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Spin-foam models and the physical scalar product

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This paper aims at clarifying the link between loop quantum gravity and spin-foam models in four dimensions. Starting from the canonical framework, we construct an operator P acting on the space of cylindrical functions $\mathrm{Cyl}(\Gamma)$, where Γ is the four-simplex graph, such that its matrix elements are, up to some normalization factors, the vertex amplitude of spin-foam models. The spin-foam models we are considering are the topological model, the Barrett-Crane model, and the Engle-Pereira-Rovelli model. If one of these spin-foam models provides a covariant quantization of gravity, then the associated operator P should be the so-called ''projector'' into physical states and its matrix elements should give the physical scalar product. We discuss the possibility to extend the action of P to any cylindrical functions on the space manifold.

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I. INTRODUCTION

Finding the physical scalar product is certainly one of the most important question of loop quantum gravity [1,2]. This is somehow equivalent to the problem of finding solutions of the remaining scalar constraint which is, so far, still an open issue. Two main and very active directions have been followed to tackle the problem: (i) formulating consistently the scalar constraint as a well-defined operator acting on the kinematical Hilbert space; (ii) making sense of the covariant quantization to compute physical transitions amplitudes between states of quantum geometry. The former has been explored mainly by Thiemann [3] and collaborators: very tricky and very nice regularizations of the scalar constraints have been found; the important question is now to extract physical solutions out of it. The master constraint program [4] has been considered to that aim. Spin-foam models [5] are the covariant alternative attempt to solve the problem: they propose a way to ''compute'' the path integral of gravity where space-time appears as a combinatorial foam which can be understood as a covariant generalization of the notion of spin networks. Then a spin foam is somehow interpreted as the structure which encodes the ''time evolution'' of a state of quantum gravity. Spin-foam models have been studied intensively these last years to answer some fundamental questions they have raised, two of the most important being the following: What is the precise link between spin-foam models and the path integral of quantum gravity? Can we establish an explicit link between spin-foam models and loop quantum gravity as in the three dimensional case [6]?

To understand the meaning of the first question, it is worth recalling that spin-foam models are only ansatz for the path integral of quantum gravity. The ansatz is based on the Plebanski formulation of general relativity [7] where gravity appears as a topological background field (BF) theory supplemented with simplicity constraints on the B field. The path integral of a (Euclidean) BF theory is a topological invariant which can be reformulated ''exactly'' as a spin-foam model which is called, in a more mathematical language, a state sum model. The natural idea is to try to impose the simplicity constraints at the level of the path integral to get a spin-foam model for gravity. Barrett and Crane (BC) [8] proposed a first model: it was studied a lot but recently it was shown not to reproduce expected behavior at the semiclassical limit [9] while computing the two-point functions of gravity in the context of loop quantum gravity (LQG) propagator calculations [10]. It was then realized that the way Barrett and Crane had imposed the simplicity constraints at the level of the spin foam would have been, in a sense, too strong. Engle, Pereira, and Rovelli (EPR) have proposed a new model [11] which seems a more promising candidate: in a subsequent paper with Livine [12], they have proposed a way to impose the simplicity constraints using the ''master constraint'' techniques introduced in the context of canonical quantization by Thiemann. One can incorporate the Immirzi parameter in the new model and it is possible to extend it to the Lorenzian case [13]. In the meanwhile another model from Freidel and Krasnov (FK) [14] has appeared. The FK model instead imposes the constraints using the coherent states techniques introduced by Livine and Speziale [15].

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All of these new models are under study at this moment [16] in order to see, in particular, if they behave correctly in the classical and semiclassical limits [17].

The second question concerning the link between canonical and covariant quantizations of gravity has been quite problematic for a long time: the Lorentzian BC model seems incompatible with loop quantum gravity because it is known that the spectra of the area operator are not identical in the two approaches. Covariant loop quantum gravity [18] was introduced to repair this problem modifying (in a covariant way) the canonical quantization: the obtained theory is unfortunately too cumbersome to be useful for the moment. Instead of modifying the canonical quantization, one could consider standard loop quantm gravity as the good framework for the canonical quantization of gravity and think about finding a spin-foam model consistent with this approach. This is exactly what the new EPR model is doing: the projected states of the new model are naturally identified with the standard spin-network states; the spectrum of the area operators in the covariant quantization is the same as the one in the canonical quantization. Therefore, the EPR model seems to be a good candidate to test if spin-foam models can explicitly realize a ‘‘projection’’ (in the sense of loop quantum gravity) into physical states. Indeed, we expect the physical scalar product between two spin-network states to be given by the spin-foam amplitude associated to a graph whose boundaries are the two given spin networks.

This article aims at clarifying this relation with a simple example. We consider Euclidean spin-foam models associated to the group $G = SU(2) \times SU(2)$. It is characterized by its vertex amplitude V: the vertex amplitude is the weight associated to a four simplex; it is therefore a function $V(I_{ij}, \omega_i)$ of the G representations I_{ij} coloring the 10 faces of the four simplex and of the G intertwiners ω_i associated to the five tetrahedra of the four simplex. The index i runs from 1 to 5 and labels the five tetrahedra in the boundary of the four simplex. We want to interpret this vertex amplitude V as the physical scalar product between two spin-network states: the one-tetrahedron state τ_1 and the four-tetrahedra state τ_4 associated to spin networks, respectively, dual to one tetrahedron and to four tetrahedra as illustrated in the Fig. 1. The free ends of these spin networks coincide and therefore τ_1 and τ_4 are particular cylindrical functions of the same graph, denoted $\tilde{\Gamma}$, as illustrated in Fig. 9 in the core of the paper. The graph $\tilde{\Gamma}$ is the union of the four-simplex graph Γ with four free edges and it was introduced to take into account the free ends of the states τ_1 and τ_4.

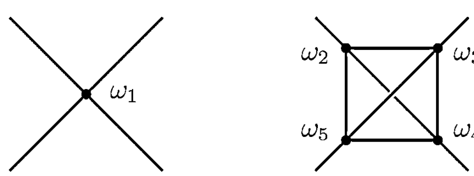


FIG. 1. Illustration of the one-tetrahedron state τ_1 on the left and the four-tetrahedron state τ_4 on the right. Vertices, labeled by $i \in \{0, 5\}$, are colored with intertwiners ω_i and edges ℓ_{ij} with representations I_{ij}. The four free ends are colored with representations I_{1i}.

More precisely, we construct on operator P acting on the space of cylindrical functions $\mathrm{Cyl}(\tilde{\Gamma})$ such that its matrix elements are related to the vertex amplitude of spin-foam models as follows:

$$\langle \tau_4, P\tau_1 \rangle = NV(I_{ij}, \omega_i), \tag{1}$$

where N is an eventual normalization factor. In that sense, the matrix element $\langle \tau_4, P\tau_1 \rangle$ would be the physical scalar product between the kinematical states τ_1 and τ_4. In fact, the bra-ket notation for the physical scalar product might be misleading because mathematically P is a linear form on the space $\mathrm{Cyl}(\tilde{\Gamma})$, i.e., $P \in \mathrm{Cyl}(\tilde{\Gamma})^*$, abusively called a ‘‘projector,’’ and the physical scalar product is $\langle \tau_4, P\tau_1 \rangle = P(\bar{\tau}_4 \tau_1)$. In the context of the Gelfand-Naimark-Segal theory (see the Ashtekar-Lewandowski review [1] and references therein), P, if it satisfies some additional properties, would be a state and would allow to construct the whole physical Hilbert space in principle.

We find a solution for the projector P for different spin-foam models: the topological $SU(2)$ BF model whose vertex V_{BF} is the $15j$ symbol of $SU(2)$ (this system has no physical relevance); the BC model whose vertex V_{BC} is the well-known $10j$ symbol; the new model whose vertex V_{EPR} has been defined recently; and also the FK model whose vertex construction is a direct extension of the EPR one (in this paper we concentrate only on the vertex amplitude without discussing the measure factors associated to the FK model, see [14]). The projector P_{BF} associated to the topological model is a multiplicative operator which acts only on the edges of the spin networks and imposes that the connection is flat. The projectors, P_{BC} and P_{EPR}, respectively, associated to the BC and the EPR models act both on the vertices (as derivative operators, in the sense that it involves left and right invariant derivatives) and on the edges of spin networks. Note that we construct one solution of P and we do not precisely address the question of the unicity in this article.

The plan of this article is the following. In Sec. II, we propose a simple and general integral formula of the vertex amplitudes of Euclidean four dimensional spin-foam models. It is quite a universal formula for it contains as particular cases the vertices of all the known models as the topological, the BC, and the EPR models. In Sec. III, we make use of this formula to construct physical operators for each model in a way similar to the three dimensional case. More precisely, we find a solution to the equation (1) for each model and we discuss the properties of these solutions. We conclude with some perspectives.

Let us finish the introduction with a point concerning the notations: all the spin networks we are considering are $SU(2)$ colored spin networks as in standard LQG.

II. THE VERTEX OF A SPIN-FOAM MODEL

In this section, we present some properties concerning the vertex amplitude of several spin-foam models. The notion of vertex amplitude is defined in the first part where we give a very brief introduction on spin-foam models. In a second part, we propose a general and rather simple integral formula for the vertex amplitude which will be useful in the next section to make a link with the canonical quantization. In the last part of this section, we illustrate this formula in the particular models we are interested in, namely, the topological, the BC, and the EPR models. Furthermore, we underline that the BC and the EPR models are particular cases of a large class of spin-foam models. We present the construction of this class of spin-foam models and we show that their vertex amplitude admits an integral formulation of the same type.

A. A brief introduction on spin-foam models

A spin-foam model is basically the assignment of a complex amplitude $\mathcal{A}(\mathcal{T})$ to any triangulation $\mathcal{T}$ of a given four dimensional manifold $\mathcal{M}$. The triangulation consists in the union $\cup_{i=2}^{4}\mathcal{T}_i$ of the set of its faces $\mathcal{T}_2$, the set of its tetrahedra $\mathcal{T}_3$ and the set of its four simplices $\mathcal{T}_4$. The amplitude $\mathcal{A}$ is constructed from the representation theory of a given Lie group G that we assume compact for simplicity. To do so, one first colors each face $f \in \mathcal{T}_2$ with an unitary irreducible representation (UIR) j_f of G and each tetrahedron $t \in \mathcal{T}_3$ with intertwiners ι_t between representations coloring its four faces. Then, one associates an amplitude $\mathcal{A}_2(j_f)$ to each face f, an amplitude $\mathcal{A}_3(\omega_t, j_{f_t})$ to each tetrahedron t which depends on the intertwiner ω_t and on the representations coloring its four faces f_t, and an amplitude $V(\omega_{t_s}, j_{f_s})$ to each four simplex s which depends on the representations j_{f_s} and ω_{t_s} coloring its ten faces f_s and five tetrahedra t_s. Finally, the spin-foam amplitude is formally defined by the series

$$\mathcal{A}(\mathcal{T}) \equiv \sum_{\{j_f\},\{\omega_t\}} \prod_{f\in\mathcal{T}_2} \mathcal{A}_2(j_f) \prod_{t\in\mathcal{T}_3} \mathcal{A}_3(\omega_t, j_{f_t}) \times \prod_{s\in\mathcal{T}_4} V(\omega_{t_s}, j_{f_s}). \tag{2}$$

where the sum runs into a certain subset of UIR and intertwiners of G. The sum is *a priori* infinite and therefore the amplitude is only defined formally at this stage unless it is convergent. Notice that in all the models that have been studied in the literature, the amplitude $\mathcal{A}_3$ is assumed to depend on the intertwiners ω_t only. The function V is precisely the vertex amplitude of the spin-foam model. To finish with this brief introduction of spin-foam models, let us mention that the previous construction could be generalized to the case where G is noncompact and to the case where G is replaced by a quantum group. Spin-foam models can also be defined for any dimensional manifold $\mathcal{M}$.

