Extremal dyonic black holes in $D = 4$ Gauss-Bonnet gravity

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We investigate extremal dyon black holes in the Einstein-Maxwell-dilaton theory with higher curvature corrections in the form of the Gauss-Bonnet density coupled to the dilaton. In the same theory without the Gauss-Bonnet term the extremal dyon solutions exist only for discrete values of the dilaton coupling constant a. We show that the Gauss-Bonnet term acts as a dyon hair tonic enlarging the allowed values of a to continuous domains in the plane (a, q_m) where q_m is the magnetic charge. In the limit of the vanishing curvature coupling (a large magnetic charge) the dyon solutions obtained tend to the Reissner-Nordström solution but not to the extremal dyons of the Einstein-Maxwell-dilaton theory. Both solutions have the same dependence of the horizon radius in terms of charges. The entropy of new dyonic black holes interpolates between the Bekenstein-Hawking value in the limit of the large magnetic charge (equivalent to the vanishing Gauss-Bonnet coupling) and twice this value for the vanishing magnetic charge. Although an expression for the entropy can be obtained analytically using purely local near-horizon solutions, its interpretation as the black hole entropy is legitimate only once the global black hole solution is known to exist, and we obtain numerically the corresponding conditions on the parameters. Thus, a purely local analysis is insufficient to fully understand the entropy of the curvature-corrected black holes. We also find dyon solutions which are not asymptotically flat, but approach the linear dilaton background at infinity. They describe magnetic black holes on the electric linear dilaton background.

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I. INTRODUCTION

In a recent paper [1] we started an investigation of extremal black holes in the four-dimensional Einstein-Maxwell-dilaton-Gauss-Bonnet model motivated by an interest in microscopic string calculations of the black hole entropy (for a review see [2,3]). In the theories with higher curvature corrections, the entropy deviates from the Bekenstein-Hawking value and can be calculated using Wald's formalism [4–9]. Remarkably, it still exhibits an agreement with the string theory predictions at the corresponding level, both in the Bogomol'nyi-Prasad-Sommerfield (BPS) [10–22] and non-BPS [23–39] cases. In some supersymmetric models with higher curvature terms exact classical solutions for static black holes were obtained [18,19,22]. Moreover, as was argued by Sen [40–42], the knowledge of the global black hole solutions is not necessary in order to compare classical and quantum results for the entropy: in the classical theory the entropy can be computed locally using the entropy function approach based on the attractor property typical for supergravity black holes [23–28]. However the question remains: Do the global black holes corresponding to local solutions used to construct the entropy function really exist? Generically, the existence of local solutions exhibiting the event horizons does not guarantee that they describe black holes which must be regular outside the horizon and asymptotically flat. Even though formally the entropy can be obtained from local considerations, the parameters involved may be subject to restrictions which are revealed only when we try to extend the local solutions to infinity. This issue is addressed in the present paper for extremal dyon black holes with the $\mathrm{AdS}_2 \times S^2$ horizon in the Gauss-Bonnet gravity.

The extremal dilatonic black hole [43–46] is a particularly interesting model associated with the heterotic string. In the Einstein-Maxwell-dilaton (EMD) theory without curvature corrections such a solution has a singular horizon of vanishing radius, the corresponding Bekenstein-Hawking entropy being zero. Typical higher curvature correction to this theory is given by the Gauss-Bonnet density coupled to the dilaton (later on referred as the Einstein-dilaton-Gauss-Bonnet (EDGB) model). In this model the local solutions with the $\mathrm{AdS}_2 \times S^2$ (Reissner-Nordström type) horizons of finite radius can be found. Presumably they should describe the extremal black holes

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possessing nonzero entropy. In [1] we have explored whether the local solutions constructed via series expansions in the vicinity of the event horizon can be extended to infinity as asymptotically flat black holes. It was shown that this is possible only within some bounded region in the space of parameters. Actually, the model contains two parameters which are worth considering as independent in the classical theory: the Gauss-Bonnet coupling constant α weighting the contribution of the Gauss-Bonnet term, and the dilaton coupling constant a. In the case of the purely electric configurations, the local solutions around the $\mathrm{AdS}_2 \times S^2$ horizon exist only for nonzero α, since no dilatonic purely electric (or purely magnetic) black holes with the $\mathrm{AdS}_2 \times S^2$ horizon are possible in the Einstein-Maxwell-dilaton theory without curvature corrections. In what follows we fix the value of α and take as the only continuously varying parameter the dilaton coupling a. If $a = 0$, the model reduces to the Einstein-Maxwell theory, in which the desired extremal solution does exist (the extremal Reissner-Nordström solution). We have studied whether similar solutions exist in the curvature-corrected theory for nonzero a, aiming to investigate the case $a = 1$ relevant to the compactified heterotic string. It turned out that the extremal black hole solutions exist only in a bounded region $0 \le a \le a_{\rm cr}$ where the critical value $a_{\rm cr}$ is close to $1/2$. For greater a, the local solutions exhibiting the $\mathrm{AdS}_2 \times S^2$ horizons cannot be extended to infinity as regular black holes, but develop singularities outside the horizon. The threshold value of a might serve as an indication that the corresponding string configuration experiences some qualitative change of state like the black hole–string transition [47].

The case of the purely electric extremal black hole in the EDGB model is special in the sense that no similar solutions exist in the EMD theory without curvature corrections (unless $a = 0$), so there is no smooth transition to the noncorrected theory. The situation becomes more flexible if we allow for both electric and magnetic charges to be present, since extremal dyons with the $\mathrm{AdS}_2 \times S^2$ horizon do exist in the EMD theory. As we will see, such solutions are possible for discrete values of the dilaton coupling constant: $a^2 = 1$ (the heterotic case), $a^2 = 3$ (the Kaluza-Klein case) and some sequence of other integer a_i^2 (coinciding with the sequence found in [48] for existence of nonextremal black holes with two horizons). Therefore in the curvature-corrected theory we have an infinite sequence of starting points for the dilaton coupling constant, not just the trivial point $a = 0$. So one can expect to have a much larger domain of existence of extremal charged dyonic black holes in the EDGB model than in the purely electric case.

We will be interested here uniquely in the extremal black holes with the degenerate event horizon. Nondegenerate black holes in the same theory were extensively studied in the past both perturbatively [49,50] and numerically [51–55]. More recently global properties of EDGB black hole solutions were studied using the dynamical system approach [56–59]. Stability issues were discussed in [60–64]. In these papers the existence of both neutral and charged asymptotically flat solutions with a nondegenerate event horizon and without naked singularities was established. These solutions have the Schwarzschild-type event horizon and they do not possess the extremal limits. The solution with the degenerate event horizon thus forms a separate branch of EDGB black holes which was not studied before.

The existence of nondegenerate dyonic black holes in the EMD theory without curvature corrections was studied in detail in [48]. It was found that such solutions generically exhibit one (nondegenerate) horizon of the Schwarzschild type, but for some discrete values of the dilaton coupling constant there are solutions with two horizons of the Reissner-Nordström type. In this latter case the limiting extremal solutions turn out to be possible. We will show that adding the Gauss-Bonnet term to the EMD action acts as a hair tonic for extremal dyons, allowing for continuously varying dilaton couplings. Still, the allowed domain is bounded (in somewhat irregular way), so the threshold behavior observed in [1] persists in the dyon case too. The solutions corresponding to the values of parameters approaching the boundary of the allowed domain exhibit interesting saturation properties similar to the BPS bounds in the EMD theory without curvature corrections.

The plan of the paper is as follows. Section II contains general definitions and investigation of the hidden symmetries of the reduced theory. It is shown that the one-dimensional theory possesses the two-parametric off-shell and the three-parametric on-shell symmetry groups. These symmetries serve as a convenient tool allowing us to describe the space of solution semianalytically, in spite of the fact that the equations of motion do not have analytic solutions. In Sec. III we investigate local series solutions around the assumed $\mathrm{AdS}_2 \times S^2$ horizon. We show that for fixed values of the electric and magnetic charges the local solutions contain only one free parameter, contrary to two parameters in the corresponding theory without curvature corrections. Examining the higher-order coefficients we find the family of curves in the parameter plane of a and q_m (the magnetic charge) on which the coefficients of the local solution are singular. These curves are parametrized by the same sequence of integers a_i^2 which correspond to the existence of dyons in the uncorrected theory. In this section we also calculate the entropy and show that it interpolates between the Bekenstein-Hawking value $A/4$ in the case of the vanishing Gauss-Bonnet coupling (or the large magnetic charge) and twice the Bekenstein-Hawking large value $A/2$ for purely electric solutions, as found in [1]. Section IV is devoted to the asymptotic solutions. We find that, similar to the uncorrected EMD theory, there are two physically interesting asymptotic patterns corresponding either to the usual asymptotically flat black holes, or to black holes on the linear dilaton background [65]. Numerical results are presented in Sec. V. Using the on-shell symmetries we can express the ratios of physical parameters as functions of the dilaton coupling constant only, which clarifies the properties of the whole family of dyonic solutions obtained. We also demonstrate their asymptotic BPS-type behavior on the boundary of the allowed domain of parameters. The parameter region for the asymptotically linear dilaton background (LDB) solutions is shown to be located on the other side of the limiting singular curve in the parameter plane. In Appendix A we give some details concerning the EMD dyons and show how analytical solutions known for two lower values of the discrete dilaton coupling sequence can be obtained by summing up the local series solutions valid in the vicinity of the horizon. In Appendix B some higher-order coefficient of the local solution in the EDGB model is listed which is necessary in deriving the existence of limiting curves in the parameter space.

