Long wavelength limit of evolution of nonlinear cosmological perturbations

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In the general matter composition where the multiple scalar fields and the multiple perfect fluids coexist, in the leading order of the gradient expansion, we construct all of the solutions of the nonlinear evolutions of the locally homogeneous universe. From the momentum constraint, we derive the constraints which the solution constants of the locally homogeneous universe must satisfy. We construct the gauge invariant perturbation variables in the arbitrarily higher order nonlinear cosmological perturbation theory around the spatially flat Friedmann-Robertson-Walker universe. We construct the nonlinear long wavelength limit formula representing the long wavelength limit of the evolution of the nonlinear gauge invariant perturbation variables in terms of perturbations of the evolutions of the locally homogeneous universe. By using the long wavelength limit formula, we investigate the evolution of nonlinear cosmological perturbations in the universe dominated by the multiple slow rolling scalar fields with an arbitrary potential. The τ function and the N potential introduced in this paper make it possible to write the evolution of the multiple slow rolling scalar fields with an arbitrary interaction potential and the arbitrarily higher order nonlinear Bardeen parameter at the end of the slow rolling phase analytically. It is shown that the nonlinear parameters such as $f_{\rm NL}$ and $g_{\rm NL}$ are suppressed by the slow rolling expansion parameters.

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I. INTRODUCTION AND SUMMARY

In the inflationary universe scenario, the quantum fluctuations of the scalar fields called inflatons are thought to be the origin of the cosmological large scale structures such as galaxies and clusters of galaxies. In the slow rolling phase, the quantum fluctuations are stretched into the superhorizon scales and stay outside the horizon until they return into the horizon in the radiation dominant universe. Therefore the method for investigating the evolutions of the long wavelength cosmological perturbations became necessary, and the long wavelength limit (LWL) method was developed [1–3]. In the LWL method, we use the LWL formulas representing the long wavelength limit of the evolutions of the cosmological perturbations in terms of the quantities of the corresponding exactly homogeneous universe. As for the adiabatic modes, the exact LWL formula had already been constructed, and it was used in order to investigate the slow rolling phase [4] and the oscillatory phase [5,6]. Taruya and Nambu suggested in the multiple scalar field system also the LWL formula exists [3]. Soon in this case the exact LWL formula was constructed [1,2], and it was used to investigate the multiple oscillatory scalar fields [7,8]. In addition, in Ref. [1], we presented the flexible way for constructing the LWL formulas in the general matter composition. It will be called the Kodama and Hamazaki (KH) construction in this paper. In the KH construction, the perturbation variables related with the exactly homogeneous universe, such as the scalar field perturbation and the energy density perturbation, are expressed in terms of the exactly homogeneous perturbations, that is, the derivative of the exactly homogeneous quantity with respect to the solution constant. By solving the spatial components of the Einstein equations, the perturbation variables not related with the exactly homogeneous quantities such as vector quantities, for example, velocity perturbation variables, are expressed in the form of the integral of the perturbation variables related with the exactly homogeneous universe which have already been determined. We pointed out that the perturbation solution constants contained in the expressions of the perturbation variables determined in the processes explained above must satisfy the constraint coming from the momentum constraint. The KH construction can be applied to the system containing the perfect fluid components having vector degrees of freedom such as the velocity perturbation variables. Therefore by using the KH construction, in the most general matter composition where multiple scalar fields and multiple perfect fluids coexist, the LWL formula was constructed and used to investigate the multiple component reheating and the multiple component curvaton decay [9]. Afterward, in the case of the nonlinear perturbation also, the existence of the LWL formula was suggested [10,11]. In this paper we give the definition of arbitrarily higher order nonlinear gauge invariant perturbation variables and the exact nonlinear LWL formula representing them in terms of the derivatives of the quantities of the locally homogeneous universe with respect to the solution constants.

In order to calculate the present density perturbations and the cosmic microwave background anisotropies from the initial seed perturbations, we need to calculate the evolution of the scalar field perturbations on superhorizon scales during the slow rolling phase. In Refs. [12,13], by

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decomposing the multiple scalar fields into the adiabatic field and the entropy fields instant by instant, the evolutionary behaviors of cosmological perturbations during the slow rolling phase were discussed. But this study is the local investigation; that is, it is based on the Taylor expansion of the evolution equations around the first horizon crossing, and the evolutions of the scalar fields in the long time interval were not solved. Then in this paper we present the method which makes it possible to trace the evolutions of the multiple slow rolling scalar fields with an arbitrary interaction potential for the long time period as in the whole slow rolling phase.

This paper is organized as follows. Section II is devoted to the nonlinear LWL formula. In Sec. II A, under the assumption that the universe is the spatial flat Friedmann-Robertson-Walker (FRW) universe in the background level, in the leading order of the gradient expansion, we give the evolution equations of the locally homogeneous universe. We point out that these locally homogeneous evolution equations are similar to the corresponding exactly homogeneous evolution equations and that deviations between the two are induced by the unimodular factor of the spatial metric $\tilde{\gamma}_{ij}$ which has a contribution from the adiabatic decaying mode. Following the philosophy of the KH construction [1], we construct all of the solutions of the evolution equations of the locally homogeneous universe. We give the constraint which comes from the momentum constraint and which must be satisfied by the solution constants contained in the solution of the locally homogeneous universe. In Sec. II B, we give the definition of the gauge invariant perturbation variables of the arbitrarily higher order nonlinear cosmological perturbation theory and prove their gauge invariance. We present the nonlinear LWL formula representing the long wavelength limit of the evolutions of the gauge invariant perturbation variables in the nonlinear perturbation theory in terms of the derivatives with respect to the solution constant of the locally homogeneous solutions. Section III is devoted to the analysis of the nonlinear evolution of the multiple slow rolling scalar fields as an application of the nonlinear LWL formula constructed in the previous section. In Sec. III A, it is shown that in the slow rolling phase the scalar field evolution equations are simplified by truncation. By estimating the truncation error, we establish the accuracy of the truncated evolution equations in the slow rolling expansion scheme. In Sec. III B, the ideas of the τ function and the N potential are introduced, and they are shown to enable us to trace the multiple slow rolling scalar fields in the whole slow rolling phase analytically. By adopting the τ function as the evolution parameter, the truncated evolution equations of the multiple slow rolling scalar fields are simplified enough for their solutions to be written analytically. From the scalar field solutions we can easily calculate the N potential, which allows us to calculate the arbitrarily higher order Bardeen parameter at the end of the slow rolling phase from the initial scalar field perturbations at the first horizon crossing. In Sec. III C, by the τ function and the N potential we calculate the various perturbation variables such as the Bardeen parameter, the entropy perturbations, the gravitational wave perturbation, and their spectrum indices in the case of the multiple quadratic potential whose truncated evolution can be exactly solved. In Sec. III D, we consider the effect of the interaction between multiple slow rolling scalar fields in the case where the masses of the scalar fields do not satisfy any resonant relations. We point out that this problem is related with the well-known Poincaré theorem about the linearization. In the concrete model, we calculate the various quantities such as the N potential and the nonlinear parameters $f_{\rm NL}$ and $g_{\rm NL}$ introduced in Ref. [14]. In Sec. III E, we investigate the resonant interaction between the multiple slow rolling scalar fields. We show that the N potential has no singular part by introducing the resonant interactions. Section IV is devoted to the discussions. The appendixes are devoted to the proofs of the propositions presented in the main content of this paper.

II. NONLINEAR LWL FORMULA

In this section, we derive the nonlinear LWL formula in the most general matter composition whose energy momentum tensor is divided into $A = (S, f)$ parts, where S represents N_S component scalar fields ϕ_a ($a = 1, 2, \ldots, N_S$) and f represents N_f component perfect fluids ρ_α ($\alpha = 1, 2, \ldots, N_f$). The content of this section is the nonlinear generalization of the content of Ref. [9]. Our results in this section can be applied not only to the multiple slow rolling scalar fields but also to the multicomponent reheating and the multicomponent curvaton decay [9,15].

In Sec. II A, we give the evolution equations of the locally homogeneous universe in the leading order of the gradient expansion, and we construct all of the solutions of these locally homogeneous universe evolution equations. In Sec. II B, we give the definition of the nonlinear gauge invariant perturbation variables, and we give the nonlinear LWL formula representing the long wavelength limit of the evolutions of the gauge invariant perturbation variables in terms of the derivative with respect to the solution constant of the evolutions of the locally homogeneous universe determined in Sec. II A.

Our theory has two small expansion parameters: One is ϵ characterizing the small spatial derivative, that is, the small wave number, and the other is δ_c characterizing the amplitudes of the higher order perturbations. In general, under our present scheme, all of the terms are classified as $O(\epsilon^k \delta_c^l)$, where k and l are appropriate non-negative integers. Since we are interested in the full nonlinear perturbations, in Sec. II A we will not Taylor expand with respect to δ_c. But since we treat the evolutions of the long wavelength perturbations only, we will expand with respect to ϵ and we will drop terms which are small compared to the leading order by $O(\epsilon^2)$ order quantity. This process corresponds to the leading order of the gradient expansion [10,11,16], and the universe treated in this way is called the locally homogeneous universe. The locally homogeneous evolution equations are very similar to the exactly homogeneous evolution equations. The evolutions of locally homogeneous physical quantities which have counterparts in the corresponding exactly homogeneous universe can be determined as easily as the exactly homogeneous evolutions are determined. This is the attractive point of our LWL method.