In this paper, we consider exclusively the case where $\mathcal{M}$ is four dimensional and we study some properties of the vertex amplitude V only. Therefore, we will not mention the amplitudes $\mathcal{A}_2$ and $\mathcal{A}_3$ when we discuss the spin-foam models in the sequel; as a result, we will omit any discussion concerning the amplitude $\mathcal{A}$ and *a fortiori* the question of its convergence. We hope to study these aspects in the future. Furthermore, we will only consider Euclidean spin-foam models that are associated to the compact Lie groups $G = SU(2)$ (for the topological model) or $G = SU(2) \times SU(2)$ (for the BC and EPR models). Letters $I, J, \cdots$ label unitary irreducible representations of the group G and the associated vector spaces are denoted $U_I, U_J, \cdots$. When $G = SU(2)$, I is a half interger whereas it is a couple of half integers when $G = SU(2) \times SU(2)$. Because of the compactness of G, each representation I is finite dimensional and associates to any $g \in G$ a finite dimensional matrix which will be denoted $R^I(g)$ when $G = SU(2) \times SU(2)$ and $D^I(g)$ in the other case. To a representation, I is associated a contragredient (or a dual) representation I^* such that $R^{I^*}(g) = {}^tR^I(g^{-1})$ and the same for the $SU(2)$ representations D^{I^*}; it is common to identify $U_I^* \equiv U_{I^*}$ to U_I. More precision concerning the representation theory of the groups G will be given later.

The vertex $V(I_{ij}, \omega_i)$ is then a function of the five intertwiners ω_i coloring the five tetrahedra (which are ordered and labeled by $i \in \{1, 5\}$) of a four simplex and of the ten representations $(I_{ij})_{i<j}$ of G coloring the ten faces at the intersections of the tetrahedra i and j; ω_i: $\otimes_{j>i} U_{I_{ij}} \rightarrow \otimes_{j<i} U_{I_{ji}}$ is an intertwiner between the representations I_{ij} "meeting" at the tetrahedron i. In the next part, we are going to show that the vertex amplitude of all the models we consider can be written as an integral over ten copies of the three sphere S^3 as follows:

$$V(I_{ij}, \omega_i) = \int\Big(\prod_{i<j} dx_{ij}\Big) C(x_{ij}) \mathcal{V}(I_{ij}, \omega_i; x_{ij}), \tag{3}$$

where $C(x_{ij})$ is a universal function, in the sense that it is model independent, which reads

$$C(x_{ij}) \equiv \int\Big(\prod_{i=1}^{5} dx_i\Big) \delta(x_{ij}^{-1} x_i x_j^{-1}). \tag{4}$$

$\mathcal{V}$ is a model dependent function of the variables x_{ij}. As we will see in the next section, such a formula will be crucial to link spin-foam models with loop quantum gravity.

B. A General expression of the vertex

There exists many equivalent ways to define the vertex amplitude of a spin-foam model. For our purposes, it is convenient to view the vertex amplitude as a "Feynman graph" evaluation of a closed oriented graph which is dual to a four simplex. The dual of a four simplex Γ is in fact topologically equivalent to a four simplex and then consists in a set of five vertices linked by ten edges: we endow the set of vertices with a linear ordering such that the vertices are labeled with an integer $i \in \{1, 5\}$; this ordering induces a natural orientation on the links, indeed the link ℓ_{ij} between the edges i and j is oriented from i to j if $i < j$. One associates a complex amplitude to this graph using the following "Feynman" rules: each oriented link ℓ_{ij}, with $i < j$, is associated to a UIR of G denoted I_{ij} (the opposite link ℓ_{ji} is associated to the contragredient representation denoted for simplicity $I_{ji} = I^*_{ij}$); each vertex i is associated to an intertwiner ω_i: $\otimes_{j>i} U_{I_{ij}} \rightarrow \otimes_{j<i} U_{I_{ji}}$. As a result, the "Feynman evaluation" of such a graph is the scalar obtained by contracting the ten propagators with the five intertwiners and gives the vertex amplitude which formally reads

$$V(I_{ij}, \omega_i) = \left\langle \bigotimes_{i=1}^{5} \omega_i \right\rangle \equiv \sum_{\{e_{ij}\}} \prod_{i=1}^{5} \left\langle \bigotimes_{j<i} e_{ji} |\omega_i| \bigotimes_{j>i} e_{ij} \right\rangle, \quad (5)$$

where e_{ij} runs over the finite set of a given orthonormal basis of $U_{I_{ij}}$ and we have used the standard bra-ket notation to denote the vectors $|e_{ij}\rangle$ of $U_{I_{ij}}$ and the dual vectors $\langle e_{ij}|$. In the language of loop quantum gravity, we would say that $V(I_{ij}, \omega_i)$ is simply the evaluation of the spin network associated to the colored graph $(\Gamma, \{I_{ij}, \omega_i\})$ when the connection is flat.

In order to have a more useful formula, it will be convenient to trivially identify ω_i with an element of $\mathrm{Hom}(\otimes_{j\neq i} U_{I_{ij}}, \mathbb{C})$ and then to notice that ω_i is completely characterized by a vector $v_i \in \otimes_{j\neq i} U^*_{I_{ij}}$. These vectors can be written in the form $v_i = \sum_{(a_{ij})} \alpha_i^{(a_{ij})} \otimes_{j\neq i} v_{a_{ij}}$ where $(a_{ij})_{j\neq i}$ is a set whose elements label vectors $v_{a_{ij}} \in U_{I_{ij}}$, $\alpha_i^{(a_{ij})}$ are complex numbers and the sum is finite. The explicit relation between ω_i and v_i is the following:

$$\omega_i = \langle v_i| \int dg \bigotimes_{j\neq i} R^{I_{ij}}(g) \in \mathrm{Hom}\left(\bigotimes_{j\neq i} U_{I_{ij}}, \mathbb{C}\right), \quad (6)$$

where we have used the $SU(2) \times SU(2)$ notations for the representations and $\int dg$ is the Haar measure of G. As a result, the vertex amplitude can be reformulated as a multi-integral over G according to the formula

$$V(I_{ij}, \omega_i) = \sum_{(a_{ij})} \prod_{i=1}^{5} \alpha_i^{(a_{ij})} \int \left(\prod_{i=1}^{5} dg_i\right) \times \langle \otimes_{i<j} v_{a_{ij}} | \bigotimes_{i<j} R^{I_{ij}}(g_i g_j^{-1}) | \otimes_{i>j} v_{a_{ij}} \rangle, \quad (7)$$

which can be written in the following more compact well-known form

$$V(I_{ij}, \omega_i) = \int \left(\prod_{i=1}^{5} dg_i\right) (\otimes_{i=1}^{5} v_i) \cdot \left(\bigotimes_{i<j} R^{I_{ij}}(g_i g_j^{-1})\right), \quad (8)$$

where the dot $\cdot$ denotes the appropriate contraction between the vectors v_i and the matrices of the representations. This vertex amplitude is in fact rather general and characterizes partially a large class of spin-foam models. It is general because we have for the moment a total freedom in the choice of the representations and the intertwiners; it is nonetheless only partial because we do not consider the amplitudes associated to faces and tetrahedra. To go further in the study of this amplitude, we need to recall some basic results on the representation theory of $SU(2) \times SU(2)$.

1. Representation theory of G: basic results

Let us start with the group $SU(2)$: its representations are labeled by a half integer, the spin I; they are finite dimensional of dimension $d_I = 2I + 1$ and we denote by $|I, i\rangle$ with $i \in [-I, I]$ the vector of an orthonormal basis of U_I. The group $G = SU(2) \times SU(2)$ is the double cover of $SO(4)$; it is also known as the spin group Spin(4). Any of its elements g can be written as a couple (g_L, g_R) of two $SU(2)$ group elements. Its UIR are labeled by a couple of (integers or half integers) spins (I, J): they are finite dimensional and the vector space $U_{IJ} = U_I \otimes U_J$ of the representation (I, J) is the tensor product of the two $SU(2)$ representation vector spaces U_I and U_J. Therefore, the family of vectors $(|I, i\rangle \otimes |J, j\rangle)_{IJij}$ form an orthonormal basis of U_{IJ}. The action of $g \in G$ in this basis is simply given by

$$\begin{aligned} R^{IJ}(g)|I, i\rangle \otimes |J, j\rangle &= R^{IJ}(g_L, g_R)|I, i\rangle \otimes |J, j\rangle \\ &= D^I(g_L)|I, i\rangle \otimes D^J(g_R)|J, j\rangle. \end{aligned} \quad (9)$$

The $SU(2)$ matrix elements $\langle I, i|D^I|I, j\rangle$ are the Wigner functions.

The space U_{IJ} admits another natural basis which will be useful in the sequel. This other basis is constructed from the remark that the vector space U_{IJ} decomposes into $SU(2)$ UIR vector spaces U_K as follows:

$$U_{IJ} \simeq \bigoplus_{K=|I-J|}^{I+J} U_K. \quad (10)$$

This decomposition provides indeed another orthonormal basis of U_{IJ}, given by the family of vectors $(|K, k\rangle)_{Kk}$ where $K \in [|I - J|, I + J]$ and $k \in [-K, K]$ as usual.

