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Supersymmetric Gödel and warped black holes in string theory

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It is observed that three-dimensional Gödel black holes can be promoted to exact string theory backgrounds through an orbifold of a hyperbolic asymmetric marginal deformation of the $SL(2, \mathbb{R})$ Wess-Zumino-Witten model. Tachyons are found in the spectrum of long strings. Uplifting these solutions in type IIB supergravity, extremal black holes are shown to preserve one supersymmetry in accordance with the Bañados-Teitelboim-Zanelli limit. We also make connections with some recently discussed warped black hole solutions of topologically massive gravity.

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I. INTRODUCTION

The nontrivial 3d part of the Gödel spacetime can be recognized as a timelike (or elliptic) deformation of anti–de Sitter (AdS) spacetime [1]. The minimal setting to describe the Gödel universe as a solution of an action consists in 3d Einstein gravity coupled either to matter fields [2] or to a gravitational Chern-Simons term [3]. Interestingly, the 3d Gödel spacetime can be embedded in string theory as an exact marginal deformation of the $SL(2, \mathbb{R})$ Wess-Zumino-Witten (WZW) model [4]. Tachyons destabilizing the background are found in the spectrum of long strings and thus lead to a stringy clue to the chronology protection conjecture [5]. Various regularizations of the geometry were proposed; see e.g. [4,6,7] and references therein.

It is intriguing that this instability occurs even though the 3d Gödel universe enjoys supersymmetry as originally found in its five-dimensional cousins [8]. More precisely, Killing spinors can be found in the $\mathcal{N} = 2$ extension of Einstein-Maxwell-Chern-Simons theory but not in the $\mathcal{N} = 1$ extension [9]. Also, it was shown that in heterotic string theory, the 3d Gödel universe breaks all supersymmetry but preserves one-half of it in type IIB [4].

In this work we would like to understand how these properties generalize to Gödel black holes. The generalization is not completely trivial because, as shown in [2], black holes are defined via periodic identifications on a background *other* than the Gödel universe, namely, what is called equivalently the tachyonic Gödel background in [2], the hyperbolic deformation of anti–de Sitter space in [10], or the spacelike warped anti–de Sitter space in [3]. This spacetime contains *no* closed timelike curves as observed in [10], as opposed to the original Gödel spacetime. In the latter background, the spectrum of long string states exhibits an infinite number of tachyonic long string states with an arbitrary number of oscillators whose origin could be traced back to the presence of closed timelike curves [4]. For the tachyonic Gödel background, it is not clear if tachyons will be found, nor are the conclusions of [4] reached for the elliptic deformation directly applicable to the Gödel black holes. Also, in the work of [9], the extremal Gödel black holes were not found as supersymmetric solutions, which is in contrast to the extremal Bañados-Teitelboim-Zanelli (BTZ) limit where supersymmetric extensions are known [11].

Because of the discrete identifications, Gödel black holes contain closed timelike curves in the asymptotic region. We thus still expect to find an instability in the string spectrum. Nevertheless, in the causally safe region close to the horizon, standard thermodynamics holds once the correct conserved charges have been identified. One can ask also if, regardless of the causal pathologies, black hole entropy can be microscopically computed as for the BTZ [12]. In fact, Gödel geometries admit, in general, an asymptotic symmetry algebra containing one copy of the Virasoro algebra [13]. When the spacetime is supported by Maxwell-Chern-Simons fields, the central charge turns out to be negative.

We will show in Sec. II how Gödel black holes describe exact string backgrounds via deformations of the $SL(2, \mathbb{R})$ WZW model. We will make contact between previous work [2,14] and the recently discussed warped geometries [3]. The spectrum of strings containing tachyons will be described. In Sec. III, we will uplift the Gödel black holes to 10d solutions of type IIB supergravity and discuss supersymmetry. The extremal Gödel black holes will be shown to admit one 3d Killing spinor. We conclude with some remarks on black hole entropy in the last paragraph.

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II. GÖDEL BLACK HOLES AS A MARGINAL DEFORMATION

A. Asymmetric marginal deformations of the SL(2, ℝ) WZW model

Let us start with an SL(2, ℝ) WZW model at level k with action $S_{\rm WZW}$ and $\mathfrak{sl}(2,\mathbb{R})$-valued currents $J(z) = J^b(z)T_b$, $\bar{J}(\bar{z}) = \bar{J}^b(z)T_b$, describing string theory on a target space whose fields are the AdS_3 metric and a given Neveu-Schwarz–Neveu-Schwarz two-form. Let us take the conventions of [3] and denote (J_0^0, J_0^1, J_0^2) [resp. $(\tilde{J}_0^0, \tilde{J}_0^1, \tilde{J}_0^2)$] the zero modes of $J^a(z)$ [resp. $\bar{J}^a(\bar{z})$] satisfying

$$\begin{aligned}&[J_0^1, J_0^2] = 2J_0^0,\\ &[J_0^0, J_0^1] = -2J_0^2, \quad \text{and} \quad [J_0^0, J_0^2] = 2J_0^1.\end{aligned} \tag{1}$$