In our scheme, we can assign $\partial_t\tilde{\gamma}_{ij} = O(\delta_c)$ and $\partial_t^2\tilde{\gamma}_{ij} = O(\delta_c)$, where $\tilde{\gamma}_{ij}$ is the unimodular factor of the spatial metric defined by (2.5), since we require only that the universe should be the spatially flat FRW universe in the background level. So in the leading order of the gradient (ϵ) expansion, the terms containing $\partial_t\tilde{\gamma}_{ij}$ and $\partial_t^2\tilde{\gamma}_{ij}$ cannot be dropped. But in Refs. [10,16], mainly in order to avoid the computational complexity, the authors assigned $\partial_t\tilde{\gamma}_{ij} = O(\epsilon^2)$ and $\partial_t^2\tilde{\gamma}_{ij} = O(\epsilon^2)$ and discarded the terms containing $\partial_t\tilde{\gamma}_{ij}$ and $\partial_t^2\tilde{\gamma}_{ij}$. But this is confusion between the different small parameters ϵ and δ_c and cannot be justified. In this section, under the more natural assumption $\tilde{\gamma}_{ij} = \delta_{ij} + O(\delta_c)$, we show that the evolution of $\tilde{\gamma}_{ij}$ can be solved analytically and that the evolution equations of the other dynamical variables are simple enough to be solved analytically in spite of the inclusion of the terms containing $\partial_t\tilde{\gamma}_{ij}$ and $\partial_t^2\tilde{\gamma}_{ij}$. By doing so, in the leading order of the gradient (ϵ) expansion, the evolutions of all of the modes $2N_S + 4N_f + 4$ are obtained consistently in the universe where N_S scalar fields and N_f perfect fluids coexist. In Refs. [10,16], the evolutions of the several dynamical variables become higher order $O(\epsilon^2)$, and therefore the initial conditions of such variables are unnaturally constrained to too small quantities.

A. Evolution of the locally homogeneous universe

First, we present the Einstein equations $G_{\mu\nu} = \kappa^2 T_{\mu\nu}$, where $\kappa^2 = 8\pi G$ is the gravitational constant, based on the 3 + 1 decomposition based on Ref. [17]. The metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \tag{2.1}$$

is written by

$$g_{00} = -\alpha^2 + \beta_k\beta^k, \tag{2.2}$$

$$g_{0i} = \beta_i, \tag{2.3}$$

$$g_{ij} = \gamma_{ij}, \tag{2.4}$$

where α is the lapse, β_i is the shift vector, and $\beta^i := \gamma^{ij}\beta_j$. Greek indices take the values $\mu, \nu = 0, 1, 2, 3$ and Latin indices take values $i, j = 1, 2, 3$. The spatial metric γ_{ij} is decomposed as

$$\gamma_{ij} = a^2\tilde{\gamma}_{ij}, \qquad \det(\tilde{\gamma}_{ij}) = 1, \tag{2.5}$$

where a can be interpreted as the nonlinear generalization of the scale factor. The extrinsic curvature of the $t = \text{const}$ hypersurface is given by

$$-K_{ij} = \alpha\Gamma^0_{ij} = \frac{1}{2\alpha}(\dot{\gamma}_{ij} - D_i\beta_j - D_j\beta_i), \tag{2.6}$$

where D_i is the covariant derivative with respect to γ_{ij}. The extrinsic curvature is decomposed as

$$K_{ij} = \tfrac{1}{3}\gamma_{ij}K + a^2\tilde{A}_{ij}, \tag{2.7}$$

$$\gamma^{ij}\tilde{A}_{ij} = 0. \tag{2.8}$$

Then we obtain

$$-K = \frac{1}{\alpha}\left(3\frac{\dot{a}}{a} - D_i\beta^i\right). \tag{2.9}$$

The energy momentum tensor is given by

$$T_{\mu\nu} = (\rho + P)u_\mu u_\nu + Pg_{\mu\nu}, \tag{2.10}$$

where ρ, P, and u_μ are the energy density, the pressure, and the 4-velocity of the total system, respectively. The 4-velocity u^μ is written as

$$u^0 = [\alpha^2 - (\beta_k + v_k)(\beta^k + v^k)]^{-1/2}, \tag{2.11}$$

$$u^i = u^0v^i, \tag{2.12}$$

where v_i is the 3-velocity of the total system and $v^i := \gamma^{ij}v_j$. By using $n_\mu := (-\alpha, 0, 0, 0)$, which is the unit vector normal to the time slices, the 3 + 1 decomposition of the energy momentum tensor is given by

$$E := T_{\mu\nu}n^\mu n^\nu = (\rho + P)(\alpha u^0)^2 - P, \tag{2.13}$$

$$J_j := -T_{\mu\nu}n^\mu\gamma^\nu_j = (\rho + P)\alpha u^0u_j, \tag{2.14}$$

$$S_{ij} := T_{ij} = (\rho + P)(u^0)^2(\beta_i + v_i)(\beta_j + v_j) + P\gamma_{ij}. \tag{2.15}$$

The Hamiltonian and momentum constraints are written as

$$R - \tilde{A}_{ij}\tilde{A}^{ij} + \tfrac{2}{3}K^2 = 2\kappa^2E, \tag{2.16}$$

$$D_i\tilde{A}^i_j - \tfrac{2}{3}D_jK = \kappa^2J_j, \tag{2.17}$$

where the indices of $\tilde{A}_{ij}$ is raised by $\tilde{\gamma}^{ij}$, which is the inverse matrix of $\tilde{\gamma}_{ij}$. The evolution equations for γ_{ij} are written as

$$(\partial_t - \beta^k\partial_k)a = \tfrac{1}{3}a(-\alpha K + \partial_k\beta^k), \tag{2.18}$$

$$(\partial_t - \beta^k\partial_k)\tilde{\gamma}_{ij} = -2\alpha\tilde{A}_{ij} + \tilde{\gamma}_{ik}\partial_j\beta^k + \tilde{\gamma}_{jk}\partial_i\beta^k - \tfrac{2}{3}\tilde{\gamma}_{ij}\partial_k\beta^k. \tag{2.19}$$

The evolution equations for K_{ij} are given by

$$(\partial_t - \beta^k\partial_k)K = \alpha\left(\tilde{A}_{ij}\tilde{A}^{ij} + \frac{1}{3}K^2\right) - D_kD^k\alpha + \frac{\kappa^2}{2}\alpha(E + S^k_k), \tag{2.20}$$

$$\begin{aligned}(\partial_t - \beta^k\partial_k)\tilde{A}_{ij} = \frac{1}{a^2}\bigg[&\alpha\left(R_{ij} - \frac{1}{3}\gamma_{ij}R\right) - \left(D_iD_j\alpha - \frac{1}{3}\gamma_{ij}D_kD^k\alpha\right)\bigg] \\ &+ \alpha(K\tilde{A}_{ij} - 2\tilde{A}_{ik}\tilde{A}^k_j) + \tilde{A}_{ik}\partial_j\beta^k + \tilde{A}_{jk}\partial_i\beta^k - \frac{2}{3}\tilde{A}_{ij}\partial_k\beta^k \\ &- \frac{\kappa^2\alpha}{a^2}\left(S_{ij} - \frac{1}{3}\gamma_{ij}S^k_k\right).\end{aligned} \tag{2.21}$$

R_{ij} is the Ricci tensor of the metric γ_{ij}, $R = \gamma^{ij}R_{ij}$, and $S^k_k = \gamma^{kl}S_{kl}$.

Next, under the $3 + 1$ decomposition, we rewrite the evolution equations of the energy momentum tensor $T^\mu_{A\nu}$ of the A component:

$$\nabla_\nu T^\nu_{A\mu} = Q_{A\mu}, \tag{2.22}$$

where $A := (a, \alpha)$, a is the scalar field index, and α is the perfect fluids index. The energy momentum transfer vector $Q_{A\mu}$ is decomposed as

$$Q_{A\mu} = Q_A u_\mu + f_{A\mu}, \tag{2.23}$$

$$u^\mu f_{A\mu} = 0, \tag{2.24}$$

where Q_A and $f_{A\mu}$ are the energy transfer and the momentum transfer vector of the A component, respectively. The energy momentum tensor of the total system T^μ_ν can be expressed as the sum of each component energy momentum tensor $T^\mu_{A\nu}$:

$$T^\mu_\nu = \sum_A T^\mu_{A\nu}. \tag{2.25}$$

The energy momentum conservation, that is, the Bianchi identity, gives

$$\sum_A Q_{A\mu} = 0. \tag{2.26}$$

We consider the scalar field component $\phi = (\phi_a)$. The energy momentum tensor of the scalar field part is given by

$$(T^\mu_\nu)_S = \nabla_\mu\phi\cdot\nabla_\nu\phi - \tfrac{1}{2}[g^{\rho\sigma}\nabla_\rho\phi\cdot\nabla_\sigma\phi + 2U]g_{\mu\nu}. \tag{2.27}$$