The changing basis formulas are given in terms of the Clebsch-Gordan coefficients $\langle Kk|IiJj\rangle$ as follows:

$$|Ii\rangle \otimes |Jj\rangle = \sum_{K,k} \langle Kk|IiJi\rangle |Kk\rangle \quad \text{and}$$
$$|Kk\rangle = \sum_{IJij} \langle Kk|IiJi\rangle |Ii\rangle \otimes |Jj\rangle. \tag{11}$$

To write the action of G on the basis elements $|Kk\rangle$, it is convenient to find the subgroup $H \subset G$ which leaves the subspaces U_K of the decomposition (10) invariant and then to identify G with the space $G \simeq H \times (G/H)$. In fact, it is immediate to see that $H \simeq SU(2)$, the coset G/H is isomorphic to the sphere S^3 and then we identify G with $SU(2) \times S^3$. Notice that the identification we have just mentioned is not canonical because G admits many $SU(2)$ subgroups; therefore, to make this identification well-defined, one has to precise which $SU(2)$ subgroup one is talking about. In our case, the $SU(2)$ subgroup is the diagonal one, i.e., it is the group of the elements (g_L, g_R) where $g_L = g_R$. As a result, the explicit mapping between G and $SU(2) \times S^3$ is

$$G \to SU(2) \times S^3$$
$$(g_L, g_R) = (u, ux) \mapsto (u, x) = (g_L, g_L^{-1} g_R). \tag{12}$$

This mapping is of course invertible and its inverse is trivially given by

$$SU(2) \times S^3 \to G \qquad (u, x) \mapsto (u, ux). \tag{13}$$

The multiplication law $(g_L, g_R)(g'_L, g'_R) = (g_L g'_L, g_R g'_R)$ induces the multiplication rule

$$(u, x)(u', x') = (uu', u'^{-1} x u' x') \tag{14}$$

in the $SU(2) \times S^3$ representation of G. In particular, the inverse of the element (u, x) is given by $(u, x)^{-1} = (u^{-1}, ux^{-1}u^{-1})$. The diagonal terms $u \equiv (u, 1)$ and the pure spherical terms $x \equiv (1, x)$ will be relevant in the following construction. Note that elements of S^3 are identified with $SU(2)$ group elements so that they can be multiplied or inverted.

Let us now come back to the action of G on the vectors $|K, k\rangle$ of the vector space U_{IJ}; this action is best written and simpler using the factorization $SU(2) \times S^3$ of G. Indeed, a simple calculation shows that

$$\begin{aligned} R^{IJ}_{KkL\ell}(u) &= R^{IJ}_{KkL\ell}(u, u) \\ &= \sum_{m_1, m_2} \langle Kk|Im_1 Jm_2\rangle D^I_{m_1,n_1}(g_L) D^J_{m_2,n_2}(u) \\ &\quad \times \langle In_1 Jn_2|L\ell\rangle \\ &= \delta_{K,L} D^K_{k\ell}(u) \\ R^{IJ}_{KkL\ell}(x) &= R^{IJ}_{KkL\ell}(1, x) = \sum_{ijj'} \langle Kk|IiJj\rangle \langle IiJj'|L\ell\rangle D^J_{jj'}(x), \end{aligned} \tag{15}$$

where we have introduced the notation $R^{IJ}_{KkL\ell}(g) \equiv \langle Kk|R^{IJ}(g)|L\ell\rangle$ for the $SU(2) \times SU(2)$ matrix elements. As expected, we see that $u \in SU(2)$ leaves any $SU(2)$ representation spaces U_K of the decomposition (10) invariant whereas x moves the vectors from one $SU(2)$ representation space to another. This closes the brief review on $SU(2) \times SU(2)$ representations theory.

2. The vertex amplitude as an integral over several copies of S^3

We make use of the basic properties on representations theory recalled above to write the general formula of the vertex amplitude (8) in the form (3). To do so, one splits the integrations over the group variables $g_i \in G$ in the formula (8) into integrations over the $x_i \in S^3$ variables and integrations over the $u_i \in SU(2)$ variables using the isomorphism (12) and one obtains

$$\begin{aligned} V(I_{ij}, \omega_i) = &\int \Big(\prod_{i=1}^5 dx_i\Big)\Big(\prod_{i=1}^5 du_i\Big)(\otimes_{i=1}^5 v_i) \\ &\cdot \Big(\bigotimes_{i<j} R^{I_{ij}}(u_i) R^{I_{ij}}(x_i x_j^{-1}) R^{I_{ij}}(u_j^{-1})\Big), \end{aligned} \tag{16}$$

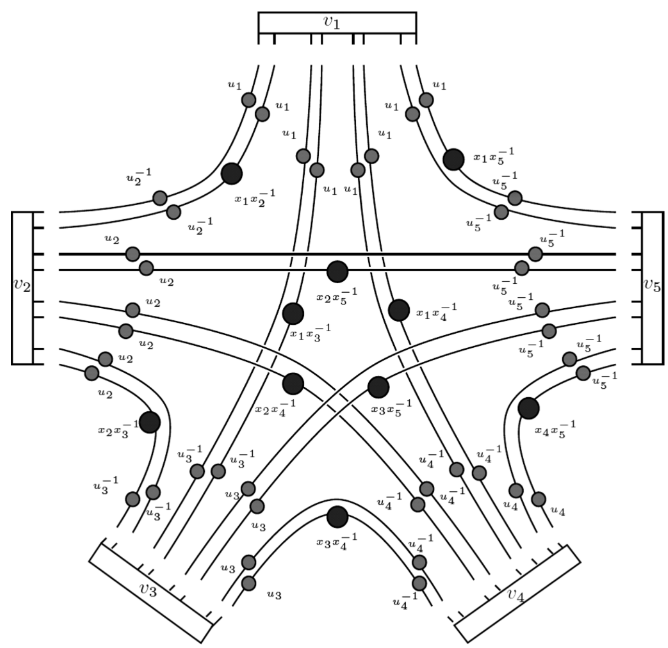


FIG. 2. This picture is a graphical representation of the integrand in the formula (16) defining the vertex amplitude. Each line is doubled because it carries a representation of $SU(2) \times SU(2)$ and the single lines in the pair colored with (I, J) are colored by I and J, separately. Furthermore, the single lines are endowed with bullets that represent the insertion of $SU(2)$ group elements: the small ones are associated to diagonal elements $u_i \in SU(2)$ whereas the big ones are associated to spherical elements $x_i x_j^{-1} \in S^3$. The vectors v_i are represented by boxes and they are contracted with the free ends of the graph.

where $R^I(u) \equiv R^I(u,1)$ (respectively, $R^I(u,u)$) and $R^I(x) \equiv R^I(1,x)$ (respectively, $R^I(1,x)$) are the matrices of $SU(2)\times SU(2)$ representations I in the $SU(2)\times S^3$ (respectively $SU(2)\times SU(2)$) formulations. To have a "geometrical" intuition of this formula, we give a graphical representation of the integrand in Fig. 2 below. In the models we are going to consider explicitly in the sequel, we can perform the integrations over the u_i variables; therefore we formally perform the integration over the u_i's in the general formula (16) and we obtain a formula for the vertex amplitude as an integral over five copies of S^3 only

$$V(I_{ij},\omega_i) = \int\Big(\prod_{i=1}^{5} dx_i\Big)(\otimes_{i=1}^5 \nu_i)\cdot\Big(\bigotimes_{i<j} R^{I_{ij}}(x_i x_j^{-1})\Big). \tag{17}$$

The integrations over the five $SU(2)$ variables u_i have been hidden in the following definition of the vectors $\nu_i \in \otimes_{j\neq i} U^*_{I_{ij}}$:

$$\nu_i \equiv \sum_{(a_{ij})} \alpha_i^{(a_{ij})} \int du (\otimes_{j>i}\langle v_{a_{ij}}| R^{I_{ij}}(u_i)) \otimes (\otimes_{j<i} R^{I_{ji}}(u)^{-1}|v_{a_{ij}}\rangle) \tag{18}$$

where we have used the explicit decomposition of the vectors $v_i \in \otimes_{j\neq i} U^*_{I_{ij}}$ given in the introductive part of Sec. II B. This formula will be much more explicit when we consider the particular spin-foam models we are interested in. For the moment, for pedagogical purposes, we propose a pictorial representation in Fig. 3 of the argument of the previous integral (18) when $i=1$.

Before considering specific examples, let us add one more important remark. The vertex amplitude can be trivially reformulated as an integral over ten copies of G as follows:

$$V(I_{ij},\omega_i) = \int\Big(\prod_{i<j} dx_{ij}\Big) C(x_{ij})(\otimes_{i=1}^5 \nu_i)\cdot\Big(\bigotimes_{i<j} R^{I_{ij}}(x_{ij})\Big), \tag{19}$$

where the constraint $C(x_{ij})$ is a distribution which imposes, roughly speaking, x_{ij} to be a "coboundary," i.e., of the form $x_i x_j^{-1}$. An explicit formula for $C(x_{ij})$ is simply given by the integral

$$C(x_{ij}) = \int\Big(\prod_{i=1}^{5} dx_i\Big)\prod_{i\neq j}\delta(x_{ij}^{-1} x_i x_j^{-1}), \tag{20}$$

where δ is the $SU(2)$ delta distribution. It is possible to perform the above integration whose result is simply given by the product of five delta distributions,

$$C(x_{ij}) = \delta(x_{123})\delta(x_{234})\delta(x_{345})\delta(x_{451})\delta(x_{512}), \tag{21}$$

where $x_{ijk} = x_{ij}x_{jk}x_{ki}$ and, by convention, $x_{ij} = x_{ji}^{-1}$. The interpretation of the constraint $C(x_{ij})$ will become clear in the last section where we make the link with the canonical quantization. To conclude, we underline that we have finally found the desired formula (3) for the vertex amplitude with the announced expression of the distribution $C(x_{ij})$ and the model dependent function $\mathcal{V}(I_{ij},\omega_i;x_{ij}) = (\otimes_{i=1}^5\nu_i)\cdot(\bigotimes_{i<j}R^{I_{ij}}(x_{ij}))$ is a particular contraction of five $SU(2)$ matrices.

C. Vertices of particular models

This part is devoted to study some aspects of the vertex amplitude (17) for the topological model, the BC model, and the EPR model. In fact, these models differ only by the choice of the intertwiners ω_i or equivalently the vectors v_i which are their building blocks. Thus, to understand the construction of these models and their differences, one has to understand the definition of their associated intertwiners. For that purpose, let us start by recalling basic properties of intertwiners. First of all, in spin-foam models, we are interested in four valent intertwiners only. The four valent intertwiners between four given representations form a (normed) vector space of finite dimension. In the case where $G = SU(2)$, one can exhibit three canonical (natural) orthogonal basis (labeled by an index $\epsilon \in \{+,-,0\}$ that indicates the "coupling channel") presented

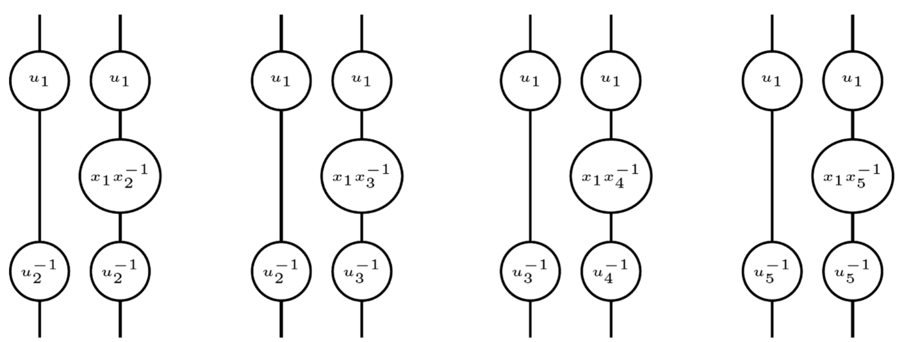


FIG. 3. Structure of the node $i=1$. Four pairs of edges are attached at each node of the graph: each edge is colored with a $SU(2)$ representation. The bullets illustrate the inclusions of $SU(2)$ variables u_i or S^3 variables $x_i x_j^{-1}$. Notice that, in the $SU(2)\times SU(2)$ formulation, each pair of lines is associated to the element (g_L, g_R), g_L corresponding to the left line and g_R to the right one.