This background is an exact string theory one, since WZW models represent two-dimensional worldsheet conformal field theories (CFTs). An interesting feature of WZW models is that they allow for integrable marginal deformations, which allows one to reach a wide variety of new exact backgrounds. The deformation is usually written as

$$S_{\delta\lambda} = S_{\rm WZW} + \delta\lambda \int d^2 z \mathcal{O}(z,\bar{z}), \tag{2}$$

where $\delta\lambda$ is a parameter being switched on continuously. A necessary condition for the operator $\mathcal{O}(z,\bar{z})$ to be exactly marginal is obviously that it is marginal, i.e. of conformal weights $(1, 1)$. In WZW models, such operators are naturally present, and appear to be truly marginal under additional conditions [15]. For our purposes, we will be interested in a particular type of deformation, named *asymmetric* deformation; see e.g. [10,14]. Such deformations are possible if one considers an $\mathcal{N} = 1$ heterotic supersymmetric extension of the WZW model (for a short review, see Appendix C of [14]). In the case of $\mathfrak{sl}(2,\mathbb{R})$, one adds 3 left-moving free fermions transforming in the adjoint representation of $\mathfrak{sl}(2,\mathbb{R})$, while leaving the right-moving sector unchanged. However, a right-moving current algebra with total central charge $c = 16$ has to be added representing the internal (compactified) bosons. As a result, we end up with a left-moving $\mathcal{N} = 1$ current algebra and a right-moving $\mathcal{N} = 0$ one (for details, see [10,16,17]). We consider the following deformation operator:

$$\mathcal{O}(z,\bar{z}) = \left(J^a(z) - \frac{i}{2}\epsilon^{abc}\psi_b(z)\psi_c(z)\right)\bar{I}^i(\bar{z}), \tag{3}$$

where $J^a(z)$ is a left-moving generator of $\mathfrak{sl}(2,\mathbb{R})$, ψ_a are the 3 left-moving worldsheet fermions, and $\bar{I}^i(\bar{z})$ is an arbitrary right-moving current belonging to the Cartan subalgebra of the heterotic gauge group. These are normalized as

$$\bar{I}^i(\bar{z})\bar{I}^j(\bar{w}) \sim \frac{k_G h^{ij}}{2(\bar{z}-\bar{w})^2}, \quad i,j = 1,\ldots,\text{rank (gauge group)}, \tag{4}$$

with $h^{ij} = f^{ik}{}_l f^{lj}{}_k / g^*$, $f^{ik}{}_l$ and g^* being the structure constants and dual Coxeter numbers of the heterotic gauge group. It can be shown that these operators are truly marginal [15]. The background fields resulting from integrating the infinitesimal asymmetric deformation (2) and (3) to a finite one with parameter $\rm H$ are written as [10,16,18–20]

$$g_{\mu\nu} = \mathring{g}_{\mu\nu} - 2{\rm H}^2 J^a_\mu J^a_\nu \quad \text{no sum}, \tag{5}$$

$$B_{\mu\nu} = \mathring{B}_{\mu\nu}, \tag{6}$$

$$A_\mu = {\rm H}\sqrt{\frac{2k}{k_G}} J^a_\mu, \tag{7}$$

where $\mathring{g}_{\mu\nu}$ and $\mathring{B}_{\mu\nu}$ are the initial anti–de Sitter background fields and $J^a = J^a_\mu dx^\mu$, $\bar{J}^a = \bar{J}^a_\mu dx^\mu$. It is worth noting that these background fields are exact to all orders in α', contrarily to what happens e.g. for the symmetric deformations (see [21] for a pedagogical review). The deformation preserves a $U(1)_L \times SL(2,\mathbb{R})_R$ isometry of the original $SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R$ isometry of AdS_3 space.

We place emphasis on the fact that, although this construction is intrinsically heterotic due to the presence of the gauge field, the same background can be obtained in type II superstrings via a Kaluza-Klein reduction. In that case, the current $\bar{I}^i(\bar{z})$ belongs to an internal compact U(1) instead, and the gauge field is produced in the dimensional reduction procedure [10]. On the other hand, since the asymmetric deformations have a constant dilaton, we might expect them to be mapped by S-duality to type IIA solutions, where in this case the geometries will be supplemented by Ramond-Ramond fields (although we will not consider these possibilities here) [21].

B. Gödel black holes as orbifolded hyperbolic deformations

The asymmetric deformations can be classified according to the nature of the current considered in the deformation (3). Deformations driven by a timelike (J^3), spacelike (J^2), or lightlike ($J^1 + J^3$) generator will be termed elliptic, hyperbolic, or parabolic, respectively. The metric of a hyperbolic asymmetric deformation of the SL(2, ℝ) WZW can be written as [10]

$$\begin{aligned} ds^2 = \frac{k}{4}[&-d\tau^2 + du^2 + d\sigma^2 + 2\sinh\sigma du d\tau \\ &- 2{\rm H}^2(du + \sinh\sigma d\tau)^2]. \end{aligned} \tag{8}$$

For $\mathrm{H} = 0$, this is simply AdS_3 space, where for $\{\tau, u, \sigma\} \in \mathbb{R}^3$ these coordinates cover the whole space exactly once.

