# Interacting field theories in de Sitter space are nonunitary

Emil T. Akhmedov[1,*] and P. V. Buividovich[1,2,+]

[1] *ITEP, B. Cheremushkinskaya street 25, Moscow, 117218, Russia*

[2] *JIPNR, National Academy of Science, Academician Krasin street 99, Minsk, 220109, Belarus*

(Received 29 August 2008; published 5 November 2008)

It is well known that there should be a total cancellation of the IR divergences in unitary interacting field theory, such as QED and gravity. The cancellation should be at all orders between loop and tree-level contributions to cross sections. This is the crucial fact related to the unitarity of the evolution operator (S-matrix) of the underlying interacting field theory. In this paper we show that such a cancellation does *not* happen in de Sitter space.

DOI: 10.1103/PhysRevD.78.104005 PACS numbers: 04.62.+v, 98.80.Cq

## I. INTRODUCTION

It is commonly accepted that de Sitter background should correspond to the lowest energy state in the gravity theory with the positive cosmological constant. The main argument behind this point of view is that de Sitter space has the largest possible isometry group with the given cosmological constant, while any deviation from de Sitter background breaks the isometry. As well, it can be shown that there are no exponentially growing linearized fluctuations in de Sitter space [1].

Let us address, however, the following question in the *interacting* field theory on de Sitter background: Does an inertially moving charged particle in de Sitter space emit radiation or not? Because the space in question is conformally flat, we propose to consider field theory which is not conformal, otherwise the behavior of fields is not much different from the one in Minkowski space. For example, free electromagnetic fields do not feel the expansion of de Sitter space and behave as if they are in the Minkowski one. However, we can consider either minimally coupled scalars and gravitons (whose free field theories already are nonconformal) or turn on interactions which break conformal invariance.

It is possible to fetch the answer on the posed question even before going into the calculational details using just general physical arguments. In fact, an inertially moving particle in de Sitter space accelerates with respect to a free floating (inertial) observer in the same space. Hence, it is tempting to think that the inertial particle should radiate from the point of view of the observer in question. We support these general comments with explicit calculation in the main body of the text. Somewhat similar phenomenon have been considered in [2].

But if the particle radiates it will do that eternally—as long as the particle and the background are left untouched. Where does the energy for the radiation come from? One can object that we should not ask about energy in a time dependent de Sitter-like background. But exactly in this objection resides the answer to our question. In fact, the Hamiltonian of an *interacting nonconformal* field theory in de Sitter background does not have a ground state, if the cosmological constant is held fixed. We argue that the radiation happens at the cost of the decrease of the cosmological constant, if the latter is not supported to be constant by any imaginary external influence.

What we are trying to point out here is that the situation in many respects is similar to the one in QED with the constant electric field if we use the creation/annihilation operators which correspond to the exact harmonics rather than just to plain waves. As is well known the similarity goes even further—up to the pair creation [3]—but to avoid any mystery of quantum field theory in curved backgrounds with horizons, we prefer to discuss a simple semiclassical (tree-level QFT) phenomenon and IR rather than UV behavior of quantum corrections, which can be completely understood with the presently existing level of knowledge. It is obvious that, even if we neglect the pair production, a charge placed in the constant electric field background will radiate at the cost of the decrease of the field in question or at the cost of work performed by the external source keeping the electric field constant. Similarly the cosmological constant should decrease if it is not held fixed by any external source.

In this note we would like to find a straightforward signal showing that there are problems with quantum fields in de Sitter background. Because of asymptotic nonflatness of de Sitter space (hence, no energy conservation) the radiation discussed above could be considered as not being a problem, but we show that it inevitably results in one. We observe that the evolution operator in de Sitter space is not unitary if we keep the cosmological constant fixed. The way to see that is through the noncancellation of the IR divergences between tree and loop contributions to the cross sections.

Let us sketch here the arguments presented in the main body of the text. The statement is that in de Sitter space a charged particle on mass shell does emit radiation. Hence, its virtuality [4] is not related to the momentum of the

*akhmedov@itep.ru
+buividovich@tut.by

 

emitted radiation. Thus, the tree-level cross sections for emitting soft radiation are finite. In fact, recall that IR divergences in cross sections (of a QFT without background fields) appear because the propagators of the particles which create radiation, being proportional to their inverse virtuality, are singular as the momentum of the emitted radiation goes to zero [5]: As follows from the energy-momentum conservation the virtuality is proportional to the momentum of the soft radiated quantum. The power of the singularity is such that, after the integration over the phase volume of the radiated soft quantum in any cross section, the logarithmic IR divergence appears. Such IR divergences cancel with the ones appearing in loop corrections to the cross sections [5]. The cancellation can be directly linked to the unitarity of the S-matrix in the QFT [6] or, more concretely, to the optical theorem.

Now in the case of de Sitter space, while tree-level cross sections are finite, the loop diagrams do have IR divergences. As a result, unlike the situation in a QFT without background fields, there is nothing which can cancel them. Hence, the evolution operator in de Sitter space is not unitary, as it should be for a nonclosed system due to the presence of a background field which is held fixed by an external source.

It is worth pointing out here that the problems with cancellation of the IR divergences appear in de Sitter space even if we respect the de Sitter isometry at every step of the calculation, i.e. de Sitter space is unstable and the isometry is broken, at least if one turns on interactions. It is unstable in the sense that the cosmological constant will decrease, which will result eventually in the Friedmann-Robertson-Walker (FRW) universe (with nonaccelerating expansion). In the latter case we are going to have an analog of the standard Minkowski vacuum for quantum fields.

## II. GENERAL DISCUSSION

We set the cosmological constant to be one and keep it fixed throughout the paper. Although our arguments are general, for simplicity we would like to consider two minimally coupled real scalar particles in $D$-dimensional de Sitter space with the Yukawa type interaction:

$$S_{\mathrm{matter}} = \frac{1}{2}\int d^D x\sqrt{-g}[g^{ab}\partial_a\Psi\partial_b\Psi + M^2\Psi^2 + g^{ab}\partial_a\psi\partial_b\psi + m^2\psi^2 + \lambda\Psi^2\psi]. \quad (1)$$

It will become clear from the discussion below that the reason for consideration of such a theory is that we would like to keep both masses $m$ and $M$ greater than zero.