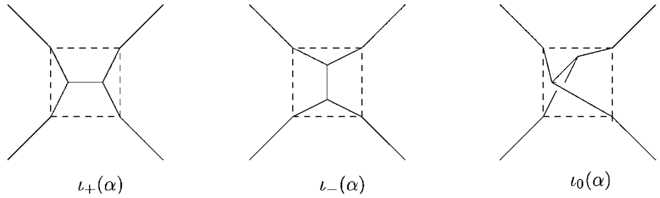


FIG. 4. The three canonical basis of the space of four-valent intertwiners. The intermediate channel is endowed with the representation α.

in Fig. 4. Whatever the basis we choose, any of its element is completely characterized by the representation appearing in the intermediate channel in the tensor product decomposition. Therefore, one often identifies the element of each basis with a representation. We will use the notations $\iota_\epsilon(\alpha)$ to denote the $SU(2)$ intertwiner in the basis ϵ with intermediate representation α. One can make uses of these results to construct the basis of $SU(2) \times SU(2)$ four valent intertwiners. In particular, one can naturally exhibit nine "tensor product" basis labeled by a couple (ϵ, ϵ'). However, we will consider in the sequel only the three basis of the type (ϵ, ϵ) which will be labeled by a single ϵ for simplicity: elements of the basis ϵ are denoted $\iota_\epsilon(\alpha)$ as in the $SU(2)$ case but with the difference that α is now a couple of $SU(2)$ representations.

Now, we are ready to define the intertwiner ω_i for the model we are interested in. Afterwards, we are going to make the general abstract formula of the vertex amplitude more concrete and more useful for studying its properties.

1. The topological model

We start with the simplest, certainly the more mathematically precise but nonphysical model. The topological model is closely related to BF theory with gauge group $SU(2)$. More precisely, given a triangulation $\mathcal{T}$ of a four-dimensional manifold $\mathcal{M}$, one can discretize the BF action to be well-defined on this triangulation and the path integral $\mathcal{Z}_{\rm BF}(\mathcal{T})$ of the discretized action can be formulated as a state sum or equivalently a spin-foam model

$$\mathcal{Z}_{\rm BF}(\mathcal{T}) = \sum_{\{j_f\},\{\omega_t\}} \prod_{f\in\mathcal{T}^2} \dim(j_f) \prod_{t\in\mathcal{T}^3} \dim(\omega_t)^{-1} \times \prod_{s\in\mathcal{T}^4} V_{\rm BF}(\omega_{t_s}, j_{f_s}), \tag{22}$$

where we have used notations of (2); we have identified the intertwiners ω_t with the associated representation and $V_{\rm BF}$ is the vertex amplitude completely defined by the graph 5. This amplitude is known as a $15j$ symbol and can be formulated as a finite sum of products of $6j$ symbols

$$V_{\rm BF}(I_{ij}, \omega_i) = \sum_K \frac{1}{d_{\omega_1} d_{\omega_5} d_K^2} \begin{Bmatrix} \omega_1 & I_{12} & I_{13} \\ \omega_2 & I_{25} & K \end{Bmatrix} \times \begin{Bmatrix} \omega_2 & \omega_3 & I_{13} \\ I_{23} & I_{13} & K \end{Bmatrix} \begin{Bmatrix} I_{35} & I_{24} & I_{34} \\ \omega_3 & \omega_4 & K \end{Bmatrix} \times \begin{Bmatrix} \omega_1 & I_{14} & I_{15} \\ I_{25} & \omega_5 & K \end{Bmatrix} \begin{Bmatrix} I_{45} & \omega_5 & I_{34} \\ I_{14} & \omega_4 & K \end{Bmatrix}. \tag{23}$$

The $6j$ symbols are the totally symmetrized $6j$ symbols defined, for example, in chapter 6 of Ref. [19]. Note that the sum is finite and then the vertex amplitude is well-defined. However, the state sum is generally divergent; it can be made convergent by gauge fixing or by turning classical groups into quantum groups. The state sum is a (formal) piecewise linear (PL) invariant, i.e., invariant under homeomorphisms.

We have voluntarily not given neither the interwiners $\omega_i^{\rm BF}$ nor the vectors v_i defining the model according to the previous section. Indeed, such a formulation is not very useful for the topological model and the description of the previous section is naturally adapted for $SU(2) \times SU(2)$ spin-foam models and not really for $SU(2)$ spin-foam models.

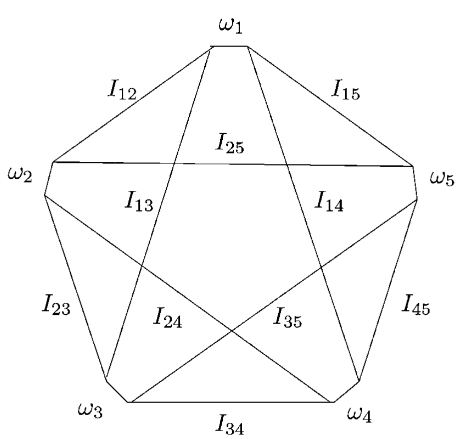


FIG. 5. Pictorial representation of a $15j$ symbol: vertices are labeled by representations ω_i and edges by representations I_{ij}.

2. The Barrett-Crane model

The Barrett-Crane model has been constructed as a step towards the covariant quantization of four dimensional pure Euclidean or Lorentzian gravity *à la* Plebanski. Here, we consider exclusively the Euclidean case. The BC model is then a state sum associated to a triangulation $\mathcal{T}$ of a four manifold $\mathcal{M}$ which is supposed to reproduce the path integral $\mathcal{Z}_{\rm Pl}(\mathcal{T})$ of a discretized version of the Plebanski action. However, the link between the BC model and gravity is somehow misleading. Indeed, the BC state sum has been constructed heuristically as a modification of the $SU(2) \times SU(2)$ topological state sum according to the following rules: representations coloring the faces of the four simplex are supposed to be simple, i.e., of the form (I_{ij}, I_{ij}); the intertwiners $\omega_i^{\rm BC}$ associated to the tetrahedra are also called simple or BC intertwiners we will recall the definition in the sequel; the vertex amplitude $V_{\rm BC}$ associated to the four simplices are the so-called $10j$ symbols whose definition will also be recalled later. The BC model does not say anything concerning the amplitudes $\mathcal{A}_2$ and $\mathcal{A}_3$ associated to the faces and the tetrahedra of the triangulation. However, many arguments lead to certain expressions of $\mathcal{A}_2$ and $\mathcal{A}_3$ and the corresponding state sums have been numerically tested [20]. Anyway, we will not consider these amplitudes in this paper.

Let us concentrate on the construction of the vertex amplitude V_{BC} whose basic ingredient is the simple intertwiner. A simple n-valent intertwiner is such that any of its decompositions into three-valent intertwiners introduce only simple representations in the intermediate channel. The simple intertwiner has been studied intensively in the literature; in particular, it was shown to be unique up to a global normalization [21]. This property makes clear that the vertex amplitude of the BC model is a function $V(I_{ij}, \omega_i^{\mathrm{BC}})$ of only ten representations and it is called a $10j$ symbol. To precisely define the simple intertwiner ω_i^{BC}, it is more convenient to start with the formula (6) which shows that ω_i^{BC} is completely determined by the choice of a "simple" vector $v_i^{\mathrm{BC}} \in \otimes_{j\neq i} U^*_{I_{ij}J_{ij}}$ where (I_{ij}, J_{ij}) is a $SU(2)\times SU(2)$ UIR. If (I_{ij}, J_{ij}) is a simple representation, i.e., $I_{ij} = J_{ij}$, then the associated vector space admits a unique normalized (diagonal) $SU(2)$ invariant vector w (or $|w\rangle$) which we identify with its dual $\langle w| \in V^*_{I_{ij}}$. In that case, indeed, the decomposition (10) of $U_{I_{ij}J_{ij}}$ into $SU(2)$ representations contains the space U_0 which is the one-dimensional space of diagonal $SU(2)$ invariant states. The simple vector is in fact the tensor product of these invariant vectors: $\omega_i^{\mathrm{BC}} = w^{\otimes 4}$. As a result, the expression of the simple intertwiner in the tensor product basis reads

$$\omega_i^{\mathrm{BC}} = \frac{1}{\prod_{j\neq i}\sqrt{d_{I_{ij}}}}\sum_\alpha d_\alpha \iota_\epsilon(\alpha), \tag{24}$$

where the sum runs over simple representations $\alpha \equiv (\alpha, \alpha)$ only and is finite. An important property is that the previous sum is independent on the choice of the basis ϵ. Using this formula of the simple intertwiner, one finds immediately the vertex amplitude of the BC model

$$V_{\mathrm{BC}}(I_{ij}, \omega_i^{\mathrm{BC}}) = \frac{1}{\prod_{i\neq j} d_{I_{ij}}}\sum_\alpha d_\alpha V_{\mathrm{BF}}(I_{ij}, \alpha)^2 \tag{25}$$

as a sum of BF amplitudes V_{BF} which are $SU(2)$ $15j$ symbols. The sum runs over simple representations only and is independent on the choice of the intertwiners defining the $15j$ symbol. Such a formula is too cumbersome to be useful and one prefers to use the integral formulation (19) of the amplitude to study its physical properties. This integral formula simplifies indeed drastically because the $SU(2)$ integral defining ν_i (18) becomes trivial due to the $SU(2)$ invariance of the vectors v_i, and reads

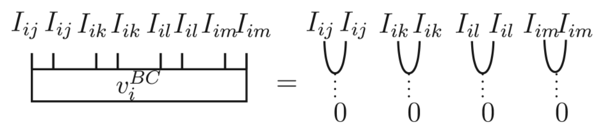
$$V_{\mathrm{BC}}(I_{ij}, \omega_i) = \int\Big(\prod_{i=1}^5 dx_i\Big)\langle w^{\otimes 10}|\bigotimes_{i<j} R^{I_{ij}}(1, x_i x_j^{-1})|w^{\otimes 10}\rangle. \tag{26}$$

Using the second equations in (15), one obtains the following integral formula for the $10j$ symbol:

$$\begin{aligned} V_{\mathrm{BC}}(I_{ij}, \omega_i^{\mathrm{BC}}) &= \int\prod_{i\neq j} dx_{ij}\frac{\chi_{I_{ij}}(x_{ij})}{d_{I_{ij}}}C(x_{ij}) \\ &= \int\prod_{i=1}^5 dx_i \prod_{i<j}\frac{\chi_{I_{ij}}(x_i x_j^{-1})}{d_{I_{ij}}}, \end{aligned} \tag{27}$$

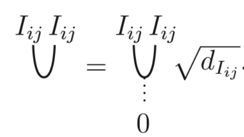
where $\chi_I(x)$ is the $SU(2)$ character of x in the representation I. Up to some normalization factors, the previous formula coincides with the Euclidean $10j$ symbols. This integral formulation was very useful to study the classical behavior of the Euclidean BC model. Let us finish this brief presentation of the BC model with two important remarks.