This geometry has been recently mentioned as a solution of topologically massive gravity; see Eq. (3.3) of [3]. The relation with their parameters $(\hat{l}, \hat{\nu})$ is

$$\mathrm{H}^2 = \frac{3(1-\hat{\nu}^2)}{2(3+\hat{\nu}^2)}, \qquad k = \frac{4\hat{l}^2}{3+\hat{\nu}^2}. \tag{9}$$

Therefore, the deformed anti–de Sitter metric for $\hat{\nu}^2 > 1$ (stretched AdS_3 space in the terminology of [3]), yielding regular black holes upon identification, can only be regarded as an exact string background if the deformation parameter, and consequently the $U(1)$ field, becomes imaginary. As we look only for real solutions of the WZW model, we will discard such solutions. On the other hand, the metric for real H, corresponding to $\hat{\nu}^2 < 1$ (squashed AdS_3 space), is the tachyonic Gödel background discussed in [2,13] and leads, after identifications, to the Gödel black holes [2,13], which we write, for convenience, as

$$ds^2_{\text{Gödel BH}} = \frac{dr^2}{f(r)} + (1-2\mathrm{H}^2)(dT - mrd\phi)^2 - f(r)d\phi^2 \tag{10}$$

where $f(r) = m^2r^2 + c_1 r + c_2$. Contact is made with [2], Eq. (31), via the substitution

$$m^2 = 2\left(\frac{1+\alpha^2l^2}{l^2}\right), \qquad \mathrm{H}^2 = \frac{1-\alpha^2l^2}{2(1+\alpha^2l^2)}, \qquad c_1 = -8G\nu, \qquad c_2 = \frac{4GJ}{\alpha} \tag{11}$$

and $T = \frac{m}{2\alpha}t$. In order to show that (10) is indeed obtained by performing discrete identifications on the metric (8), we first remark that (10) is exactly the metric (4.1) of [3] with the following substitution (hatted quantities are the ones of [3]):

$$\hat{\nu}^2 = \frac{3\alpha^2l^2}{2+\alpha^2l^2}, \qquad \hat{l}^2 = \frac{3l^2}{2+\alpha^2l^2}, \tag{12}$$

$$\nu = \frac{-3(1+\alpha^2l^2)}{8\alpha lG(2+\alpha^2l^2)}\left(\alpha l(\hat{r}_+ + \hat{r}_-) - \sqrt{2(1+\alpha^2l^2)\hat{r}_+\hat{r}_-}\right), \tag{13}$$

$$J = \frac{9\sqrt{2\hat{r}_+\hat{r}_-}(1+\alpha^2l^2)}{32\alpha G(2+\alpha^2l^2)^2}\left((1+3\alpha^2l^2)\sqrt{2\hat{r}_+\hat{r}_-} - 2\alpha l\sqrt{1+\alpha^2l^2}(\hat{r}_+ + \hat{r}_-)\right), \tag{14}$$

and the change of coordinates $t = \hat{l}\hat{t}$, $\hat{r} = -\frac{2\alpha}{\hat{\nu}\hat{l}}r + \frac{1}{2\hat{\nu}} \times \sqrt{\hat{r}_+\hat{r}_-(\hat{\nu}^2+3)}$, $\phi = \hat{\phi}$. The region of parameter space where closed timelike curves appear, $\hat{\nu}^2 < 1$, is exactly the black hole sector of [2] with $\alpha^2l^2 < 1$. We can then use the change of coordinates (5.3)–(5.5) of [3], also valid in the parameter range $\hat{\nu}^2 < 1$, to show that the metric (10) (but with $\phi \in \mathbb{R}$, while for Gödel black holes ϕ is periodic) can be written in a coordinate patch as (8). The Killing vector used to perform the identifications that make ϕ periodic is given by

$$\partial_\phi = \frac{\hat{\nu}^2+3}{8}\left[\left(\hat{r}_+ + \hat{r}_- - \frac{\sqrt{(\hat{\nu}^2+3)\hat{r}_+\hat{r}_-}}{\hat{\nu}}\right)L_2 - (\hat{r}_+ - \hat{r}_-)R_2\right] \tag{15}$$

where L_2 and R_2 are the $SL(2,\mathbb{R})$ Killing vectors associated with the currents J_2 and $\bar{J}_2$, respectively (given explicitly e.g. in [3], Appendix A). We note that quotients of (8) had already appeared in [14], but these were not studied further because of the absence of a causally safe asymptotic region.

For completeness, we provide a list of the real asymmetric deformations of anti–de Sitter space in Table I in order to make a larger contact between the works of [2,3,14].

In conclusion, we have shown that Gödel black holes supplemented with the appropriate background fields represent an *exact string theory background* through an orbifold of a hyperbolic asymmetric deformation of the $\mathfrak{sl}(2,\mathbb{R})$ WZW model in complete continuation of [4]. In particular, it solves the beta function equations to all orders in the inverse string tension α' [4,10].

C. String spectrum

The power of marginal deformations of WZW models lies in the fact that, besides being able to read off the deformed background fields, it is in theory also possible to determine the deformed partition function from the original one (see [21] Chap. 3 for an overview and an

TABLE I. List of $SL(2,\mathbb{R}) \times U(1)$ deformations of 3d anti–de Sitter space. In each case, the Einstein tensor is equal to a cosmological constant term plus a direct product of the $U(1)$ Killing vector K. Identifications in the Gödel universe lead to conical singularities (Gödel particles), and identifications in the tachyonic Gödel universe lead to Gödel black holes.