II. GENERAL SETTING

We consider the four-dimensional dilatonic Gauss-Bonnet theory which is the Einstein-Maxwell-dilaton theory with an arbitrary dilaton coupling constant a modified by the Gauss-Bonnet (GB) term:

$$S = \frac{1}{16\pi}\int\{R - 2\partial_\mu\phi\partial^\mu\phi - e^{2a\phi}(F^2 - \alpha\mathcal{L}_{GB})\} \times \sqrt{-g}d^4x, \tag{1}$$

where $\mathcal{L}_{GB}$ is the Gauss-Bonnet density

$$\mathcal{L}_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}. \tag{2}$$

This action contains two parameters (we use the units $G = c = 1$): the dilaton coupling constant a and the Gauss-Bonnet coupling constant α. We assume $a \geq 0$, $\alpha \geq 0$, solutions for negative a can be obtained changing the sign of the dilaton. Note that in this action the Maxwell term is not multiplied by α to facilitate decoupling of the Gauss-Bonnet term from the EMD action.

Consider the static spherically symmetric metrics,[1] parametrized by two functions $w(r)$ and $\rho(r)$:

$$ds^2 = -w(r)dt^2 + \frac{dr^2}{w(r)} + \rho^2(r)d\Omega_2^2, \tag{3}$$

the scalar curvature and the Gauss-Bonnet density then read

$$R = \frac{1}{\rho^2}[-(4w\rho\rho' + w'\rho^2)' + 2\rho'(w\rho)' + 2], \tag{4}$$

$$\mathcal{L}_{GB} = \frac{4}{\rho^2}[w'(w\rho'^2 - 1)]'. \tag{5}$$

The corresponding ansatz for the Maxwell oneform is

$$A = -f(r)dt - q_m\cos\theta d\varphi, \tag{6}$$

where $f(r)$ is the electrostatic potential and q_m is the magnetic charge. Note that the Gauss-Bonnet term breaks the discrete S-duality of the EMD theory without curvature corrections which is described by the transformation

$$g_{\mu\nu} \rightarrow g_{\mu\nu}, \qquad F \rightarrow e^{-2a\phi*}F, \qquad \phi \rightarrow -\phi, \tag{7}$$

where $F = dA$. It can be expected therefore that the properties of electric or magnetic black holes in this theory will be essentially different.

A. Reduced action and equations of motion

The one-dimensional Lagrangian associated with the ansatz is obtained by dropping the total derivative in the dimensionally reduced action:

$$L = \frac{1}{2}[\rho'(w\rho)' + 1] - 2\alpha a w'(w\rho'^2 - 1)\phi' e^{2a\phi} - \frac{1}{2}w\rho^2\phi'^2 + \frac{1}{2}\rho^2 f'^2 e^{2a\phi} - \frac{1}{2}\frac{q_m^2}{\rho^2}e^{2a\phi}. \tag{8}$$

The corresponding equations of motion read

$$4\alpha a[(w\rho'^2 - 1)\phi' e^{2a\phi}]' - 4\alpha a w'\rho'^2\phi' e^{2a\phi} - \rho\rho'' - \rho^2\phi'^2 = 0, \tag{9}$$

$$4\alpha a(ww'\rho'\phi' e^{2a\phi})' - \frac{1}{2}w''\rho - (w\rho')' - w\rho\phi'^2 + \rho f'^2 e^{2a\phi} + \frac{q_m^2}{\rho^3}e^{2a\phi} = 0, \tag{10}$$

$$(w\rho^2\phi')' + 2\alpha a[w'(w\rho'^2 - 1)]' e^{2a\phi} + a\rho^2 f'^2 e^{2a\phi} - a\frac{q_m^2}{\rho^2}e^{2a\phi} = 0, \tag{11}$$

$$(\rho^2 f' e^{2a\phi})' = 0. \tag{12}$$

Integrating once the last equation for the form field (12),

$$f'(r) = q_e\rho^{-2}e^{-2a\phi}, \tag{13}$$

where q_e is the electric charge parameter, one can then insert (13) into the previous equations to obtain the six-order system consisting of three differential equations of the second order with the electric and magnetic charges q_e, q_m entering as fixed parameters.

B. Symmetries and conserved quantities

The action (1) is invariant under the following two-parametric group of global transformations:

[1] A detailed discussion of the gauge fixing can be found in [1]. In this paper we adopt the gauge $g_{tt} = 1/g_{rr}$.

$$w \to w e^{-2\delta}, \qquad \rho \to \rho e^{\delta}, \qquad \phi \to \phi + \frac{\delta}{a}, \qquad f \to f e^{-2\delta}; \qquad r \to r + \nu, \tag{14}$$

which generate two conserved Noether currents,

$$J_g := \left(\frac{\partial L}{\partial \Phi'^A}\Phi'^A - L\right)\partial_g r \Big|_{g=0} - \frac{\partial L}{\partial \Phi'^A}\partial_g \Phi^A \Big|_{g=0}, \quad \partial_r J_g = 0, \tag{15}$$

where Φ^A stands for the set w, ρ, ϕ, f, and $g = \delta$, ν being the transformation parameters. The conserved quantity corresponding to ν is the Hamiltonian

$$H = \frac{1}{2}[\rho'(w\rho)' - 1] - 2\alpha a w'(3w\rho'^2 - 1)\phi' e^{2a\phi} - \frac{1}{2}w\rho^2\phi'^2 + \frac{1}{2}\rho^2 f'^2 e^{2a\phi} + \frac{1}{2}\frac{q_m^2}{\rho^2}e^{2a\phi}. \tag{16}$$

This quantity must vanish on-shell for diffeomorphism invariant theories, so $H = 0$. The Noether current corresponding to δ leads to the conservation equation $\partial_r J_\delta = 0$ with the current

$$J_\delta = \frac{w\rho^2\phi'}{a} - \frac{w'\rho^2}{2} + 2q_e f + 2\alpha[(w\rho'^2 - 1)(w' - 2aw\phi') + 2aww'\rho\rho'\phi']e^{2a\phi}, \tag{17}$$

which is an Abelian counterpart of the integral given in [66].

The integrals of motion allow one to reduce the order of the system by 2 leading to the fourth-order system. Moreover, for $q_e = 0$ one can further reduce the order to 3 introducing the new variables

$$w \to \exp(w), \qquad \rho \to \exp\left(\rho - \frac{w}{2}\right), \qquad \phi \to \phi - \frac{1}{2a}w. \tag{18}$$

Using them we can exclude from the system w, while w' and w'' still remain. For numerical integration we will still use the initial six-dimensional system, applying the integrals of motion to control accuracy of the calculation.

The symmetry group is enlarged on-shell on the space of the solutions of the equations of motion. It can be easily seen that the solution space is invariant under a *three-parametric* group of global transformations which consists in rescaling of the electric charge

$$q_e \to q_e e^{2\delta}, \qquad q_m \to q_m, \tag{19}$$

(leaving the magnetic charge invariant), rescaling and shift of an independent variable

$$r \to r e^{(\mu/2)+\delta} + \nu, \tag{20}$$

and the following transformation of the field functions:

$$w \to w e^{\mu}, \qquad \rho \to \rho e^{\delta}, \qquad \phi \to \phi + \frac{\delta}{a}, \qquad f \to f e^{(\mu/2)-\delta}. \tag{21}$$

These do not leave the action (1) invariant, unless a condition

$$\mu = -2\delta \tag{22}$$

is imposed in which case we go back to the transformations (14). However, dealing with the solutions, we can assume both parameters μ and δ to be independent.

III. LOCAL SOLUTIONS WITH $AdS_2 \times S^2$ HORIZON

We are looking for extreme black holes carrying both electric and magnetic charges in the EDGB theory for which the metric function $w(r)$ has double zero at some point $r = r_H$ (the event horizon) and is nonsingular for $r > r_H$. Already in the EMD theory without curvature corrections the analytical dyonic solutions are known only for two special values of the dilaton coupling, namely $a^2 = 1$ and $a^2 = 3$ (for a more detailed discussion see Appendix A) while for generic values of a only a numerical analysis is possible. So we can hope to solve the problem of dyonic black holes in the EDGB theory only numerically. Meanwhile, already from a local analytical analysis of the solution near the event horizon we obtain important restrictions on the parameters.