This theory in Minkowski space does possess the cancellation of the IR divergences if $m = 0$ and $M > 0$ or does not have them at all if $M, m > 0$. As well one can consider nonminimally coupled scalars if they will make such substitutions as $m^2 \to m^2 + \zeta R$ for all scalar masses, where $R$ is the de Sitter curvature, and $\zeta$ is a parameter. Conformal coupling corresponds to the case $m = 0$, $\zeta = (D-2)/4(D-1)$. It is important that the interaction term breaks the conformal invariance in any case.

At first sight the most convenient reference frame where one can do all the calculations is the planar one

$$ds^2 = -dt^2 + e^{2t}dx_i dx_i = \frac{1}{\tau^2}(-d\tau^2 + dx_i dx_i), \quad (2)$$

where $\tau = e^{-t}$, because then we have to deal with the noncompact spacial sections and the formulas for amplitudes are very similar to those in Minkowski space QED with constant electric field. Unfortunately in these coordinates we encounter problems. To see them, consider the Klein-Gordon equation describing propagation of free waves in these coordinates:

$$(\tau^2\partial_\tau^2 - (D-2)\tau\partial_\tau - \tau^2\partial_i\partial_i + m^2)\psi = 0 \quad (3)$$

and similarly for $\Psi$. Because this is a free wave equation, its solutions obey the superposition principle. Hence, from the point of view of the observer, seeing the corresponding metric, particles (single waves) are just solutions of such an equation, having a finite flux to be defined below. But we would like to respect the de Sitter isometry, which restricts the choice of the basis of harmonics. A particular basis which leads to the de Sitter invariant vacuum state is as follows [7]:

$$\psi_k \propto e^{i\vec{k}\vec{x}}\tau^{(D-1)/2}H_\nu^{(2)}(k\tau), \qquad \nu = \sqrt{\left(\frac{D-1}{2}\right)^2 - m^2}, \qquad k = |\vec{k}|. \quad (4)$$

These harmonics correspond to the positive energy states, while their complex conjugates—to the negative. Here $H^{(2)}$ is the Hankel function: $H_\nu^{(1)} = (H_\nu^{(2)})^*$.

Let us stress here the main problem with these harmonics. The term linear in the differential over $\tau$ under the brackets in (3), which we refer to as a "friction" term, has a wrong sign as $\tau$ goes to $+\infty$ (past infinity). As the result among the harmonics present in the complete basis of the solutions of this equation we have those which are exponentially big (in $t$) in the past, when $D \geq 3$. In fact, the solution presented in (4) behaves, when $\tau \to +\infty$ ($t \to -\infty$), as follows:

$$\psi_k \propto \tau^{(D-2)/2}e^{-ik\tau + i\vec{k}\vec{x}}. \quad (5)$$

This happens because the metric in (2) is singular as $\tau \to +\infty$. As the result all loop diagrams have divergences in the $\tau \to +\infty$ corner of the time integration axis. Such a divergence is present along with the IR one which appears at the $\tau = 0$ corner.

We postpone the discussion of the problems with the definition of the S-matrix in de Sitter space to the following sections and define here the mass-shell three leg amplitude as follows. It is proportional to the integral:

$$A \propto \left\langle k, q \left| \int d^D x \sqrt{-g} \hat{\Psi}^2 \hat{\psi} \right| p \right\rangle$$
$$\propto \int_0^{+\infty} \frac{d\tau}{\tau^D} \int d^{(D-1)}\vec{x} e^{i(\vec{p}-\vec{k}-\vec{q})} \tau^{(3/2)(D-1)} H^{(1)}_{\nu_1}(p\tau)$$
$$\times H^{(2)}_{\nu_1}(k\tau) H^{(2)}_{\nu_2}(q\tau) = \delta^{(D-1)}(\vec{p}-\vec{k}-\vec{q})$$
$$\times \int_0^{+\infty} d\tau \tau^{(D-3)/2} H^{(2)}_{\nu_1}(p\tau) H^{(1)}_{\nu_1}(k\tau) H^{(1)}_{\nu_2}(q\tau). \qquad (6)$$

Here $|k, q\rangle = \hat{a}^+_k \hat{b}^+_q |\text{vac}\rangle$, etc. Here $\hat{a}$ and $\hat{b}$ are creation operators for the harmonics (4) of the fields $\Psi$ and $\psi$, correspondingly; $|\text{vac}\rangle$ is the de Sitter invariant Bunch-Davies vacuum [7] and $\nu_1 = \sqrt{(\frac{D-1}{2})^2 - M^2}$, $\nu_2 = \sqrt{(\frac{D-1}{2})^2 - m^2}$. Because of such a behavior as in (5), the integral in (6) looks like

$$A \propto \int^{+\infty} d\tau\, \tau^{(D-6)/2} e^{i(p-k-q)\tau}$$

at the upper integration limit. Hence, for $D \geq 5$ we have a divergence even in the tree-level amplitude independently of the value of $m$ and $M$. Similar problems appear in the loop diagrams starting with $D = 4$.

One can try to avoid the divergence by turning on the interactions at some finite $\tau_0$ and evolve to a future $\tau < \tau_0$. This is explicitly done in loop amplitudes in the papers [8] and implicitly in the papers [9,10]. However, in this way we break the de Sitter invariance by hand, because the latter acts on $\tau_0$. Hence, it is not an occasion that in the quoted papers a perturbation of the de Sitter metric which does not respect the invariance was observed. It is not that we completely disagree with such an approach, taking into account that de Sitter invariance is going to be dynamically broken anyway, but we just would like to show here that one will encounter problems even if they will always try to respect the invariance.

Before going further let us point out the meaning of the amplitude (6). First, let us stress that the calculation of the amplitude gives a generally covariant (and gauge invariant) way to address the question of radiation. In fact, if it is not vanishing for given directions of external momenta and when all its external legs are on mass shell, it just means that a single wave can split into two waves from the point of view of the observer corresponding to the background metric, in which all calculations have been done.

