1 R_2 (\text{Tr} \hat{\Sigma}_1, \text{Tr} \hat{\Sigma}_2)$ takes values in the Newton polytope $\Delta(P)$, as explained previously. Interestingly, this result, that the total integral (or “mass” in the standard mathematics literature) of the real Monge-Ampère measure is given by $\text{Area}(\Delta(P))$, has

²⁴More precisely, when $P_1 \neq P_2$, this is called a real mixed Monge-Ampère measure.

appeared previously in the mathematics literature ([58], theorem 4).

From the above calculation, we have shown that the evaluation of the complex Monge-Ampère measure reduces to the evaluation of the real Monge-Ampère measure. Since the Ronkin function is described by the zero mode in the KK decomposition, we learn from this fact that contributions from KK modes, although present, are canceled out in the final expression. This is just the same as in the discussion in Sec. II: topological charge is determined only from the boundary condition and all KK modes vanish at spatial infinity.

We can provide one more different explanation of the formula $I_{\text{intersection}} = \text{Area}(\Delta(P))$, which is much easier to understand (although strictly speaking, this is also just a restatement of the previous explanation). Taking the limit $R_1, R_2 \rightarrow \infty$, the intersection charge $I_{\text{intersection}}$ is given by the half of the intersection number of the tropical varieties of P_1 and P_2 , and it is easy to see that the number is given by $2 \text{Area}(\Delta(P))$ (see Fig. 8 for example). In tropical geometry, this statement is known as the tropical Bernstein theorem ([64], theorem 9.5).

Although the computation here applies only to $R_1, R_2 \rightarrow \infty$ limit, we expect that the intersection charge is still given by the same formula for finite R_1, R_2 as well, since the intersection charge is quantized and does not change continuously depending on R_1, R_2 . For the same reason, although all the arguments so far are in the strong gauge coupling limit $g \rightarrow \infty$, we expect the same formula is kept in the finite gauge coupling as well.

So far, we have concentrated on the case of $N_C = N_F = N$ called local theory. We expect that a similar formula

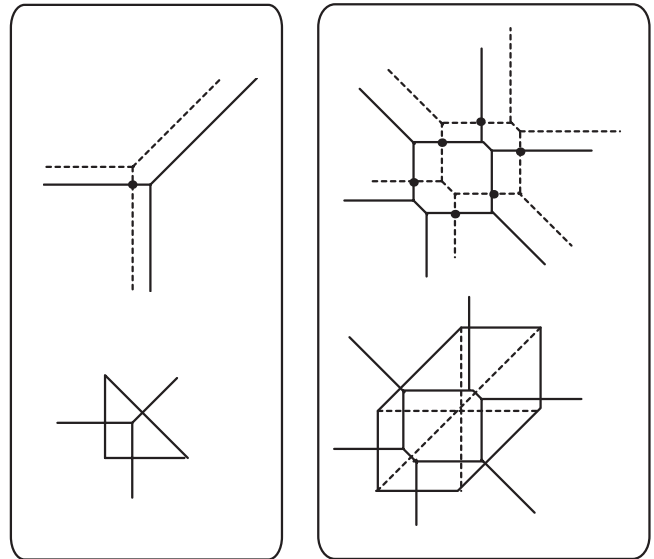


FIG. 8. Intersection of one tropical variety and its shift in generic directions. The corresponding Newton polytope $\Delta(P)$ is given below. It is easy to see that the number of intersection points is given by $2 \text{Area}(\Delta(P))$.

holds even for $N_F > N_C$, which is often called semilocal theory. For $N_C = 1$, $N_F = 2$, we can actually show this rigorously.²⁵ In the strong gauge coupling limit the solution of the master equation (2.14) for $H_0 = (P_1, P_2)$ is given by

$$\Omega \rightarrow \Omega_0 = |P|^2 \equiv |P_1|^2 + |P_2|^2. \quad (4.32)$$

In this case, $I_{\text{intersection}}$ needs no regularization. We use two representations of the current (intersection charge density) $[P]$ for $P = (P_1, P_2)$. One is King's formula (also known as the vector Poincaré, or the Poincaré-Martinelly formula [61,65])

$$[P] = dd^c \log|P| \wedge dd^c \log|P|, \quad (4.33)$$

and another is

$$[P] = dd^c \log|P_1| \wedge dd^c \log|P_2|, \quad (4.34)$$

which follows from the Poincaré-Lelong formula applied to P_2 on X_1 . We thus have

$$\begin{aligned} I_{\text{intersection}} &= \frac{1}{8\pi^2} \int dd^c \log|P| \wedge dd^c \log|P| \\ &= \frac{1}{8\pi^2} \int dd^c \log|P_1| \wedge dd^c \log|P_2| \end{aligned} \quad (4.35)$$

$$= \text{Area}(\Delta(P_1), \Delta(P_2)), \quad (4.36)$$

where in the last line we have again used Bernstein's theorem, and $\text{Area}(\Delta(P_1), \Delta(P_2))$ is a mixed volume defined by

$$\begin{aligned} 2 \text{Area}(\Delta(P_1), \Delta(P_2)) &= \text{Area}(\Delta(P_1) + \Delta(P_2)) \\ &\quad - \text{Area}(\Delta(P_1)) - \text{Area}(\Delta(P_2)) \end{aligned} \quad (4.37)$$

and the sum (Minkowski sum) $\Delta(P_1) + \Delta(P_2)$ is defined by

$$\Delta(P_1) + \Delta(P_2) = \{x + y | x \in \Delta(P_1), y \in \Delta(P_2)\}. \quad (4.38)$$

For general N_C and $N_F (> N_C)$, the solution of the master equation (2.14) in the strong coupling limit is given by

$$\det \Omega \rightarrow \det \Omega_0 = |P|^2 \equiv \sum_i |P_i|^2, \quad (4.39)$$

$$\begin{aligned} P_i &= \epsilon_{r_1 r_2 \dots r_{N_C}} H_0^{r_1 A_1} H_0^{r_2 A_2} \dots H_0^{r_{N_C} A_{N_C}} \\ (i &= 1, \dots, N_F! / N_C! (N_F - N_C)!). \end{aligned} \quad (4.40)$$