A. Horizon expansions

Local solutions in the vicinity of the event horizon $r = r_H$ can be constructed expanding them in terms of the deviation $x = r - r_H$:

$$w(r) = \sum_{k=2}^{\infty} w_k x^k, \qquad \rho(r) = \sum_{k=0}^{\infty}\rho_k x^k, \qquad P(r) := e^{2a\phi(r)} = \sum_{k=0}^{\infty} P_k x^k. \tag{23}$$

According to the assumption of extremality, the function $w(r)$ starts with the quadratic term: vanishing of w_0 means that $r = r_H$ is a horizon and vanishing of w_1 means that the horizon is degenerate. Such an expansion contains only one free parameter P_1 for fixed values of charges:

$$w(r) = \frac{x^2}{\rho_0^2} - \frac{P_1}{6\alpha a^2\rho_0^4}[3(a^2-1)q_m^4 + 6\alpha(3a^2-2)q_m^2 + 4\alpha^2(5a^2-3)]x^3 + O(x^4),$$

$$\rho(r) = \rho_0 + \frac{P_1}{4\alpha a^2\rho_0}[(a^2-1)q_m^4 + 2\alpha(3a^2-2)q_m^2 + 4\alpha^2(a^2-1)]x + O(x^2),$$

$$P(r) = \frac{\rho_0^2}{2(2\alpha + q_m^2)} + P_1 x + O(x^2). \tag{24}$$

The physical value of the horizon radius ρ_0 is not a free parameter here: it is fixed by the charges as follows:

$$\rho_0^2 = \frac{2q_e(2\alpha + q_m^2)}{\sqrt{4\alpha + q_m^2}}. \tag{25}$$

Note that the dilaton coupling constant enters the expansions only through a^2, so the space of solutions is symmetric under $a \to -a$, $\phi \to -\phi$ (we assume $a \geq 0$). When $a \to 0$, some of the expansion coefficients diverge unless $P_1 = 0$. If $P_1 = 0$, the higher-order coefficients in the dilaton expansion also vanish and the dilaton is constant. Then the Gauss-Bonnet terms in the action become a total derivative, so the theory reduces to the Einstein-Maxwell one. One can then show that the series expansions for w and ρ then combine indeed into the extremal Reissner-Nordström solution.

Higher-order expansion coefficients (see Appendix B) contain in denominators a sequence of the following combinations of parameters:

$$\Upsilon_i(a, q_m) := (a^2 - a_i^2)q_m^4 + 2\alpha(3a^2 - 2a_i^2)q_m^2 + 4\alpha^2[a^2(a_i^2 + 2) - a_i^2], \qquad i \geq 2, \tag{26}$$

where a_i^2 are the integers:

$$a_i^2 = 1 + 2 + \cdots + i = \frac{i(i+1)}{2}. \tag{27}$$

Therefore the expansions do not exist for the values of a and q_m satisfying the equations

$$\Upsilon_i = 0, \qquad i \geq 2, \tag{28}$$

which define the sequence of limiting curves in the parameter plane a, q_m. The only possibility to avoid the divergences of the expansion coefficients would be to choose the parameter P_1 as the product $P_1 = c\prod_{i=2}^{\infty}\Upsilon_i$ with finite c. However, in this case there will be a turning point outside the horizon in which $d\rho/dr = 0$, so that the solution does not extend to infinity but ends up in a singularity.

Remarkably, the sequence of integers (27) coincides with that found by Poletti *et al.* [48] as the condition of the existence of asymptotically flat dyons with two (nondegenerate) horizons in the EMD theory without curvature corrections. In our case this sequence enters in the definition of the set of functions $\Upsilon_i(a, q_m)$, whose vanishing marks nonexistence of the extremal local solution. In the limit $q_m \to \infty$ which is equivalent to $\alpha \to 0$ (decoupling of the Gauss-Bonnet term) the solution of the equation $\Upsilon_i(a, q_m) = 0$ is just $a = a_i$. Specializing the path in the a, q_m plane one can in principle make higher-order coefficients nonsingular, but the procedure is somewhat subtle (a more detailed discussion will follow).

Another characteristic curve in the parameter plane a, q_m is defined by vanishing of the linear term in the expansion of the radial function $\rho(r)$:

$$\Upsilon_1 := (a^2 - 1)q_m^4 + 2\alpha(3a^2 - 2)q_m^2 + 4\alpha^2(a^2 - 1) = 0. \tag{29}$$

This signals a potential singularity of the corresponding global solution (the horizon itself is a turning point for the radial variable). Note that the curve $a(q_m)$ obtained as the solution of the equation $\Upsilon_1 = 0$ reaches the value $a = 1$ for $q_m = 0$, and $q_m \to \infty$, and it has a local minimum $a^2 = 4/5$ for $q_m = \sqrt{2\alpha}$. It is worth noting that the expression for Υ_1 does not follow the general formula (26) for Υ_i (valid for $i \geq 2$).

Therefore, the family of curves $\Upsilon_i = 0$ divide the two-dimensional parameter space of a (vertical axis) and q_m (horizontal axis) of the global solutions for a fixed α into the disconnected regions. This situation will be described in detail in Sec. IV.

The values of the integrals of motion corresponding to the series expansions (24) are

$$H = \frac{1}{2\rho_0^2}[q_m^2 P_0 + q_e^2 P_0^{-1} - \rho_0^2] = 0, \qquad J_\delta = 2q_e f_0, \tag{30}$$

where P_0 is given explicitly in (24) and f_0 is the value of the electrostatic potential on the horizon.

The unique free parameter P_1 in the near-horizon expansions will be fixed by asymptotic flatness. This means that the extremal dyonic solutions are completely characterized by the charges. It is convenient to absorb the Gauss-Bonnet coupling constant α into the redefinition of the parameters $q_e = \hat{q}_e/\sqrt{\alpha}$, $q_m = \sqrt{\alpha}\hat{q}_m$ and $P_1 = \hat{P}_1/\alpha$. Then the near-horizon expansions will read

$$\begin{aligned} w(r) &= \frac{x^2}{\rho_0^2} - \frac{\hat{P}_1}{6a^2\rho_0^4}[3(a^2 - 1)\hat{q}_m^4 + 6(3a^2 - 2)\hat{q}_m^2 \\ &\quad + 4(5a^2 - 3)]x^3 + O(x^4), \\ \rho(r) &= \rho_0 + \frac{\hat{P}_1}{4a^2\rho_0}[(a^2 - 1)\hat{q}_m^4 + 2(3a^2 - 2)\hat{q}_m^2 \\ &\quad + 4(a^2 - 1)]x + O(x^2), \\ \hat{P}(r) &= \alpha e^{2a\phi} = \frac{\rho_0^2}{2(2 + \hat{q}_m^2)} + \hat{P}_1 x + O(x^2), \end{aligned} \tag{31}$$

and the relation (25) becomes

$$\rho_0^2 = \frac{2\hat{q}_e(2 + \hat{q}_m^2)}{\sqrt{4 + \hat{q}_m^2}}. \tag{32}$$

It is important to determine the correct sign of $\hat{P}_1$. To be able to interpret the region $r > r_H$ as an exterior of a black hole, one has to ensure positiveness of the derivative ρ' at the horizon. From the near-horizon expansion of ρ one finds

$$\rho'|_{x=0} = \frac{\hat{P}_1}{4a^2\rho_0}\hat{\Upsilon}_1 > 0, \quad \hat{\Upsilon}_1 := (a^2-1)\hat{q}_m^4 + 2(3a^2-2)\hat{q}_m^2 + 4(a^2-1). \tag{33}$$

Thus, we should take positive $\hat{P}_1$ for $\hat{\Upsilon}_1 > 0$ and negative $\hat{P}_1$ for $\hat{\Upsilon}_1 < 0$. Introducing the sign parameter $\varsigma = \frac{\hat{P}_1}{|\hat{P}_1|}$, we find, therefore,

$$\varsigma = \frac{\hat{\Upsilon}_1}{|\hat{\Upsilon}_1|}. \tag{34}$$

Another useful redefinition is based on the observation that ρ_0 and P_1 enter the expansions in the combination $b = |\hat{P}_1|/(a^2\rho_0^2)$. Consider now the transformations of the expansion parameters under the symmetries of the solution space (20) and (21). It is easy to see that the full set of local solutions can be generated from one particular solution with $\rho_0 = 1$, $b = 1$, which we will call the normalized local solution, by the symmetry transformations with $\delta = -\ln\rho_0$ and $\mu = 2\ln(b\rho_0)$. The normalized local solution does not contain free parameters (for fixed q_m):

$$\begin{aligned} w(r) &= x^2 - \varsigma\frac{1}{6}[3(a^2-1)\hat{q}_m^4 + 6(3a^2-2)\hat{q}_m^2 \\ &\quad + 4(5a^2-3)]x^3 + O(x^4), \\ \rho(r) &= 1 + \varsigma\frac{1}{4}[(a^2-1)\hat{q}_m^4 + 2(3a^2-2)\hat{q}_m^2 \\ &\quad + 4(a^2-1)]x + O(x^2), \\ \hat{P}(r) &= \frac{1}{2(2+\hat{q}_m^2)} + \varsigma a^2 x + O(x^2). \end{aligned} \tag{35}$$