The energy momentum tensor of the scalar field part $(T_{\mu\nu})_S$ can be written as the perfect fluid form by identifying

$$\rho_S = -\tfrac{1}{2}g^{\rho\sigma}\nabla_\rho\phi\cdot\nabla_\sigma\phi + U, \tag{2.28}$$

$$P_S = -\tfrac{1}{2}g^{\rho\sigma}\nabla_\rho\phi\cdot\nabla_\sigma\phi - U, \tag{2.29}$$

$$u_{a\mu} = -\frac{1}{\mathrm{sgn}(\dot{\phi}_a)(-g^{\rho\sigma}\nabla_\rho\phi\cdot\nabla_\sigma\phi)^{1/2}}\partial_\mu\phi_a, \tag{2.30}$$

where the minus sign of $u_{a\mu}$ is adopted by requiring $u_{a0} = -\alpha + O(\epsilon^2)$ in the gradient expansion scheme, where ϵ is the small parameter characterizing the spatial derivative defined below. When we assume that the energy momentum transfer vector of the scalar field part $(Q_\mu)_S$ is given by

$$(Q_\mu)_S = S_a\nabla_\mu\phi_a, \tag{2.31}$$

where the source function S_a describes the energy transfer from the scalar field ϕ_a to other components, the evolution equation of the scalar field components $\nabla_\nu(T^\nu_\mu)_S = (Q_\mu)_S$ holds if the scalar field ϕ_a satisfies the phenomenological equations of motion of the scalar field ϕ_a:

$$\Box\phi_a - \frac{\partial U}{\partial\phi_a} = S_a, \tag{2.32}$$

where

$$\Box\phi_a = (-g)^{-1/2}\partial_\mu[(-g)^{1/2}g^{\mu\nu}\partial_\nu\phi_a], \tag{2.33}$$

$$g := \det(g_{\mu\nu}). \tag{2.34}$$

Since we are interested in the cosmological perturbations on superhorizon scales, we put the gradient expansion assumption defined by

$$\partial_i = O(\epsilon), \qquad \beta_i = O(\epsilon), \qquad v_i = O(\epsilon), \qquad f_{Ai} = O(\epsilon), \tag{2.35}$$

where ϵ is the small parameter characterizing the small wave number of the cosmological perturbations. By the assumption $\partial_i = O(\epsilon)$, we assume that the spatial scale of all of the inhomogeneities is of the order of $1/\epsilon$; that is, all of the physical quantities which are approximately homogeneous on each horizon can vary on the superhorizon scales. The local homogeneity and isotropy in the horizon guarantee that the vector quantities such as β_i, v_i, and f_{Ai} are of the order of $O(\epsilon)$. Unlike Refs. [10,16], where $\partial_t\tilde{\gamma}_{ij} = O(\epsilon^2)$ is assumed, from the requirement that our locally homogeneous universe should be the spatially flat FRW universe in the background level, we assume that

$$\tilde{\gamma}_{ij} = \delta_{ij} + O(\delta_c), \tag{2.36}$$

where δ_c is the small parameter characterizing the higher order perturbations. Since we consider the leading order of the gradient (ϵ) expansion without expanding with respect to δ_c, we must keep the terms containing $\partial_t\tilde{\gamma}_{ij} = O(\delta_c)$

and $\partial_t^2 \tilde{\gamma}_{ij} = O(\delta_c)$ as explained in the beginning of this section.

Under the gradient (ϵ) expansion scheme, the relations between the total system quantities and the component quantities are given by

$$\rho = \sum_A \rho_A, \tag{2.37}$$

$$P = \sum_A P_A, \tag{2.38}$$

$$h = \sum_A h_A, \tag{2.39}$$

$$h v_i = \sum_A h_A v_{Ai} + O(\epsilon^3), \tag{2.40}$$

$$0 = \sum_A Q_A, \tag{2.41}$$

$$0 = \sum_A f_{Ai}, \tag{2.42}$$

where h_A is the A component enthalpy defined by $h_A := \rho_A + P_A$.

As for the fluid component α, $\nabla_\mu T^\mu_{\alpha 0} = Q_{\alpha 0}$ gives

$$\dot{\rho}_\alpha = -3H(\rho_\alpha + P_\alpha) + Q_\alpha \alpha + O(\epsilon^2), \tag{2.43}$$

where H is the Hubble parameter defined by $H := \dot{a}/a$, and integrating $\nabla_\mu T^\mu_{\alpha i} = Q_{\alpha i}$ with respect to t gives

$$h_\alpha(\beta_i + v_{\alpha i}) = \frac{\alpha}{a^3} C_{\alpha i} + \frac{\alpha}{a^3} \int_{t_0} dt \alpha a^3 \left[-\partial_i P_\alpha - \frac{1}{\alpha} D_i \alpha h_\alpha + Q_{\alpha i} \right] + O(\epsilon^3), \tag{2.44}$$

where t_0 is the initial time and $C_{\alpha i} := C_{\alpha i}(\boldsymbol{x})$ are the integration constants.

As for the scalar field components, from (2.32), the equation of motion of the scalar field ϕ_a is given by

$$\frac{1}{\alpha^2}\left[\ddot{\phi}_a + 3\frac{\dot{a}}{a}\dot{\phi}_a - \frac{\dot{\alpha}}{\alpha}\dot{\phi}_a \right] + \frac{\partial U}{\partial \phi_a} + S_a = O(\epsilon^2). \tag{2.45}$$

Since the source function S_a is the scalar quantity, we can assume that S_a is the function of other scalar quantities. As such scalar quantities, we can adopt ϕ_a,

$$T_{2a} := \mathrm{sgn}(\partial_0 \phi_a)(-g^{\rho\sigma}\nabla_\rho \phi_a \nabla_\sigma \phi_a)^{1/2} = \frac{1}{\alpha}\dot{\phi}_a + O(\epsilon^2), \tag{2.46}$$

or

$$T_{2a} := u^\mu \nabla_\mu \phi_a = \frac{1}{\alpha}\dot{\phi}_a + O(\epsilon^2), \tag{2.47}$$

where u_μ is the arbitrary unit timelike vector field and

$$T_3 := \nabla_\mu u^\mu = 3\frac{1}{\alpha}\frac{\dot{a}}{a} + O(\epsilon^2). \tag{2.48}$$

Therefore, in the leading order of the ϵ expansion, the form of the source functions S_a can be given by

$$S_a = S_a\left(\phi_a, \frac{1}{\alpha}\dot{\phi}_a, 3\frac{1}{\alpha}\frac{\dot{a}}{a}\right) + O(\epsilon^2). \tag{2.49}$$

For example,

$$S_a = \Gamma_a \frac{1}{\alpha}\dot{\phi}_a, \tag{2.50}$$

where Γ_a is the decay constant of the scalar field ϕ_a, is the most simple source function.

Since until now we have presented the leading order of the gradient (ϵ) expansion of all of the locally homogeneous evolution equations, we will construct all of the solutions of the evolutions of the locally homogeneous universe. From (2.19), we obtain

$$\tilde{A}_{ij} = -\frac{1}{2\alpha}\frac{d}{dt}\tilde{\gamma}_{ij} + O(\epsilon^2). \tag{2.51}$$

By substituting the above equation into (2.21), we obtain

$$\frac{d^2}{dt^2}M = \left(\frac{d}{dt}\ln\frac{\alpha}{a^3}\right)\dot{M} + \dot{M}M^{-1}\dot{M} + O(\epsilon^2), \tag{2.52}$$

where $M := (\tilde{\gamma}_{ij})$. By neglecting $O(\epsilon^2)$ order terms, we obtain the solution as

$$M = R\exp\left[\int_{t_0} dt \frac{\alpha}{a^3} T\right], \tag{2.53}$$

where $R = R(\boldsymbol{x})$ and $T = T(\boldsymbol{x})$ are time-independent matrices. Since M is a unimodular symmetric matrix for an arbitrary t, R is unimodular symmetric, T is traceless, and RT is symmetric. By using (2.53), we obtain

$$\tilde{A}_{ij}\tilde{A}^{ij} = \frac{1}{4\alpha^2}\mathrm{tr}(\dot{M}M^{-1}\dot{M}M^{-1}) = \frac{c_T}{a^6}, \tag{2.54}$$

$$c_T := \tfrac{1}{4}\mathrm{tr}(T^2), \tag{2.55}$$

where $c_T = c_T(\boldsymbol{x})$ is a time-independent constant. By using the above results, (2.16) gives

$$H^2 = \alpha^2\left(\frac{\kappa^2}{3}\rho + \frac{1}{6}\frac{c_T}{a^6}\right), \tag{2.56}$$

and (2.20) gives

$$\dot{H} = \frac{\dot{\alpha}}{\alpha}H - \frac{\alpha^2}{2}\frac{c_T}{a^6} - \frac{\alpha^2\kappa^2}{2}(\rho + P). \tag{2.57}$$

By eliminating H in (2.56) and (2.57), we obtain the well-known continuity equation of the total system in the expanding universe as

$$\dot{\rho} = -3H(\rho + P). \tag{2.58}$$

Equations (2.43) and (2.45) and the above three evolution equations agree with the exactly homogeneous evolution equations if the proper time slicing $\alpha = 1$ [18] and $c_T = 0$. But we cannot assume $c_T = 0$ since the solution constants must satisfy the constraint originating from the momentum constraint (2.17) as

$$\kappa^2 \sum_\alpha C_{j\alpha} + \frac{1}{2}\partial_i T^i_j - \frac{1}{4}\,\mathrm{tr}[R^{-1}\partial_j R T] - \left[2a^3\partial_j\left(\frac{H}{\alpha}\right) + \kappa^2\frac{a^3}{\alpha}\sum_a \dot{\phi}_a \partial_j \phi_a\right]_{t_0} = 0. \qquad (2.59)$$

For the derivation of (2.59), see Appendix A.