Therefore the intersection charge $I_{\text{intersection}}$ can be evaluated by integrating $(dd^c \log|P|)^2$ with P being an $N_F! / N_C! (N_F - N_C)!$ -dimensional vector. Unfortunately, we have no mathematical estimate for $N_F \geq 3$, and only

²⁵We thank Alexander Rashkovskii for providing us with the following proof.

the upper bound [66] for the integral of the complex Monge-Ampère measure in \mathbb{C}^2 [not $(\mathbb{C}^*)^2$] is known. We conjecture that the result holds in this more general case in the same manner.²⁶

This concludes our discussion of the topological charges. Our discussion mainly concentrates on the case $N = 1$. As we have seen, however, even for $N > 1$ case, the overall $U(1)$ part represented by $c_1 = -\frac{1}{2\pi} \text{Tr} F$ is still described by the language of the amoeba and tropical geometry. At the same time, we should not think that is the whole story. Out of all the topological charges (2.5), (2.8), and (2.9), the instanton number

$$I_{\text{instanton}} = \int c_2 = \frac{1}{8\pi^2} \int [\text{Tr} F \wedge \text{Tr} F - \text{Tr}(F \wedge F)] \quad (4.41)$$

vanishes in $U(1)$ theory and appears only in non-Abelian theory. We will discuss this in the next section, but before that let us discuss the metric of moduli space for the web of vortices.

D. Metric on moduli space

In this section, we discuss the metric of the moduli space. The metric of the moduli space is given by a formula similar to the one-dimensional case, Eqs. (3.11) and (3.12),

$$\begin{aligned} K_{ij} &= c \int \left(\frac{\omega^2}{2!} \partial_i \bar{\partial}_j \log \det \Omega \right. \\ &\quad + 2l^2 \omega \wedge i \text{Tr} [\bar{\partial}(\Omega \partial \Omega^{-1}) \bar{\partial}_j (\Omega \partial_i \Omega^{-1}) \\ &\quad \left. - \bar{\partial}(\Omega \partial_i \Omega^{-1}) \bar{\partial}_j (\Omega \partial \Omega^{-1}) \right]. \end{aligned} \quad (4.42)$$

This is a new result which is valid for arbitrary values of N_C and N_F .

Here we focus on the case $N_C = N_F = 1$ in which all the moduli parameters are contained in the Laurent polynomial. Let us consider a Newton polytope $\Delta(P)$ associated

²⁶At least, we can prove that intersection charge $I_{\text{intersection}}$ is quantized. We thank Alexander Rashkovskii for providing us with this argument. The argument goes as follows. Let P be an N -dimensional vector $\vec{P} = (P_1, P_2, \dots, P_N)$. For a two-dimensional subspace A of the N -dimensional complex space $(\mathbb{C}^*)^N$, let $P^A = (P_1^A, P_2^A)$ be $P_j^A = \sum_{i=1}^N \lambda_{ij} P_i$, where λ_{ij} is the $2 \times N$ -matrix of orthonormal basis of P . Then by [61,67], the Monge-Ampère measure is represented as

$$(dd^c \log|P|)^2 = \int_{G(N,2)} (dd^c \log|P_A|)^2 d\mu(A),$$

where $d\mu(A)$ is the Haar measure on the complex Grassmannian $G(N, 2)$. Note $(dd^c \log|P_A|)^2$ is well-defined for all A except for an algebraic subset of $G(N, 2)$ which is zero measure. Since P_A has exactly two components, it follows from the discussion of $N = 2$ case that $\int (dd^c \log|P_A|)^2$ is given by the intersection number of P_1^A and P_2^A , which is an integer. This means $A \mapsto \int (dd^c \log|P_A|)^2$ is an integer-valued continuous function. We have now proved that the value of $I_{\text{intersection}}$ is quantized.

with a Laurent polynomial P . Let $V_{\text{ex}}(Q)$ and $V_{\text{in}}(Q)$ be sets of external and internal vertices of $\Delta(P)$, respectively. The coefficients of the Laurent polynomial P are identified with the moduli parameters and there exist zero modes corresponding to these moduli parameters

$$P(u_1, u_2) = \sum_{(n_1, n_2) \in V(Q)} a_{n_1, n_2} u_1^{n_1} u_2^{n_2}. \quad (4.43)$$

The coefficients a_{n_1, n_2} , $(n_1, n_2) \in V_{\text{ex}}(Q)$ determine the positions of the external legs and the size of the loops of the vortex web, while the coefficients a_{n_1, n_2} , $(n_1, n_2) \in V_{\text{in}}(Q)$ determine only the size of the loops of the vortex web. Since the zero modes associated with the motion of the external legs are non-normalizable, we cannot define the metric for these zero modes and we must fix the moduli parameters a_{n_1, n_2} , $(n_1, n_2) \in V_{\text{ex}}(Q)$.

First let us consider the case where the loop sizes and the radii of two circles of the torus are much larger than the width of the vortex sheets $l \equiv 1/g\sqrt{c}$. In this case we can evaluate the leading terms in the Kähler metric by taking the thin vortex sheet limit $l \rightarrow 0$

$$\lim_{l \rightarrow 0} K_{ij} = 2c \int d^4x \partial_i \bar{\partial}_j \log |P|. \quad (4.44)$$

From Eq. (4.9) we obtain the Kähler potential in the thin vortex sheet limit $l \rightarrow \infty$ as

$$K \approx 8\pi^2 c R_1 R_2 \int d^2x (N_P(x_1, x_2, a, \bar{a}) - f(x_1, x_2, a) - \overline{f(x_1, x_2, \bar{a})}). \quad (4.45)$$

Here $f(x_1, x_2, a)$ is a holomorphic function with respect to a_{n_1, n_2} , $(n_1, n_2) \in V_{\text{in}}(Q)$ and it should be chosen to make the Kähler potential finite. A possible choice of the function $f(x_1, x_2, a)$ is

$$f(x_1, x_2) = \frac{1}{2} N_{\bar{P}}(x_1, x_2), \quad (4.46)$$

$$\tilde{P}(u_1, u_2) = \sum_{(n_1, n_2) \in V_{\text{ex}}(Q)} a_{n_1, n_2} u_1^{n_1} u_2^{n_2}. \quad (4.47)$$

Therefore the Kähler potential can be evaluated by integrating the Ronkin functions.