Note the presence of the sign function ς in the odd power terms. The electric charge corresponding to the normalized local solution is given by

$$\hat{q}_e = \frac{\sqrt{4+\hat{q}_m^2}}{4+2\hat{q}_m^2}. \tag{36}$$

B. Reissner-Nordström limit

One case in which the extremal Reissner-Nordström solution is valid is the already mentioned limit $a \to 0$. However, there is another limit in which our local solutions make contact with the Reissner-Nordström solution: $\hat{q}_m \to \infty$. This can be implemented by taking either $q_m \to \infty$ or $\alpha \to 0$. In this limit, the near-horizon expansions (31) reduce to

$$\begin{aligned} w(r) &= \frac{x^2}{\rho_0^2}[1 - 2F_1x + 3F_1^2x^2 - 4F_1^3x^3 + O(x^4)], \\ \rho(r) &= \rho_0\bigg[1 + F_1x + \frac{a^2-2}{2a^2}\frac{1}{\hat{q}_m^4}\left(\frac{\hat{P}_1\hat{q}_m^4}{\rho_0^2}\right)^2 x^2 \\ &\quad + \frac{(a^2-1)(2a^4+19a^2-36)}{24a^4(a^2-6)}\frac{1}{\hat{q}_m^4}\left(\frac{\hat{P}_1\hat{q}_m^4}{\rho_0^2}\right)^3 x^3 \\ &\quad + O(x^4)\bigg], \\ \hat{P}(r) &= \alpha e^{2a\phi} = \frac{\rho_0^2}{2\hat{q}_m^2} + \hat{P}_1\bigg[x - F_1x^2 - \frac{(a^2-1)^2(a^2+4)}{16a^4(a^2-6)} \\ &\quad \times \left(\frac{\hat{P}_1\hat{q}_m^4}{\rho_0^2}\right)^2 x^3 + O(x^4)\bigg], \end{aligned} \tag{37}$$

where

$$F_1 = \frac{(a^2-1)\hat{P}_1\hat{q}_m^4}{4a^2\rho_0^2}. \tag{38}$$

This is valid for all values of a except $a = a_i$ located on the curves of $\Upsilon_i = 0$ when $\hat{q}_m \to \infty$. To ensure the $AdS_2 \times S^2$ structure of the horizon and the regularity of the expansions we assume ρ_0 to be finite and impose the condition $F_1 = \text{const} \neq 0$. Since $\rho_0^2 = 2\hat{q}_e\hat{q}_m$, this implies $\hat{P}_1 \sim \sqrt{\alpha}\hat{q}_m^{-3} \to 0$. Then, the expansions simplify and can be summed up to the closed expressions

$$w(r) = \frac{x^2}{\rho_0^2}(1+F_1x)^{-2}, \qquad \rho(r) = \rho_0(1+F_1x), \qquad \hat{P}(r) = \hat{P}_0. \tag{39}$$

Now, the asymptotic flatness means $F_1 = 1/\rho_0$, and finally assuming $r_H = \rho_0$ we will have $x = r - \rho_0$ which leads to the Reissner-Nordström dyonic black hole

$$w(r) = \left(1 - \frac{\rho_0}{r}\right)^2, \qquad \rho(r) = r, \qquad \hat{P}(r) = \hat{P}_0 = \alpha e^{a\phi_h}, \tag{40}$$

with the electric and magnetic charge parameters $\hat{q}_e = \sqrt{\frac{\hat{P}_0}{2}}\rho_0$ and $\hat{q}_m = \rho_0/\sqrt{2\hat{P}_0}$ (corresponding to the equal dilaton-rescaled charges $Q_e = Q_m = \rho_0/\sqrt{2}$).

Note again that there are two different ways to implement the limit $\hat{q}_m \to \infty$: either $\alpha \to 0$, $\hat{P}_0 \to 0$ and the radius of the horizon remaining finite $\rho_0 = 2\hat{q}_e\hat{q}_m = 2q_eq_m$, or $q_m \to \infty$, implying the infinitely large radius of the horizon.

C. Relation to EMD dyons

The dyon solutions of the EMD theory without curvature corrections for a generic a exhibit one horizon and do not admit an extremal limit. For a discrete sequence $a = a_i$ there are solutions with two horizons which may have such limits [48]. This sequence is the same as found above from a different reasoning. We expect to have a relationship between our solution and those of the Ref. [48] (in the extremal limit) when $\hat{q}_m \to \infty$ (equivalent to $\alpha \to 0$). However, the situation is more subtle. Restarting with the set of equations of motion for $\alpha = 0$ and considering the series solution near the event horizon, we obtain in the lowest order:

$$P_0 = \frac{q_e}{q_m}, \qquad \rho_0^2 = 2q_e q_m. \tag{41}$$

This corresponds to the limiting form of the coefficients in the EDGB theory. Analyzing the higher-order equations for the expansions' coefficients we find the following. In general ρ_1 is a free parameter (which has to be fixed by the asymptotic conditions) while all w_k and ρ_k with $k \geq 2$ are completely determined by the equations order by order. However, when one tries to solve the equations with respect to P_k for $k \geq 1$, an interesting bifurcation behavior is observed. There are two possible cases: either $a^2 = 1$ (then P_1 is a free parameter) or $P_1 = 0$ (we leave aside the special case $a = 0$.) In the first case, all higher-order P_k are fixed by the equations. In the second case we observe another bifurcation: either $a^2 = 3$ (and P_2 is then free) or $P_2 = 0$. Again, in the first case $P(r)$ is fixed by P_2, but in the second case we have a further bifurcation: either $a^2 = 6$ or $P_3 = 0$. The analogous bifurcations exist in any order. This branching procedure reproduces the value $a = a_i$ at the i-th step. This indicates that the extremal dyonic black holes can exist only for this discrete sequence.

Thus, for any $i \geq 1$, there are two independent parameters ρ_1, P_i in the local solution (ρ_0, P_0 being related to the charge parameters), and the expansion coefficients ρ_j, $j = 2, \ldots, 2i - 1$ and P_j, $j = 1, \ldots, i - 1$ are *all zero*. Moreover, the expansion of $w(r)$ differs from the corresponding expansion of the Reissner-Nordström solution ($a = 0$)

$$\begin{aligned} w_{\mathrm{RN}}(r) &= \frac{x^2}{(\rho_0 + \rho_1 x)^2} \\ &= \frac{1}{\rho_0^2}x^2 - \frac{2\rho_1}{\rho_0^3}x^3 + \frac{3\rho_1^2}{\rho_0^4}x^4 - \frac{4\rho_1^3}{\rho_0^5}x^5 + \frac{5\rho_1^4}{\rho_0^6}x^6 \\ &\quad + O(x^7), \end{aligned} \tag{42}$$

only starting with the term w_{2i+2}.

In view of such a behavior, to reach the series expansions arising in the EMD theory, which could be expected to arise in the limit $\alpha = 0$ of the EDGB model, is somewhat problematic. First, in the EDGB case there is only one free parameter, P_1, while there are *two* parameters, ρ_1 and P_i, in the EMD theory. So the solutions emerging in the limit $\hat{q}_m \to \infty$, if they exist, can contact the corresponding solutions in the EMD theory only for a special value of ρ_1. Second, the limit $a \to a_i$, $\hat{q}_m \to \infty$ in the parameter space depends on the direction chosen. In particular, one gets essentially different results taking first $a \to a_i$ and then $\hat{q}_m \to \infty$, or first $\hat{q}_m \to \infty$ and then $a \to a_i$. And both of these two do not seem to give the result of the EMD theory.

We refer the reader to Appendix A for details concerning the cases of lower values of the sequence a_i for which a closed form summation is possible.

D. Entropy and temperature

Following the Sen's entropy function approach, the entropy of extremal dyonic black holes can be calculated straightforwardly:

$$S = 2\pi q_e \sqrt{q_m^2 + 4\alpha} = \pi\rho_0^2 + \frac{2\pi\alpha\rho_0^2}{2\alpha + q_m^2}. \tag{43}$$

Technical details are similar to those in the pure electric case treated in [1]. There are two interesting limits of the above expression. First, the Bekenstein-Hawking entropy-area relation, $S = A/4$, $A = 4\pi\rho_0^2$, is recovered when $\alpha = 0$ or $q_m \to \infty$. Second, if the magnetic charge parameter q_m is vanishing, we recover the result obtained for the pure electric case [1,67], namely, the double Bekenstein-Hawking value. Such an enhancement of the entropy for small black holes was discussed in Ref. [68]. Note that for a generic extremal dyonic solution, the black hole entropy can not be expressed through the horizon area only.

It is worth noting that although the value of the entropy can be calculated using only the local solution valid in the vicinity of the $\mathrm{AdS}_2 \times S^2$ event horizon, its interpretation as the entropy of a black hole presumes an existence of the global solutions extending to infinity. We will see later on that this imposes certain restrictions on the values of the magnetic charge and the dilaton coupling constant a. A purely local analysis is therefore insufficient for drawing conclusions about the correspondence between the string and the geometric values of the entropy.

The temperature of the extremal EDGB black hole is zero, as for the extremal solution without the Gauss-Bonnet term:

$$T = \frac{1}{2\pi}\left(\sqrt{g^{rr}}\frac{\partial\sqrt{g_{tt}}}{\partial r}\right)\Bigg|_{r=r_H} = \frac{1}{2\pi\rho_0^2}(r - r_H)|_{r=r_H} = 0. \tag{44}$$

IV. ASYMPTOTIC BEHAVIOR

Another region where the local solutions can be constructed analytically is the asymptotic zone $r \to \infty$. Similar to the case on the uncorrected EMD theory, we find that the black holes of two types can exist: the usual asymptotically flat solutions, and the black holes on the linear dilaton background [65]. In the latter case the dilaton diverges at infinity (linearly for a proper choice of the radial coordinate), but the ADM mass of the black hole itself is finite, so that the solution can be interpreted as the black hole on the linear dilaton background.