We consider the long wavelength limit of all of the solutions of the evolution equations of the locally homogeneous universe. The unimodular factor of the spatial metric $\tilde{\gamma}_{ij}$ is obtained by (2.53). These $\tilde{\gamma}_{ij}$ induce the deviation between the true locally homogeneous universe and the corresponding exactly homogeneous universe, for example, c_T terms in (2.56) and (2.57). They contain the scalar adiabatic decaying mode which was carefully treated in Refs. [1,9], since in the scalar part in the linear cosmological perturbation theory it induces the deviation between the long wavelength limit of the true perturbation solutions and the derivative of the exactly homogeneous universe with respect to the solution constant. According to the philosophy of the KH construction [1], the expressions of ρ_α and ϕ_a related with the exactly homogeneous quantities are obtained by solving the evolution equations (2.43) and (2.44) under the Hamiltonian constraint (2.56) and the proper time slicing $\alpha = 1$. Under the proper time slicing $\alpha = 1$, (2.43), (2.45), and (2.56) are almost the same as the counterparts of the exactly homogeneous universe. Therefore, solving the former true locally homogeneous evolution equations requires as little labor as solving the latter exactly homogeneous evolution equations. As for the velocity perturbations of the fluid components $h_\alpha(\beta_i + v_{\alpha i})$, which are the vector quantities not related with the exactly homogeneous quantities, their evolutions are given by (2.44) in whose right-hand side the second integral term contains P_α, h_α, and α which have already been determined by the previous process. The solution constants must satisfy the momentum constraint (2.59).

Let us count the degrees of freedom. As for (2.53), R is unimodular symmetric, T is traceless, and RT is symmetric; therefore, R and T have 5 degrees of freedom, respectively. But by the coordinate transformation $\bar{x}^i = f^i(x)$ ($i = 1, 2, 3$), the 3 degrees of freedom of R can be made vanishing. According to (2.43) and (2.45), the densities ρ_α and the scalar fields ϕ_a have N_f and $2N_S$ degrees of freedom, respectively. According to (2.44), the fluids velocities $v_{\alpha i}$ have $3N_f$ degrees of freedom. The momentum constraint (2.59) gives 3 constraints. Therefore, the total degrees of freedom is $5 + 5 - 3 + N_f + 2N_S + 3N_f - 3 = 4 + 4N_f + 2N_S$. Then we have obtained all of the solutions of the evolution equations of the locally homogeneous universe in the leading order of the gradient (ϵ) expansion.

From the time dependence of all of the solutions, we can interpret the physical roles of all of the solutions. Two from R can be interpreted as the gravitational wave growing modes. Five from T can be interpreted as the two gravitational wave decaying modes, the one adiabatic scalar decaying mode, and the two adiabatic vector decaying modes. By the three momentum constraints (2.59), three of $C_{j\alpha}$ are adjusted. In the remaining $(3N_f - 3)$ $C_{j\alpha}$'s, the $(2N_f - 2)$ entropic vector decaying modes and the $(N_f - 1)$ entropic scalar decaying modes are contained. The N_f densities ρ_α have the N_f scalar growing modes, and the N_S scalar fields ϕ_a have the N_S scalar field growing modes and the N_S scalar field decaying modes.

B. Gauge invariant variables and the derivation of the LWL formula

In this subsection, we give the definitions of the gauge invariant perturbation variables in the arbitrary higher order perturbation theory in the leading order of the gradient (ϵ) expansion and the LWL formula representing the evolutions of these gauge invariant perturbation variables in terms of derivative with respect to the solution constant of the corresponding physical quantities of the locally homogeneous universe.

As for $M = (\tilde{\gamma}_{ij})$, since the homogeneous parts of R and T defined by (2.53) are determined by the fact that the background metric is the spatially flat FRW universe, R and T are expanded as

$$R_{ij}(\boldsymbol{x}) = \delta_{ij} + \sum_{k=1}^{\infty}\frac{1}{k!}\delta^k R_{ij}(\boldsymbol{x}), \qquad (2.60)$$

$$T_{ij}(\boldsymbol{x}) = \sum_{k=1}^{\infty}\frac{1}{k!}\delta^k T_{ij}(\boldsymbol{x}), \qquad (2.61)$$

where δ in the above is the operator generating the higher order perturbation quantities.

As shown in the previous subsection, in the leading order of the gradient (ϵ) expansion, the evolution equations of the locally homogeneous universe which have counterparts in the evolution equations of the exactly homogeneous universe do not contain the spatial derivative. Therefore, in the expression of the solution of the evolution of an arbitrary locally homogeneous physical quantity (related with the exactly homogeneous quantity) A such as the scalar fields ϕ_a and the fluid energy densities ρ_α, all of the dependences on the spatial coordinate $\boldsymbol{x}$ are contained in the (time-independent) spatial-dependent integration constants $C(\boldsymbol{x})$:

$$A = A(t, C(\boldsymbol{x})). \qquad (2.62)$$

Since the solutions of the evolution equations of the locally homogeneous universe contain the nonlinear effect in the full order, the locally homogeneous physical quantity A can be expanded as

$$A(t, \boldsymbol{x}) = A(t) + \sum_{k=1}^{\infty} \frac{1}{k!} \delta^k A(t, \boldsymbol{x}). \tag{2.63}$$

The higher order perturbation effects are induced by the dependences on the spatial coordinate $\boldsymbol{x}$. Therefore, each solution constant $C_i(\boldsymbol{x})$ can be expanded as

$$C_i(\boldsymbol{x}) = C_i + \sum_{k=1}^{\infty} \frac{1}{k!} \delta^k C_i(\boldsymbol{x}), \tag{2.64}$$

whose background part is spatially independent. The kth order nonlinear perturbation $\delta^k A$ can be expressed in terms of the nonlinear perturbations of the spatial-dependent integration constants $\delta^l C(\boldsymbol{x})$. In this paper, the expressions representing $\delta^k A$ in terms of $\delta^l C(\boldsymbol{x})$ are called the LWL formula. In order to derive the LWL formula, we propose a simple mathematical trick. In the leading order of the gradient (ϵ) expansion, the expression of the solution of an arbitrary locally homogeneous physical quantity A can be written as a function of the time t and the integration constants C as shown in (2.62). We assume that all of the integration constants C depend on one parameter λ imaginarily instead of $\boldsymbol{x}$. As understood below, λ represents the spatial coordinate dependence symbolically. The physical quantity A can be expanded as

$$A(\lambda) = \sum_{k=0}^{\infty} \frac{1}{k!} \lambda^k \frac{d^k}{d\lambda^k} A(\lambda)|_{\lambda=0}. \tag{2.65}$$

The full order nonlinear solution is formally recovered by setting $\lambda = 1$. Since both the perturbation δ and the λ differentiation $d/d\lambda$ are the derivative operators satisfying the same chain and product rules, and the kth order λ differentiation $d^k/d\lambda^k \ldots|_{\lambda=0}$ is multiplied by λ^k, we can use the λ differentiation to track the algebraic behaviors of the perturbation δ. By comparing (2.65) with (2.63), we can read the correspondences given by

$$\frac{d^k}{d\lambda^k} A(\lambda)|_{\lambda=0} \leftrightarrow \delta^k A(t, \boldsymbol{x}). \tag{2.66}$$

The gauge transformation can be expressed by the Lie derivative $L(T)$ as

$$A(\lambda, \mu) = \exp\{\mu L(T)\} A(\lambda, \mu = 0), \tag{2.67}$$

where μ is the parameter characterizing the size of the gauge transformation, $A(\lambda, \mu = 1)$ is the transformed variable, and $A(\lambda, \mu = 0) = A(\lambda)$ is the original variable. When the vector field $T := T^\mu \partial_\mu$ in the Lie derivative $L(T)$ can be expanded in terms of λ, the zeroth order term $T(\lambda = 0)$ is zero since we consider the infinitesimal gauge transformation. By differentiating (2.67) with respect to λ, afterward putting $\lambda = 0$, we obtain the well-known expressions of the gauge transformations of $\delta^k A$ [19]:

$$\widetilde{\delta A} = L_1 A + \delta A, \tag{2.68}$$

$$\widetilde{\delta^2 A} = L_2 A + L_1 L_1 A + 2 L_1 \delta A + \delta^2 A, \tag{2.69}$$

where $\widetilde{\delta^n A}$ is the gauge transformed perturbation variables of $\delta^n A$ and L_k is the Lie derivative induced by $\delta^k T$; $L_k := L(\delta^k T)$. The gauge transformation law (2.67) is the solution of the differential equation:

$$\frac{d}{d\mu} A(\lambda, \mu) = L(T) A(\lambda, \mu), \tag{2.70}$$

which is much simpler than the individual gauge transformation expressions of $\delta^k A$. As for the vector field $T := T^\mu \partial_\mu$ inducing the Lie derivative $L(T)$, we can assume that

$$T^i = O(\epsilon), \tag{2.71}$$

since we consider only the gauge transformations keeping the local homogeneity and isotropy in the horizon.

By using the differential equation (2.70), we can show the following proposition.