Next let us consider the case where the loop sizes are much larger than the radius of torus $R \equiv R_1 = R_2 \gg l$. In this case the leading terms in the Kähler potential can be evaluated by taking the small radius limit of Eq. (4.45). Since $K \approx \mathcal{O}(R)$ in the limit $R \rightarrow 0$, the leading term in the Kähler potential is given by

$$K \approx 8\pi^2 c R \lim_{R \rightarrow 0} \int d^2x R (N_P(x_1, x_2, a, \bar{a}) - N_{\bar{P}}(x_1, x_2)) \\ = 8\pi^2 c R \int d^2x (F_P(x_1, x_2) - F_{\bar{P}}(x_1, x_2)), \quad (4.48)$$

where F_P and $F_{\bar{P}}$ are piecewise linear functions defined by

$$F_P = \lim_{R \rightarrow 0} R N_P(x_1, x_2) = \max_{(n_1, n_2) \in V(Q)} (n_1 x_1 + n_2 x_2 + r_{n_1, n_2}), \quad (4.49)$$

$$F_{\bar{P}} = \lim_{R \rightarrow 0} R N_{\bar{P}}(x_1, x_2) \\ = \max_{(n_1, n_2) \in V_{\text{ex}}(Q)} (n_1 x_1 + n_2 x_2 + r_{n_1, n_2}), \quad (4.50)$$

with $r_{n_1, n_2} \equiv R \log |a_{n_1, n_2}|$.

Let us consider the simplest example of one-loop configuration associated with the Laurent polynomial

$$P(u_1, u_2) = u_1 + u_2 + u_1^{-1} u_2^{-1} + a_{0,0}. \quad (4.51)$$

In this case the vertices of the Newton polytope are

$$V(Q) = \{(-1, -1), (0, 0), (1, 0), (0, 1)\} \quad (4.52)$$

$$V_{\text{in}}(Q) = \{(0, 0)\} \quad (4.53)$$

$$V_{\text{ex}}(Q) = \{(-1, -1), (1, 0), (0, 1)\}. \quad (4.54)$$

There exists only one normalizable moduli parameter $a_{0,0}$ for which we can define the metric. This moduli parameter $a_{0,0}$ is related to the size of the loop in Fig. 9(b), which is proportional to $r \equiv R \log |a_{0,0}|$. The nonnormalizable moduli $a_{-1, -1}$, $a_{1,0}$, $a_{0,1}$ have already been fixed to 1 in Eq. (4.51). The Laurent polynomial defined in Eq. (4.47) is given by

$$\tilde{P}(u_1, u_2) = u_1 + u_2 + u_1^{-1} u_2^{-1}. \quad (4.55)$$

The piecewise linear functions $F_P(x_1, x_2)$ and $F_{\bar{P}}(x_1, x_2)$ defined in Eq. (4.50) are now given by

$$F_P(x_1, x_2) = \max(-x_1 - x_2, x_1, x_2, r), \quad (4.56)$$

$$F_{\bar{P}}(x_1, x_2) = \max(-x_1 - x_2, x_1, x_2).$$

By integrating Eq. (4.48), we obtain the asymptotic Kähler potential as the volume of a tetrahedron in Fig. 10, which is given by

$$K \approx 12\pi^2 c R r^3. \quad (4.57)$$

Differentiating the Kähler potential Eq. (4.57), we obtain the metric of the moduli space and then the effective Lagrangian

$$L_{\text{eff}} = 18\pi^2 c r R (\dot{r}^2 + R^2 \dot{\theta}^2). \quad (4.58)$$

Here θ is the phase of the moduli parameter $a = e^{(r/R) + i\theta}$. This asymptotic Lagrangian can be interpreted as the kinetic energy associated with the motion of the three vortices composing the loop. Since they can be interpreted as walls with tension $T_1 = T_2 = \hat{c}/R$ and $T_3 = \sqrt{2}\hat{c}/R$, two of three vortices have the mass $m_{1,2} = T_{1,2} \times \text{length} = 3\hat{c}r/R$ and the other vortex has the mass $m_3 = T_3 \times \text{length} = 6\hat{c}r/R$. If the moduli parameter varies with velocity \dot{r} , the vortices move in the (x_1, x_2) plane with velocities $v_{1,2} = \dot{r}$ and $v_3 = \dot{r}/\sqrt{2}$. Then the total

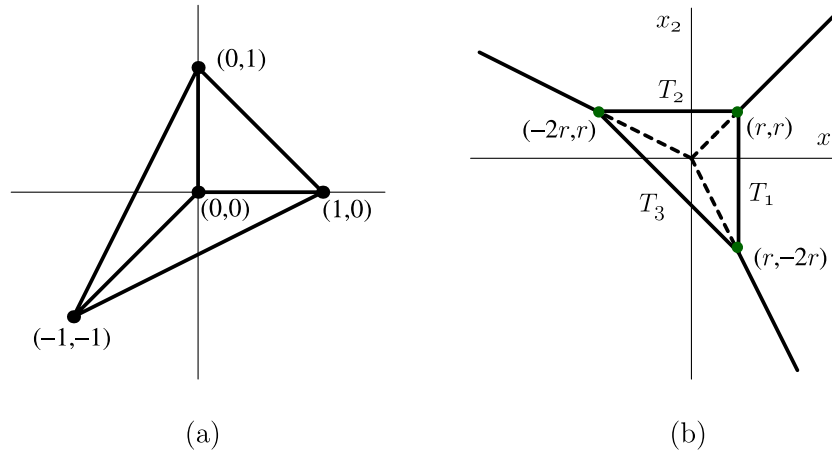


FIG. 9 (color online). (a) Newton polytope and (b) web diagram (tropical variety) for $P = u_1 + u_2 + u_1^{-1}u_2^{-1} + a_{0,0}$. The loop consists of three vortices, which can be interpreted as walls with tension $T_1 = T_2 = \hat{c}/R$ and $T_3 = \sqrt{2}\hat{c}/R$ [see Eq. (4.19)].