A. Asymptotically flat solutions

Looking for asymptotically flat global solutions we have to ensure $w \to 1$, $\rho/r \to 1$, $\phi \to \phi_\infty$ (constant) as $r \to \infty$. The subleading terms should be expandable in the power series of $1/r$. The asymptotic solution with these properties contains five parameters: the ADM mass M, the electric and magnetic charges Q_e, Q_m (rescaled), the dilaton charge D, and the asymptotic value of the dilaton ϕ_∞:

$$\begin{aligned} w(r) &= 1 - \frac{2M}{r} + \frac{\alpha Q_e^2 + \alpha^{-1} Q_m^2}{r^2} + O(r^{-3}), \\ \rho(r) &= r - \frac{D^2}{2r} - \frac{D(2MD - \alpha a Q_e^2 + \alpha^{-1} a Q_m^2)}{3r^2} \\ &\quad + O(r^{-3}), \\ \phi(r) &= \phi_\infty + \frac{D}{r} + \frac{2DM - \alpha a Q_e^2 + \alpha^{-1} a Q_m^2}{2r^2} + O(r^{-3}), \end{aligned} \tag{45}$$

where

$$Q_e = q_e e^{-a\phi_\infty}, \qquad Q_m = q_m e^{a\phi_\infty}. \tag{46}$$

The dilaton charge can be also read off from an asymptotic expansion of the dilaton exponential:

$$\begin{aligned} e^{2a(\phi-\phi_\infty)} &= 1 + \frac{2aD}{r} \\ &\quad + \frac{2aD(aD+M) - \alpha a^2 Q_e^2 + \alpha^{-1} a^2 Q_m^2}{r^2} \\ &\quad + O(r^{-3}). \end{aligned} \tag{47}$$

The values of two integrals of motion in terms of the asymptotic parameters are

$$H = \frac{1}{2}(w_\infty \rho_\infty'^2 - 1), \qquad J_\delta = 2q_e f_\infty - M - \frac{D}{a}. \tag{48}$$

The constant f_∞ in the electric potential is usually fixed to be zero.

Behavior of the global solution which starts with the normalized local solution (35) at the horizon depends only on the dilaton coupling constant a and the magnetic charge parameter $q_m = \sqrt{\alpha}\hat{q}_m$. Its existence for all a is not guaranteed *a priori*. But, in some intervals of a whose boundaries depend on q_m, as can be shown numerically, there exist solutions varying smoothly with increasing x such that the function w and the derivative ρ' stabilize at infinity on some constant values $w_\infty \neq 1$, $\rho'_\infty \neq 1$. Then, using the symmetries (20) and (21) of the solution space, one can rescale the global solution obtained to achieve the desired unit values for these parameters. As we have argued, two parameters μ, δ effectively replace the parameters ρ_0, P_1 of the (non-normalized) local solution (24) or (31). So one could expect that the rescaling of the solution ensuring the asymptotic conditions $w_\infty = 1$, $\rho'_\infty = 1$ would fix both quantities ρ_0, P_1 on the horizon. But from the Hamiltonian constraint equation $H = 0$ with H given by the Eq. (48) it is clear that one must have $w_\infty \rho_\infty'^2 = 1$ for any asymptotically flat solution. Therefore it is enough to perform *one* but not *two* independent rescalings in order to achieve $w_\infty = 1$, $\rho'_\infty = 1$. Indeed, under the transformation (20) and (21), the relevant functions and parameters are transformed as follows:

$$\begin{aligned} &w \to w e^{\mu}, \qquad \rho' \to \rho' e^{\mu/2}, \qquad w\rho'^2 \to w\rho'^2; \\ &\rho_0 \to \rho_0 e^{\delta}, \qquad P_1 \to P_1 e^{\delta - \mu/2}. \end{aligned} \tag{49}$$

Since the choice of μ, δ is equivalent to the choice of ρ_0, P_1, an invariance of the product $w\rho'^2$ means that the solution starting on the horizon with *any* ρ_0, P_1 will reach at infinity the values w_∞, ρ'_∞ satisfying $w_\infty \rho_\infty'^2 = 1$. Therefore, taking $\mu = -\ln w_\infty$, we will achieve simultaneously $w_\infty = 1$ and $\rho'_\infty = 1$. This means that the asymptotically flat solutions still form a two-parameter family, two parameters being the electric charge q_e and the magnetic charge q_m. Five asymptotic parameters M, D, ϕ_∞, Q_e, Q_m are functions of q_e, q_m which can be found numerically.

B. Black holes on the linear dilaton background

There is another type of physically interesting black hole solution which asymptotically approaches the linear dilaton background [65]. For such black holes, some metric functions diverge asymptotically, so in order to be able to recognize them numerically we have to pass to some conformally rescaled metric [69] (the dual frame),

$$ds^2_{\rm dual} = e^{-2a(\phi-\phi_\infty)} ds^2, \tag{50}$$

which has the following explicit form in our case:

$$ds^2_{\rm dual} = P^{-1}\left(-w dt^2 + \frac{dr^2}{w} + \rho^2 d\Omega^2\right). \tag{51}$$

In the dual frame the asymptotic metric for LDB solutions could be either $M_2 \times S^2$ (only for a special value of the dilation coupling, $a^2 = 1$ for the four-dimensional theory) or $AdS_2 \times S^2$ [69,70]. Therefore, if we rewrite the metric in the following form:

$$ds^2_{\rm dual} = -\frac{w}{P} dt^2 + \frac{P}{w} du^2 + P^{-1}\rho^2 d\Omega^2, \tag{52}$$

where $du = dr/P$, then the metric functions should have the following asymptotic limit (determining the radii of the S^2):

$$P^{-1}\rho^2 \sim R_0, \qquad \frac{w}{P} \sim \tilde{R}_0^2 u^2, \tag{53}$$

where R_0 and $\tilde{R}_0$ denote the curvature radii of the S^2 and AdS_2 respectively. The differentiation of the second equation gives the relation

$$\left(\frac{P}{w}\right)^{1/2} w' - \left(\frac{w}{P}\right)^{1/2} P' \sim \tilde{R}_0, \tag{54}$$

which can be used in the numerical procedure.

V. NUMERICAL ANALYSIS

In this section we present the numerical results for the global extremal dyonic black hole solutions. We extend the local solutions constructed via the series expansions near the horizon to the asymptotic region by a numerical integration. As expected, the free parameter P_1 turns out to be fixed by the asymptotic conditions, flatness or LDB.

A. Asymptotically flat dyons

For the pure electric extremal black holes [1], the global solutions were found to exist only in a limited range of the dilaton couplings less than the critical value $a_{\rm cr}$. By turning on the magnetic charge, the range of a for domain of existence can be extended from the single interval $0 \leq a < a_{\rm cr}$ to an infinite sequence of disconnected intervals, located between the limiting curves $\Upsilon_i = 0$ in the two-dimensional parameter plane a, q_m. More precisely, the regular solutions exist for a satisfying the inequality

$$a_i^-(q_m) < a < a_i^+(q_m), \tag{55}$$

where a_i^- is located above the curve $\Upsilon_{i-1} = 0$ and a_i^+ below the curve $\Upsilon_i = 0$. The domains of existence obtained numerically are shown in Fig. 1. The most surprising feature of this plot is that in the limit $q_m \to \infty$, which is equivalent to turning off the Gauss-Bonnet term, the domain of existence of the dyon solutions does not reduce to the discrete sequence $a = a_i$. As we have seen, already the series solutions in the near-horizon region are essentially different in the EDGB model and in the curvature uncorrected EMD theory: the former being one-parametric, while the latter—two-parametric. Transition between these

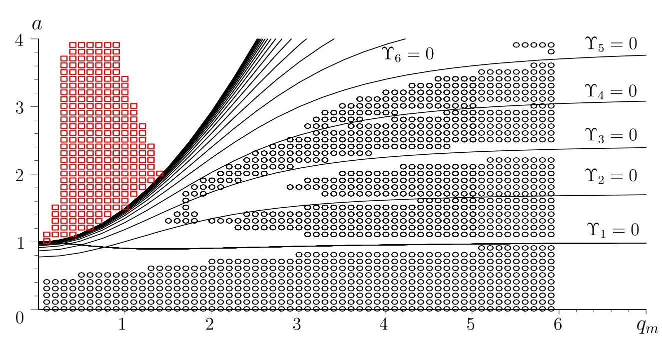

FIG. 1 (color online). The domains of existence of EDGB dyons in the two-dimensional parameter plane a, q_m for $q_m \leq 6$. The family of the limiting curves corresponds to the solutions of the equations $\Upsilon_i = 0$ implying singularities of the higher-order coefficients in the horizon series. Black circles correspond to asymptotically flat solutions, red squares to black holes on the linear dilaton background.

two series solutions is unclear, the limiting form of the series in the EDGB model depends on the choice of direction in the a, q_m plane in which the limit $q_m \to \infty$ is taken. Numerical solutions were checked to exist up to some large values of q_m, that is, for a small Gauss-Bonnet coupling. They still exist for continuously varied a not for the discrete values as in the pure EMD theory. Therefore the curvature-corrected theory gives qualitatively different predictions for dyons even when the Gauss-Bonnet coupling is small.