Remark 1. The previous calculation can be done in a completely graphical way. Indeed, the "black" boxes representing the vectors v_i^{BC} in Fig. 2 reduce to the following form



(28)

where the dashed lines represent spin 0 representation. We see explicitly that v_i^{BC} project into diagonal $SU(2)$ invariant vectors. Furthermore, the $3j$ vectors involving a spin 0 representation are proportional to the "identity" according to the following pictorial rule



(29)

As a result, one immediately obtains the pictorial representation of the BC vertex amplitude which is given by the product of the normalization factor $\prod_{i<j} d_{I_{ij}}^{-1}$ and the graph in Fig. 6. The graph consists in ten disconnected loops colored by representations I_{ij} which makes obvious that the vertex amplitude integrand is, up to a normalization, the product of ten characters $\chi_{I_{ij}}(x_{ij})$.

Remark 2. There is another equivalent expression for the vertex amplitude which was very useful to study the classical behavior of the vertex amplitude found by Freidel and Louapre [22]. This formula will not be used in this paper but it is still interesting to mention it at least to ask the question whether a similar formula exists for the EPR model. This formula is based on the simple fact that the character $\chi_I(x)$ depends only on the conjugacy class

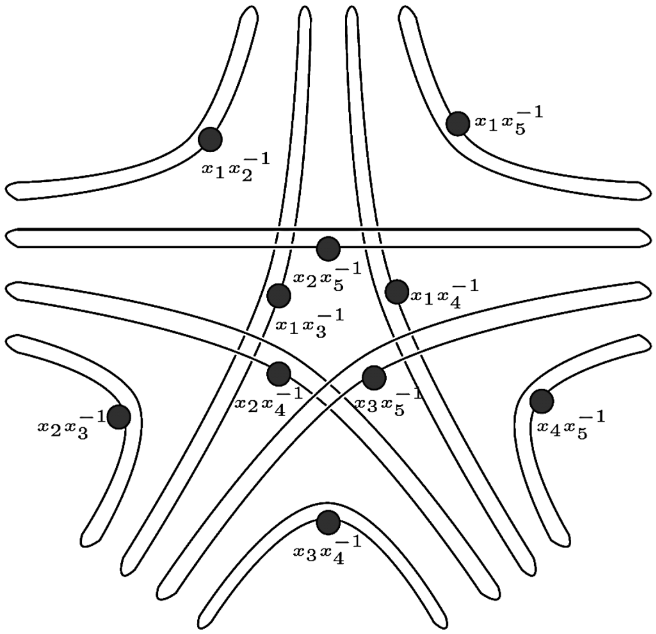


FIG. 6. Pictorial representation of the BC vertex integrant up to the normalization factor $\prod_{i<j} d_{I_{ij}}^{-1}$. The graph is made of ten disconnected unknots colored with representations I_{ij}. In each loop is inserted a S^3 element of the form $x_i x_j^{-1}$.

$\theta \in [0, \pi]$ of $x = \Lambda h(\theta)\Lambda^{-1}$: $\Lambda \in SU(2)/U(1)$ and $h(\theta)$ is in the Cartan torus of $SU(2)$. This fact leads after some calculations to an expression of the vertex amplitude as an integral over the conjugacy classes

$$V_{\mathrm{BC}}(I_{ij}, \omega_i^{\mathrm{BC}}) = \int \left(\prod_{i \neq j} d\theta_{ij} \frac{\sin(d_{I_{ij}} \theta_{ij})}{d_{I_{ij}}} \right) \tilde{C}(\theta_{ij}). \tag{30}$$

The notation $\tilde{C}$ holds for the "Fourier transform" of the distribution C; it is a distribution as well given by

$$\tilde{C}(\theta_{ij}) \equiv \frac{2^{10}}{\pi^{10}} \int \left(\prod_{i<j} \sin\theta_{ij} d\Lambda_{ij} \right) C(\Lambda_{ij} h(\theta_{ij}) \Lambda_{ij}^{-1}) = \delta(G[\cos(\theta_{ij})]), \tag{31}$$

where G holds for the Gramm matrix. Such a relation is in fact a particular example of a much more general duality relation [23].

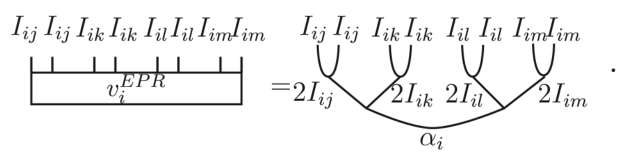
3. The Engle-Pereira-Rovelli model

The BC model has been considered as the most promising spin-foam model for a long time: its definition is simple, it has a quite appealing physical interpretation and admits the good classical limit [20,22,24] in the sense that the associated vertex amplitude tends to the Regge action in the classical limit, apart from a term due to degenerate contributions, and it was also successful in reproducing the correct asymptotic behavior of the diagonal components of the graviton propagator [10,25]. Nevertheless, it has been recently realized that the model does not satisfy the required properties to reproduce at the semiclassical limit the nondiagonal components of the propagator [9]. The reasons of this failure have been deeply investigated and a quest for a new model have been started. Recent researches have led to the so-called EPR model which has been argued to be a serious candidate. However, the model was discussed a lot and some criticisms can be found in [16,26]. This section is devoted to recall the basis of this model in the Euclidean sector with no Immirzi-Barbero parameter $\gamma = 0$.

As in the BC framework, Engle, Pereira, and Rovelli have proposed a formula for the vertex amplitude V_{EPR} only. To construct V_{EPR}, one starts by coloring the faces of the four simplex by simple representations and the tetrahedra i by specific intertwiners denoted ω_i^{EPR}. We propose to define ω_i^{EPR} through its associated vector v_i^{EPR} according to the formula (6). To do so, to each simple representation (I_{ij}, I_{ij}), we associate the projector $\mathbb{I}_{2I_{ij}}: U_{I_{ij}I_{ij}} \rightarrow U_{2I_{ij}}$ from the $SU(2) \times SU(2)$ representation's vector space $U_{I_{ij}I_{ij}}$ into the vector space of the $SO(3)$ representation of spin $2I_{ij}$. In the standard bra-ket notation, the projector reads $\mathbb{I}_{2I_{ij}} = \sum_m |m2I_{ij}\rangle\langle m2I_{ij}|$; it is clear that it can be trivially identified to its dual $\mathbb{I}^*_{2I_{ij}} = \mathbb{I}_{2I_{ji}}$. Then, the vector v_i is constructed from this projector as follows:

$$v_i^{\mathrm{EPR}} \equiv \iota_\epsilon(\alpha_i) \left(\bigotimes_{j \neq i} \mathbb{I}_{2I_{ij}} \right), \tag{32}$$

where $\iota_\epsilon(\alpha_i)$ is a $SO(3)$ intertwiner, viewed as an element of the tensor product $\otimes_{j\neq i} V_{2I_{ij}} *$, characterized by $\epsilon \in \{0, +, -\}$ and the $SO(3)$ representation α_i as illustrated in Fig. 4. As the vector v_i^{EPR} is totally determined by a $SO(3)$ representation α_i and a choice of basis ϵ, we will identify in the sequel the vector v_i^{EPR} with the couple (α_i, ϵ). The pictorial representation of v_i is the following:



(33)

Note that we made a particular choice for ϵ to draw the picture; another choice would lead to a different contraction of the four edges colored by the representations $2I_{ij}$. Contrary to the BC model, the EPR intertwiner between four given representations I_{ij} is not unique for it depends on α_i and ε, both belonging to a finite set.

Now, it is possible to decompose the EPR intertwiner in any tensor product basis of the space of four-valent $SU(2) \times SU(2)$ intertwiners. We are interested in its decomposition in the basis of the type (ϵ, ϵ) whose elements

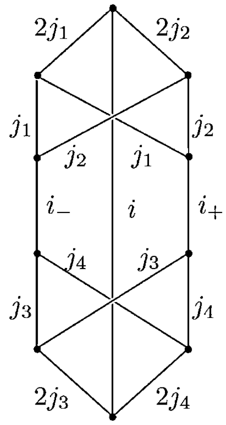


FIG. 7. EPR fusion coefficients. The edges are colored with $SU(2)$ representations and the vertices with symmetric $SU(2)$ $3j$ symbols. The picture illustrates the coefficient $f(\omega_i, I_{ij}, \iota_\epsilon(\alpha))$ for $I_{ij} = \{j_1, j_2, j_3, j_4\}$, ω_i is characterized by i (and some ϵ) and $\alpha = (i_+, i_-)$.

are denoted $\iota_\epsilon(\alpha)$. After some simple calculation, we recover the following expression of the EPR intertwiner given in the literature

$$\omega_i^{\mathrm{EPR}} = \sum_\alpha f(\omega_i, I_{ij}, \iota_\epsilon(\alpha))\iota_\epsilon(\alpha), \tag{34}$$

where the coefficient f is graphically "represented" in Fig. 7 and the sum is finite and runs over $SU(2) \times SU(2)$ representations α with a fixed chosen basis ϵ. In the notation of Engle-Pereira-Rovelli, α is denoted (i_+, i_-) and the representation defining ω_i is denoted i. Note that the sum (34) is not restricted to simple representations.

Now, we have all the ingredients to compute the vertex amplitude $V_{\mathrm{EPR}}(I_{ij}, \omega_i^{\mathrm{EPR}})$ for the EPR model. From the expression (34), we show immediately that

$$V_{\mathrm{EPR}}(I_{ij}, \omega_i^{\mathrm{EPR}}) = \sum_{\alpha=(i_+,i_-)} f(\omega_i, I_{ij}, \iota_\epsilon(i_+, i_-)) \times V_{\mathrm{BF}}(I_{ij}, i_+)V_{\mathrm{BF}}(I_{ij}, i_-), \tag{35}$$

where $V_{\mathrm{BF}}(I_{ij}, i_\pm)$ are the $SU(2)$ $15j$ symbols which depends on the representations I_{ij} and α but also on the choice of the basis ϵ which has not been explicitly written. The sums runs over $SU(2) \times SU(2)$ representations α with a fixed ϵ. Such a formula is rather complicated and one might prefer working instead with an integral formula of the form (5). To obtain such a formula, one has to separate in the integral (8) the variables u_i from the variables x_i as in (16) and then to perform the integration over the variables u_i. These last integrations are very simple to compute: the integration over u_3 is trivial and those over the remaining variables u_i give a simple normalization factor $N = (d_{2I_{12}} d_{2I_{45}} d_{\omega_1} d_{\omega_5})^{-1}$.