Proposition 1.—For an arbitrary scalar quantity A, the perturbation quantities $D^n A$ and $D^n(\dot{A}/\alpha)$, where D is defined by

$$D := \frac{d}{d\lambda} - \frac{da}{d\lambda} \frac{1}{\dot{a}} \frac{d}{dt}, \tag{2.72}$$

are gauge invariant up to order $O(\epsilon^2)$ error. For the lapse function α, $\mathcal{A}_n$ defined by

$$\mathcal{A}_n := \left(\frac{d}{d\lambda} - \frac{d}{dt} \frac{1}{\dot{a}} \frac{da}{d\lambda} \right)^n \alpha \tag{2.73}$$

is gauge invariant up to order $O(\epsilon^2)$ error.

For the proof, see Appendix C.

$D^n A$, $D^n(\dot{A}/\alpha)$, and $\mathcal{A}_n$ are higher order generalizations of DA, $D\dot{A}$, and $\mathcal{A}$ defined in Ref. [9], respectively. $D^n(\dot{A}/\alpha)$ and $\mathcal{A}_n$ are not independent. For example,

$$D\left(\frac{\dot{A}}{\alpha}\right) = \frac{1}{\alpha^2} \{ (DA)^{\cdot} \alpha - \dot{A} \mathcal{A}_1 \}. \tag{2.74}$$

Therefore, by putting $A = \phi_a$ and $A = \rho_\alpha$, we can use $D^n \phi_a$, $D^n(\dot{\phi}_a/\alpha)$, and $D^n \rho_\alpha$ as the independent perturbation variables. Our DA is almost the same as the well-known gauge invariant perturbation variable, for example, for $A = \phi_a$ the Sasaki Mukhanov variable in the linear perturbation theory [9,18,20,21]. Our $D^2 A$ agrees almost perfectly with the gauge invariant quantities introduced by the second order gauge invariant perturbation theory [22–24]. The reason of the tiny deviation between our gauge invariant perturbation quantity and the gauge invariant perturbation quantity of the first and the second order gauge invariant perturbation theory is that we assume $T^i = O(\epsilon)$ based on the gradient expansion scheme. In our scheme (2.71), we can regard the scale factor a as the scalar quantity up to $O(\epsilon^2)$. Therefore, in the same way as in the proof of Proposition 1, we can show that

$$\zeta_n := \bar{\delta}^n \ln a, \tag{2.75}$$

$$\bar{\delta} := \frac{d}{d\lambda} - \frac{d\rho}{d\lambda}\frac{1}{\dot{\rho}}\frac{d}{dt} \tag{2.76}$$

are gauge invariant up to $O(\epsilon^2)$. ζ_n is the higher order generalization of the well-known Bardeen parameter [25,26].

We discuss the influence of the replacement of the evolution parameter. Concretely, we consider replacing the old evolution parameter t with the new evolution parameter as the scale factor a. With the scale factor a as the evolution parameter, the D operation defined by (2.72) can be written in the more simple form. $d/d\lambda$ in (2.72) is the λ derivative with t fixed, that is, operating on the locally homogeneous physical quantity such as ρ_α and ϕ_a expressed by using t as the evolution parameter. However, arbitrary locally homogeneous quantities can be also expressed by using the scale factor a as the evolution parameter instead of t. In this case, we can consider the differentiation $(\partial/\partial\lambda)_a$ taken at fixed a. The differential operators in the two groups $(d/d\lambda, d/dt)$ and $((\partial/\partial\lambda)_a, (\partial/\partial a)_a)$ are commutative in each group, but the differential operators belonging to the different groups, for example, $d/d\lambda$ and $(\partial/\partial a)_a$, are not commutative. By using the relation

$$\frac{d}{d\lambda} = \left(\frac{\partial}{\partial\lambda}\right)_a + \frac{da}{d\lambda}\left(\frac{\partial}{\partial a}\right)_a, \tag{2.77}$$

the D operation (2.72) can be expressed much more simply as

$$D = \left(\frac{\partial}{\partial\lambda}\right)_a. \tag{2.78}$$

Therefore $D^n A$ and $D^n(\dot{A}/\alpha)$ can be written in the very simple way as

$$D^n A = \left(\frac{\partial^n}{\partial\lambda^n}\right)_a A, \qquad D^n\left(\frac{\dot{A}}{\alpha}\right) = \left(\frac{\partial^n}{\partial\lambda^n}\right)_a\left(\frac{\dot{A}}{\alpha}\right). \tag{2.79}$$

In this way, in the LWL formalism where the scale factor a, not t, is used as the evolution parameter [7–9], the expressions of the solutions of $D^n A$ and $D^n(\dot{A}/\alpha)$ can be obtained by calculating only a single term written with the higher order $(\partial/\partial\lambda)_a$ derivative of the solution of the corresponding locally homogeneous physical quantity. In the same way, $\bar{\delta}$ defined by (2.76) can be expressed in the word of the scale factor a as

$$\bar{\delta} = \left(\frac{\partial}{\partial\lambda}\right)_a - \left\{\left(\frac{\partial\rho}{\partial\lambda}\right)_a \Big/ \left(\frac{\partial\rho}{\partial a}\right)_a\right\}\left(\frac{\partial}{\partial a}\right)_a. \tag{2.80}$$

As for the Bardeen parameter defined by (2.75), we can show the following propositions.

Proposition 2.—If ζ_1 is conserved for arbitrary values of integration constants $C(\lambda = 0)$, all ζ_n $(n \geq 2)$ are also conserved, and they can be expressed as

$$\zeta_n = \left(\frac{\partial}{\partial\lambda}\right)_a^{n-1}\zeta_1. \tag{2.81}$$

For the proof, see Appendix D.

Proposition 3.—When $P = P(\rho)$ holds, ζ_n $(n \geq 1)$ can be expressed as

$$\zeta_n = \frac{1}{3}\left(\frac{\partial^n}{\partial\lambda^n}\right)_a\left[\int d\rho\,\frac{1}{\rho + P(\rho)}\right], \tag{2.82}$$

and all ζ_n $(n \geq 1)$ are conserved.

For the proof, see Appendix E.

All of the definitions of the nonlinear gauge invariant perturbation variables in this subsection are written in terms of the λ differentiations. As explained at the beginning of this subsection, the λ differentiation implies not only the symbol of the higher order nonlinear perturbation of the physical quantity but also the process of taking the derivative with respect to the integration constants of the corresponding locally homogeneous quantity. Therefore, all of the definitions of the nonlinear gauge invariant perturbation variables in terms of the λ differentiations can also be regarded as the LWL formulas themselves, so from now on they will be called the LWL formulas.

III. NONLINEAR EVOLUTION OF THE MULTIPLE SLOW ROLLING SCALAR FIELDS

In this section, as the application of the LWL formula derived in the previous section, we consider the evolutions of the long wavelength nonlinear cosmological perturbations in the universe dominated by the multiple slow rolling scalar fields. The τ function and the N potential introduced in this section are useful tools for tracing analytically the evolutions of the multiple slow rolling scalar fields in the long time interval. We calculate spectral indices of the linear cosmological perturbations. In the interacting system, we derive the formulas giving the amplitudes of the nonlinear Bardeen parameters at the end of the slow rolling phase in terms of the initial scalar field perturbations and calculate the nonlinear parameters $f_{\rm NL}$ and $g_{\rm NL}$ [14] representing the non-Gaussianity of the Bardeen parameter.

A. Evolution of the multiple slow rolling scalar fields

In the slow rolling phase, the scalar fields ϕ_a roll slowly on the potential U. The potential energy U, which hardly changes, triggers the exponential expansion of the universe. The Hubble parameter is large compared to the masses of the scalar fields. The ratio of the kinetic energy part in the whole energy density ρ is small compared to the contribution from the potential energy U. In the investigations of the evolutions of the scalar fields under this situation, it is effective to use the transformation by which the evolution equations of the slow rolling scalar field system are greatly simplified; that is, the effects of the time derivatives of the scalar fields $p_a := \dot{\phi}_a/\alpha$ on the evolutions of the scalar fields ϕ_a are eliminated.