Second, notice that the amplitude (6) is proportional to the spacial $\delta$ function imposing the momentum conservation law $\vec{p} = \vec{k} + \vec{q}$. In Minkowski space the time integral would impose as well the energy conservation law $p_0 = k_0 + q_0$ via $\delta$ function. At the same time on mass shell in Minkowski space we obtain $p_0 = \sqrt{\vec{p}^2 + M^2}$, $k_0 = \sqrt{\vec{k}^2 + M^2}$, $q_0 = \sqrt{\vec{q}^2 + m^2}$ and the energy conservation condition does not have a solution. Hence, the amplitude is zero and mass-shell (inertial) particles in Minkowski space cannot emit radiation.

Now, in (6) we do not have the energy conservation due to the presence of the external gravitational field originating from the cosmological constant—similar to the QED in an external constant electric field. Thus, the amplitude is not zero on mass shell. For the case when it is convergent (i.e. when $D \leq 4$ and $M, m > 0$) this can be seen explicitly via numerical calculation of the integral in (6) using MATHEMATICA or MAPLE.

In our opinion, the aforementioned divergencies in loop diagrams at the past infinity ($\tau = +\infty$) are simply sort of boundary effects, which emerged because the planar coordinates cover only half of de Sitter space. To avoid the aforementioned divergencies in loop diagrams at the past infinity ($\tau = +\infty$) we propose to consider the global coordinate system:

$$ds^2 = -dt^2 + \cosh^2 t d\Omega^2_{D-1}, \qquad (7)$$

where $d\Omega^2_{D-1} = d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \cdots + \sin^2\theta_1 \ldots \sin^2\theta_{D-2} d\theta^2_{D-1}$. Unlike the planar coordinates, the global ones cover de Sitter space completely. An important feature of these and the planar coordinates is that they are seen by the inertial observer. The Klein-Gordon equation in these coordinates is as follows:

$$\left(\partial_t^2 + (D-2)\tanh t\, \partial_t + m^2 - \frac{\Delta_{D-1}(\Omega)}{\cosh^2 t}\right)\psi = 0. \qquad (8)$$

Here $\Delta_{D-1}(\Omega)$ is the Laplacian on the $(D-1)$-dimensional sphere.

The friction term in (8) is proportional to $\tanh t$ and changes sign from $-(D-1)$ in the past infinity ($t \to -\infty$) to $+(D-1)$ in the future infinity ($t \to +\infty$). As the result we obtain a complete set of modes which are finite at every value of $t$ at any given mass $m$. To solve (8) explicitly we can use the separation of variables:

$$\psi_{jn}(t, \Omega) = \varphi_j(t) Y_{jn}(\Omega). \qquad (9)$$

Here $\Delta_{D-1}(\Omega) Y_{jn}(\Omega) = -j(j + D - 2) Y_{jn}(\Omega)$ and $n$ is the multi-index $(n_1, \ldots, n_{D-2})$.

The spherical harmonics $Y_{jn}(\Omega)$ have obvious properties presented in [11]. The field $\varphi_j(t)$ obeys the obvious equation following from (8) and (9). This equation has two designated complete sets of solutions: the so-called "in" and "out" modes [12] (see as well [11]). The complete set of in modes is

$$\varphi^{\pm}_j(t) \propto \cosh^j(t) e^{(j+(D-1)/2 \mp i\mu)t} F\bigg(j + \frac{D-1}{2}, j + \frac{D-1}{2} \mp i\mu; 1 \mp i\mu; -e^{2t}\bigg), \qquad (10)$$

where $\mu = \sqrt{m^2 - (\frac{D-1}{2})^2}$ and $F(a, b; c; z)$ is the hypergeometric function. The solution (10) can be continued to the case when $m < (D-1)/2$. The out modes $\bar{\varphi}^{\pm}_j(t)$ are related to the in modes as follows $\bar{\varphi}^{\pm}_j(t) = (\varphi^{\pm}_j(-t))^*$.

The in or out wave functions in question [as well as (4)] are orthonormal with respect to the norm $i\int_X \psi_1\sqrt{-g}g^{00}\partial_0\psi_2^* d^{D-1}\Omega$, which is invariant under the change of the spacial section $X$, as the consequence of Eq. (8) [or (3)]. This norm defines the flux. Hence, any solution of Eq. (8) which has a definite finite flux (i.e. corresponds to a propagating particle) can be decomposed in the complete basis of either in or ''out'' modes. The particular choice of the in or out modes as the basis of harmonics leads to the de Sitter invariant vacuum state [12].

For the future references let us discuss here the asymptotic behavior of the in modes. The hypergeometric function $F(a, b; c; z)$ does not have any poles on the negative $z$ axis ($z = -e^{2t}$ in our case). Hence, the in modes (10) are regular at any value of $t$ and $m$ and behave in the past infinity ($t \to -\infty$) as

$$\varphi_j^{\pm} \to e^{((D-1)/2 \mp i\mu)t} \tag{11}$$

because $F \to 1$ as $z = -e^{2t} \to 0$. In the future infinity ($t \to +\infty$, $z = -e^{2t} \to -\infty$) they look like

$$\varphi_j^{\pm} \to e^{-((D-1)/2)t}(c_1 e^{\mp i\mu t} + c_2 e^{\pm i\mu t}) \tag{12}$$

with some complex constants $c_1$ and $c_2$. Such a behavior follows from

$$\lim_{z\to-\infty} F(a, b; c; z) \to c_1(-z)^{-a} + c_2(-z)^{-b}. \tag{13}$$

Note that if $m = 0$ we have harmonics which approach nonvanishing constants as $t \to \pm\infty$, but there are not any modes which are exponentially growing.

