

Schwinger pair production in dS_2 and AdS_2

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We study Schwinger pair production in scalar QED from a uniform electric field in dS_2 with scalar curvature $R_{dS} = 2H^2$ and in AdS_2 with $R_{AdS} = -2K^2$. With suitable boundary conditions, we find that the pair-production rate is the same analytic function of the scalar curvature in both cases.

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I. INTRODUCTION

Recently anti-de Sitter spacetime (AdS) has attracted much attention because of the AdS/conformal field theories correspondence. The de Sitter spacetime (dS) is also widely studied because of the accelerating phase of the present Universe and the inflationary period in the early Universe. The AdS and dS spacetimes have constant scalar curvatures and are the maximally symmetric spacetimes for any given dimension.

The Minkowski vacuum becomes unstable with a strong electric field and decays into pairs of charged particles, known as Schwinger pair production [1–3]. The vacuum also may be unstable when the spacetime expands or contracts, leading to particle creation [4,5], particularly in the de Sitter spacetime [6–11]. Schwinger pair production in curved spacetimes may be an interesting issue, combining the two effects. Pair production by a uniform electric field in dS_2 was studied in Refs. [12–14] and in AdS_2 in Ref. [15]. Also pair production was considered in an anisotropically expanding universe [16,17]. Though the instantons in dS_2 and AdS_2 have been known independently, the relation between the pair-production rates has not been examined before.

The purpose of this paper is to study scalar QED in a uniform electric field in dS_2 and AdS_2 and to calculate the Schwinger pair-production rates from the exact solutions of the massive charged Klein-Gordon equation. Since the geometry of dS_2 with scalar curvature $R_{dS} = 2H^2$ can be analytically continued to AdS_2 with $R_{AdS} = -2K^2$, both of which are conformally flat, the Schwinger pair-production rates are shown to have the same kind of analytical continuation from one geometry to another.

The organization of this paper is as follows. In Sec. II, we formulate the Klein-Gordon equation in a uniform electric field in dS_2 and AdS_2 . In Sec. III, we find the pair-production rate from the exact mode solutions in dS_2 . In

Sec. IV, we find the exact mode solutions and pair-production rate in AdS_2 . In Sec. V, we discuss the Schwinger pair-production rates and show that they are both the same analytic function of the scalar curvatures in both geometries.

II. KLEIN-GORDON EQUATION IN dS_2 AND AdS_2

The two-dimensional de Sitter spacetime has positive constant curvature and may be assigned the topology $R^1 \times S^1$, while the anti-de Sitter spacetime has negative constant curvature and may be assigned the topology $S^1 \times R^1$. The two-dimensional de Sitter spacetime (dS_2) with a scalar curvature $R = 2H^2$ can be embedded into a three-dimensional hyperboloid with radius $1/H$ and has the metric [18]

$$ds_{dS}^2 = -dt^2 + \cosh^2(Ht)dx^2, \quad (1)$$

where $-\infty < t < \infty$ and x is identified periodically with period $2\pi/H$ to give the topology $R^1 \times S^1$, while the two-dimensional anti-de Sitter spacetime (AdS_2) with scalar curvature $R = -2K^2$ has the metric of the form

$$ds_{AdS}^2 = -\cosh^2(Kx)dt^2 + dx^2, \quad (2)$$

where this time $-\infty < x < \infty$ and t is identified periodically with period $2\pi/K$ to give the topology $S^1 \times R^1$. Here H and K have the dimension of inverse length. Alternatively, one may take x to have infinite range in (1) and t to have infinite range in (2) to get the covering spaces for dS_2 and AdS_2 , respectively.

To study Schwinger pair production by a uniform electric field in dS_2 and AdS_2 , we consider scalar QED described by the Klein-Gordon equation for bosons with mass m and charge q (in units of $\hbar = c = 1$):

$$\left[\frac{1}{\sqrt{-g}} (i\partial_\mu + qA_\mu)(\sqrt{-g}g^{\mu\nu}(i\partial_\nu + qA_\nu)) + m^2 \right] \Phi(t, x) = 0. \quad (3)$$

In a two-dimensional curved spacetime, a uniform electric field E leads to the electromagnetic field two-form [19]

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$$\mathbf{F} = E\sqrt{|g|}dx \wedge dt. \quad (4)$$

Since the electromagnetic potential satisfies $d\mathbf{A} = \mathbf{F}$, for dS_2 we may take the potential of the form

$$\mathbf{A} = -\frac{E}{H} \sinh(Ht)dx. \quad (5)$$

Note that the potential (5) respects one of the Killing symmetries, ∂_x . In AdS_2 the electromagnetic potential is given by

$$\mathbf{A} = \frac{E}{K} \sinh(Kx)dt. \quad (6)$$

AdS_2 has the Killing vector, ∂_t , and allows separation of variables.