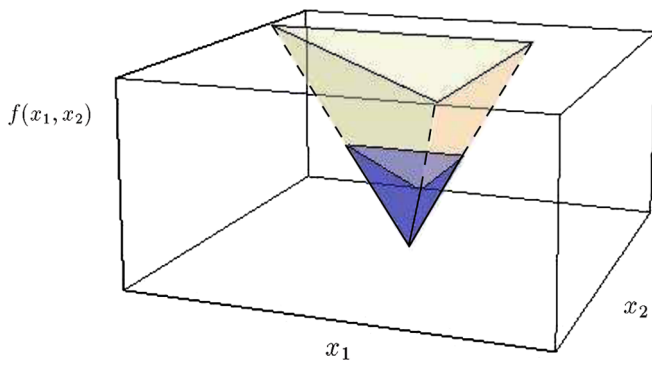


FIG. 10 (color online). The asymptotic Kähler potential is proportional to the volume of the tetrahedron surrounded by four planes $f(x_1, x_2) = -x_1 - x_2$, $f(x_1, x_2) = x_1$, $f(x_1, x_2) = x_2$, $f(x_1, x_2) = r$.

kinetic energy associated with the motion of the vortices is $\sum_{i=1}^3 \frac{m_i}{2} v_i^2 = \frac{9}{2} \hat{c} r / R r^2 = 18\pi^2 c r R r^2$.

V. INSTANTONS INSIDE NON-ABELIAN VORTEX WEBS

In this section, we discuss the instanton number $I_{\text{instanton}} = \int c_2$ as promised in the previous section. We consider the case of $U(2)$ gauge theory with $N_F = 2$ scalar fields as the simplest model which admits the BPS configuration with $I_{\text{instanton}} \neq 0$. The generalization to the $U(N)$ gauge group should be straightforward.

A. Instanton number on a planar vortex plane: a review

Let us first review the 1/2 BPS vortex moduli space. The vacuum of $U(2)$ gauge theory with $N_F = 2$ scalar fields breaks the color and flavor symmetry $U(2)_C \times SU(2)_F$ into color-flavor locked symmetry $SU(2)_{C+F}$. The 1/2 BPS single vortex in this theory further breaks $SU(2)_{C+F}$ into $U(1)_{C+F}$. Therefore there appear Nambu-Goldstone

modes of the complex projective space $\mathbb{C}P^1 \simeq SU(2)_{C+F}/U(1)_{C+F}$ localized around the vortex [2,3]. The orientation moduli $\mathcal{M}_{\text{orientation}} \equiv \mathbb{C}P^1$ form a part of the moduli space.

The moduli matrix for the vortex at $z_1 = 0$ with an orientational moduli parameter b is given by [5]

$$H_0 = \sqrt{c} \begin{pmatrix} 1 & b \\ 0 & z_1 \end{pmatrix} \sim \sqrt{c} \begin{pmatrix} z_1 & 0 \\ 1/b & 1 \end{pmatrix}, \quad (5.1)$$

where \sim represents the V -equivalence relation. These two moduli matrices provide two patches b and $1/b$ of $\mathbb{C}P^1$ [5]. For this moduli matrix, the solution of the master equation is given by

$$\Omega = \begin{pmatrix} 1 + |b|^2 & b\bar{z}_1 \\ \bar{b}z_1 & \frac{\Omega_* - |z_1|^2}{1 + |b|^2} + |z_1|^2 \end{pmatrix}, \quad (5.2)$$

where Ω_* is a solution of the master equation in the case of the Abelian-Higgs model ($N_C = N_F = 1$), such that

$$\partial_{\bar{z}_1}(\Omega_* \partial_{z_1} \Omega_*^{-1}) = -\frac{g^2 c}{4} (1 - |z_1|^2 \Omega_*^{-1}). \quad (5.3)$$

Therefore once the vortex solution Ω_* of the Abelian-Higgs model is given, one can construct the whole solution of the non-Abelian model.

Next let us discuss the 1/4 BPS configuration of instantons and vortices on $(\mathbb{C}^*)^2$. The 1/2 BPS vortex plays the role of a host soliton in the 1/4 BPS vortex-instanton configuration. It has an internal degree of freedom parameterized by an orientational moduli parameter b , which is interpreted as an inhomogeneous coordinate of $\mathcal{M}_{\text{orientation}} = \mathbb{C}P^1$ as denoted above. The instantons inside the vortex can be constructed as lumps in the vortex effective theory which is the $\mathbb{C}P^1$ sigma model [22,23]. By using the moduli matrix (5.1) for 1/2 BPS single vortex, we can construct the moduli matrix for some 1/4 BPS configurations as follows. The moduli matrix for the 1/4 BPS vortex-instanton configuration is obtained by

promoting the orientational moduli b to a holomorphic function of the other holomorphic coordinate z_2 [23]

$$H_0 = \sqrt{c} \begin{pmatrix} 1 & b(z_2) \\ 0 & z_1 \end{pmatrix}, \quad b(z_2) = a \prod_{i=1}^k (z_2 - z_2^{(i)}), \quad (5.4)$$

where the moduli parameter $z_2^{(i)}$ denotes the position of the i -th instanton and a determines the overall size of the instantons on the vortex sheet defined by $z_1 = 0$. This function $b(z_2)$ is regarded as a holomorphic map from the vortex sheet to $\mathcal{M}_{\text{orientation}} = \mathbb{C}P^1$.