Our numerical results reveal the following behavior of the dyon solutions for large q_m (or small Gauss-Bonnet coupling). If we fix q_m and then take the limit $a \to a_i^\pm$ for the dilaton coupling, the parameter P_1 (which is fixed by the asymptotic flatness conditions $w_\infty \to 1$, $\rho'_\infty \to 1$) diverges, and the solution is ill-defined. On the other hand, if we fix the value $a \neq a_i$ of the dilaton coupling and then take the limit $q_m \to \infty$, the parameter P_1 goes to zero. In this case, the limiting solution $q_m \to \infty$ corresponds to the dyonic *Reissner-Nordström* extremal black hole (with the frozen dilaton). Such solutions, however, do not exist in the pure EMD theory. Moreover, the subsequent limit $a^2 \to a_i^2$ always gives the dyonic Reissner-Nordström solution. Therefore, the extremal dyons in the pure EMD theory look like a set of discrete points in the two-dimensional continuum which is difficult to resolve. Nevertheless, the limiting value of the radius of the horizon is the same both in the EDGB and the EMD theories (see also Appendix A).

Using the symmetry of the solution space under the δ-transformation (19), one can generate the sequence of solutions with different electric charges q_e and correspondingly with different masses, dilaton charges and the asymptotic values of the dilaton ϕ_∞. Since variation of the electric charge is essentially equivalent to variation of the unique parameter ρ_0 (for fixed q_m) in the horizon expansion, it is clear that using the δ-transformation we will generate *all* extremal solutions. Under this transformation the mass and the dilaton charge scale as e^{δ}, while the electric charge and the dilaton exponent $e^{2a\phi_\infty}$ scale as $e^{2\delta}$. Therefore the ratios,

$$k_M = \frac{M^2}{q_e}, \qquad k_D = \frac{D^2}{q_e}, \qquad k_\phi = \frac{e^{2a\phi_\infty}}{q_e}, \tag{56}$$

depend only on a. Their numerical plots are presented in Fig. 2.

With growing magnetic charge, two end points of the allowed interval of a tend to the boundary curves. When the dilaton coupling approaches the critical values (the end points $a_i^\pm$ of each regularity interval with fixed q_m), the global physical quantities such as the mass M, the dilaton charge D, the electric charge $Q_e = q_e e^{-a\phi_\infty}$ and the magnetic charge $Q_m = q_m e^{a\phi_\infty}$ diverge and the only free parameter P_1 also goes to infinity (in order to ensure an asymptotic flatness). However, the ratio of the following physical quantities

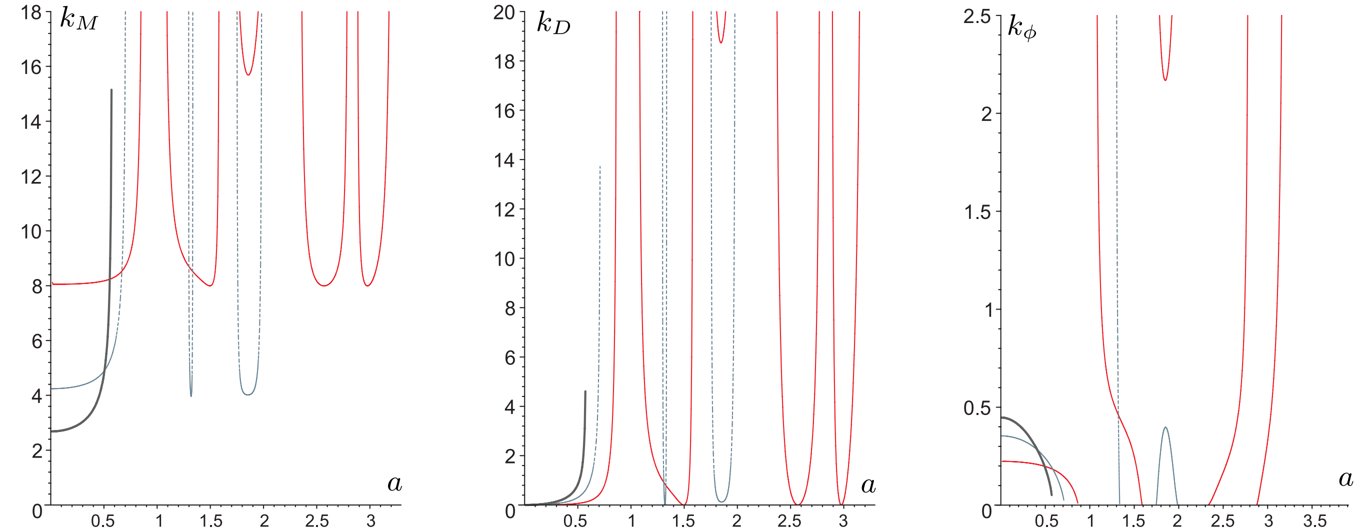

FIG. 2 (color online). Asymptotic parameters M, D, ϕ_∞ in terms of the horizon parameters q_e, q_m as given by the ratios $k_M = \frac{M^2}{q_e}$, $k_D = \frac{D^2}{q_e}$ and $k_\phi = \frac{e^{2a\phi}}{q_e}$ depending on q_m. These ratios are shown as functions of the dilaton coupling constant a for some values of the magnetic charge: $q_m = 1$ (black, thick), $q_m = 2$ (gray, dashed) and $q_m = 4$ (red, thin).

$$k_{\rm BPS} = \frac{1 + a^2}{2a^2} \frac{a^2M^2 + D^2}{Q_e^2 + Q_m^2}, \tag{57}$$

has a simple limit $k_{\rm BPS} \to 1$ (Fig. 3). Remarkably, as $a \to a_i^\pm$, the associated global parameters tend to satisfy the following relation:

$$a^2M^2 + D^2 = \frac{2a^2}{1 + a^2}(Q_e^2 + Q_m^2), \tag{58}$$

which coincides with the BPS condition for charged black holes in the EMD theory without curvature corrections. One can also see that in this limit the discrete S-duality of the EMD theory is restored. This feature is similar to that observed in another stringy generalization of the EMD theory in which the Maxwell action is replaced by the Born-Infeld action but no Gauss-Bonnet term is introduced [71].

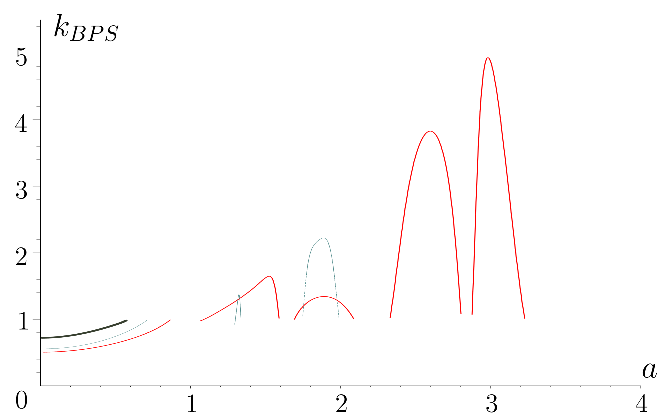

FIG. 3 (color online). The ratios $k_{\rm BPS} = \frac{(1+a^2)(a^2M^2+D^2)}{2a^2(Q_e^2+Q_m^2)}$ as the functions of a for some values of the magnetic charge: $q_m = 1$ (black, thick), $q_m = 2$ (gray, dashed) and $q_m = 4$ (red, thin). One can see the pseudo-BPS behavior at the end points of the domains of existence.

B. Magnetic black holes on the LDB background

The domain of existence of black holes on the linear dilaton background satisfying the condition (53) is located to the left from the condensing family of limiting curves $\Upsilon_i = 0$ on the parameter plane a, q_m in Fig. 1. The boundary curve is given by $\Upsilon_\infty = 0$ or, explicitly,

$$q_m^4 + 4\alpha q_m^2 - 4\alpha^2(a^2 - 1) = 0. \tag{59}$$

Domains of existence of all asymptotic flat solutions are located to the right from this curve while those with the LDB asymptotics—to the left. LDB dyons do not exist for dilaton couplings less than one. Moreover, for larger magnetic charges, the asymptotically LDB solutions require a larger value of the dilaton coupling.

From an analysis of similar solutions in the EMD theory [70] it follows that one of the charges of the dyonic configuration defines the strength of the background electric or magnetic fields in the LDB, while another is associated with the charge of a black hole. So one can have electric black holes on the magnetic LDB or magnetic black holes on the electric LDB. The asymptotic form of the metric is

$$w \sim \frac{r}{\xi_0}, \qquad \rho \sim (\xi_0 r)^{1/2}, \tag{60}$$

where ξ_0 is the scaling parameter associated with the field strength of the background, and the dilaton for the electric black hole on a magnetic background is

$$P = e^{2a\phi} \sim \frac{r}{\xi_0}, \tag{61}$$

while for the magnetic black hole on an electric background

$$P = e^{2a\phi} \sim \frac{\xi_0}{r}. \tag{62}$$

The solutions presented in this paper are consistent with the dilaton field (62) and the scaling parameters $\mu = -2\delta = -\ln\xi_0$. Thus, our solutions can be interpreted as magnetically charged black holes on the electric LDB (recall that the discrete S-duality is broken in the EDGB theory). In the case of the vanishing magnetic charge they are physically expected to reduce to the pure electric LDB without a black hole. Technically, however, our solutions cannot have such a limit since we have assumed the existence of the horizon *a priori*.