For the future reference let us define here the propagator for the in modes [11,12]. Consider the de Sitter invariant function depending on two points [11,12]:

$$Z(z, z') = -\sinh t \sinh t' + \cosh t \cosh t' \cos\Delta\Omega, \tag{14}$$

where $\Delta\Omega$ is the angle between the spacial parts of the coordinates $z$ and $z'$. It can be shown that $Z(z, z') = \cos L$, where $L$ is the geodesic distance between $z$ and $z'$ for spacial separations, or $i$ times the geodesic proper time difference for timelike separations. The Green function, being de Sitter invariant, should depend only on such a combination of the two points. Hence, the equation for the Green function, which is (8) with the appropriate $\delta$-functional sources on the right-hand side (RHS), can be converted into

$$[(1 - Z^2)\partial_Z^2 - DZ\partial_Z - m^2]G(Z) = A\delta(Z + 1) + B\delta(Z - 1) \tag{15}$$

by the direct change of variables (14), from $(t, \Omega)$ to $Z$. Here $A$ and $B$ are some constants. The RHS of this equation is singular when the $z$ and $z'$ points coincide (i.e. when $Z = 1$) and when $z$ coincides with the antipodal point of $z'$ (i.e. when $Z = -1$) [12]. The in Feynman type propagator obeys this equation with $A = B = 1$ and is given by [12]:

$$G_{\mathrm{in}}(Z) \propto \frac{\mu(\mu+1)}{\sinh(\pi\mu)} \times \left[F\left(\frac{D-1}{2} + i\mu, \frac{D-1}{2} - i\mu; \frac{D}{2}; \frac{1+Z}{2}\right) + F\left(\frac{D-1}{2} + i\mu, \frac{D-1}{2} - i\mu; \frac{D}{2}; \frac{1-Z}{2}\right)\right], \tag{16}$$

where $\mu$ is defined above. We are going to use this function in the loop calculations below.

## III. ON QFT WITH COMPACT SPACIAL SECTIONS

We see that spacial sections in global coordinates are compact $(D-1)$-dimensional spheres, which is less convenient than flat sections in planar coordinates. To see that compactness of spacial sections does not spoil all the picture, in this chapter we would like to consider the general features of QFT on such a space-time as, for example, $ds^2 = -dt^2 + R^2 d\Omega_{D-1}^2$ with fixed radius $R$. We are going to show that such a QFT as (1) on the background in question has similar properties to the theory in Minkowski space: such properties as the impossibility for inertial particle to emit radiation as measured by an inertial observer and the cancellation of IR divergences. Let us stress here that our confidence in the fact that there are no problems with the theory (1) on the background in question is relaying on the obvious observation that it has the unitary evolution operator.

Definite energy mass-shell harmonics in such a theory look like $\phi_j \propto e^{-ik_0 t} Y_{jm}(\Omega)$, where $k_0 = \sqrt{m^2 + \frac{1}{R^2} j(j + D - 2)}$. Then, the three leg mass-shell amplitude is

$$A \propto \int_{-\infty}^{+\infty} dt\, e^{-i(p_0 - k_0 - q_0)t} \int d\Omega\, Y_{j_1 n_1}(\Omega) Y_{j_2 n_2}^*(\Omega) Y_{j_3 n_3}^*(\Omega) \tag{17}$$

and is proportional to $\delta(p_0 - k_0 - q_0)$. The second integral (over the angles) gives the generalization of the $3j$ symbols to the $SO(D-1)$ group with $D \geq 4$, which is not quite a convenient object in comparison with the $\delta$ function appearing for the case of the flat spacial sections. The $3j$ symbols are nonzero if $j_2 - j_3 \leq j_1 \leq j_2 + j_3$.

On mass shell we have that

$$p_0 = \sqrt{M^2 + \frac{1}{R^2} j_1(j_1 + D - 2)},$$

$$k_0 = \sqrt{M^2 + \frac{1}{R^2} j_2(j_2 + D - 2)},$$

$$q_0 = \sqrt{m^2 + \frac{1}{R^2} j_3(j_3 + D - 2)}.$$

With such $p_0$, $k_0$, and $q_0$ the condition $p_0 = k_0 + q_0$ cannot be saturated for $j_2 - j_3 \leq j_1 \leq j_2 + j_3$. Hence, the argument of the $\delta$ function imposing the energy conservation is always nonzero, i.e. the amplitude itself is zero. Thus, similar to the Minkowski space, we arrive at the obvious conclusion that in the space in question an inertial (mass-shell) particle cannot emit radiation.

Recall that in Minkowski space IR divergences in the cross section appear only when $m = 0$. The amplitude of a hard process containing emission of one soft $\psi$ mass-shell quantum by the heavy $\Psi$ particle has the three leg multiplicative contribution

$$\int_{-\infty}^{+\infty} dt \int d\Omega G_M(t', \Omega'; t, \Omega) e^{-i(p_0 - q_0)t} Y_{j_1 n_1}(\Omega) Y^*_{j_3 n_3}(\Omega), \tag{18}$$

where $G_M$ is the propagator of the $\Psi$ field, i.e. one of the legs in the amplitude is off shell. The propagator is

$$G(t, \Omega; t', \Omega') = \sum_\lambda \frac{\Psi_\lambda(t, \Omega)\Psi^*_\lambda(t', \Omega')}{\lambda}, \tag{19}$$

where $[\Box - M^2]\Psi_\lambda = \lambda\Psi_\lambda$ and $\Box = -\partial_t^2 + \frac{1}{R^2}\Delta_{D-1}(\Omega)$. Obviously $\Psi_\lambda \propto e^{-ik_0 t} Y_{j_2 n_2}(\Omega)$, where now $k_0 = \sqrt{M^2 + \lambda + \frac{1}{R^2} j_2(j_2 + D - 2)}$, i.e. $\sum_\lambda = \int dk_0 \sum_{j_2=0}^{+\infty}$. The integral over $t$ in (18) leads to the energy conservation of the form:

$$\sqrt{M^2 + \frac{1}{R^2} j_1(j_1 + D - 2)} = \sqrt{M^2 + \lambda + \frac{1}{R^2} j_2(j_2 + D - 2)} + \frac{1}{R}\sqrt{j_3(j_3 + D - 2)}. \tag{20}$$

For the big $j_1 \sim j_2$ and small (soft) $j_3$ we have the solution of this equation as follows: $\lambda \approx -2(p_0 q_0 - \frac{1}{R^2} j_1 j_3)$, where $p_0 = \sqrt{M^2 + \frac{1}{R^2} j_1(j_1 + D - 2)}$ and $q_0^2 \approx \frac{1}{R^2} j_3(D - 2)$. Hence, the amplitude is divergent as $1/\lambda \propto 1/\sqrt{j_3}$ when $j_3 \to 0$, while the cross section is divergent as $1/j_3$. Similar divergences (with the opposite sign) appear in one loop contributions to the cross sections.