In order to compute the instanton number, we require the information upon the solution of the master equation Ω for the moduli matrix Eq. (5.4). Since the topological charges are determined only from the boundary condition, it can be calculated from the asymptotic behavior of the solution Ω which is obtained by the following procedure. Solving the BPS equation in the 2 + 1-dimensional effective theory on the vortex worldvolume, we obtain the lump solution $b(z_2)$ corresponding to the instanton inside the vortex sheet. Substituting the solution of the effective theory $b(z_2)$ back into the 1/2 BPS vortex solution Eq. (5.2), we find

$$\Omega \equiv \begin{pmatrix} 1 + |b(z_2)|^2 & b(z_2)\bar{z}_1 \\ \bar{b}(z_2)z_1 & \frac{b(z_2)\bar{z}_1}{1+|b(z_2)|^2} + |z_1|^2 \end{pmatrix}. \quad (5.5)$$

Although this is not an exact solution of the master equation since $b(z_2)$ is not a constant, we can see that this matrix Ω possesses the correct asymptotic behavior at spatial infinity by substituting Ω into the master equation. Therefore the topological charge, which is determined by behavior of the fields at spatial infinity, can be evaluated from this matrix Ω . Inserting $F = -iS^{-1}\bar{\partial}(\Omega\partial\Omega^{-1})S$ into Eq. (2.4), we obtain

$$\begin{aligned} I &= \frac{1}{8\pi^2} \int \text{Tr}(F \wedge F) \\ &= \frac{1}{8\pi^2} \int dd^c \log \Omega_*(z_1, \bar{z}_1) \wedge dd^c \log(1 + |b(z_2)|^2) = k. \end{aligned} \quad (5.6)$$

Therefore the instanton charge is measured by the degree of the function $b(z_2)$, namely, the degree of the holomorphic map from the vortex sheet \mathbb{C}^* to $\mathcal{M}_{\text{orientation}} = \mathbb{C}P^1$.

B. Instanton number on non-Abelian vortex webs

As we have seen in the example above, the instantons can exist on the planar vortex sheets. Now we work out more general 1/4 BPS configurations explicitly. The planar host vortex sheet can be extended to a general web of the vortex sheets characterized by $P(u_1, u_2)$ as discussed in the previous sections. We therefore replace the lower-right component z_1 in the moduli matrix (5.4) by $P(u_1, u_2)$. Moreover the holomorphic function b in (5.4) is also replaced with a function of two holomorphic coordinates

z_1 and z_2 . The moduli matrix for such 1/4 BPS configuration of the instantons and vortex sheets on $(\mathbb{C}^*)^2$ thus becomes

$$H_0 = \begin{pmatrix} 1 & b(u_1, u_2) \\ 0 & P(u_1, u_2) \end{pmatrix}, \quad (5.7)$$

where $P(u_1, u_2)$ and $b(u_1, u_2)$ are Laurent polynomials

$$\begin{aligned} P(u_1, u_2) &= \sum a_{n_1, n_2} u_1^{n_1} u_2^{n_2}, \\ b(u_1, u_2) &= \sum b_{n_1, n_2} u_1^{n_1} u_2^{n_2}. \end{aligned} \quad (5.8)$$

Although Eq. (5.7) is not the most general form of the moduli matrix for the 1/4 BPS configurations, we treat this simple form of the moduli matrix for essential explanation. The coefficients of the Laurent polynomials a_{n_1, n_2} and b_{n_1, n_2} are the moduli parameters which give the location of vortex sheets, and the positions and sizes of the instantons, respectively.

For the moduli matrix Eq. (5.7), the instanton charge I in Eq. (2.4) is computed as follows. Since the topological charge should not change under continuous deformations, we can take a strong coupling limit $g \rightarrow \infty$ in which the solution of the master equation Ω approaches to

$$\Omega \rightarrow \Omega_0 = \frac{1}{c} H_0 H_0^\dagger = \begin{pmatrix} 1 + |b|^2 & b\bar{P} \\ P\bar{b} & |P|^2 \end{pmatrix}. \quad (5.9)$$

However a direct calculation does not work since the vortex sheets become singular in this limit. To avoid the calculation involving the singular vortex sheets, we perform another deformation of the configuration. Similarly to the case of the planar vortex sheet, let Ω be a 2×2 matrix given by

$$\Omega \equiv \begin{pmatrix} 1 + |b|^2 & b\bar{P} \\ P\bar{b} & \frac{\Omega_* - |P|^2}{1 + |b|^2} + |P|^2 \end{pmatrix}, \quad (5.10)$$

where Ω_* is a solution to the following equation:

$$\begin{aligned} \bar{\partial}_{\bar{z}_1}(\Omega_* \partial_{z_1} \Omega_*^{-1}) + \bar{\partial}_{\bar{z}_2}(\Omega_* \partial_{z_2} \Omega_*^{-1}) \\ = -\frac{g^2 c}{4} (1 - |P|^2 \Omega_*^{-1}). \end{aligned} \quad (5.11)$$

Although the matrix Ω is not a solution of the master equation unless the holomorphic function $b(u_1, u_2)$ is constant everywhere, this matrix has the correct topological information about the configuration similarly to the planar case. For this matrix Ω , we can show that

$$\begin{aligned} \int ch_2 &= \frac{1}{8\pi^2} \int \text{Tr}[\bar{\partial}(\Omega\partial\Omega^{-1}) \wedge \bar{\partial}(\Omega\partial\Omega^{-1})] \\ &= \frac{1}{16\pi^2} \int \left(dd^c \log \Omega_* \wedge dd^c \log(1 + |b|^2) \right. \\ &\quad \left. - \frac{1}{2} dd^c \log \Omega_* \wedge dd^c \log \Omega_* \right). \end{aligned} \quad (5.12)$$

If we take the strong gauge coupling limit $g \rightarrow \infty$ in Eq. (5.11), then Ω_* approaches to $|P|^2$. Therefore we obtain the instanton charge I in Eq. (2.4) as

$$I = \frac{1}{8\pi^2} \int (dd^c \log|P| \wedge dd^c \log(1 + |b|^2) - dd^c \log|P| \wedge dd^c \log|P|), \quad (5.13)$$

where the first term gives the instanton number $I_{\text{instanton}}$ in Eq. (2.9) and the second term gives the intersection charge $I_{\text{intersection}}$ in Eq. (2.8) which has been computed in Sec. IV C. The instanton number is rewritten by using the Poincaré-Lelong formula

$$I_{\text{instanton}} = \frac{1}{8\pi^2} \int_{(\mathbb{C}^*)^2} dd^c \log|P| \wedge dd^c \log(1 + |b|^2) = \frac{1}{4\pi} \int_X dd^c \log(1 + |b|^2), \quad (5.14)$$

where X denotes the zero locus of P corresponding to the vortex sheets. Therefore the instanton number is given by the degree of the map $b|_X: X \rightarrow \mathbb{C}^{\mathbb{P}^1}$.