VI. DISCUSSION

In this paper, we have shown that the EDBG four-dimensional gravity admits extremal dyonic black hole solutions with the horizon of the $AdS_2 \times S^2$ type. Somewhat surprisingly, adding the Gauss-Bonnet term to the Einstein-Maxwell-dilaton theory leads to an enhancement of the domain of parameters for which the global solutions exist. Namely, the asymptotically flat dyon solutions in the EMD theory exist only for a discrete sequence of the dilaton coupling constant values, while in the model with the Gauss-Bonnet term the continuously varying parameters are allowed. An effective parameter space is a two-dimensional plane which is split into the sequence of regions separated by the limiting curves, marking singularities of the coefficients of the local power series solutions. These curves are related to the above discrete sequence of parameters of the EMD dyons, and when approaching them, the solutions of the EDGB model exhibit saturation features similar to the BPS conditions of the EMD theory.

The relationship between the extremal EDGB dyons and those in the EMD theory is nontrivial. Dyon solutions of the EDGB theory exist only with nonzero electric charge, and for large values of the magnetic charge they tend to the Reissner-Nordström solution with a frozen dilaton and not to the discrete family of extremal dyons in the EMD theory as could be expected. Therefore, the pure EMD theory without curvature corrections predicts different black hole solutions than the corresponding curvature-corrected theory in the limit of vanishing curvature coupling. Remarkably, the latter limiting theory predicts the same dependence of the horizon radius on charges, as the pure EMD theory.

The entropy of the extremal EDGB dyons interpolates between the Bekenstein-Hawking value in the limit of the large magnetic charges (equivalent to the vanishing curvature coupling) and the doubled Bekenstein-Hawking value in the limit of purely electric solutions. The entropy can be calculated using only local analytical solutions valid in the vicinity of the event horizon. The expression obtained, however, does not bear any sign of the bounds on the parameters for which the global black hole solutions exist. Such bounds can be revealed once we try to extend the local solutions to infinity as asymptotically flat ones. This lesson is worth keeping in mind in the discussion of the entropy of the curvature-corrected black holes in string theory. Purely local analysis of classical solutions is still insufficient to fully understand the entropy of black holes.

We also found the second family of the EDGB dyons which asymptotically approach the linear dilaton background. Our solutions may be interpreted as the magnetically charged extremal black holes on the electric linear dilaton background. For them, the metric is not asymptotically flat, but the value of the black hole mass and the other parameters can be extracted by subtracting the corresponding background values. In string theory such black holes correspond to thermalized states of the quantum theory through the quantum field theory/domain wall correspondence.

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APPENDIX A: EXTREMAL DYONS IN 4D EINSTEIN-MAXWELL-DILATON THEORY

The action and the equations of motion of the four-dimensional Einstein-Maxwell-dilaton theory with arbitrary coupling a are obtained setting $\alpha = 0$ in Eqs. (9)–(11). Consider again the series expansions in the form (24) around the horizon $r = r_H$ in terms of the deviation $x = r - r_H$. In the lowest order we obtain

$$P_0 = \frac{q_e}{q_m}, \qquad \rho_0^2 = 2q_e q_m, \tag{A1}$$

which coincides with the $\alpha \to 0$ limit of the corresponding EDGB relations. However, the computation of the higher-order coefficients shows that one deals with the local solution containing *two* free parameters (with the fixed electric and magnetic charges), which can be conveniently chosen as ρ_1 and P_1. Recall, that the corresponding local solution in the EDGB model constructed in Sec. III contained only one free parameter P_1. Another new feature is the presence of bifurcations in the higher-order expansion coefficients as was described in Sec. III. These bifurcations give rise to a sequence of discrete values of the dilaton coupling constant $a_i^2 = i(i+1)/2$ for which one obtains the nontrivial expansions. For each i the series expansion for the dilaton function starts with the term of the i-th order, P_i, with all lower coefficients being zero, $P_{j<i} = 0$ (except for P_0). This discrete sequence was also observed in the Ref. [48] where it was shown that the asymptotically flat solutions exhibit *two horizons* if $a = 0, 1, \sqrt{3}, \sqrt{6}, \cdots, \sqrt{n(n+1)/2}$, and one horizon otherwise. Since the solutions with the double zero of the metric function g_{tt} at the horizon can be obtained only starting with the nonextremal solutions with two horizons, this is consistent with the special values we have found here for genuinely extremal solutions. Note that only in the first two cases $a = 1$ and $a = \sqrt{3}$ the exact dyonic solutions have been obtained analytically. In what follows we show how this can be done via direct summation of the series solutions.

1. $a^2 = 1$

The series solution reads

$$w(r) = \frac{1}{\rho_0^2}x^2 - \frac{2\rho_1}{\rho_0^3}x^3 + \left(\frac{3\rho_1^2}{\rho_0^4} + \frac{P_1^2 q_m^4}{\rho_0^6}\right)x^4 - \left(\frac{4\rho_1^3}{\rho_0^5} + \frac{4\rho_1 P_1^2 q_m^4}{\rho_0^7}\right)x^5 + \left(\frac{5\rho_1^4}{\rho_0^6} + \frac{10\rho_1^2 P_1^2 q_m^4}{\rho_0^8} + \frac{P_1^4 q_m^8}{\rho_0^{10}}\right)x^6 - \left(\frac{6\rho_1^5}{\rho_0^7} + \frac{20\rho_1^3 P_1^2 q_m^4}{\rho_0^9} + \frac{6\rho_1 P_1^4 q_m^8}{\rho_0^{11}}\right)x^7 + O(x^8), \tag{A2}$$

$$\rho(r) = \rho_0 + \rho_1 x - \frac{P_1^2 q_m^4}{2\rho_0^3}x^2 + \frac{\rho_1 P_1^2 q_m^4}{2\rho_0^4}x^3 - \left(\frac{\rho_1^2 P_1^2 q_m^4}{2\rho_0^5} + \frac{P_1^4 q_m^8}{8\rho_0^7}\right)x^4 + \left(\frac{\rho_1^2 P_1^2 q_m^4}{2\rho_0^6} + \frac{3\rho_1 P_1^4 q_m^8}{8\rho_0^8}\right)x^5 - \left(\frac{\rho_1^4 P_1^2 q_m^4}{2\rho_0^7} + \frac{6\rho_1^2 P_1^4 q_m^8}{8\rho_0^9} + \frac{P_1^6 q_m^{12}}{16\rho_0^{11}}\right)x^6 + O(x^7), \tag{A3}$$

$$P(r) = P_0 + P_1 x - P_1\left(\frac{\rho_1}{\rho_0} - \frac{P_1 q_m^2}{\rho_0^2}\right)x^2 + P_1\left(\frac{\rho_1}{\rho_0} - \frac{P_1 q_m^2}{\rho_0^2}\right)^2 x^3 - P_1\left(\frac{\rho_1}{\rho_0} - \frac{P_1 q_m^2}{\rho_0^2}\right)^3 x^4 + O(x^5). \tag{A4}$$

The summation is performed by collecting terms of different order in ρ_1 inside the series in terms of $P_1^2 q_m^4$. For example, for $\rho(r)$ one can rewrite the series as

$$\rho(r) = \rho_0 + \rho_1 x - \frac{P_1^2 q_m^4}{2\rho_0^3}x^2\left(1 - \frac{\rho_1}{\rho_0}x + \frac{\rho_1^2}{\rho_0^2}x^2 - \cdots\right) + \frac{P_1^4 q_m^8}{8\rho_0^7}x^4\left(1 - 3\frac{\rho_1}{\rho_0}x + 6\frac{\rho_1^2}{\rho_0^2}x^2 - \cdots\right) + \cdots$$
$$= (\rho_0 + \rho_1 x)\left(1 - \frac{P_1^2 q_m^4}{2\rho_0^2}\frac{x^2}{(\rho_0 + \rho_1 x)^2} + \frac{P_1^4 q_m^8}{8\rho_0^4}\frac{x^4}{(\rho_0 + \rho_1 x)^4} - \cdots\right) = \sqrt{(\rho_0 + \rho_1 x)^2 - \hat{P}_1^2 q_m^4 x^2}. \tag{A5}$$

A similar pattern can be found in the series expansion for $w(r)$, the result being as simple as

$$w(r) = \frac{x^2}{\rho^2(r)}. \tag{A6}$$

For $P(r)$, introducing a new parameter $\bar{P}_1$ defined via $P_1 = \bar{P}_1 \rho_0$ one gets just a geometric recurrence:

$$P(r) = P_0 + P_1 x\left[1 - \left(\frac{\rho_1}{\rho_0} - \frac{P_1 q_m^2}{\rho_0^2}\right)x + \left(\frac{\rho_1}{\rho_0} - \frac{P_1 q_m^2}{\rho_0^2}\right)^2 x^2 - \left(\frac{\rho_1}{\rho_0} - \frac{P_1 q_m^2}{\rho_0^2}\right)^3 x^3 + \cdots\right] \tag{A7}$$

$$= P_0 + \frac{\rho_0^2 \bar{P}_1 x}{(\rho_0 + \rho_1 x) - \bar{P}_1 q_m^2 x}$$
$$= P_0 \frac{(\rho_0 + \rho_1 x) + \bar{P}_1 q_m^2 x}{(\rho_0 + \rho_1 x) - \bar{P}_1 q_m^2 x}. \tag{A8}$$

Thus, we have obtained an exact solution in a closed form.