In this section, we consider the multiple slow rolling scalar fields ϕ_a under the conditions:

$$S_a = 0, \qquad U = \sum_a \frac{1}{2} m_a^2 \phi_a^2 + U_{\rm int}, \tag{3.1}$$

where $U_{\rm int}$ is the sum of mth order monomials ($m \geq 3$). As the independent variables, we adopt ϕ_a and $p_a := \dot{\phi}_a/\alpha$. By nondimensionalizing the dynamical variables as

$$\frac{a}{a_0} \to a, \qquad \frac{\phi_a}{\phi_0} \to \phi_a, \qquad \frac{p_a}{p_0} \to p_a, \qquad \frac{c_T}{c_{T0}} \to c_T, \tag{3.2}$$

and the parameter as

$$\frac{m_a}{m_0} \to m_a, \tag{3.3}$$

that is,

$$\frac{\rho}{m_0^2 \phi_0^2} \to \rho, \tag{3.4}$$

we obtain the dimensionless parameters:

$$\epsilon_* := \frac{\sqrt{3}}{\kappa \phi_0}, \qquad \eta^2 := \frac{p_0^2}{m_0^2 \phi_0^2}, \qquad \nu^2 := \frac{1}{2} \frac{c_{T0}}{\kappa^2 m_0^2 \phi_0^2} \frac{1}{a_0^6}. \tag{3.5}$$

ϵ_* is the small constant representing the ratio of the mass scale to the Hubble parameter. This ϵ_* is different from ϵ characterizing the small wave number in the previous section. η^2 is positive constant since we consider only the scalar fields with positive definite kinetic parts. We assume that $\epsilon_* \sim \eta \sim \nu \ll 1$. Then the evolution equations of ϕ_a, p_a, and c_T are given by

$$\frac{d}{dN} \phi_a = \frac{\epsilon_*}{\eta} \frac{1}{\rho^{1/2}} \eta^2 p_a, \tag{3.6}$$

$$\frac{d}{dN} p_a = -3 p_a - \frac{\epsilon_*}{\eta} \frac{1}{\rho^{1/2}} \frac{\partial U}{\partial \phi_a}, \tag{3.7}$$

$$c_T = \text{const}, \tag{3.8}$$

where the evolution parameter is the e-folding number of the scale factor $N := \ln a$ and the energy density ρ is given by

$$\rho = \frac{\eta^2}{2} \sum_a p_a^2 + U + \nu^2 c_T \frac{1}{a^6}. \tag{3.9}$$

By using $p_a^{(1)}$ defined by

$$p_a^{(1)} := p_a + \frac{1}{3} \frac{\epsilon_*}{\eta} \frac{1}{\rho^{1/2}} \frac{\partial U}{\partial \phi_a}, \tag{3.10}$$

which represents the deviation of p_a from the truncated slow rolling solution, the evolution equations (3.6) and (3.7) can be rewritten as

$$\frac{d}{dN} \phi_a = \eta^2 F_a(\phi) + \eta^2 f_a(\phi, p^{(1)}, c_T, N), \tag{3.11}$$

$$\frac{d}{dN} p_a^{(1)} = -3 p_a^{(1)} + \eta^2 g_a(\phi, p^{(1)}, c_T, N), \tag{3.12}$$

where

$$F_a(\phi) := -\frac{1}{3} \frac{\epsilon_*^2}{\eta^2} \frac{1}{U} \frac{\partial U}{\partial \phi_a} \tag{3.13}$$

and

$$|f_a| \leq |p^{(1)}| + \eta^2, \qquad |g_a| \leq 1, \tag{3.14}$$

for an appropriate complex domain containing the real interval where we consider the motion of ϕ_a, $p_a^{(1)}$, and N. In this section, in all of the inequalities we omit all of the finite constants, and $|p^{(1)}|$ is interpreted as the quantity bounded by $M|p^{(1)}|$ for some positive constant M and

$$|p^{(1)}| := \sum_a |p_a^{(1)}|. \tag{3.15}$$

From now on, we will simply write $p_a^{(1)}$ as p_a for notational simplicity. In this section, we consider the evolution for $0 \leq N \leq 1/\eta^2$ during which the scalar fields roll slowly on the potential U. For a function $f(N)$, let us define $\|f\|$ by

$$\|f\| := \sup_{0 \leq N \leq 1/\eta^2} |f(N)|. \tag{3.16}$$

Under these notations, the following propositions hold.

Proposition 4.—Let k be a non-negative integer and δ_c a small positive constant. Under the initial conditions

$$\frac{\partial^k}{\partial \lambda^k} \phi(0), \qquad \frac{\partial^k}{\partial \lambda^k} p(0), \qquad \frac{\partial^k}{\partial \lambda^k} c_T \sim \delta_c^k, \tag{3.17}$$

for $0 \leq N \leq 1/\eta^2$, the upper bounds of the independent variables are given by

$$\left| \left(\frac{\partial^k}{\partial \lambda^k} \right)_a \phi \right| \leq \delta_c^k, \tag{3.18}$$

$$\left| \left(\frac{\partial^k}{\partial \lambda^k} \right)_a p \right| \leq \delta_c^k (e^{-3N} + \eta^2). \tag{3.19}$$

For the proof, see Appendix F.

In the above and from now on, as for λ differentiations of the physical quantities at the initial time $N = 0$, the suffixes a implying "a fixed" are omitted since what the λ differentiations operate on do not contain any $a = e^N$ dependent parts.

Proposition 5.—Let k be a non-negative integer. The differences $\Delta\phi_a := \phi_a - \bar{\phi}_a$, where ϕ obey the exact evolution equations (3.11) and (3.12) and $\bar{\phi}$ obey the truncated evolution equations as

$$\frac{d}{dN}\bar{\phi} = \eta^2 F(\bar{\phi}), \tag{3.20}$$

are bounded as

$$\left\|\left(\frac{\partial^k}{\partial \lambda^k}\right)_a \Delta\phi\right\| \leq \eta^2 \delta_c^k \sim \eta^2 \left\|\left(\frac{\partial^k}{\partial \lambda^k}\right)_a \phi\right\|, \tag{3.21}$$

under the initial conditions

$$\frac{\partial^k}{\partial \lambda^k}\Delta\phi(0) = 0. \tag{3.22}$$

For the proof, see Appendix G.

According to Proposition 5, if we want to investigate the evolution of the leading order in the slow rolling (η^2) expansion, we have only to solve rather simple evolution equations (3.20). In the following subsections, we study the evolutionary behaviors of the above evolution equations (3.20).

B. The τ function and the N potential

In this subsection, we introduce the τ function and the N potential which enable us to trace the multiple slow rolling scalar fields in the long time interval analytically. By using the τ function as the evolution parameter, the evolutions of the scalar fields ϕ_a and the old evolution parameter N can be expressed in the form of the simple analytic function of τ. From these expressions of ϕ_a in terms of the τ function, we can calculate the N potential in the simple way. The N potential is written in terms of the initial values of the scalar fields $\phi_a(0)$ only. It not only represents the difference between the e-folding number of the scale factor at the end of the slow rolling phase and that at the first horizon crossing but also has the complete information as to the nonlinear curvature perturbations. The λ differentiations of the N potential generate the S formulas connecting the amplitudes of all of the higher order Bardeen parameters at the end of the slow rolling phase with the scalar field perturbations at the first horizon crossing.

Under the conditions (3.1), we will investigate the evolutionary behaviors of the solution of the evolution equations which are obtained by the truncation explained in the previous subsection:

$$\frac{d}{dN}\phi_a = -\frac{1}{\kappa^2}\frac{1}{U}\frac{\partial U}{\partial \phi_a}. \tag{3.23}$$

From the present subsections, we will call off the nondimensionalization in the previous subsection, since the truncated evolution equations (3.23) have already been made sufficiently simple. In the multiple scalar field case, the evolution equations (3.23) cannot be solved analytically, and the scalar fields ϕ_a cannot be expressed in the form of the well-known function of N. So by replacing the old evolution parameter N with the new evolution parameter τ, we decompose (3.23) into two parts:

$$\frac{d}{d\tau}\phi_a = -\frac{\partial U}{\partial \phi_a}, \tag{3.24}$$

$$\frac{d}{d\tau}N = \kappa^2 U. \tag{3.25}$$

The new evolution parameter τ introduced in the above equations will be called the τ function from now on. By introducing the τ function, the evolution equations become simple enough to be solved analytically. In the potential (3.1), we can easily solve (3.24) by iteration, and then we can get $\phi_a(\tau)$ expressed in the form of the analytic functions of the τ function. By substituting these $\phi_a(\tau)$ into U in (3.25), and integrating (3.25) with respect to the τ function, we obtain the expression of the old evolution parameter N in terms of the new evolution parameter τ:

$$N = \int_0^\tau d\tau \kappa^2 U. \tag{3.26}$$

The expressions of ϕ_a and N in terms of the τ function describe the dynamical evolutions of our multiple slow rolling scalar field system completely.

From this subsection, we adopt the τ function as the evolution parameter. We introduce $(\partial/\partial\lambda)_\tau$ as the λ differentiation taken at the fixed τ, that is, operating on the locally homogeneous quantities expressed by using the τ function as the evolution parameter. In order to exaggerate the fact that the τ derivative and $(\partial/\partial\lambda)_\tau$ are commutative, we use $(\partial/\partial\tau)_\tau$ as the τ derivative from now on. By using $(\partial/\partial\lambda)_\tau$ and $(\partial/\partial\tau)_\tau$, D defined by (2.72) can be expressed as

$$D = \left(\frac{\partial}{\partial\lambda}\right)_\tau - \left\{\left(\frac{\partial a}{\partial\lambda}\right)_\tau \Big/ \left(\frac{\partial a}{\partial\tau}\right)_\tau\right\}\left(\frac{\partial}{\partial\tau}\right)_\tau, \tag{3.27}$$

and $\bar{\delta}$ defined by (2.76) can be expressed as

$$\bar{\delta} = \left(\frac{\partial}{\partial\lambda}\right)_\tau - \left\{\left(\frac{\partial U}{\partial\lambda}\right)_\tau \Big/ \left(\frac{\partial U}{\partial\tau}\right)_\tau\right\}\left(\frac{\partial}{\partial\tau}\right)_\tau, \tag{3.28}$$

where we used

$$\frac{d}{d\lambda} = \left(\frac{\partial}{\partial\lambda}\right)_\tau + \frac{d\tau}{d\lambda}\left(\frac{\partial}{\partial\tau}\right)_\tau, \tag{3.29}$$

and $\rho = U$ which holds under the present truncation (3.23) . Then by using the τ function as the evolution parameter, the Bardeen parameter ζ_n defined by (2.75) can be decomposed as

$$\zeta_n = \frac{\partial^n}{\partial\lambda^n}\bar{N} - \frac{\kappa^2}{4}\bar{\delta}^n A(0, 0). \tag{3.30}$$