Name	Deformation type	$G_{\mu\nu} + \Lambda g_{\mu\nu} \sim K_\mu K_\nu$	Real deformations
Timelike warped AdS	Elliptic	$K^2 = -1$	Gödel universe (stretched)
Spacelike warped AdS	Hyperbolic	$K^2 = +1$	Tachyonic Gödel universe (squashed)
Null warped AdS	Parabolic	$K^2 = 0$	···

extensive list of references). In the case at hand, however, determining the deformed partition function in a straightforward way would require one to decompose the $SL(2,\mathbb{R})_k$ partition function in a hyperbolic basis of characters, which is to date an unsolved problem. Also, having to deal with the $\widehat{SL(2,\mathbb{R})}_k$ current algebra in a basis diagonalizing a noncompact operator leads to additional technical complications (see [22–27] for related discussions in the context of the BTZ black hole). Nevertheless, the spectrum of heterotic string states in orbifolds of the asymmetric hyperbolic deformations including the twisted sectors originating from the orbifold procedure [25] has been obtained in [14]. It reads as [28]

$$L_0 = -\frac{j(j-1)}{k} - \frac{\lambda^2}{k+2} - \frac{k+2}{2k}\left(\frac{2\lambda}{k+2}+\nu\right)^2 + L_0^{\rm tw} + N + h_{\rm int},$$
$$\bar{L}_0 = -\frac{j(j-1)}{k} - \frac{\bar{\lambda}^2}{k+2} + \bar{L}_0^{\rm tw} + \bar{N} + \bar{h}_{\rm int} \quad (16)$$

where $L_0^{\rm tw}$ and $\bar{L}_0^{\rm tw}$ are the contributions to the weights of the heterotic super WZW primaries touched by the deformation and the orbifold:

$$L_0^{\rm tw} = \left(\frac{k}{2\sqrt{2}}w\Delta_- + \frac{1}{\sqrt{k}}(\mu+\nu)\cosh x + \bar{\nu}\sqrt{\frac{2}{k_g}}\sinh x\right)^2,$$
$$\bar{L}_0^{\rm tw} = \left(\bar{\nu}\sqrt{\frac{2}{k_g}}\cosh x + \frac{1}{\sqrt{k}}(\lambda+\nu)\sinh x\right)^2 + \left(\frac{k+2}{2\sqrt{2}}w\Delta_+ + \bar{\lambda}\sqrt{\frac{2}{k+2}}\right)^2.$$

In these expressions, the deformation parameter $\rm H$ is related to x through $\cosh x = \frac{1}{1-2{\rm H}^2}$, with $x > 0$ so as to have ${\rm H}^2 \leq 1/2$ (see [10]). The $SL(2,\mathbb{R})$ representations are parametrized by j, which is related to the second Casimir c_2 as $c_2 = -j(j-1)$. The spectrum contains continuous representations with $j = \frac{1}{2} + is$, $s \in \mathbb{R}^+$, as well as discrete representations with $j \in \mathbb{R}^+$ lying within the unitarity range $1/2 < j < (k+1)/2$, which are related to long and short string states in the WZW spectrum, respectively [29]. The parameters $(\lambda, \bar{\lambda}) \in \mathbb{R}^2$ are the (continuous) eigenvalues of the operators J^2 and $\bar{J}^2$, while ν and $\bar{\nu}$ are the corresponding eigenvalues with respect to $i\psi_1\psi_3$ and the internal fermions on the gauge sector considered in the deformation operator (3) ($\nu = n + a/2$, $\bar{\nu} = \bar{n} + \bar{a}/2$, $n, \bar{n} \in \mathbb{N}$, $a, \bar{a} = 0$ for the NS sector and $a, \bar{a} = 1$ for the Ramond one). The oscillator numbers and contributions from the internal CFT in the left- and right-moving sectors are given by $(N, h_{\rm int})$ and $(\bar{N}, \bar{h}_{\rm int})$, respectively. The winding sectors with winding number $w \in \mathbb{Z}$ originate from the orbifold along the Killing vector $\Delta_- L_2 + \Delta_+ R_2$ [14,25].

One may now use these expressions to demonstrate that the spectrum of the orbifolded hyperbolic asymmetric deformation contains tachyonic long strings, along the lines of [4,24,29]. The analysis presented here is very rough and only aims at pointing out the presence of at least one tachyonic state, as has been done in [4] for the asymmetric elliptic deformation. First, we note that the inclusion of winding or spectral flowed sectors should, in principle, be extended to the fermions of the left-moving super WZW model [30], as well as to those in the gauge sector [4]. Then, the contributions of the internal CFTs and the oscillator numbers have to be such that the level matching condition is satisfied. From this, the energy spectrum $E = \Delta_-\lambda - \Delta_+\bar{\lambda}$ [22,24] can be determined from the mass-shell condition. Considering a state with $N = 1/2$, $n = \bar{n} = a = \bar{a} = 0$, and $\lambda = \bar{\lambda}$, the condition $L_0 - 1/2 = 0$ for a state in a continuous representation leads to

$$E = \frac{(\Delta_- - \Delta_+)}{2\sqrt{2}\sinh^2 x}\left(k^{3/2}w\Delta_- \cosh x \pm \sqrt{w^2k^3\Delta_-^2 - 2(1+4hk+4s^2)\sinh^2 x}\right). \quad (17)$$

Therefore, we conclude that for a state sufficiently excited in the internal CFT or with s large enough, the energy could become imaginary, pointing at an instability of the background. One could conjecture that the endpoint of the tachyon decay could correspond to the double deformation of [14], free of closed timelike curves, obtained by superposing a symmetric deformation to the asymmetric one, but we will not expand further in that direction.