To clarify the situation let us consider the Minkowski space ($ds^2 = -dt^2 + d\vec{x}^2$) variant of the theory in question and the part of an amplitude responsible for the radiation of a soft $\psi$ particle by the hard $\Psi$ one. For simplicity here we restrict ourselves to the four space-time dimensions. If we insert the Minkowski space analog of the propagator (19) into the amplitude, we can use the $\delta$ functions, imposing energy-momentum conservation at the vertex, to fix $\lambda = k_0^2 - \vec{k}^2 - M^2 = (p_0 - q_0)^2 - (\vec{p} - \vec{q})^2 - M^2$. Here $p$ is the four-momentum of the on-shell incoming $\Psi$ particle, $q$ is the four-momentum of the on-shell outgoing radiated $\psi$ particle, and $k$ is the four-momentum of the off-shell (with virtuality $\lambda$) outgoing, after the radiation, $\Psi$ particle.

If we consider radiation of the very soft particle, i.e. the modulus of the corresponding $D$ momentum is $|q| \to 0$, then $k$ is very close to the mass shell, while $p^2 = M^2$ and $q^2 = 0$, i.e. $\lambda = -2pq$, because $m = 0$. Thus, the propagator is singular as $|q| \to 0$. Moreover, such a dependence of $\lambda$ on $q$ is important for the factorization of the IR divergences in the cross sections for radiation of many soft quanta [13], which, in its own right, is crucial for the total cancellation of all divergences.

As a result, after the integration of the differential cross section for the radiation of one soft quantum over its invariant phase volume, we obtain

$$\int |A|^2 \frac{d^3\vec{q}}{|\vec{q}|} \propto \int \frac{1}{(pq)^2} \frac{d^3\vec{q}}{|\vec{q}|} \propto \log m_0.$$

This is the IR divergence with the cutoff $m_0 \to 0$. Because all IR divergences are of the same order, we have to sum such contributions over all external legs of the hard process in question [5,13]. Then similar divergences (with the opposite sign) appear in the loop contributions to the cross section of the hard process:

$$\text{loop IR divergence} \propto \int d^4q \frac{1}{(pq)^2 q^2}.$$

All such contributions (from loops and tree-level diagrams) add up, so that every divergence does cancel [13]. Higher loop contributions cancel with the divergences coming from multiple soft quantum radiations [5]. It is important to stress here that we can choose another basis of harmonics for $\Psi$ (dressed with the cloud of $\psi$'s) such that IR divergences will not be present, neither in trees nor in loops [6], but it is impossible to get rid of the divergences only in trees without cancelling them in loops or vise versa.

## IV. RADIATION AND IR DIVERGENCES IN DE SITTER SPACE

In this section we are going to show that in de Sitter space the IR divergences do not cancel already at the leading order. But before going into the calculational details let us note that we avoid using the term S-matrix in de Sitter space. The latter is built on the basis of the matrix element of the evolution operator $\hat{S} = Te^{-i\int_{-\infty}^{+\infty} dt H_{\rm int}(t)}$ where $H_{\rm int}$ is the interaction Hamiltonian. The matrix elements in question are defined with respect to the basis of states $\langle {\rm out}|\hat{a} \ldots \hat{a}$ and $\hat{a}^+ \ldots \hat{a}^+|{\rm in}\rangle$ while it is assumed that $|{\rm out}\rangle \equiv e^{-i\int_{-\infty}^{+\infty} H_0 dt}|{\rm in}\rangle = ({\rm phase})|{\rm in}\rangle$, where $H_0$ is the free Hamiltonian. In the odd space-time dimensional de Sitter spaces $|{\rm out}\rangle = |{\rm in}\rangle$. Hence, in odd dimensions we are safe and our arguments pass smoothly [11].

But it appears that in even dimensional de Sitter spaces $|{\rm in}\rangle \neq |{\rm out}\rangle$ [11,12]. In fact, in even dimensions $|{\rm out}\rangle \equiv e^{-i\int_{-\infty}^{+\infty} H_0 dt}|{\rm in}\rangle \neq ({\rm phase})|{\rm in}\rangle$, because $|{\rm in}\rangle$ is not an eigenstate of the free Hamiltonian. The latter has the form $\hat{H}_0 \propto \Sigma_k[A(t)\hat{a}_k^+\hat{a}_k + B(t)\hat{a}_{-k}\hat{a}_k + B^*(t)\hat{a}_{-k}^+\hat{a}_k^+]$ with some functions of time $A(t)$ and $B(t)$ [14]. Exactly due to the presence of the nondiagonal terms $\hat{a}_{-k}\hat{a}_k$ and $\hat{a}_{-k}^+\hat{a}_k^+$ in the Hamiltonian we have to make the Bogolyubov transformation to diagonalize it and to observe the particle production [3].

Thus, the $|\text{out}\rangle$ state differs from $|\text{in}\rangle$ by the presence of the created particles, which can be explicitly established by the following relation $|\text{out}\rangle = \hat{V}(\hat{a}, \hat{a}^+)|\text{in}\rangle$ with some operator $\hat{V}$ [14]. In this paper we would like to consider the matrix elements of the evolution operator which are of the form $\langle\text{in}|\hat{a}\dots\hat{a}\,\hat{S}\,\hat{a}^+\dots\hat{a}^+|\text{in}\rangle$. Physically this means that we neglect the particle production by the external field and consider only scattering amplitudes in such a background. We strongly believe that this is sufficient to make a statement about (non)unitarity of the evolution operator $\hat{S}$ itself. To see that our arguments are meaningful one can consider the similar situation appearing in QED in the background of the constant electric field.