III. PAIR PRODUCTION IN dS_2

In the coordinates (1) for dS_2 , the Klein-Gordon equation minimally coupled with the potential (5) takes the form

$$\left[\partial_t^2 + H \tanh(Ht) \partial_t + \frac{1}{\cosh^2(Ht)} \left(i \partial_x - \frac{qE}{H} \sinh(Ht) \right)^2 + m^2 \right] \Phi(t, x) = 0. \quad (7)$$

Then the Fourier component, $\Phi(t, x) = e^{ikx} \phi_k(t) / \sqrt{\cosh(Ht)}$, satisfies the one-dimensional equation

$$[-\partial_t^2 + V_{\text{ds}}(t)] \phi_k(t) = 0, \quad (8)$$

where

$$V_{\text{ds}}(t) = -\frac{1}{\cosh^2(Ht)} \left(k + \frac{qE}{H} \sinh(Ht) \right)^2 - m^2 + \frac{H^2}{4} \left(1 + \frac{1}{\cosh^2(Ht)} \right). \quad (9)$$

In quantum mechanics, Eq. (8) is a scattering problem of a particle with a negative potential but with zero energy. In the two asymptotic regions $t = \pm\infty$, there is an asymptotic frequency

$$\omega_0^2 = -V_{\text{ds}}(\pm\infty) = \left(\frac{qE}{H} \right)^2 + m^2 - \frac{H^2}{4}, \quad (10)$$

so the positive frequency solutions of $\Phi_k(t) = \phi_k(t) / \sqrt{\cosh(Ht)}$ at early and late times are given asymptotically by

$$\begin{aligned} u_{\text{in}}(-\infty) &\sim \frac{e^{-i\omega_0 t}}{\sqrt{2\omega_0 \cosh(Ht)}}, \\ u_{\text{out}}(\infty) &\sim \frac{e^{-i\omega_0 t}}{\sqrt{2\omega_0 \cosh(Ht)}}, \end{aligned} \quad (11)$$

and the negative frequency solutions asymptotically by u_{in}^* and u_{out}^* . The initial vacuum and final vacuum are defined with respect to $\Phi_{\text{in}}(t, x) = e^{ikx} u_{\text{in}}(t)$ and $\Phi_{\text{out}}(t, x) = e^{ikx} u_{\text{out}}(t)$, respectively. These are different definitions from the de Sitter invariant vacua [9], so our initial and final vacua are not the same, leading to pair production with respect to them.

From Ref. [20] we find the general solution to Eq. (8) with two linearly independent solutions given by

$$\begin{aligned} \phi_k(t) &= z^{n/2} (1-z)^{n^*/2} [c_1 F(\mu, \nu; \gamma; z) \\ &\quad + c_2 z^{1-\gamma} F(\mu - \gamma + 1, \nu - \gamma + 1; 2 - \gamma; z)], \end{aligned} \quad (12)$$

where F is the hypergeometric function [21], and c_1 and c_2 are integration constants, and

$$\begin{aligned} n &= \frac{1}{2} - \frac{k}{H} + i \frac{qE}{H^2}, & \mu &= \frac{n+n^*}{2} - i \frac{\omega_0}{H}, \\ \nu &= \mu^*, & \gamma &= n + \frac{1}{2}, & z &= \frac{1 + i \sinh(Ht)}{2}. \end{aligned} \quad (13)$$

From the asymptotic formula [22] for $|z| \gg 1$, the solution at early and late times can be written as

$$\begin{aligned} \Phi_k(t) &= D_1 u_{\text{in}}(t) + D_2 u_{\text{in}}^*(t) \\ &= D_3 u_{\text{out}}(t) + D_4 u_{\text{out}}^*(t), \end{aligned} \quad (14)$$

where D 's are constants determined by c 's, n and μ only. Eliminating the u_{in}^* part and normalizing the remaining part to $u_{\text{in}}(t)$, for each momentum we obtain the frequency mixing

$$u_{\text{in}}(t) = \alpha_k u_{\text{out}}(t) + \beta_k u_{\text{out}}^*(t), \quad (15)$$

and the Bogoliubov transformation

$$\hat{a}_{\text{out}}(k) = \alpha_k \hat{a}_{\text{in}}(k) + \beta_k^* \hat{a}_{\text{in}}^\dagger(k), \quad (16)$$