By $\bar{N}$, we represent

$$\bar{N} := \int_0^\infty d\tau \kappa^2 U, \tag{3.31}$$

and this $\bar{N}$ will be called the N potential from now on. In the above and from now on, as for λ differentiations of the physical quantities at the initial time $\tau = 0$, the suffixes τ

implying "τ fixed" are omitted since what the λ differentiations operate on do not contain any τ dependent parts. $A(2n, k)$ is defined by

$$A(0, 0) := 4 \int_\tau^\infty d\tau U, \tag{3.32}$$

$$A(2n, k) := \left(-\frac{1}{2}\right)^n \left(\frac{\partial^n}{\partial \tau^n} \frac{\partial^k}{\partial \lambda^k}\right)_\tau A(0, 0). \tag{3.33}$$

By using $A(2n, k)$, $\bar{\delta}$ can be expressed as

$$\bar{\delta} = \left(\frac{\partial}{\partial \lambda}\right)_\tau + \frac{1}{2} \frac{A(2, 1)}{A(4, 0)} \left(\frac{\partial}{\partial \tau}\right)_\tau. \tag{3.34}$$

Afterward we will prove the fact that, as for the Bardeen parameter ζ_n at the end of the slow rolling phase, the first term in (3.30) has a leading contribution in the slow rolling expansion scheme. In order to prove this statement, we put forth several assumptions. All of the investigations in this section will be established under the following assumptions:

(i) All of the masses of the scalar fields are of the same order:

$$m_a^2 \sim m^2, \tag{3.35}$$

where

$$m^2 := \min_a \{m_a^2\}. \tag{3.36}$$

(ii) The interaction potential $U_{\rm int}$ is the sum of the mth order monomials ($m \geq 3$) and satisfies

$$|U_{\rm int}| \leq \mu_c m^2 \frac{1}{\phi_0} |\phi|^3 \tag{3.37}$$

for $|\phi| \leq \phi_0$. ϕ_0 is defined by the positive constant of the order of

$$\phi_0 \sim \phi_a(0), \tag{3.38}$$

$|\phi|$ is defined by

$$|\phi|^2 := \sum_a |\phi_a|^2, \tag{3.39}$$

and μ_c is the small positive constant characterizing the interaction strength.

(iii) The nth order perturbation at the initial time $\tau = 0$ is of the order of

$$\frac{\partial^k}{\partial \lambda^k} \phi_a(0) \sim \delta_c^k \phi_0, \tag{3.40}$$

where δ_c is the small constant characterizing the perturbation size.

Under these assumptions, the following proposition holds.

Proposition 6.—The following estimations hold:

$$\frac{\partial^k}{\partial \lambda^k} \bar{N} = \frac{\kappa^2}{4} \sum_a \frac{\partial^k}{\partial \lambda^k} [\phi_a^2(0)] + \langle \kappa^2 \mu_c \delta_c^k \phi_0^2 \rangle, \tag{3.41}$$

$$A(2n, k) = \sum_a (m_a^2)^n \frac{\partial^k}{\partial \lambda^k} [\phi_a^2(0)] \exp[-2m_a^2 \tau] + \langle \mu_c m^{2n} \delta_c^k \phi_0^2 \exp[-2m^2 \tau] \rangle, \tag{3.42}$$

where $\langle M \rangle$ is the quantity bounded by M.

For the proof, see Appendix H.

At the end of the slow rolling phase when the Hubble parameter is of the order of the scalar field mass, the values of the scalar fields are of the order of the Planck mass $\phi_a(\tau) \sim 1/\kappa$. According to Proposition 6, at the end of the slow rolling phase, we obtain the estimations as

$$\frac{\partial^k}{\partial \lambda^k} \bar{N} \sim \kappa^2 \delta_c^k \phi_0^2, \tag{3.43}$$

$$A(2n, k) \sim m^{2n} \delta_c^k \phi_0^2 \left(\frac{m}{H_0}\right)^2, \tag{3.44}$$

where H_0 is the Hubble parameter at the initial time $\tau = 0$. By using the above estimations, the second term of (3.30) can be estimated as

$$-\frac{\kappa^2}{4} \bar{\delta}^n A(0, 0) \sim \kappa^2 \delta_c^n \phi_0^2 \left(\frac{m}{H_0}\right)^2 \sim \frac{\partial^n}{\partial \lambda^n} \bar{N} \cdot \left(\frac{m}{H_0}\right)^2. \tag{3.45}$$

Since the second term of (3.30) is suppressed by the slow rolling parameter $(m/H_0)^2$ compared to the first term, the main part of the nonlinear nth order Bardeen parameter ζ_n at the end of the slow rolling phase is given by the λ derivative of the N potential:

$$\zeta_n = \frac{\partial^n}{\partial \lambda^n} \bar{N}. \tag{3.46}$$

So in order to obtain the final amplitude of the Bardeen parameter ζ_n, we have only to calculate the N potential $\bar{N}$.

In particular case, the N potential can be calculated very easily. When the scalar field potential is written in the separable form

$$U = \sum_a U_a(\phi_a), \tag{3.47}$$

the N potential can also be expressed in the separable form as

$$\bar{N} = \sum_a \bar{N}(a), \tag{3.48}$$

where

$$\bar{N}(a) := \kappa^2 \int_0^{\phi_a(0)} d\phi_a \left(U_a \Big/ \frac{\partial U_a}{\partial \phi_a}\right). \tag{3.49}$$

C. Exactly solvable model; noninteracting case

In this subsection, in the multiple free field case, we give the Bardeen parameter, the entropic perturbations, the gravitational wave perturbations, and their spectral indices. Since we can obtain the analytic expression of $\phi_a(\tau)$, all of the results obtained in this subsection have no black boxes originating from the S formulas connecting the amplitudes of the final physical quantities and those of the initial physical quantities, which were unknown in Ref. [12].

We consider the exactly solvable model defined by

$$U = \sum_a \frac{1}{2} m_a^2 \phi_a^2 \tag{3.50}$$

and calculate the various physical quantities. The N potential is calculated as

$$\bar{N} = \frac{\kappa^2}{4} \sum_a \phi_a^2(0), \tag{3.51}$$

and then the Bardeen parameter ζ_n at the end of the slow rolling phase is calculated by (3.46). From now on, we consider the linear perturbations and calculate their spectral indices. We consider the entropy perturbation between ϕ_a and ϕ_b defined by

$$S_{ab} := -3H\left(\frac{1}{\dot{\rho}_a} D\rho_a - \frac{1}{\dot{\rho}_b} D\rho_b\right). \tag{3.52}$$

In the present case (3.50), S_{ab} is given by

$$S_{ab} = 3\kappa^2 U\left(\frac{1}{m_a^2 \phi_a(0)} \frac{\partial \phi_a(0)}{\partial \lambda} - \frac{1}{m_b^2 \phi_b(0)} \frac{\partial \phi_b(0)}{\partial \lambda}\right). \tag{3.53}$$

In order to calculate spectral indices, we have to know the amplitudes of the quantum fluctuations of the scalar fields and the gravitational waves at the first horizon crossing $\tau = 0$:

$$\left\langle\!\!\left\langle \frac{\partial \phi_a(0)}{\partial \lambda} \frac{\partial \phi_b(0)}{\partial \lambda} \right\rangle\!\!\right\rangle \sim H^2 \delta_{ab} \sim \kappa^2 U \delta_{ab}, \tag{3.54}$$

$$\frac{1}{\kappa^2} \left\langle\!\!\left\langle \frac{\partial \tilde{\gamma}_{ij}(0)}{\partial \lambda} \frac{\partial \tilde{\gamma}_{ij}(0)}{\partial \lambda} \right\rangle\!\!\right\rangle \sim H^2 \sim \kappa^2 U. \tag{3.55}$$

When we calculate the above correlation functions, for simplicity we do not take into account the first order slow rolling corrections unlike Ref. [12], so the above correlation functions become diagonal. From the horizon crossing relation as

$$k = aH = \frac{\kappa}{\sqrt{3}} e^N U^{1/2}, \tag{3.56}$$

we obtain

$$d \ln k = \left(\kappa^2 U + \frac{1}{2U} \frac{dU}{d\tau}\right) d\tau = \kappa^2 U \left\{1 + O\left(\frac{m^2}{H_0^2}\right)\right\} d\tau \approx \kappa^2 U d\tau. \tag{3.57}$$