It is straightforward to find the basic tree-level *mass-shell* amplitude describing the process when the $\Psi$ particle radiates the $\psi$ particle. In the global coordinates the amplitude is proportional to

$$\begin{aligned} A \propto \int d\Omega Y_{j_1n_1}(\Omega)Y^*_{j_2n_2}(\Omega)Y^*_{j_3n_3}(\Omega)\int_{-\infty}^{+\infty} dt\cosh^{D-1}(t) \\ \times\left[\cosh^{j_1}(t)e^{(j_1+((D-1)/2)+i\mu_1)t}F\left(j_1+\frac{D-1}{2}, j_1+\frac{D-1}{2}+i\mu_1; 1+i\mu_1; -e^{2t}\right)\right] \\ \times\left[\cosh^{j_2}(t)e^{(j_2+((D-1)/2)-i\mu_1)t}F\left(j_2+\frac{D-1}{2}, j_2+\frac{D-1}{2}-i\mu_1; 1-i\mu_1; -e^{2t}\right)\right] \\ \times\left[\cosh^{j_3}(t)e^{(j_3+((D-1)/2)-i\mu_2)t}F\left(j_3+\frac{D-1}{2}, j_3+\frac{D-1}{2}-i\mu_2; 1-i\mu_2; -e^{2t}\right)\right], \end{aligned} \tag{21}$$

where $\mu_1 = \sqrt{M^2 - (\frac{D-1}{2})^2}$, $\mu_2 = \sqrt{m^2 - (\frac{D-1}{2})^2}$.

Note that (21) is valid even if $m$ or $M$ are less than $(D-1)/2$. Such an amplitude is just an analytical continuation of the corresponding generalized $3j$ symbol, which follows from the continuation $S^D \to dS^D$. From this we can already argue that the mass-shell amplitude (21) is nonzero.

However, if either one of the masses, $m$ or $M$, is vanishing, the integral for the amplitude is divergent (see below). We discuss the meaning of these divergences in the concluding section. At this stage we would like to avoid such problems with divergences of the tree-level amplitudes and to keep our discussion as transparent as it is possible for the case in question. It happens that, if we keep both masses $M$ and $m$ nonzero, the integral (21) is convergent. Unfortunately, even MATHEMATICA and MAPLE refuse to take such an integral analytically. Let us show explicitly that it is really convergent.

The integrand expression in (21) can hypothetically blow up only if $t \to \pm\infty$, because the hypergeometric function is regular for the negative argument, i.e. for any finite value of $t$. As $t \to -\infty$ we can use the behavior of the in harmonics from (11) to obtain that the integrand expression in (21) approaches $e^{(((D-1)/2)-i\mu_2)t}$ for the lower integration limit. Hence, the integral is convergent in this corner of the integration axis even if $\mu_2$ is purely imaginary, i.e. when $m < (D-1)/2$. Indeed in the latter case $|\mu_2| \leq (D-1)/2$. This inequality is saturated only when $m = 0$. It is only in the latter case there can be the perfect cancellation of the exponential suppression $e^{(((D-1)/2)-i\mu_2)t} \to 1$ as $t \to -\infty$, and we have the divergent amplitude.

In the other corner of the integration axis, i.e. when $t \to +\infty$, we can use the asymptotics as in (12) and find that the integrand behaves as

$$\begin{aligned} e^{-((D-1)/2)t}(c_1e^{+i\mu_1t} + c_2e^{-i\mu_1t})(c'_1e^{-i\mu_1t} + c'_2e^{+i\mu_1t}) \\ \times (c''_1e^{-i\mu_2t} + c''_2e^{+i\mu_2t}). \end{aligned} \tag{22}$$

Hence, in this corner of the $t$ integration axis the integral (21) is convergent if $\mu_1$ is real and $m \neq 0$.

As well, using MATHEMATICA and MAPLE for numerical calculation of the integral (21), one can explicitly see that it is not zero for $j_2 - j_3 \leq j_1 \leq j_2 + j_3$. The integrand expression in (21) is plotted in Fig. 1 for several different values of $M$, $m$ and $j_1$, $j_2$, $j_3$. Thus, all our considerations so far at least mean that a massive particle can radiate another massive particle on mass shell in de Sitter space.

Based on these considerations we can make a general conclusion that mass-shell particles can radiate fields under which they carry charges, unless the corresponding theory, describing interactions between "matter" and "radiation," is conformal. For example, all particles can radiate gravitons, we just have to appropriately understand the corresponding divergent amplitudes as the generalized functions. With similar reasoning one can arrive at the conclusion that an eternally accelerating charged particle in Minkowski space (e.g. under the action of the constant electric field) does emit radiation.

As a side remark let us point out here that one may object to the conclusions made in the previous paragraph based on the following considerations. It is known that if one drops a spherically symmetric massive ($m$) body in de

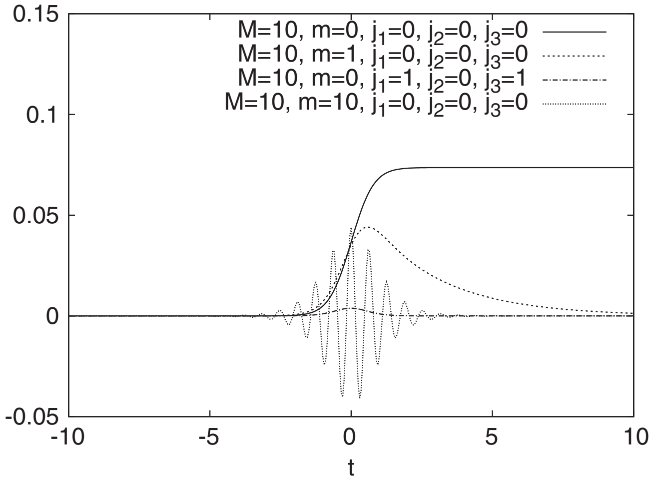


FIG. 1. The real part of the integrand in (21) for different values of $M$, $m$ and $j_1$, $j_2$, $j_3$. The imaginary part is equal to zero for the first three plots. The presence of the nonzero imaginary contribution to the leading amplitude is already a sign favoring that the theory in question is nonunitary.