where [23]

$$\beta_k = e^{i\pi(n^* - n - 2\mu)/2}. \quad (17)$$

The coefficients satisfy the relation for bosons

$$|\alpha_k|^2 - |\beta_k|^2 = 1. \quad (18)$$

Then the mean number of produced pairs with k [4,5],

$$\mathcal{N}_k = \langle \text{in} | \hat{a}_{\text{out}}^\dagger(k) \hat{a}_{\text{out}}(k) | \text{in} \rangle = |\beta_k|^2, \quad (19)$$

is given by the instantonlike ‘‘action,’’ defined so that $|\beta_k|^2 = e^{-S_{\text{ds}}}$, as

$$\begin{aligned} \mathcal{S}_{dS} &= \frac{2\pi}{H^2} \left[\sqrt{(qE)^2 + (mH)^2 - \frac{H^4}{4}} - qE \right] \\ &= \frac{\pi m^2}{qE} \frac{2 - \frac{R}{4m^2}}{1 + \sqrt{1 + \frac{m^2 R}{2(qE)^2} - \frac{R^2}{16(qE)^2}}}. \end{aligned} \quad (20)$$

Here R is the scalar curvature.

In the zero-electric field limit ($E = 0$), we recover the particle-production probability in de Sitter space [6–11]

$$|\beta_k|^2 = e^{-(2\pi\sqrt{m^2 - \frac{H^2}{4}})/H}, \quad (21)$$

which is the Boltzmann factor with the Gibbons-Hawking temperature $T_{GH} = H/(2\pi)$ when $H \ll m$. In the $H = 0$ limit, we recover the Schwinger pair-production rate in the two-dimensional Minkowski spacetime. The instanton actions in Refs. [12,13] are the limiting case of Eq. (20) when $H \ll m$. The pair-production rate of scalar QED may be compared with that of spinor QED in Ref. [14]. A direct calculation using the worldline instanton [24–26] and the WKB instanton action [27] also gives this limiting result, with only the $-H^2/4$ term missing from the square root in the first expression (20) above for \mathcal{S}_{dS} .

IV. PAIR PRODUCTION IN AdS_2

In AdS_2 with the electromagnetic potential (6), the Fourier component, $\Phi(t, x) = e^{-i\omega t} \varphi_\omega(x)/\sqrt{\cosh(Kx)}$, satisfies the equation

$$[-\partial_x^2 + V_{AdS}(x)]\varphi_\omega(x) = 0, \quad (22)$$

where

$$\begin{aligned} V_{AdS}(x) &= -\frac{(\omega + \frac{qE}{K} \sinh(Kx))^2}{\cosh^2(Kx)} + m^2 \\ &\quad + \frac{K^2}{4} \left(1 + \frac{1}{\cosh^2(Kx)} \right). \end{aligned} \quad (23)$$

The formalism for dS_2 cannot be applied to this static problem since there do not exist ingoing states at past infinity and outgoing states at future infinity. However, we may apply the tunneling idea for pair production: virtual pairs are created from a tunneling barrier, which is the Dirac sea lowered by the electric potential, and then move to spatial infinity to be real pairs along the electric field [27–35]. For virtual pairs to be real ones, an asymptotic momentum given by

$$k_0^2 = -V_{AdS}(\pm\infty) = \left(\frac{qE}{K}\right)^2 - m^2 - \frac{K^2}{4} \quad (24)$$

should be real, thus requiring $qE > \sqrt{(mK)^2 + K^4/4}$. Then the solution takes asymptotically the form

$$\begin{aligned} v_{in}(-\infty) &\sim \frac{e^{ik_0x}}{\sqrt{2k_0 \cosh(Kx)}}, \\ v_{out}(\infty) &\sim \frac{e^{ik_0x}}{\sqrt{2k_0 \cosh(Kx)}}. \end{aligned} \quad (25)$$

The v_{in} is the ingoing wave to the barrier and v_{out} is the outgoing wave from the barrier.