To see a distribution of the topological charge density, we take two limits of the parameters: one is a small instanton limit and the other is a small radius limit $R \rightarrow 0$. The small instanton limit is realized by taking the limit $b_{n_1, n_2} \rightarrow \infty$ with fixed ratios $b_{n_1, n_2}/b_{\bar{n}_1, \bar{n}_2}$. Then the two form $dd^c \log(1 + |b|^2) \rightarrow dd^c \log|b|^2$ has a delta-func-

tion-like support on the zeros of $b(u_1, u_2)$. From this fact we find that in the small instanton limit the instantons are localized at common zeros of $b(u_1, u_2)$ and $P(u_1, u_2)$, and that the vortex sheets are located at $P(u_1, u_2) = 0$. We also find that instantons are localized at the positions of lumps from the viewpoint of effective theory on the vortex sheets.

Next, let us consider the small radius limit $R \rightarrow 0$. In this limit, the function $\log(1 + |b|^2)$ becomes

$$\begin{aligned} \frac{R}{2} \log(1 + |b|^2) &\rightarrow \tilde{F}_b(x_1, x_2) \\ &\equiv \max_{(n_1, n_2)} (n_1 x_1 + n_2 x_2 + s_{n_1, n_2}), \end{aligned} \quad (5.15)$$

where $s_{0,0} = \frac{R}{2} \log(1 + |b_{0,0}|^2)$ and $s_{n_1, n_2} = R \log|b_{n_1, n_2}|$ for $(n_1, n_2) \neq 0$ are fixed in the limit. Then the instanton number takes the form

$$\begin{aligned} I_{\text{instanton}} &= \frac{1}{8\pi^2} \int_{(\mathbb{C}^*)^2} dd^c \log|P| \wedge dd^c \log(1 + |b|^2) \\ &= \int_{\mathbb{R}^2} d^2x \epsilon_{ij} \epsilon_{kl} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} F_P(x_1, x_2) \\ &\quad \times \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_l} \tilde{F}_b(x_1, x_2). \end{aligned} \quad (5.16)$$

Therefore the instanton number density is localized at the intersection of the tropical variety of the Laurent polynomial P and the lines on which the piecewise linear function

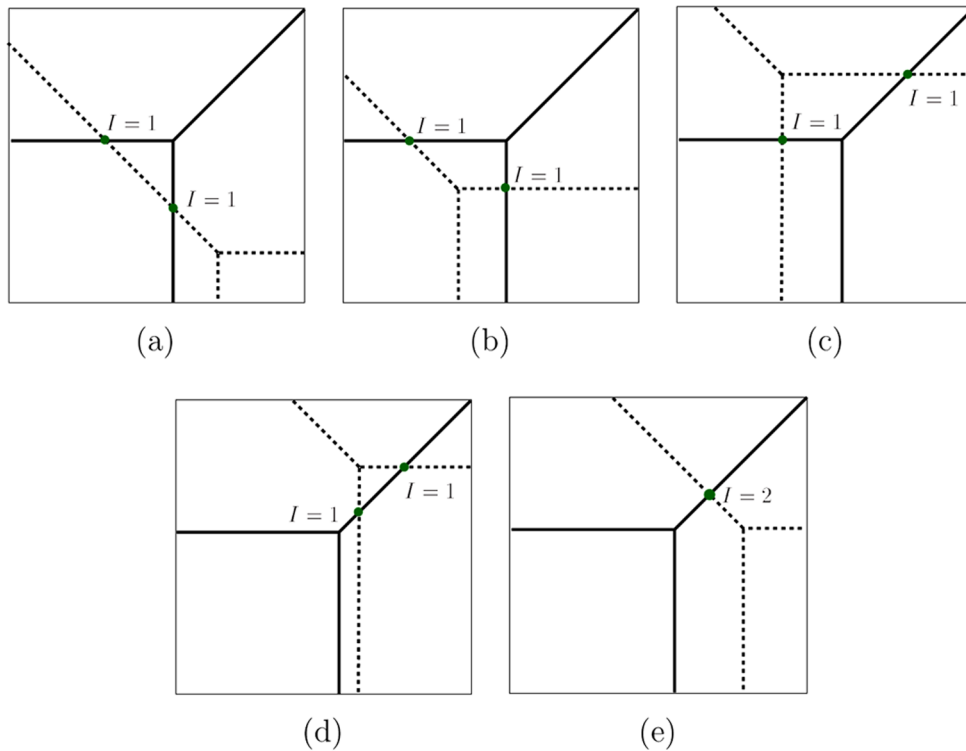


FIG. 11 (color online). The instanton number density in the small radius limit for $P = u_1 + u_2 + 1$ and $b = b_{1,1}u_1u_2 + b_{1,0}u_1 + b_{0,0}$. The instanton number density is localized at the intersections of the tropical variety for P (solid line) and the lines on which $\tilde{F}_b(x_1, x_2)$ is not differentiable (dashed lines).

$\tilde{F}_b(x_1, x_2)$ is not differentiable. When the instanton number density is localized at the intersection of the lines $n_1x_1 + n_2x_2 + r = 0$ and $\tilde{n}_1x_1 + \tilde{n}_2x_2 + s = 0$, the instanton number is given by

$$\begin{aligned}
 I_{\text{instanton}} &= \int d^2x \epsilon_{ij} \epsilon_{kl} n_i n_k \delta(n_1x_1 + n_2x_2 + r) \\
 &\quad \tilde{n}_j \tilde{n}_l \delta(\tilde{n}_1x_1 + \tilde{n}_2x_2 + s) \\
 &= \left| \det \begin{pmatrix} n_1 & n_2 \\ \tilde{n}_1 & \tilde{n}_2 \end{pmatrix} \right| = |n_1 \tilde{n}_2 - \tilde{n}_1 n_2|. \quad (5.17)
 \end{aligned}$$

Figure 11 shows examples of the instanton number density on \mathbb{R}^2 for $P = u_1 + u_2 + 1$ and $b = b_{1,1}u_1u_2 + b_{1,0}u_1 + b_{0,0}$. Varying the moduli parameters, the instantons localized at the intersection move along the tropical variety of P . For each intersection in Fig. 11(a)–11(d) the instanton number is $I = 1$. In Fig. 11(e) $I = 2$ instanton, which can be interpreted as the coincident instantons, is localized at the intersection of the lines $x_1 - x_2 = 0$ and $x_1 + x_2 + R \log|b_{1,1}| - \frac{R}{2} \log(1 + |b_{0,0}|^2) = 0$.