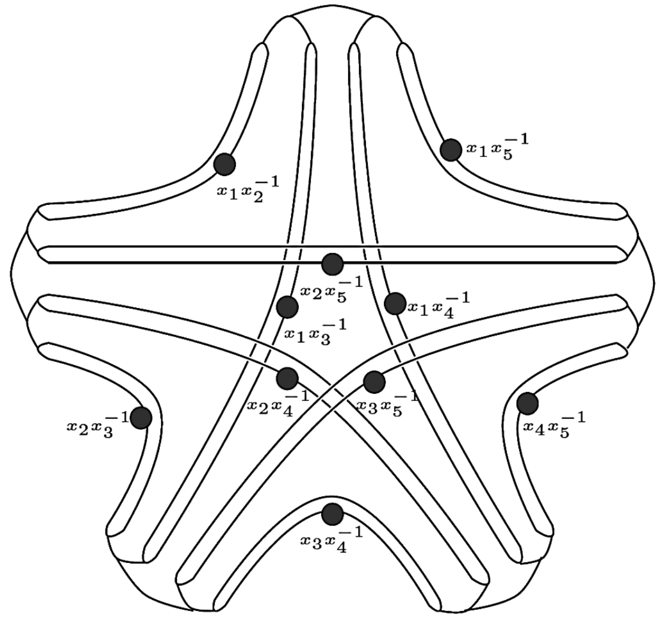


FIG. 8. Pictorial representation of the EPR argument in the integral formula: vertices are labeled by $i = 1, \cdots, 5$ where $i = 1$ is the top vertex and the others are enumerated according to the anticlockwise orientation; edges are then oriented and are labeled by (ij) with $i < j$. The doubled lines are colored with simple representations (I_{ij}, I_{ij}). The lines (ij) in the same pair are linked to a line colored with the representation $2I_{ij}$. At each vertex, the four single lines are linked with a line of representation ω_i.

Afterwards, the vertex amplitude reduces to the formula

$$V_{\mathrm{EPR}}(I_{ij}, \omega_i) = N \int \prod_{i\neq j} dx_{ij} C(x_{ij}) \mathcal{V}(I_{ij}, \omega_i; x_{ij}), \tag{36}$$

where the amplitude $\mathcal{V}$ is a function of the ten variables x_{ij} and is graphically represented in Fig. 8. This formula is the EPR counterpart of the formula (27) for the BC model. It will appear very useful in the next section to make a contact with loop quantum gravity. It might also be useful to study the classical and semiclassical properties of the EPR model as it is the case for the BC model.

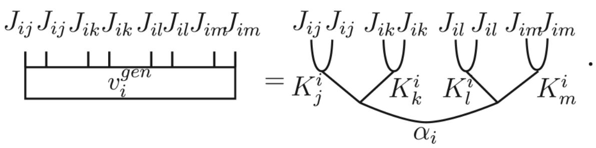
4. *A direct generalization: the Freidel-Krasnov models*

This section is devoted to present a very direct generalization of the EPR model. This generalization leads to a large class of spin-foam models to which both the EPR and the BC models belong. This generalization is very well-known and corresponds to considering generic projected spin networks, as boundary states. The corresponding vertex amplitude has been studied and computed in [26].

To motivate the construction of FK models, let us recall that the vector v_i^{EPR}, necessary to define the EPR intertwiner ω_i^{EPR}, has been constructed making use of a projector $\mathbb{I}_{2I_{ij}}$ from the vector space of the $SU(2) \times SU(2)$ simple representation (I_{ij}, I_{ij}) into the $SO(3)$ vector space representation $U_{2I_{ij}}$. A direct generalization would be to define a vector $v_i^{\rm gen}$ using instead, at each vertex i, projectors $\mathbb{I}_{K_j^i}$ from $V_{I_{ij}I_{ij}}$ into the $SO(3)$ representation $U_{K_j^i}$ for any representation $K_j^i \in [0, 2I_{ij}]$. The formal expression of the general vector is then the following:

$$v_i^{\rm gen} \equiv \iota_\epsilon(\alpha_i)\left(\bigotimes_{j\neq i}\mathbb{K}_j^i\right). \tag{37}$$

The vector $v_i^{\rm gen}$ so defined depends on the choice of the intertwiner $\iota_\epsilon(\alpha_i)$ and on the representations K_j^i. It is represented by the following diagram



(38)

This leads to a vertex amplitude very similar to the EPR one. In particular, its integral formula takes the same form of (36) where the normalization factor is changed into $N = (d_{I_2^1} d_{I_5^4} d_{\omega_1} d_{\omega_1})^{-1}$ and the function $\mathcal{V}$ is represented by the same graph drawn in Fig. 8 with different spin labels.

As a consequence, we get a large class of spin-foam models vertex amplitudes $V_i^{\rm gen}$ which depends not only on the ten representations I_{ij} coloring the faces of the four simplex but also depends on five other representations per tetrahedron i which have been denoted α_i, K_j^i. Up to now, only special cases of such models have been studied: the BC model where $K_j^i = \alpha_i = 0$, the EPR model where $K_j^i = K_i^j = 2I_{ij}$ and α_i is a free parameter. Thus, either we choose to project into the trivial representation either into the high test representation. The FK model consists in another choice of the representations K_j^i and α_i.

Many arguments lead to the fact that the EPR intertwiners define the good physical model, namely, the one which should reproduce the discretized path integral of the Euclidean Plebanski theory.

III. THE VERTEX AND THE PHYSICAL SCALAR PRODUCT

In this section we are proposing a link between (covariant) spin-foam models and (canonical) loop quantum gravity. To explain our strategy, we start by recalling some needed basic results of LQG. One of the main points of LQG is the assumption that physical states can be constructed from the so-called kinematical Hilbert space $\mathcal{H}_{\rm kin}$ which consists in the space of cylindrical functions endowed with the kinematical scalar product $\langle , \rangle$ defined from the $SU(2)$ Haar measure. The spin-network states form an orthonormal basis of $\mathcal{H}_{\rm kin}$. Then, the idea is basically to impose the constraints of gravity to extract physical states out of the kinematical space. So far, we know how to impose the Gauss constraint and the space-diffeomorphisms constraints and this leads to the construction of the diffeomorphism invariant states: they form the space $\mathcal{H}_{\rm diff}$ which is endowed with the Ashtekar-Lewandowski measure [27]. To be more correct, diffeomorphism invariant states are not elements of the kinematical spaces but rather dual elements. Their precise construction is well described in [1]. The physical Hilbert space $\mathcal{H}_{\rm phys}$ is still unknown but expected to be constructed from the Ashtekar-Lewandowski measure. Up to now, we do not how to solve the remaining Hamiltonian constraint. Spin-foam models have been introduced as an alternative to find physical states and the physical scalar product in the sense that the amplitude of a spin-foam models should reproduce the physical scalar product between the states at the boundary of the spin foam. This section aims precisely at clarifying this last point in a simple case.

More precisely, we consider the spin foam associated to the four-simplex graph denoted Γ. Its amplitude is given, up to some eventual normalization factors, by the vertex amplitude V. From the general boundary (covariant) formulation point of view, Γ is viewed as a graph interpolating between two kinematical boundary states which are τ_1 and τ_4 as schematically depicted in Fig. 9. In fact, as shown in Fig. 9, τ_1 and τ_4 belong to the space $\mathrm{Cyl}(\tilde{\Gamma})$ where $\tilde{\Gamma}$ is the union of Γ with four free ends. These free ends have been added for technical purposes only. Notice that Γ can be equivalently interpreted as the graph interpolating between two different graphs that would be denoted τ_2 (with two vertices) and τ_3 (with three vertices). For that, one would

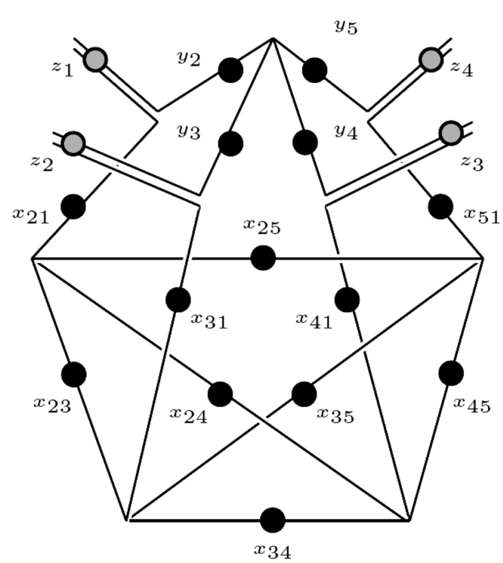


FIG. 9. Representation of the graph $\tilde{\Gamma}$. The subgraphs associated to τ_1 and τ_4 have been underlined and the group variables associated to each edge have been emphasized.

need to introduce also some free ends at the graph Γ. From the canonical point of view, the states τ_1 and τ_4 are considered schematically as cylindrical functions on the graph $\tilde{\Gamma}$. Therefore, one naturally asks the question whether it exists a "physical projector" P acting on the space $\mathrm{Cyl}(\tilde{\Gamma})$ such that its matrix element $\langle \tau_4, P\tau_1\rangle$ constructed from the kinematical scalar product gives the vertex amplitude. The notation $\langle \tau_4, P\tau_1\rangle$ can be misleading because P has in fact to be viewed as a state in the sense of Gelfand-Naimark-Segal (GNS), i.e., P is a linear form on $\mathrm{Cyl}(\tilde{\Gamma})$, and the physical scalar product reads $\langle \tau_4, P\tau_1\rangle = P(\bar{\tau}_4\tau_1)$. We abusively use the same notation for the projector viewed as a " matricial operator" or a linear form. To be interpreted as a GNS state, P has to satisfy additional properties, like the positivity, that we will not discuss here. We show that it is possible to construct explicitly such an operator P for the topological, the BC, and the EPR models. The projector for the FK model can also be obtained immediately generalizing the construction in the EPR case. We will use the obvious notations P_{BF}, P_{BC}, and P_{EPR} to denote the physical projector in the different cases.

There are two important points to clarify. The first one is the issue of uniqueness of the solution: we find one (class of) solution(s) for P in each model but we do not know if it is unique (in some precise sense of course). Second, we work in the kinematical Hilbert space and we expect P to behave correctly with respect to diffeomorphisms invariance in order to extend it to $\mathcal{H}_{\mathrm{diff}}$. We hope to address these important mathematical issues in the future.

A. The topological model

The topological model is the simplest case to consider. Even if it is not of a great physical interest, it is a good toy model to test the possibility of constructing a "physical projector" P. Furthermore, we will see that this construction will be useful to study the other more physical cases. Let us emphasize that the construction of P_{BF} is very similar to the construction of the projector into physical states in three dimensions as expected from the topological nature of the model.

As we said in the introduction of this section, the boundary states τ_1 and τ_4 are elements of $\mathrm{Cyl}(\tilde{\Gamma})$: τ_1 is a function of the eight group variables y_h, z_k, with $k = 1, \cdots, 4$ and $h = 2, \cdots, 5$, as shown in Fig. 10; τ_4 is a function of 14 group variables, ten of them are denoted x_{ij} with i, $j = 1, \cdots, 5$ and $i \neq j$, and the four remaining are the z_k variables as shown in Fig. 10. Note that the z_k group variables are those associated to the free ends of $\tilde{\Gamma}$ which are common to the spin-network graphs associated to τ_1 and τ_5.