As in the case of dS_2 , we find the exact solution

$$\begin{aligned} \varphi_\omega(t) &= z^{\tilde{n}/2} (1-z)^{\tilde{n}^*/2} [\tilde{c}_1 F(\tilde{\mu}, \tilde{\nu}; \tilde{\gamma}; z) \\ &\quad + \tilde{c}_2 z^{1-\tilde{\gamma}} F(\tilde{\mu} - \tilde{\gamma} + 1, \tilde{\nu} - \tilde{\gamma} + 1; 2 - \tilde{\gamma}; z)], \end{aligned} \quad (26)$$

where \tilde{c}_1 and \tilde{c}_2 are integration constants, and

$$\begin{aligned} \tilde{n} &= \frac{1}{2} - \frac{\omega}{K} + i \frac{qE}{K^2}, & \tilde{\mu} &= \frac{\tilde{n} + \tilde{n}^*}{2} - i \frac{k_0}{K}, \\ \tilde{\nu} &= \tilde{\mu}^*, & \tilde{\gamma} &= \tilde{n} + \frac{1}{2}, & z &= \frac{1 + i \sinh(Kx)}{2}. \end{aligned} \quad (27)$$

Imposing the tunneling boundary condition by eliminating v_{out}^* and appropriately normalizing the solution, we obtain

$$v_{out}(x) = \tilde{\alpha}_\omega v_{in}(x) + \tilde{\beta}_\omega v_{in}^*(x), \quad (28)$$

where

$$\tilde{\beta}_\omega = e^{i\pi(\tilde{n} - \tilde{n}^* + 2\tilde{\mu})/2}. \quad (29)$$

The mean number can be expressed in terms of the instantonlike ‘‘action’’ from the exact solution

$$\mathcal{N}_\omega = |\tilde{\beta}_\omega|^2 = e^{-\mathcal{S}_{AdS}}, \quad (30)$$

where

$$\begin{aligned} \mathcal{S}_{AdS} &= \frac{2\pi}{K^2} \left[qE - \sqrt{(qE)^2 - (mH)^2 - \frac{K^4}{4}} \right] \\ &= \frac{\pi m^2}{qE} \frac{2 - \frac{R}{m^2}}{1 + \sqrt{1 + \frac{m^2 R}{2(qE)^2} - \frac{R^2}{16(qE)^2}}}. \end{aligned} \quad (31)$$

The instanton ‘‘action’’ (31) agrees with that obtained from the one-loop effective action in Ref. [15].

V. CONCLUSION

We have studied Schwinger pair production by a uniform electric field in dS_2 and AdS_2 . We solved the Klein-Gordon equation in these curved spacetimes and found the pair-production rate by appropriately imposing boundary conditions. A number of interesting points have been observed.

First, there is an analytical continuation of the Schwinger pair-production rate between dS_2 with $R_{dS} = 2H^2$ and AdS_2 with the scalar curvature $R_{AdS} = -2K^2$. In fact, the exact results are invariant under the correspondence $K = iH$ and $\omega = ik$. This is because the metric and the electromagnetic fields for dS_2 are analytically continued to AdS_2 under the transformations $t \leftrightarrow ix$ and $x \leftrightarrow it$ together with $K \leftrightarrow iH$. The wave function, say the right-moving free wave, is properly transformed from one space into another. This means that the pair production is given by the same analytic function of the scalar curvature in both cases.

Second, the pair-production rate does not depend on the frequency or momentum. A physical interpretation is that the frequency or momentum depends on the Lorentz reference frame, whereas the spacetimes and electromagnetic fields are maximally symmetric, so that the result can only depend on the invariants m^2 , qE , and R (and actually only on their two ratios, by dimensional analysis). It is interesting to notice in both cases the expected numbers e^{-S} are given by the instantonlike ‘‘action’’ of the form

$$\begin{aligned} S_{dS} &= \frac{2\pi}{H^2} \left[\sqrt{(qE)^2 + (mH)^2 - \frac{H^4}{4}} - qE \right], \\ S_{AdS} &= \frac{2\pi}{K^2} \left[qE - \sqrt{(qE)^2 - (mH)^2 - \frac{K^4}{4}} \right], \end{aligned} \quad (32)$$

both of which can be written in terms of the scalar curvature R as

$$S = \frac{\pi m^2}{qE} \frac{2 - \frac{R}{4m^2}}{1 + \sqrt{1 + \frac{m^2 R}{2(qE)^2} - \frac{R^2}{16(qE)^2}}}. \quad (33)$$

Third, given the same magnitude (but not sign) of the scalar curvature, $|R| = 2K^2 = 2H^2$, and the strength of electric field, one has in general $S_{AdS} > S_{dS}$ and hence a larger pair-production rate in dS_2 than in AdS_2 . The gravitational confinement within AdS_2 suppresses the Schwinger pair production, and there is a minimal strength $E_0 = \sqrt{(mK)^2 + K^4}/4/q$ to be able to produce pairs. Without the electric field, the pair-production rate is that of a scalar field in the de Sitter spacetime, but vanishes in the anti-de Sitter spacetime due to the boundary condition at asymptotic regions, as expected.

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