So far we have treated the specific configuration of the non-Abelian vortex sheets and instantons on $(\mathbb{C}^*)^2$ and the specific gauge group $U(2)$. For more general cases, the computation of the instanton number seems to be complicated and difficult, but the essence should be similar to the above calculations.

VI. CONCLUSION AND DISCUSSION

In this paper, we have investigated generic intersections (or webs) of vortices with instantons inside, which is a 1/4 BPS state in the Higgs phase of (the bosonic part of) five-dimensional $\mathcal{N} = 1$ supersymmetric $U(N_C)$ gauge theory on $\mathbb{R}_t \times (\mathbb{C}^*)^2 \sim \mathbb{R}^{2,1} \times T^2$ with $N_F = N_C$ Higgs scalars in the fundamental representation. We have found that this vortex-instanton system can be beautifully and naturally understood in the mathematical framework of the amoeba and tropical geometry, and have proposed a dictionary relating the solitons and gauge theory to the amoeba and tropical geometry (summarized in Table I).

In this discussion, the moduli matrix formalism has played crucial roles. The solutions to 1/4 BPS equations are parametrized by a holomorphic function (Laurent polynomial) $H_0(z_1, z_2)$ of two complex parameters z_1, z_2 of

$(\mathbb{C}^*)^2$. This Laurent polynomial can also be considered as a Newton polynomial of some convex polytope Δ , namely, the grid diagram. In the strong gauge coupling limit, the position of vortices is exactly given by the zero of H_0 , while the projection of the shape of vortex sheet is the amoeba of Δ . Moreover, we can relate Wilson loops in T^2 , or the zero modes of gauge fields in Kaluza-Klein decomposition, to the derivatives of the Ronkin function $N_{H_0}(x_1, x_2)$, which is a convex function and is defined from Newton polynomial H_0 .

The relation with the tropical limit and tropical geometry has also been discussed. In the discussion of solitons, it is natural to consider dimensional reduction of the theory in order to obtain BPS solitons in lower-dimensional field theories. This limit is known in the mathematical literature as the tropical limit. In this limit, the shape of amoeba degenerates into a tropical variety, which is nothing but the so-called (p, q) web of the grid diagram. We have shown that tropical geometry provides a simple and elegant method to understand not only the dimensionally reduced theory but also the original vortex-instanton system on $(\mathbb{C}^*)^2$.

We have also discussed the topological charges, which are divided into three types. They are the vortex charge V , the intersection charge $I_{\text{intersection}}$ (negative contribution of the instanton charge), and the instanton number $I_{\text{instanton}}$. First, the vortex charge is uniformly distributed along vortex sheet $\det H_0 = 0$, and its density is given by the Laplacian of the Ronkin function in the tropical limit. Its total charge is the area of the vortex sheet multiplied by $2\pi c$. Second, in the strong gauge coupling limit the intersection charge density is given by the complex Monge-Ampère measure of a plurisubharmonic function $\log|H_0(z_1, z_2)|$, and the total intersection charge is given by the area of the grid diagram Δ with a suitable regularization. Third, the instanton number $I_{\text{instanton}}$ appears only in the non-Abelian case, and we have discussed the case of $N_F = N_C = 2$ as an example. Our discussion simplifies in two limits. In the small instanton limit, instantons are localized at intersections of $P(z_1, z_2)$ and another Laurent polynomial $b(z_1, z_2)$, which parametrizes an orientational moduli $\mathbb{C}P^1$. In the small radius limit, the instanton number density $I_{\text{instanton}}$ is localized at the intersection of the tropical varieties corresponding to P and b . We have also

TABLE I. Dictionary relating soliton/gauge theory to amoeba/tropical geometry.

Soliton/gauge theory	Amoeba/tropical geometry
moduli matrix $H_0(z_1, z_2)$	Newton polynomial $P(u_1, u_2)$
projection of vortex sheet	amoeba \mathcal{A}_P
$R \rightarrow 0$	tropical limit
position of stepwise kinks	tropical variety
Wilson loop $\text{Tr} \Sigma_i$	derivative of Ronkin function: $\partial_i N_P$, (4.9)
intersection charge $I_{\text{intersection}}$	(total mass of) complex Monge-Ampère measure
vortex charge density \mathcal{V}	Laplacian of Ronkin function (4.18)

obtained the general form of the Kähler potential and the asymptotic metric of the moduli space of a vortex loop as a by-product of the discussions above. In the tropical limit, the Kähler potential is given by the volume of a convex polytope, and the effective Lagrangian can be interpreted as the kinetic energy associated with the motion of vortices composing the loop.

In our discussion, we mainly focused on the case of Abelian-Higgs model ($N_F = N_C = 1$), but as far as the overall $U(1)$ part is concerned the story is exactly the same in non-Abelian $U(N_C)$ case. We also obtained new results by going to a non-Abelian gauge group, such as the instanton number. This seems to suggest a non-Abelian generalization of the amoeba and tropical geometry.

Despite such impressive success, there are still many points which need further exploration.