III. SUPERSYMMETRY PROPERTIES

A. Embedding in type II supergravity

Let us consider the consistent truncation of both type II supergravities to fields in the Neveu-Schwarz sector. The action reads as (see e.g. [31], p. 29)

$$S = \frac{1}{16\pi G_{10}}\int d^{10}x\sqrt{-\hat{g}}\left[\hat{R} - \frac{1}{2}\partial_\mu\hat{\phi}\partial^\mu\hat{\phi} - \frac{1}{12}e^{-\phi}\hat{H}_3^2\right]. \quad (18)$$

It turns out that the Gödel black holes can be uplifted to solutions of this action. In that case, the dilaton vanishes and the three-form and metric are given by

$$\hat{H}_3 = m({\rm vol}_{S^3} + dr\wedge dT\wedge d\phi + \sqrt{2}{\rm H}dr\wedge dz\wedge d\phi),$$
$$\hat{ds}^2 = ds^2_{S^3} + ds^2_{\mathbb{R}^3} + ds^2_{\text{Gödel BH}} + (dz + \sqrt{2}{\rm H}(dT - mrd\phi))^2, \quad (19)$$

where the metric (10) is used.

It is known that in (1,1) 3d supergravity, the nonzero-mass extremal BTZ black holes have only one periodic Killing spinor in the (1,0) or (0,1) representations of the gamma matrices, depending on the sign of the angular momentum [11]. In the zero-mass vacuum, these spinors add up, and therefore the so-called Ramond vacuum preserves two supersymmetries.

Let us develop a quick and informal argument in favor of supersymmetry for Gödel black holes. First, it seems that the analysis of [11] is left unchanged if one analytically continues to $t \to it$ and $\phi \to i\phi$, which indicates that the analytically continued BTZ admits the same Killing spinors. Now, uplifting to 10 dimensions, one obtains the solution $\mathrm{BTZ}_{\mathrm{an.cont.}} \times S^3 \times T_4$ of type II supergravity where the supersymmetries are also uplifted and enhanced by the $S^3 \times T_4$ factors. It turns out that the solution (19) can be obtained from a change of variables twisting one of the flat directions z with time T. Equivalently, the Gödel metric can be "untwisted" by adding an extra dimension z with the appropriate metric. It is only when z is periodically identified that the solution cannot be joined to $\mathrm{BTZ}_{\mathrm{an.cont.}} \times S^3 \times S^1 \times T_3$ by a diffeomorphism.

Since the Killing spinors depend only on the radial coordinate r, they are unaffected by the change of variables twisting one of the flat directions z with time and leading to the solution (19). Compactifying this metric on $S^3 \times T_4$, one obtains the Gödel black holes, and since the Killing spinors do not depend on the variables of $S^3 \times T_4$, they should appear as supersymmetries of the Gödel black holes.

However, a subtlety arises which invalidates part of this argument. It turns out that there is only one extremal BTZ black hole that is related to Gödel black holes in the limit $\mathrm{H} \to 0$. Indeed, when $\mathrm{H} = 0$, the metric $ds^2_{\text{Gödel BH}|_{\mathrm{H}=0}} = ds^2_{\text{an.cont. BTZ}}$ reduces to the double analytic continuation of the BTZ metric

$$ds^2_{\mathrm{BTZ}} = \frac{dr^2}{f(-r)} - (dT + mrd\phi)^2 + f(-r)d\phi^2 \qquad (20)$$

with the continuation $T \to -iT$, $\phi \to -i\phi$, and $r \to -r$. The BTZ metric is written in terms of the standard asymptotically anti–de Sitter coordinates $(t_{\mathrm{BTZ}}, r_{\mathrm{BTZ}}, \phi_{\mathrm{BTZ}})$ [32] as

$$\phi_{\mathrm{BTZ}} = \phi + \frac{2}{l_{\mathrm{AdS}} c_1} T, \qquad t_{\mathrm{BTZ}} = -\frac{2}{c_1} T, \qquad r^2_{\mathrm{BTZ}} = -c_1 r + c_2, \qquad (21)$$

and the standard parameters are given by

$$l_{\mathrm{AdS}} = \frac{2}{m}, \qquad M_{\mathrm{BTZ}} = \frac{2c_2}{l^2} - \frac{c_1^2}{4}, \qquad J_{\mathrm{BTZ}} = -2\frac{c_2}{l_{\mathrm{AdS}}}. \qquad (22)$$

Now, the extremal BTZ black hole $l_{\mathrm{AdS}} M_{\mathrm{BTZ}} = J_{\mathrm{BTZ}}$ corresponds to $c_1^2 = 4c_2 m^2$. However, the counter-rotating extremal black hole $l_{\mathrm{AdS}} M_{\mathrm{BTZ}} = -J_{\mathrm{BTZ}}$ corresponding to $c_1 = 0$ is not covered by the (T, r, ϕ) coordinates because the metric is not related by a diffeomorphism to the extremal BTZ metric.