Sitter space it produces the static de Sitter black hole metric outside its own volume:

$$ds^2 = -\left(1 - r^2 - \frac{2m}{r^{D-3}}\right)dt^2 + \frac{dr^2}{(1 - r^2 - \frac{2m}{r^{D-3}})} + r^2 d\Omega_{D-2}^2 \tag{23}$$

and does not produce any (nonstatic) gravitational waves on top of that. Hence, it seems that this argument precludes our conclusion that an inertial massive body in de Sitter space will produce gravitational waves. But the important point is that such a metric as (23) is seen by a noninertial observer which is *fixed* above the surface of the spherical body, i.e. the body and the observer compose a bound state and do not move with respect to each other. Thus, it is not an occasion that such an observer does not see any radiation from the massive body. At the same time our statement is that *it is* the inertial observer which sees the radiation from free floating bodies in de Sitter space.

Let us see now what happens with the cross section of a hard process containing the radiation of the soft $\psi$ quantum by $\Psi$ in de Sitter space. Because the amplitude (21) is nonvanishing on shell the virtuality of the $\Psi$ particle is *not* related to the mass-shell momentum ($j_3$) of the radiated $\psi$ particle. Indeed, to obtain the multiplicative factor in the amplitude of a hard process, corresponding to the radiation of the soft quantum by the hard one, we have to multiply by $1/\lambda$ the same amplitude as (21) with the only change of $\mu_1$ in the second wave function under the integral in (21) by $\sqrt{M^2 + \lambda - (\frac{D-1}{2})^2}$. Obviously the amplitude is *not* singular as $j_3 \to 0$, because $\lambda$ is not related to $j_3$. In fact, we do not have the $\delta$ function (imposing energy conservation) which fixes the value of $\lambda$ as it was in Minkowski space. Hence, we just have to integrate over all possible values of the virtuality in the amplitude, unlike the Minkowski space case. Thus, the cross section is not divergent as well, which can be explicitly seen via similar reasoning to the one presented after Eq. (21). But even if it was divergent, we would not have had the factorization of the divergences due to such a behavior of $\lambda$. The latter fact anyway spoils the cancellation of the IR divergences at *all orders* [13].

At the same time it happens that loop diagrams in de Sitter space do have IR divergences even for massive particles. Consider the one loop self-energy diagram for the $\Psi$ particle. It has the contributions of the form

$$\begin{aligned}\delta\Sigma \propto \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 \int d\Omega_1 \int d\Omega_2 \cosh^{D-1}(t_1) \\ \times \cosh^{D-1}(t_2) \\ \times F\left(\frac{D-1}{2} + i\mu_1, \frac{D-1}{2} - i\mu_1; \frac{D}{2}; \frac{1 \pm Z}{2}\right) \\ \times F\left(\frac{D-1}{2} + i\mu_2, \frac{D-1}{2} - i\mu_2; \frac{D}{2}; \frac{1 \pm Z}{2}\right),\end{aligned} \tag{24}$$

where $Z$ is given by (14) and we have borrowed propagators from (16). It is straightforward to see [using (13)] that such an integral divergent in the IR, i.e. as $Z(z, z') \to \infty$ (see e.g. similar discussion in [15]). Indeed, if say $t_1 \to -\infty$, while $t_2 \to +\infty$, then according to (14), $Z(z, z') \to e^{t_2 - t_1}(1 + \cos\Omega)/4$ and the integrand expressions in (23) behave as

$$\begin{aligned}&[c_1 e^{-i\mu_1(t_2-t_1)} + c_2 e^{i\mu_1(t_2-t_1)}] \\ &[c_1' e^{-i\mu_2(t_2-t_1)} + c_2' e^{i\mu_2(t_2-t_1)}].\end{aligned} \tag{25}$$

Hence, the integrals over $t_1$, $t_2$ in (24) are divergent if $\mu_2$ is pure imaginary, i.e. when $m < (D-1)/2$. Furthermore, we have the IR divergence in the causally connected region, because any points with $t \to -\infty$ and $t \to +\infty$ are causally connected in de Sitter space. Hence, restricting oneself to the region within the cosmological horizon does not help to cut or get rid of such IR divergences.

Thus, for any $M > 0$, but $0 < m < (D-1)/2$, we obtain finite tree-level contributions to cross sections and there is nothing which can cancel the IR divergences in the loop diagrams. It is probably worth pointing out here that we cannot interpret the divergence in (24) as an analog of the collinear one [5] for many obvious reasons. At least it is present for any value of the mass $M$ of the particle emitting the radiation.

All the considerations above make us conclude that the evolution operator leading to such a diagram technic is not unitary and the system of de Sitter background plus QFT is not closed if the cosmological constant is held fixed. It is important to stress that in the circumstances under consideration one cannot make the IR divergences to cancel by a unitary change of the basis of the creation-annihilation operators.

## V. CONCLUSIONS

Before drawing any conclusions let us make a few side remarks about the divergences of (21) when either of the masses $M$ or $m$ is vanishing. Similar divergences appear in anti-de Sitter (AdS) space: note the similarity of the anti-de Sitter metric $ds^2 = \frac{1}{z^2}(dz^2 + dx_a dx^a)$ to the one in (2) with the crucial difference, however, that $z$ is not timelike. Hence, anti-de Sitter space is not globally hyperbolic (because there is a timelike boundary) but does not have an event horizon (because there is a globally defined timelike Killing vector). Because of the lack of the global hyperbolicity, which results in such a well-known effect that in anti-de Sitter space waves can repel from the spacial boundary, one cannot define the Cauchy problem in such a space [16]. Thus, while it is possible to define the unique anti-de Sitter invariant vacuum state, one *cannot* define appropriately the evolution operator for a QFT in such a background. According to the AdS/conformal field theory correspondence [17–19] one treats the IR divergences in the QFT on anti-de Sitter space, which appear in the wave functional rather than in the S-matrix, as the UV divergences in the QFT on its boundary.