2. $a^2 = 3$

This case corresponds to the Kaluza-Klein theory. The series expansions are

$$w(r) = \frac{1}{\rho_0^2}x^2 - \frac{2\rho_1}{\rho_0^3}x^3 + \frac{3\rho_1^2}{\rho_0^4}x^4 - \frac{4\rho_1^3}{\rho_0^5}x^5 + \left(\frac{5\rho_1^4}{\rho_0^6} + \frac{2P_2^2 q_m^4}{9\rho_0^6}\right)x^6 - \left(\frac{6\rho_1^5}{\rho_0^7} + \frac{4\rho_1 P_2^2 q_m^4}{3\rho_0^7}\right)x^7 + O(x^8), \tag{A9}$$

$$\rho(r) = \rho_0 + \rho_1 x - \frac{P_2^2 q_m^4}{9\rho_0^3} x^4 + \frac{\rho_1 P_2^2 q_m^4}{3\rho_0^4} x^5 - \frac{2\rho_1^2 P_2^2 q_m^4}{3\rho_0^5} x^6 + O(x^7), \tag{A10}$$

$$P(r) = P_0 + P_2 x^2 - \frac{2\rho_1 P_2}{\rho_0} x^3 + \frac{P_2(P_2 q_m^2 + 3\rho_1^2)}{\rho_0^2} x^4 + O(x^5). \tag{A11}$$

Rearranging an expansion for $\rho(r)$ in the same way we obtain the exact sum

$$\rho(r) = \left[(\rho_0 + \rho_1 x)^4 - \frac{4}{9} P_2^2 q_m^4 x^4\right]^{1/4}. \tag{A12}$$

The function w is given again by $w(r) = x^2/\rho^2(r)$. However, it is more difficult to sum up the series expansion for the dilaton function $P(r)$. An easier way to find $P(r)$ is to solve the equations of motion directly using the above results for $\rho(r)$ and $w(r)$. This gives

$$P(r) = P_0\left[\frac{(\rho_0 + \rho_1 x)^2 + \frac{2}{3} P_2 q_m^2 x^2}{(\rho_0 + \rho_1 x)^2 - \frac{2}{3} P_2 q_m^2 x^2}\right]^{3/2}. \tag{A13}$$

Again, we have obtained an exact solution from the near-horizon expansion.

3. $a^2 = 6$ and beyond

One can try to repeat the same procedure for higher a_i. In particular, for $a^2 = 6 (i = 3)$ one has the following series expansions:

$$w(r) = \frac{1}{\rho_0^2} x^2 - \frac{2\rho_1}{\rho_0^3} x^3 + \frac{3\rho_1^2}{\rho_0^4} x^4 - \frac{4\rho_1^3}{\rho_0^5} x^5 + \frac{5\rho_1^4}{\rho_0^6} x^6 - \frac{6\rho_1^5}{\rho_0^7} x^7 + \left(\frac{7\rho_1^6}{\rho_0^8} + \frac{P_3^2 q_m^4}{10\rho_0^6}\right) x^8 - \left(\frac{8\rho_1^7}{\rho_0^9} + \frac{4\rho_1 P_3^2 q_m^4}{5\rho_0^7}\right) x^9 + O(x^{10}), \tag{A14}$$

$$\rho(r) = \rho_0 + \rho_1 x - \frac{P_3^2 q_m^4}{20\rho_0^3} x^6 + \frac{\rho_1 P_3^2 q_m^4}{4\rho_0^4} x^7 + O(x^8),$$

$$P(r) = P_0 + P_3 x^3 - \frac{3P_3\rho_1}{\rho_0} x^4 + \frac{6P_3\rho_1^2}{\rho_0^2} x^5 + O(x^6).$$

Note, that the function $w(r)$ differs from the Reissner-Nordström $w(r)$ only in terms of the order x^8 and higher (for higher i from the terms of the order $2i + 2$). For the dilaton function an expansion starts from the cubic term (generally, from the i-th terms). But we were not able to find a closed form for these expansions as a whole.

APPENDIX B: HIGHER-ORDER TERMS IN THE LOCAL SOLUTION NEAR THE HORIZON

Here we list some higher-order coefficients in the near-horizon expansions for extremal black holes in the EDGB theory considered in Sec. III. The parameters are ρ_0 which is determined by the electric and magnetic charges, and P_1 (free) which must be fixed by asymptotic conditions:

$$P_0 = \frac{\rho_0^2}{2(2\alpha + q_m^2)}, \tag{B1}$$

$$P_2 = \frac{P_1^2}{4\alpha\rho_0^2 a^2[(a^2-3)q_m^4 + 6\alpha(a^2-2)q_m^2 + 4\alpha^2(5a^2-3)]}[-(a^2-3)(a^2-1)q_m^8 + 4\alpha(5a^2-6)q_m^6 + 4\alpha^2(5a^4+3a^2-18)q_m^4 - 32\alpha^3(2a^2+3)q_m^2 + 16\alpha^4(a^4-5a^2-3)], \tag{B2}$$

$$P_3 = \frac{P_1^3}{48\alpha^2 a^4\rho_0^4[(a^2-3)q_m^4 + 6\alpha(a^2-2)q_m^2 + 4\alpha^2(5a^2-3)][(a^2-6)q_m^4 + 6\alpha(a^2-4)q_m^2 + 4\alpha^2(8a^2-6)]}$$
$$\times[-3(a^2-1)^2(a^2-3)(a^2+4)q_m^{16} - 6\alpha(a^2-1)(24a^6 - 91a^4 - 9a^2 + 96)q_m^{14}$$
$$- 4\alpha^2(317a^8 - 1828a^6 + 2444a^4 + 45a^2 - 1008)q_m^{12} - 8\alpha^3(483a^8 - 3006a^6 + 6283a^4 - 2145a^2 - 2016)q_m^{10}$$
$$- 16\alpha^4(355a^8 - 960a^6 + 6647a^4 - 5625a^2 - 2520)q_m^8 + 32\alpha^5(165a^8 - 1762a^6 - 1844a^4 + 6975a^2 + 2016)q_m^6$$
$$+ 64\alpha^6(1156a^8 - 1361a^6 + 1922a^4 + 4785a^2 + 1008)q_m^4 + 128\alpha^7(369a^8 - 49a^6 + 1573a^4 + 1755a^2 + 288)q_m^2$$
$$+ 512\alpha^8(23a^8 + 55a^6 + 165a^4 + 135a^2 + 18)]; \tag{B3}$$

$$\rho_1 = \frac{P_1}{4\alpha a^2\rho_0}[(a^2-1)q_m^4 + 2\alpha(3a^2-2)q_m^2 + 4\alpha^2(a^2-1)], \tag{B4}$$

$$\rho_2 = \frac{P_1^2}{2a^2\rho_0^3[(a^2-3)q_m^4+6\alpha(a^2-2)q_m^2+4\alpha^2(5a^2-3)]}[(a^2-3)(a^2-2)q_m^8-6\alpha(5a^2-8)q_m^6 - 4\alpha^2(5a^4+15a^2-36)q_m^4-8\alpha^3(5a^2-24)q_m^2-16\alpha^4(a^4-6)], \tag{B5}$$

$$\rho_3 = \frac{P_1^3}{24\alpha a^4\rho_0^5[(a^2-3)q_m^4+6\alpha(a^2-2)q_m^2+4\alpha^2(5a^2-3)][(a^2-6)q_m^4+6\alpha(a^2-4)q_m^2+4\alpha^2(8a^2-6)]} \times[(a^2-1)(a^2-3)(2a^4+19a^2-36)q_m^{16}+2\alpha(66a^8-259a^6-206a^4+1293a^2-864)q_m^{14} + 4\alpha^2(304a^8-1613a^6+1085a^4+3327a^2-3024)q_m^{12}+8\alpha^3(486a^8-2716a^6+3599a^4+4065a^2-6048)q_m^{10} + 16\alpha^4(438a^8-687a^6+2704a^4+1605a^2-7560)q_m^8-32\alpha^5(24a^8-2273a^6+2200a^4+1689a^2+6048)q_m^6 - 64\alpha^6(1098a^8-2261a^6+4467a^4+2427a^2+3024)q_m^4-128\alpha^7(342a^8-758a^6+2409a^4+1173a^2+864)q_m^2 - \alpha^8(9728a^8-21504a^6+110592a^4+53760a^2+27648)]; \tag{B6}$$

$$w_2 = \frac{1}{\rho_0^2}, \tag{B7}$$

$$w_3 = -\frac{P_1}{6\alpha a^2\rho_0^4}[3(a^2-1)q_m^4+6\alpha(3a^2-2)q_m^2+4\alpha^2(5a^2-3)], \tag{B8}$$

$$w_4 = \frac{P_1^2}{48\alpha^2a^4\rho_0^6[(a^2-3)q_m^4+6\alpha(a^2-2)q_m^2+4\alpha^2(5a^2-3)]}[9(a^2-3)(a^2-1)^2q_m^{12}+18\alpha(a^2-1)(9a^4-31a^2 + 18)q_m^{10}+12\alpha^2(106a^6+411a^2-396a^4-135)q_m^8+8\alpha^3(741a^6-2150a^4+1782a^2-540)q_m^6 + 16\alpha^4(1010a^6-2220a^4+1413a^2-405)q_m^4+96\alpha^5(223a^6-404a^4+195a^2-54)q^2 + 64\alpha^6(173a^6-269a^4+99a^2-27)]. \tag{B9}$$

An inspection of these and higher-order coefficients shows the systematic appearance of the combinations Υ_i in denominators, as described in the Sec. III.

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