Therefore, we expect to find only one Killing spinor for the class of extremal Gödel black holes $c_1^2 = 4c_2 m^2$. We will now show directly that Killing spinors exist by explicitly solving the Killing spinor equations for the solution (19) compactified on $S^3 \times T_4$.

B. Killing spinor equations

We will follow the notations of [33] throughout. Requiring the variations of the dilatino and gravitino to vanish and using the simplification trick shown in (7.4) of [34] leads to the Killing spinor equations

$$H_{(3)ABC}\Gamma^{ABC}\eta = 0, \qquad (23)$$

$$\left(D_A + \frac{i}{48} H_{(3)BCD}\Gamma^{BCD}\Gamma_A \mathcal{B}^{-1}\mathfrak{C}\right)\eta = 0. \qquad (24)$$

$\mathfrak{C}$ is the complex conjugation operator, and the reality matrix satisfies $\mathcal{B}\mathcal{B}^* = 1$, $\Gamma_A^* = \rho\mathcal{B}\Gamma_A\mathcal{B}^*$ where $\rho = \pm 1$ depends on the representation. We have set the dilaton to zero. The real parts of the spinor $\eta_\pm = P_\pm \eta$ are obtained from the projectors $P_\pm = \frac{1}{2}(1 \pm i\mathcal{B}^*\mathfrak{C})$ and obey

$$H_{(3)ABC}\Gamma^{ABC}\eta_\pm = 0, \qquad \left(D_A \pm \frac{1}{48} H_{(3)BCD}\Gamma^{BCD}\Gamma_A\right)\eta_\pm = 0. \qquad (25)$$

Let us choose the vielbein as

$$e^0 = -\sqrt{f(r)}d\phi, \qquad e^1 = \frac{1}{\sqrt{f(r)}}dr, \; e^2 = \sqrt{1 - 2\mathrm{H}^2}(dT - mrd\phi), \qquad e^9 = dz + \sqrt{2}\mathrm{H}(dT - mrd\phi) \qquad (26)$$

with e^i, $i = 3, \ldots, 5$ parametrizing the three-sphere and e^6, e^7, e^8 the flat directions. We choose a spinor of the form $\eta_{T^3} \otimes \eta_{M_7} \otimes \epsilon_0$ where ϵ_0 is a two-component spinor and η_{T^3} is a two-component constant spinor which factorizes from the equations. We will still denote η and Γ_A as the resulting seven-dimensional spinors and Gamma matrices. The first Killing equation reads explicitly as

$$(\Gamma^{345} + \sqrt{1 - 2\mathrm{H}^2}\Gamma^{012} + \sqrt{2}\mathrm{H}\Gamma^{019})\eta_\pm = 0. \qquad (27)$$

The ansatz for the spinor η consists in the following direct product: $\eta_\pm = \eta_{M_4\pm} \otimes \eta_{S^3\pm}$ where $\eta_{M_4\pm} = \epsilon_\pm \otimes \eta_{M_2}$ is a four-component spinor depending only on t, r, ϕ, and z, and η_{S^3} is a spinor on the sphere. We are mainly interested in the part η_{M_2} of the spinor which captures the supersymmetry properties of the three-dimensional Gödel subspace. We represent the Clifford algebra as

$$\Gamma_0 = i\sigma_3 \otimes \sigma_1 \otimes \mathbb{I}, \qquad \Gamma_1 = \sigma_3 \otimes \sigma_2 \otimes \mathbb{I}, \qquad \Gamma_2 = \sigma_1 \otimes \mathbb{I} \otimes \mathbb{I}, \qquad \Gamma_9 = \sigma_2 \otimes \mathbb{I} \otimes \mathbb{I}, \qquad (28)$$

$$\Gamma_3 = \varepsilon\sigma_3 \otimes \sigma_3 \otimes \sigma_1, \quad \Gamma_4 = \varepsilon\sigma_3 \otimes \sigma_3 \otimes \sigma_2, \quad \Gamma_5 = \varepsilon\sigma_3 \otimes \sigma_3 \otimes \sigma_3, \tag{29}$$

where, for completeness, we allowed for two inequivalent representations: $\varepsilon = \pm 1$. The four first matrices form a representation for the four-dimensional subspace (0129) of interest.

Now, the matrix $\Gamma^{345} = -i\varepsilon\Gamma_* \otimes I$ is proportional to the chirality matrix $\Gamma_* = -\sigma_3 \otimes \sigma_3$ in the space (0129). Therefore, the first Killing equation reduces to

$$(\varepsilon I + \sqrt{1 - 2\mathrm{H}^2}\sigma_2 - \sqrt{2}\mathrm{H}\sigma_1)\epsilon_\pm = 0, \tag{30}$$

which is a chirality condition on $\epsilon_\pm$ for arbitrary H.