On the contrary, although we do not have Poincaré invariance in de Sitter space, it *is* globally hyperbolic and one can define the evolution operator there. Let us stress here that we do not have Poincaré invariance, for example, in the presence of the constant electric field in QED, but we still can define the evolution operator, because the evolution problem can be correctly formulated in such circumstances.

At this point we can and should address the question why anti-de Sitter space is stable. It seems that a free floating particle in anti-de Sitter space will emit radiation as well. However, this question and the question of the cancellation of the IR divergences cannot be formulated in anti-de Sitter space because of the impossibility to define the time evolution operator in such a background due to the lack of the global hyperbolicity.

Let us now come back to the conclusions. We see that QFT in de Sitter space (i.e. if we fix the cosmological constant) behaves as if it is formulated in a background of an external quasiclassical gravitational field (excitation above the correct vacuum) due to a cosmological constant—analogously to the QED in a constant (in space and time) electric field, i.e. as the nonclosed system. That is the only interpretation of the nonunitarity, which we can give. Hence, the only conclusion which we can make here, without performing a direct calculation, is that the cosmological constant will relax to zero creating particles via Gibbons-Hawking pair production (which we do not discuss in our paper) and via performing work on accelerating created particles, which leads to the radiation in its own right.

The conceptual question is how fast will be the relaxation of the cosmological constant. If the relaxation goes fast enough we can hope to explain via this mechanism the cosmological constant problem along with obtaining natural inflation without any inflaton field [8]. We think that the rate of the decay should just depend on the actual magnitude of the field. For the big enough field the rate of the decay should be big: the bigger the energy pool is the easier to create light particles. The technical question is how to calculate the rate of the decay of the cosmological constant.

If the cosmological constant is very big we have to use the theory of quantum gravity. Unfortunately string theory does not have a formulation on de Sitter space, because such a background spoils conformal invariance of the theory on the string world sheet. In this case we have to deal with the off-shell formulation of the closed string theory, which is very hard to do with the presently existing first quantized variant of the theory (see, however, [15]).

Thus, the only hope is that we can use the ordinary Einstein-Hilbert theory, which dominates in the IR, if the cosmological constant is smaller than the Plank scale. In this respect we should stress that the issue of the instability of de Sitter space, of the IR divergences, and of the nonunitarity have been discussed in various places [2,8,12,15,20,21]. As well, the running of the cosmological constant due to the quantum fluctuations have been found in [8–10] using either Heisenberg or Schwinger-Keldish technics. Here, apart from presenting the listed above new phenomena supporting the conclusion that de Sitter space is unstable, we should criticize the actual calculation of the decay rate of the cosmological constant performed in [8–10], because, as we just pointed out, the rate was found with the use of the nonunitary evolution operator. Hence, the questions posed above remain to be answered.

## ACKNOWLEDGMENTS

The Authors would like to thank V. Zakharov for collaboration at the initial stage of work on this project. E. T. A. would like to thank M. Voloshin, L. Kofman, V. Lukash, A. Mironov, A. Dolgov, V. Rubakov, N. Nekrasov, M. Polikarpov, S. Theisen, P. Arseev, B. Voronov, I. Polyubin, and especially A. Rosly and O. Kancheli for valuable discussions. E. T. A. would like to thank MPI-AEI, Golm, Germany for the hospitality during the final period of work on this project. As well, authors would like to thank the Galileo Galilei Institute for Theoretical Physics for the hospitality and the INFN for partial support during the completion of this work. The work was partially supported by the Federal Agency of Atomic Energy of Russian Federation and by the grant for scientific schools No. NSh-679.2008.2.

[1] E. Lifshitz, J. Phys. (Moscow) **10**, 116 (1946).
[2] N. Myrhvold, Phys. Rev. D **28**, 2439 (1983).
[3] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2738 (1977).
[4] E.g. in a QFT without background field the virtuality $\lambda$ is just the difference between the energy squared and the momentum squared of the particle ($\lambda = k_0^2 - \vec{k}^2$), i.e. the measure of its off-shell position.
[5] S. Weinberg, Phys. Rev. **140**, B516 (1965).
[6] T. D. Lee and M. Nauenberg, Phys. Rev. **133**, B1549 (1964).
[7] T. S. Bunch and P. C. W. Davies, Proc. R. Soc. A **360**, 117 (1978).
[8] N. C. Tsamis and R. P. Woodard, Nucl. Phys. **B474**, 235 (1996); Commun. Math. Phys. **162**, 217 (1994); Phys. Lett. B **292**, 269 (1992); Classical Quantum Gravity **11**, 2969 (1994); Ann. Phys. (N.Y.) **238**, 1 (1995); Phys. Rev. D **54**, 2621 (1996); Ann. Phys. (N.Y.) **253**, 1 (1997).
[9] A. D. Dolgov, M. B. Einhorn, and V. I. Zakharov, Phys. Rev. D **52**, 717 (1995); Acta Phys. Pol. B **26**, 65 (1995).
[10] S. Weinberg, Phys. Rev. D **74**, 023508 (2006); **72**, 043514 (2005).
[11] R. Bousso, A. Maloney, and A. Strominger, Phys. Rev. D **65**, 104039 (2002).
[12] E. Mottola, Phys. Rev. D **31**, 754 (1985).
[13] A. V. Smilga, Report No. ITEP-35-1985.
[14] A. A. Grib, S. G. Mamaev, and B. M. Mostepanenko, *Quantum Effects in Intensive Background Fields* (Atomizdat, Moscow, 1980) (in Russian).
[15] A. M. Polyakov, Nucl. Phys. **B797**, 199 (2008).
[16] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, England, 1973).
[17] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); Int. J. Theor. Phys. **38**, 1113 (1999).
[18] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998).
[19] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[20] I. Antoniadis and E. Mottola, Phys. Rev. D **45**, 2013 (1992).
[21] L. H. Ford, Phys. Rev. D **31**, 710 (1985).