We now address the concrete question of finding the projector P_{BF} such that $\langle \tau_4, P_{\mathrm{BF}}\tau_1\rangle$ is, up to some eventual irrelevant normalization factors, the vertex amplitude V_{BF}. Of course, we have implicitly assumed that τ_1 and τ_4 are

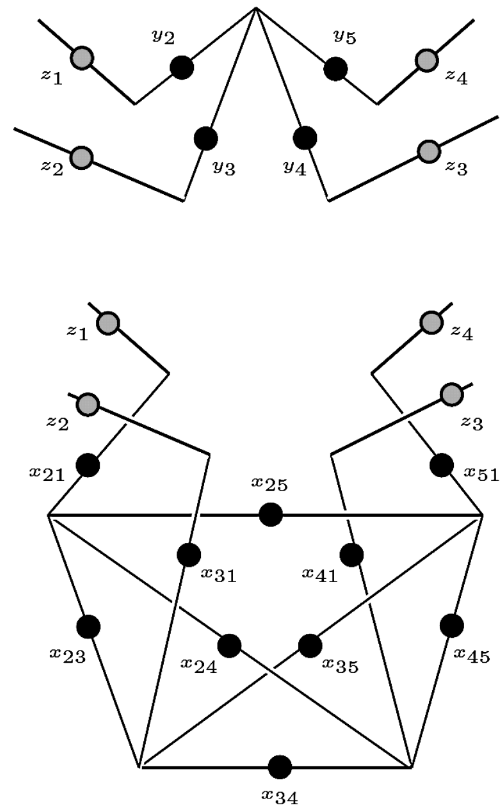


FIG. 10. Pictorial representation of the graph associated to τ_1 and τ_4, separately. The free edges are oriented from the vertices to the free ends; the internal edges are oriented according to the order on the vertices. The variables associated to the free ends are denoted z_k for the two graphs; those associated to the internal edges of τ_4 are denoted x_{ij} with $i, j = 2, \cdots, 5$; those associated to the internal edges of τ_1 are denoted y_h.

spin-network states, i.e., they are associated to a coloring of the edges and the vertices of their associated graphs. Concerning τ_1, its vertex is colored with an intertwiner denoted ω_1 and each edge associated to the variables y_k are colored with a representation denoted J_k. Concerning τ_4, its vertices i are colored with intertwiners ω_i and each edge associated to the variables x_{ij} are colored with representations I_{ij}.

The operator P_{BF} has to be a discretization of the flatness condition on the connection: it is a cylindrical distribution on Γ which imposes that the holonomies around the closed faces of Γ are trivial. One candidate which realizes such a requirement is given by

$$P_{\mathrm{BF}} = \delta(x_{123})\delta(x_{234})\delta(x_{345})\delta(x_{451})\delta(x_{512}) \qquad (39)$$

with the notation of (21). We need only five delta distributions to impose the flatness condition on the ten faces of the four simplex. Furthermore, we see that P_{BF} is nothing but the distribution $C(x_{ij})$ we have previously introduced (21). To show that this operator is indeed a solution of our problem, let us compute its matrix element between the states τ_1 and τ_4 making use of the kinematical scalar product

$$\langle\tau_4, P_{\rm BF}\tau_1\rangle = \int\Big(\prod_{i\neq j}^{5} dx_{ij}\Big)\Big(\prod_{k=2}^{5} dy_k\Big) \times\Big(\prod_{\ell=1}^{4} dz_\ell\Big)\overline{\tau_4(x, z_\ell)}C(x, y)\tau_1(y, z) = \prod_{k=1}^{4}\frac{\delta_{J_k,I_{1k}}}{d_{I_{1k}}}\int\Big(\prod_{i\neq j}dx_{ij}\Big)C(x_{ij})\tau_5(x_{ij}), \quad (40)$$

where $\tau_5(x_{ij})$ is the spin-network state associated to the four-simplex graph. To obtain this result, we have performed the integration over the z variables first, then we have absorbed the y variables using the invariance of the Haar measure to get as a final result an integral involving only the variables x_{ij}. At this point, it is immediate to see that the previous integral simplifies and we have

$$\langle\tau_4, P_{\rm BF}\tau_1\rangle = \Big(\prod_{k=1}^{4}\frac{\delta_{J_k,I_{1k}}}{d_{I_{1k}}}\Big)V_{\rm BF}(I_{ij}, \omega_i). \quad (41)$$

Up to a renormalization factor, the physical scalar product gives exactly the desired vertex amplitude of the topological model. Therefore, we found a projector P into the physical states of the topological model.

Let us finish the study of this case with some remarks. First, the construction of $P_{\rm BC}$ can be easily generalized to the space of all cylindrical functions: we only have to impose the flatness condition around the closed loops of the spin networks, but taking into account the fact that one has to avoid redundant delta distributions in order to have a finite amplitude. Second, as we have already said, the projector $P_{\rm BC}$ has a clear physical interpretation in the sense that it is a discretization of the first class constraints of the BF theory. For that reason, one can suppose that the solution we found is unique. As a final remark, let us emphasize that, even if the topological model is not physically interesting, it will appear very useful to understand the gravitational models, namely, the BC and the EPR models. Indeed, the three models admit the same kinematical Hilbert space and, as we will see, the operators $P_{\rm BC}$ and $P_{\rm EPR}$ are constructed from the operator $P_{\rm BF}$ we have just constructed. In other words, the physical scalar products of the gravitational models are obtained from the physical scalar product of the topological model. This aspect will be precisely described in the next section.

B. The Barrett-Crane model

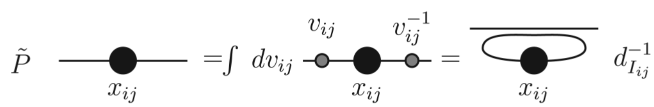
This section is devoted to the construction of the operator $P_{\rm BC}$. For that purpose, we use the same notations as in the previous section concerning the space of cylindrical functions ${\rm Cyl}(\tilde\Gamma)$, in particular, concerning the states τ_1 and τ_4. This makes sense because the topological and the BC models possess the same kinematical Hilbert space. By kinematical Hilbert space, we mean the space of $SU(2)$ gauge invariant functions which is needed to construct physical states. Thus, we look for an operator $P_{\rm BC}$ acting on the space of cylindrical functions ${\rm Cyl}(\tilde\Gamma)$ such that

$$\langle\tau_4, P_{\rm BC}\tau_1\rangle = N\Big(\prod_{k=1}^{4}\delta_{J_k,I_{1k}}\Big)V_{\rm BC}(I_{ij}, \omega_{\rm BC}), \quad (42)$$

where N is an eventual normalization factor. We propose a solution where the projector is the product $P_{\rm BC} = P_{\rm BF}\tilde P_{\rm BC}$ of the projector $P_{\rm BF} = C(x_{ij})$ of the topological model and another operator $\tilde P_{\rm BC}$ we are going to define. First, $\tilde P_{\rm BC}$ has a nontrivial action on ${\rm Cyl}(\Gamma)$ but can be trivially extended to the space ${\rm Cyl}(\tilde\Gamma)$. Then, its action on any function $F \in {\rm Cyl}(\Gamma)$ is explicitly given by

$$(\tilde P_{\rm BC}F)(x_{ij}) = \int\Big(\prod_{i<j}dv_{ij}\Big)F(v_{ij}x_{ij}v_{ij}^{-1}), \quad (43)$$

where we used the obvious notation x_{ij} for the group variable associated to the oriented edge (ij) of Γ. Thus, $\tilde P_{\rm BC}$ acts nontrivially on the internal edges of $\tilde\Gamma$; this action can be graphically represented as follows:



(44)

Let us now see that $P_{\rm BC}$ reproduces the physical scalar product in the sense of the equation (42). Indeed, an immediate calculation leads to the result

$$\langle\tau_4, P_{\rm BC}\tau_1\rangle = \Big(\prod_{k=1}^{4}\frac{\delta_{J_k,I_{1k}}}{d_{I_{1k}}}\Big)\int\Big(\prod_{i<j}dx_{ij}\Big)(P_{\rm BC}\tau_5)(x_{ij}) \quad (45)$$

$$= \Big(\prod_{k=1}^{4}\frac{\delta_{J_k,I_{1k}}}{d_{I_{1k}}}\Big)\tau_5(1)\int\Big(\prod_{i\neq j}dx_{ij}\Big)\frac{\chi_{I_{ij}}(x_{ij})}{d_{I_{ij}}}C(x_{ij}), \quad (46)$$

where $\tau_5(1)$ is the spin network τ_5 evaluated at the identity $x_{ij} = 1$, then it is the vertex amplitude of the topological model, i.e., a $SU(2)$ $15j$ symbol. Thus, the previous equation can be recasted as follows:

$$\frac{\langle\tau_4, P_{\rm BC}\tau_1\rangle}{\langle\tau_4, P_{\rm BF}\tau_1\rangle} = \Big(\prod_{k=1}^{4}\frac{\delta_{J_k,I_{1k}}}{d_{I_{1k}}}\Big)V_{\rm BC}(I_{ij}, \omega_i^{\rm BC}). \quad (47)$$

Up to some normalization factor, the operator $P_{\rm BC}$ reproduces the vertex amplitude of the BC model. Thus, $P_{\rm BC}$ can be interpreted as a projection into physical states of the BC model. Note however that the normalization factor does not have a clear interpretation for it cannot be reabsorbed into the whole spin-foam sum as a face or a tetrahedron amplitude.

The construction we are proposing rises many important remarks.

Remark 1. The operators $P_{\rm BF}$ and $\tilde{P}_{\rm BC}$ do not commute and therefore the order of their product clearly matters. The operator $P_{\rm BF}$ is a multiplicative operator that impose the discrete analogous of a flatness—connection condition. This operator might be related to a projector into space-diffeomorphism invariant states. This interpretation is based on the fact that, in three dimensions, the flatness constraint on the connection generates diffeomorphisms even if, in 4D, the situation is more complicated. The operator $\tilde{P}_{\rm BC}$ is a kind of "derivative" operator for its action involves $SU(2)$ right and left derivatives. Its physical interpretation is not clear.

Remark 2. If one believes that the BC model is related to gravity, then it is clear that $P_{\rm BF}$ is the projection into $\mathcal{H}_{\rm diff}$ and $\tilde{P}_{\rm BC}$ should contain the projection into the kernel of the Hamiltonian constraint. This is far from being obvious and that conjecture is even false if the BC model is not the one that discretizes gravity as it is suspected. Let us notice that, in our construction, $\tilde{P}_{\rm BC}$ acts first and then acts $P_{\rm BC}$ which is contrary to what one usually does in LQG where the projection into $\mathcal{H}_{\rm diff}$ arises before the projection into the kernel of the Hamiltonian constraint.