First, in this paper, we have found one-to-one correspondence between the amoeba/tropical geometry and solitons in *Abelian* gauge theory. However the configuration of instantons inside non-Abelian vortex-webs in non-Abelian gauge theory discussed in Sec. V does not correspond to the amoeba and tropical geometry so far. This configuration suggests non-Abelian generalization of amoeba and tropical geometry. Furthermore the non-Abelian vortices have been recently extended to the case of gauge group $G = U(1) \times G'$ with G' arbitrary simple group [12]. This is the case of complex one dimension. It should be extendible to the case of complex two dimensions such as $(\mathbb{C}^*)^2$. That may suggest further generalization of the amoeba and tropical geometry associated to an arbitrary group, which contains the usual one as a special case of $U(1)$ gauge group.²⁷

Second, let us pursue a possibility to generalize the space where the amoeba lives. In this paper we have considered the amoeba on $(\mathbb{C}^*)^2$. This is because the maximal spacetime dimension of supersymmetric gauge theory with eight supercharges is $d = 5 + 1$. Since space dimension 5 is odd we have studied the four-dimensional case $(\mathbb{C}^*)^2$. If we abandon implementing supersymmetry, we can extend the bosonic Lagrangian (2.1) to space dimensions higher than 5, and study higher codimensional composite solitons of the vortices and instantons extending to various directions. In fact the generalized vortex equations on arbitrary Kähler manifold of arbitrary dimensions were obtained in (bosonic) Yang-Mills-Higgs theory [39]. Therefore we expect those equations to give further corre-

spondence of the amoeba and gauge theory on $(\mathbb{C}^*)^n$ or on general Kähler manifolds.

The third topic is the relation with the dimer model [68,69]. Dimers do not appear directly in our discussion, but it is known that the dimer model is intimately connected with the amoeba and tropical geometry. For example, the Ronkin function as defined in (4.9) coincides with the thermodynamic limit of the partition function of a dimer model. Moreover, a spectral curve of the dimer model is known [68] to parametrize Harnack curve, whose amoeba has good properties. The dimers also appear in discussion of the brane tilings [70,71] and four-dimensional $\mathcal{N} = 1$ superconformal quiver gauge theories. The brane tilings now have an interpretation as configuration of D5-branes and NS5-brane [72], whose brane configuration is shown in Table II, wherein Σ is a two-dimensional surface in 4567-directions. In the weak gauge coupling limit, this Σ is the zero locus of a Newton polynomial with respect to two complex variables in $(\mathbb{C}^*)^2$. Hence the mathematical structure of the NS5-brane is exactly the same as that of the vortex sheets we have considered in this paper. The toric diagram and the grid diagram are identical, and the meaning of the tropical limit and tropical variety also coincides. Of course, we should keep in mind that there exists important differences between the soliton systems in this paper and the brane tilings. First, in this paper we have assumed systems with eight supercharges, but in brane tiling, we have only four supercharges. Related to this fact is that we do not have an analogue of the D5-brane in the soliton side, and have the instanton charge instead. Still, we might obtain something new from this analogy. For example, in the discussion of brane tilings, the projection of $(\mathbb{C}^*)^2$ on to T^2 directions, which is called alga or coamoeba, plays crucial roles [37,73]. It would be interesting to see whether coamoeba has any significance in our setup. Perhaps we can understand these points better if we can find a D-brane realization of the vortex-instanton system. For this direction, the work on D-brane configuration of vortices on cylinder [7,11], its T-dual to the D-brane configuration of the domain walls [15,16], and the D-brane configurations of the domain wall webs [28] should be a useful guideline.

The amoeba and tropical geometry also appear in the computation of the topological string amplitude and the instanton counting in 4D and 5D supersymmetric gauge theories. Their perturbative dynamics are ruled by asymptotic behavior of the (plane) partitions where the amoeba and the Ronkin function appear. It is also pointed out that the Kähler structure behind the theories is closely related to the volume of the convex cone of the Ronkin function similarly to our discussion on the Kähler potential. These relations suggest that the intersecting soliton system also admits an interpretation of the microscopic partitions (dimers) and that the Kähler geometry of the moduli space of the solitons is determined by a suitable asymptotic limit of the microscopic interpretation.

²⁷In fact, the motto of tropical geometry is to extend usual algebraic geometry by replacing the commutative ring with a commutative semiring. Another generalization is to replace a commutative ring with a noncommutative ring, which is noncommutative geometry. Perhaps our discussion of non-Abelian vortices suggests further generalization by combining two above, which should be called “noncommutative/non-Abelian tropical geometry.”

TABLE II. The five-brane configuration described by brane tilings. In the weak string coupling limit, the surface Σ becomes a zero locus of a Newton polynomial corresponding to the toric diagram. This setup is analogous to our vortex-instanton systems, although details are different.

	0	1	2	3	4	5	6	7	8	9
D5	◦	◦	◦	◦		◦		◦		
NS5	◦	◦	◦	◦	Σ (2-dim surface)					

Finally, statistical partition functions of the vortices on a cylinder were studied by using D-brane configurations [11]. There the integration over the moduli space of vortices is drastically simplified in the T-dual picture: vortices are mapped to domain walls and the integration reduces to a problem of rods. The limit of parameters employed in [11] is a little different from that in this paper. There the limit $g \rightarrow \infty$, $R \rightarrow 0$ was taken with fixed wall width $d \equiv \frac{2}{g^2 c R}$, while in this paper we have taken the strong coupling limit $g \rightarrow \infty$ first and then the small radius limit $R \rightarrow 0$. This is nothing but the nonlinear sigma model version of the limit in [11]. In the case of $N_F > N_C$, the gauged linear sigma model reduces to the nonlinear sigma model and the vortex solution reduces to the lump solution [10]. The lump solution on the cylinder can be mapped to kinks on one-dimensional space. The small radius limit $R \rightarrow 0$, which corresponds to dequantization (ultradiscretization) limit $\lim_{R \rightarrow 0} R \log(e^{A/R} + e^{B/R}) = \max(A, B)$, enable us to identify the kinks with free particles in one-dimensional space as in Sec. III. The partition function of the free particles gives the exact volume of the moduli space of the sigma-model lumps, namely, the partition function for a multi-lump system. This fact and the result of [11] suggest that the procedure of the dequantization is powerful enough to give the exact partition function of the solitons in the nonlinear sigma model and the variant of it makes the computation of the partition function very simple even in the case of finite gauge couplings. This should be extendible to the vortex webs discussed in this paper, to obtain a

partition function of the instanton-vortex system. We expect that it will be reduced to the Nekrasov's partition function in the limit of $g\sqrt{c} \rightarrow 0$ where the vortices disappear while the instantons still remain. The integration over the moduli space of the instanton-vortex system may provide a systematic method to compute the symplectic Gromov-Witten invariant, which is a combination of the Donaldson invariant and the Gromov-Witten invariant [74].

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