Using $\{\Gamma^i, \Gamma^{abc}\} = 0$ and $\{\Gamma^i, \Gamma^{345}\} = 2i$ for $a, b, c \in 0, 1, 2, 9$, it is straightforward to write the components on the sphere of the second Killing equation (25) as the usual Killing spinor equation on the sphere for η_{S^3}. The remaining components can be written as

$$\left(D_a \pm \frac{m}{8}\{\Gamma_a, \sqrt{1 - 2\mathrm{H}^2}\Gamma^{012} + \sqrt{2}\mathrm{H}\Gamma^{019}\}\right)\eta_\pm = 0. \tag{31}$$

Using $\{\Gamma_a, \Gamma^{012}\} = 2i\Gamma_*\Gamma_{a9}$, $\{\Gamma_a, \Gamma^{019}\} = -2i\Gamma_*\Gamma_{a2}$, and $\Gamma_*\Gamma^{ab} = -\frac{i}{2}\varepsilon^{abcd}\Gamma_{cd}$, the equation can be written in the familiar form

$$\left(d + \frac{1}{4}\tilde{\omega}^{ab}\Gamma_{ab}\right)\eta_{M_4\pm} = 0, \quad \tilde{\omega}^{ab} = \omega^{ab} \pm \frac{m}{2}e_c(\sqrt{1 - 2\mathrm{H}^2}\varepsilon^{c9ab} - \sqrt{2}\mathrm{H}\varepsilon^{c2ab}) \tag{32}$$

where the removal of the last identity factor of the Gamma matrices (28) is understood.

Up to now, we have solved the trivial flat and spherical parts of the Killing spinor equations. The only remaining four equations involve the four-dimensional spinor $\eta_{M_4\pm}$. Now, we expect that there will be only three nontrivial equations involving the Gödel metric. Indeed, the combination of $\sqrt{2\mathrm{H}}$ times the equation for index 2 minus $\sqrt{1 - 2\mathrm{H}^2}$ times the equation for index 9 gives

$$(\sqrt{2}\mathrm{H}D_2 - \sqrt{1 - 2\mathrm{H}^2}D_9)\eta_{M_4\pm} = 0, \tag{33}$$

which, expressed in the coordinate basis, gives the following dependence on the variables: $\eta_{M_4\pm} = \eta_{M_4\pm}(T + \sqrt{2}\mathrm{H}z, \phi, r)$. Solving the remaining equations is the object of the next section.

C. Gödel Killing spinors

In fact, the Eq. (32) for η_{M_4+} is very simple. We have

$$\left(d + \frac{1}{2}\Gamma_{01}d\tilde{\phi}\right)\eta_{M_4+} = 0, \quad \tilde{\phi} = -\frac{c_1}{2}\phi - m(T + \sqrt{2}\mathrm{H}z). \tag{34}$$

It admits the solution

$$\eta_{M_4+} = \exp\left(-\frac{\tilde{\phi}}{2}\Gamma_{01}\right)\eta_{(0)+} \tag{35}$$

where $\eta_{(0)+} = \epsilon_+ \otimes \eta^{(0)}_{M_2}$ is a constant spinor with ϵ also satisfying the chirality condition (30). However, for $c_1 \neq 0$, since the spinor is not periodic nor antiperiodic in ϕ, we have to reject this solution. In any case, the spinor is z dependent and therefore is not a Killing spinor of the three-dimensional relevant spacetime.

The equation for η_{M_4-} is more involved. Since the integrability conditions hold, a local solution always exists. One obtains

$$\eta_{M_4-} = (\cosh(K(r)) + (\sqrt{1 - 2\mathrm{H}^2}\Gamma_{02} + \sqrt{2}\mathrm{H}\Gamma_{09}) \times \sinh(K(r)))\exp\left(-\frac{\phi}{16m}M\right)\eta_{(0)-} \tag{36}$$

where

$$K(r) = \frac{1}{2}\ln\left(\frac{c_1 + 2m^2r}{2m} + \sqrt{m^2r^2 + c_1r + c_2}\right),$$
$$M = (4m^2(c_2 - 1) - c_1^2)\Gamma_{01} + (4m^2(c_2 + 1) - c_1^2) \times (\sqrt{1 - 2\mathrm{H}^2}\Gamma_{12} + \sqrt{2}\mathrm{H}\Gamma_{19}).$$

The chirality condition (30) has broken half of the supersymmetries. We expect that the tachyonic Gödel geometry is topologically trivial, similarly to the Gödel geometry. If it indeed turns out to be the case, the solution (36) is also a global spinor.

Since Gödel black holes are obtained as identifications along $\partial/\partial\phi$ on the tachyonic Gödel geometry, the Killing spinors exist globally only if they are periodic or antiperiodic under this identification. This amounts to imposing $M^2 = 0$ and $M\eta_{(0)-} = 0$. This statement is equivalent to imposing $c_1^2 = 4m^2c_2$ and the following chirality condition:

$$(-\Gamma_{01} + \sqrt{1 - 2\mathrm{H}^2}\Gamma_{12} + \sqrt{2}\mathrm{H}\Gamma_{19})\eta^{(0)}_{M_4-} = 0. \tag{37}$$

Using the definition of conserved charges [2], the relation between c_1 and c_2 is in fact the condition for extremal black holes. The condition (37) can be simplified by splitting $\eta^{(0)}_{M_4-}$ as $\epsilon_- \otimes \eta^{(0)}_{M_2}$ and using the chirality condition (30) on ϵ_-. One then gets a condition on $\eta^{(0)}_{M_2}$ only: $\sigma_1\eta^{(0)}_{M_2} = \varepsilon\eta^{(0)}_{M_2}$. Assuming the topological triviality of the background metric, these spinors exist globally.