Remark 3. The operator $\tilde{P}_{\rm BC}$:Cyl$(\Gamma) \rightarrow C(SU(2))^{\times 10}_{\rm Ad}$ is in fact a projector from the space of cylindrical functions to ten copies of the space of functions on the conjugacy classes $C(SU(2))_{\rm Ad}$ of the group $SU(2)$ where $F \in C(SU(2))_{\rm Ad}$ if and only if $F(gxg^{-1}) = F(x)$ for any x and g in $SU(2)$. Its action on a τ_5 spin-network state is given by

$$(\tilde{P}_{\rm BC}\tau_5)(x_{ij}) = \tau_5(1)\prod_{i<j}\frac{\chi_{I_{ij}}(x_{ij})}{d_{I_{ij}}}, \tag{48}$$

where I_{ij} are the representations coloring the edges (ij) of the graph Γ. It is straightforward to check that $\tilde{P}^2_{\rm BC} = \tilde{P}_{\rm BC}$. As a consequence, for the definition of $P_{\rm BC}$ to make sense, one has to extend $P_{\rm BF}$ as an operator acting on $C(SU(2))^{\times 10}_{\rm Ad}$ which is trivial.

Remark 4. In fact, the decomposition of $P_{\rm BC}$ as the product of $P_{\rm BF}$ and $\tilde{P}_{\rm BC}$ is not canonical. Our construction provides an equivalent class of functions $\tilde{P}_{\rm BC}$ according to the trivial relation $\tilde{P}_{\rm BC} \sim \tilde{Q}_{\rm BC}$ if and only if $P_{\rm BF}\tilde{P}_{\rm BC} = P_{\rm BF}\tilde{Q}_{\rm BC}$. Another natural choice for the derivative operator is $\tilde{Q}_{\rm BC}$ defined by its following action on τ_5 spin-network states:

$$(\tilde{Q}_{\rm BC}\tau_5)(x_{ij}) \equiv \tau_5(x_{ij})\prod_{i<j}\frac{\chi_{I_{ij}}(x_{ij})}{d_{I_{ij}}}. \tag{49}$$

This representative is clearly a multiplicative operator.

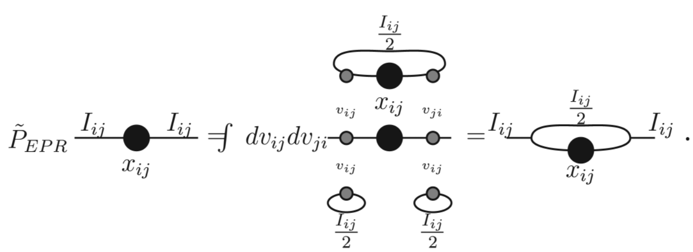
Remark 5. As a last remark, let us underline that the physical scalar product between two states in the BC model (47) can be viewed as the matrix element of the operator $\tilde{P}_{\rm BC}$ with respect to the physical scalar product of the topological model up to the "norm" $\langle \tau_4, P_{\rm BF}\tau_1\rangle$. In that sense, the BC model is very closely related to the topological model.

C. The Engle-Pereira-Rovelli model

In this section, we propose an operator $P_{\rm EPR}$ which reproduces the vertex amplitude of the EPR model. The construction of $P_{\rm EPR}$ is very similar to the construction of $P_{\rm BC}$. As for the BC model, $P_{\rm EPR}$ is the product of the noncommuting operators, $P_{\rm EPR} = P_{\rm BF}\tilde{P}_{\rm EPR}$, one of them being the projector of the topological model as well. The operator $\tilde{P}_{\rm EPR}$ is defined by its action on spin-network states $\tau_5(x_{ij})$ explicitly given by

$$\begin{aligned}(\tilde{P}_{\rm EPR}\tau_5)(x_{ij}) = \int&\Big(\prod_{i<j}dv_{ij}dv_{ji}\Big)\\ &\times\Big[\prod_{i<j}\chi_{\tilde{I}_{ij}}(v_{ij}x_{ij}v_{ji})\chi_{\tilde{I}_{ij}}(v_{ij})\chi_{\tilde{I}_{ij}}(v_{ji})\Big]\\ &\times\tau_5(v_{ij}x_{ij}v_{ji}),\end{aligned} \tag{50}$$

where we have introduced the notation $\tilde{I}_{ij} = I_{ij}/2$. As in the BC model, $\tilde{P}_{\rm EPR}$ acts on each edge of the spin network and this action can be pictured as follows:



(51)

In this figure, the closed loops represent $SU(2)$ characters. The last equality has been obtained after integrating over the v_{ij} variables. Using this pictorial representation, it is quite easy to compute the matrix elements of $P_{\rm EPR}$ between the states τ_1 and τ_4

$$\langle \tau_4, P_{\rm EPR}\tau_1\rangle = \Big(\prod_{k=1}^{4}\frac{\delta_{J_k,I_{1k}}}{d_{I_{1k}}}\Big)\int\prod_{i<j}dx_{ij}(P_{\rm EPR}\tau_5)(x_{ij}) \tag{52}$$

$$= \left(\prod_{k=1}^{4} \frac{\delta_{J_k, I_{1k}}}{d_{I_{1k}}}\right) N^{-1} V_{\mathrm{EPR}}(I_{ij}, \omega_i^{\mathrm{EPR}}), \tag{53}$$

where N is the normalization factor introduced in (36). As a consequence, we claim that the matrix elements of P_{EPR} reproduce the vertex amplitude of the EPR model. Let us now finish this section with some important remarks.

Remark 1. The previous remarks (1 and 2) concerning the BC model can be transposed to the EPR model. In particular, if the EPR model is a discretization of the path integral of gravity, $\tilde{P}_{\mathrm{EPR}}$ should be closely related to the Hamiltonian constraint. If this is true, our formula could give some hints about the regularization of the Hamiltonian constraint. Furthermore, we can easily generalize the construction to any cylindrical functions with no restriction on the underlying graph.

Remark 2. The operator $\tilde{P}_{\mathrm{EPR}}$ is constructed making use of an integration over 20 variables v_{ij} with $i \neq j$ because $v_{ij} \neq v_{ji}$ in the formula (50). Contrary to the BC model, $\tilde{P}_{\mathrm{EPR}}$ is not a projector neither an operator from $\mathrm{Cyl}(\Gamma)$ to the space of functions on the $SU(2)$ conjugacy classes. The integral (50) can be reduced to an integral over only ten variables v_{ij} with $i < j$ as follows:

$$(\tilde{P}_{\mathrm{EPR}}\tau_5)(x_{ij}) = \int \left(\prod_{i<j} dv_{ij}\right) \times \left[\prod_{i<j} \frac{\chi_{\tilde{I}_{ij}}(v_{ij})\chi_{\tilde{I}_{ij}}(x_{ij}^{-1}v_{ij})}{d_{\tilde{I}_{ij}}}\right] \tau_5(v_{ij}). \tag{54}$$

To obtain such a formula, we have first integrated over the variables v_{ij} with $i > j$ and then we have performed some changing of variables.

Remark 3. Our construction can be generalized immediately to the FK models presented in Sec. II C 4. The resulting operator P_{FK} would take exactly the same form as P_{EPR} with some differences in the representations of the characters in the integrand of (50).

Remark 4. Concerning the unicity of $\tilde{P}_{\mathrm{EPR}}$, we can make the same remark 4 as in the BC model, namely, our construction provides a certain equivalent class of solutions for $\tilde{P}_{\mathrm{EPR}}$ and the decomposition of P_{EPR} as a product of P_{BF} and $\tilde{P}_{\mathrm{EPR}}$ is not canonical.

IV. CONCLUSIONS AND PERSPECTIVES

On the first hand, this article opens one way towards the understanding of an eventual link between loop quantum gravity and spin-foam models. We have shown that the vertex amplitudes of some spin-foam models can be precisely interpreted as a "physical" scalar product between two spin networks, only if one of the spin-foam models we have studied is a quantization of gravity and this point is still under active discussions. If this is the case, then our work would make a relation between the canonical and covariant quantizations of four dimensional Euclidean gravity. It is indeed possible to construct operators P acting on the space $\mathrm{Cyl}(\Gamma)$ of cylindrical functions on the (extended) four-simplex graph $\tilde{\Gamma}$ such that its matrix elements between spin-networks states gives, up to some eventual normalization, the vertex amplitudes for spin-foam models. In a formal language, we have shown that

$$\langle s, Ps' \rangle = \mathcal{A}(s, s'), \tag{55}$$

where $\langle , \rangle$ is the kinematical scalar product; s and s' belong to $\mathrm{Cyl}(\tilde{\Gamma})$ and $\mathcal{A}(s, s')$ is the spin-foam amplitude of a graph interpolating between s and s' which is, here, proportional to the vertex amplitude. The construction works for the topological model, the Barrett-Crane model, the Engle-Pereira-Rovelli model, and their direct generalizations, namely, the Freidel-Krasnov models.

On the other hand, the same article opens questions that certainly deserve to be investigated. The first one concerns the possibility to extend our construction to the case where the spin-networks s and s' (55) are any cylindrical functions and not restricted to $\mathrm{Cyl}(\tilde{\Gamma})$ as this was the case in this article. It is clear that the action of the operators P we have constructed can be easily extended to any spin-networks with no assumption on the underlying graph defining the spin networks. It would be very nice to first compute the matrix elements of P between these general states and to check if the result is related to a spin-foam amplitude associated to a graph interpolating between the two associated spin-network graphs. We hope to study this very exciting problem in the close future.

The second question concerns the link between the operators P we have constructed and the regularization of the Hamiltonian constraint *à la* Thiemann. Indeed, one would expect that, if the spin-foam models are a discretized version of the path integral of gravity, then P should be related to the Hamiltonian constraint. It is nonetheless intriguing to notice an important difference between the ways the constraints are imposed in LQG and in the spin-foam models through the operators P: indeed, in LQG, one imposes the vectorial constraint before imposing the scalar constraint whereas the operator $P = P_{\mathrm{BF}}\tilde{P}$ is the noncommutative product of two operators, the second one P_{BC} imposes clearly the space-diffeomorphisms invariance and "projects" into the vectorial constraint kernel. Of course, it is too early to conclude anything but its seems to have a quite important discrepancy between the two approaches. To understand more precisely these aspects, one could start by understanding the link between the projector P and the classical constraints of gravity. Another disturbing point concerning the order of the two operators defining the projector P is that it gives the feeling that only flat connections contribute to the dynamics whereas this is obviously not true in general relativity. Indeed, the flatness condition is imposed at the last step. This observation rises the question whether the BC and even the EPR or the FK spin-foam models are really related to gravity. This point certainly deserves to be deeply investigated. We have already found in the literature arguments against the fact that these spin-foam models are related to gravity [26].

The third question is more mathematical: is P a GNS state? Indeed, it is quite misleading to view P as an operator acting on cylindrical functions for it is a linear form on $\mathrm{Cyl}(\tilde{\Gamma})$. Thus it seems that the GNS theory is the good mathematical framework to study P. But, if one wants to interpret that P has a GNS state, one has to check that it satisfies all the required property, among others the positivity.

We finish this conclusion by mentioning the possibility that our work could give some hints to study the classical and semiclassical behaviors of the EPR model.

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