Finally, we found that for each representation of the Clifford algebra, parametrized by ε, extremal Gödel spacetimes admit a Killing spinor,

$$\eta_{M_4-} = (\cosh(K(r)) + (\sqrt{1-2\mathrm{H}^2}\Gamma_{02} + \sqrt{2}\mathrm{H}\Gamma_{09}) \times \sinh(K(r)))\begin{pmatrix} \sqrt{2}\mathrm{H} + i\sqrt{1-2\mathrm{H}^2} \\ \varepsilon \end{pmatrix} \otimes \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix}.$$

We conclude that one class of extremal black holes ($c_1 = 0$) does not have any Killing spinor, while the other class ($c_1^2 = 4m^2 c_2$) has one supersymmetry generator. This is to be contrasted with the BTZ case, where Killing spinors were found in each class of extremal black holes.

This result fits nicely with the fact that the Gödel universe breaks one of the two $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ exact symmetries, which at the level of asymptotic symmetries breaks one of the two Virasoro algebras. It is then natural that one of the two supersymmetric extensions of the Virasoro algebras also gets broken, as we just showed. The existence of a Killing spinor shows that the $SL(2,\mathbb{R})$ algebra gets enhanced to an $Osp(1|2)$ algebra. Since the Killing spinors are periodic, the supersymmetry generators are taken in the Ramond representation.

IV. DISCUSSION

The identification of supersymmetry for extremal Gödel black holes in type IIB supergravity can be used to complement the analysis of [13]. There we derived the central extensions in the algebra of charges associated with the asymptotic symmetries of these spaces in the Einstein-Maxwell-Chern-Simons theory. Even though we have not repeated the analysis for the present matter fields, we expect that the right central charge associated with the unbroken copy of a Virasoro algebra will be the same,

$$c_R = -\frac{3\alpha l^2}{(1+\alpha^2 l^2)G} = -\frac{6\hat{\nu}\hat{l}}{(3+\hat{\nu}^2)G} = -\frac{3\sqrt{k}}{2G}\sqrt{1-2\mathrm{H}^2}. \tag{38}$$

Indeed, a close analysis shows that the central charge arises only from the Einstein part of the Lagrangian in [13]. The central charge is negative when the Gödel black holes have positive mass. It is interesting to note that the central charges vanish in the limit $\mathrm{H}^2 = 1/2$, where the deformed geometry becomes locally $AdS_2 \times \mathbb{R}$ [10].

The missing step in the argument to be able to match the macroscopic entropy with the one derived from the Cardy formula, at least in the left sector, was the knowledge of the minimal value for the L_0 eigenvalue. Given the supersymmetric energy bound,

$$L_0 \geq 0, \tag{39}$$

this minimal value Δ_0 is zero and is reached for the extremal black hole solutions. This provides a firmer ground on the use of the Cardy formula to count the microstates of Gödel black holes in the unbroken sector. It shows that even though the central charge of the Virasoro algebra is negative, there is enough structure (a Virasoro algebra and supersymmetry) to make the counting work.

An alternative approach to compute the entropy has been used in [3,35]. One can deduce from the vector (15) what can be interpreted as left- and right-moving temperatures in the dual CFT,

$$T_R \equiv \frac{\sqrt{2G\alpha(2+\alpha^2 l^2)(\alpha l^2\mu - (1+\alpha^2 l^2)J)}}{\sqrt{3}\pi\alpha l^2}, \tag{40}$$

$$T_L \equiv \frac{\sqrt{2G(2+\alpha^2 l^2)\mu}}{\sqrt{3}\pi l}. \tag{41}$$

The Bekenstein-Hawking entropy is then equal to

$$S = \frac{\mathcal{A}}{4G} = \frac{\pi^2 l}{3}(|c_L|T_L + |c_R|T_R) \tag{42}$$

where $|c_R| = |c_L|$. The advantage of this formula is that it allows one to conjecture the (absolute) value of the left central charge.

We have mentioned that Gödel black holes can also be obtained as quotients of spacelike squashed AdS_3 geometries in topologically massive gravity [3]. One can then ask if the $\mathcal{N} = 1$ supersymmetric extension [36,37] of this theory admits Gödel supersymmetric solutions. It turns out that it is not the case since all supersymmetric solutions admit a null Killing vector [38]. They all fall in the class of null/parabolic deformations of anti–de Sitter space; see Table I. Therefore, extremal Gödel black holes are not supersymmetric in $\mathcal{N} = 1$ topologically massive gravity.

We also observed that Gödel black holes represent exact string theory backgrounds, like AdS_3 space and the BTZ black holes do, though with a tachyonic spectrum. It is, however, not clear if these backgrounds could be obtained as the near-horizon geometry of some branes or fundamental string configurations. If this would be the case, it would be interesting to identify the corresponding nongravitational theory. This question has been investigated, namely, for the parabolic symmetric deformation of the $SL(2,\mathbb{R})$ WZW model [39], but to our knowledge no such analysis exists for asymmetric deformations.

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