Then we can calculate the spectral indices as

$$\frac{\partial}{\partial \ln k} \ln\langle\!\langle \zeta_1^2 \rangle\!\rangle = -\frac{4}{\kappa^2} \frac{1}{B(2)} \left(\frac{B(2)}{B(0)} + \frac{B(4)}{B(2)}\right), \tag{3.58}$$

$$\frac{\partial}{\partial \ln k} \ln\langle\!\langle S_{ab}^2 \rangle\!\rangle = \frac{4}{\kappa^2} \frac{1}{B(2)} \times \left(m_a^2 m_b^2 \frac{m_a^2 \phi_a^2(0) + m_b^2 \phi_b^2(0)}{m_a^4 \phi_a^2(0) + m_b^4 \phi_b^2(0)} - \frac{B(4)}{B(2)}\right), \tag{3.59}$$

$$\frac{\partial}{\partial \ln k} \ln\langle\!\langle \tilde{\gamma}_{ij}^2 \rangle\!\rangle = -\frac{4}{\kappa^2} \frac{B(4)}{B(2)^2}, \tag{3.60}$$

where

$$B(2n) := \sum_a (m_a^2)^n \phi_a^2(0). \tag{3.61}$$

As for the correlation between the adiabatic and the entropic perturbations defined by

$$\mathcal{C}_{ab} := \frac{\langle\!\langle \zeta_1 S_{ab} \rangle\!\rangle}{\sqrt{\langle\!\langle \zeta_1^2 \rangle\!\rangle} \sqrt{\langle\!\langle S_{ab}^2 \rangle\!\rangle}}, \tag{3.62}$$

we obtain

$$\frac{\partial}{\partial \ln k} \ln \mathcal{C}_{ab} = \frac{2}{\kappa^2} \frac{1}{B(0)} - \frac{2}{\kappa^2} \times \frac{1}{B(2)} m_a^2 m_b^2 \frac{m_a^2 \phi_a^2(0) + m_b^2 \phi_b^2(0)}{m_a^4 \phi_a^2(0) + m_b^4 \phi_b^2(0)}. \tag{3.63}$$

Since we succeed in solving the scalar field evolutions completely, our expressions of the spectral indices of the perturbation variables are expressed in terms of the initial field values and the masses of the scalar fields only, unlike Ref. [12], where the author did not solve the scalar field evolutions and their expressions of the spectral indices contain unknown factors originating from the scalar field evolutions.

D. Effect of the nonresonant interactions

In the previous subsection, we considered the free scalar fields. In this subsection, we will consider the interacting case. All of the interacting systems are classified into nonresonant cases and resonant cases. By the word "resonance," we mean all of the factors inducing the small denominators in the perturbative expansion [27]. Therefore, the resonance phenomena are not confined in the oscillatory dynamical system. In fact, the resonance can also occur in the present slow rolling system. In the nonresonant case, the masses of the scalar fields do not satisfy any resonant relations as shown in Proposition 7 presented below. In the resonant case, the masses of the scalar fields satisfy more than one resonant relation. In the nonresonant case, the Poincaré theorem greatly simplifies the treatment of the interactions. In the nonresonant case, all of the interaction terms can be removed from the evolution equations by performing the suitable transformation of the field variables. This process is called the linearization of the evolution equations. Since the linearized, that is, transformed, evolution equations are the free field equations, they can be solved easily. The effects of all of the interaction terms are contained in the transformation law of the field variables. The fact that the masses of the scalar fields do not satisfy any resonant relations guarantees the convergence of the transformation law of the field variables. In this way, the evolutions of the interacting scalar fields ϕ_a can be solved completely in the form of the analytic functions of the τ function. As for the concrete nonresonant model, we determine the evolutions of the scalar fields $\phi_a(\tau)$ and the N potential. By using this N potential, we can calculate the nonlinearity parameters $f_{\rm NL}$ and $g_{\rm NL}$ representing the non-Gaussianity of the Bardeen parameter. These nonlinearity parameters $f_{\rm NL}$ and $g_{\rm NL}$ are shown to be suppressed by the slow rolling parameter. Therefore, it is difficult to observe such small non-Gaussianity.

First, we present the proposition about the nonresonant condition. In order to express the nonresonant condition, we introduce several notations. By α and k, we represent

$$\alpha := (m_1^2, m_2^2, \ldots, m_{N_S}^2), \tag{3.64}$$

$$k := (k_1, k_2, \ldots, k_{N_S}), \tag{3.65}$$

where all k_a belong to the non-negative integer set $\mathbb{Z}_{0+}$:

$$k_a \in \mathbb{Z}_{0+}, \tag{3.66}$$

$$\mathbb{Z}_{0+} := \{k \in \mathbb{Z} | k \geq 0\}. \tag{3.67}$$

By $|k|$, we represent

$$|k| := \sum_a |k_a|. \tag{3.68}$$

Then the following proposition holds.

Proposition 7.—Suppose that, for an arbitrary $(a; k) \in \{1, 2, \ldots, N_S\} \times \mathbb{Z}_{0+}^N$ satisfying $|k| \geq 2$,

$$(k \cdot \alpha) - \alpha_a \neq 0 \tag{3.69}$$

holds and, for an arbitrary $a \in \{1, 2, \ldots, N_S\}$,

$$m_a^2 > 0 \tag{3.70}$$

holds. Then there exists a positive constant δ_m satisfying

$$|(k \cdot \alpha) - \alpha_a| \geq \delta_m \tag{3.71}$$

for an arbitrary $(a; k) \in \{1, 2, \ldots, N_S\} \times \mathbb{Z}_{0+}^N$ satisfying $|k| \geq 2$.

For the proof, see Appendix I.

Under the condition of Proposition 7, we can prove the following proposition.

Proposition 8.—We consider the evolution equation

$$\frac{\partial}{\partial \tau} \phi_a = -\alpha_a \phi_a + \tilde{f}_a(\phi), \tag{3.72}$$

where $\tilde{f}_a$ is the sum of mth order monomials of ϕ_a ($m \geq 2$) satisfying

$$|\tilde{f}_a(\phi)| < \mu_c \frac{\alpha_M}{\phi_0} |\phi|^2, \tag{3.73}$$

where α_M is the maximum of α_a, for $|\phi| \leq \phi_0$. Under the condition that α satisfies (3.71), there exists a positive constant R such that, for φ satisfying $|\varphi| \leq R$, there exists the transformation

$$\phi_a = \varphi_a + w_a(\varphi), \tag{3.74}$$

where w_a is the sum of mth order monomials of φ_a ($m \geq 2$) and satisfies

$$|w_a(\varphi)| \leq \frac{\mu_c}{\phi_0} |\varphi|^2. \tag{3.75}$$

By this transformation law (3.72), (3.74) is transformed into the linear differential equation as

$$\frac{\partial}{\partial \tau} \varphi_a = -\alpha_a \varphi_a. \tag{3.76}$$

Since Proposition 8 is well known as the Poincaré theorem, we omit the proof. As for the Poincaré theorem, please see the appendix of Ref. [27] and references therein. This theorem can be applied only to the differential equations containing the linear terms plus the small perturbations. Since they are not even the Hamiltonian dynamical systems in general, Proposition 8 does not cover the anharmonic oscillator. It is well known that the evolution equation of the nonlinear anharmonic oscillator cannot be transformed into the linear evolution equation. But for the present purpose of treating the multiple slow rolling scalar field systems, Proposition 8 is enough. All of the terms in $w_a(\varphi)$ in (3.74) have factors as

$$\frac{1}{(k \cdot \alpha) - \alpha_a}. \tag{3.77}$$

In order to prove the convergence of this transformation law (3.74) by the majorant method, the nonresonant inequalities (3.71) are essential. According to Proposition 8, by (3.74) and

$$\varphi_a = \varphi_a(0) \exp[-\alpha_a \tau], \tag{3.78}$$

the evolutions of the scalar fields ϕ_a can be expressed as an analytic function of τ.

Next, we consider the concrete example whose potential U is given by

$$U = \frac{1}{2}m_1^2\phi_1^2 + \frac{1}{2}m_2^2\phi_2^2 + \frac{g}{4}\phi_1^2\phi_2^2, \tag{3.79}$$

where m_1^2 and m_2^2 are assumed to be nonresonant as in Proposition 7. We obtain

$$\phi_1 = \phi_1(0)\exp(-m_1^2\tau) - \frac{g}{4m_2^2}\phi_1(0)\phi_2^2(0)[\exp(-m_1^2\tau) - \exp\{-(m_1^2 + 2m_2^2)\tau)\}] + \cdots, \tag{3.80}$$

$$\phi_2 = \phi_2(0)\exp(-m_2^2\tau) - \frac{g}{4m_1^2}\phi_2(0)\phi_1^2(0)[\exp(-m_2^2\tau) - \exp\{-(m_2^2 + 2m_1^2)\tau)\}] + \cdots. \tag{3.81}$$

Then we obtain the N potential as

$$\frac{1}{\kappa^2}\bar{N} = \frac{1}{4}\phi_1^2(0) + \frac{1}{4}\phi_2^2(0) - \frac{g}{8(m_1^2 + m_2^2)}\phi_1^2(0)\phi_2^2(0) + \cdots. \tag{3.82}$$

In this model, we consider the nonlinear parameters $f_{\rm NL}$ and $g_{\rm NL}$ [14] defined by

$$f_{\rm NL} = \frac{5}{6}\frac{\bar{N}_{ab}\bar{N}^a\bar{N}^b}{(\bar{N}_a\bar{N}^a)^2}, \tag{3.83}$$

$$g_{\rm NL} = \frac{25}{54}\frac{\bar{N}_{abc}\bar{N}^a\bar{N}^b\bar{N}^c}{(\bar{N}_a\bar{N}^a)^3}, \tag{3.84}$$

where

$$\bar{N}_a := \frac{\partial}{\partial\phi_a(0)}\bar{N}, \qquad \bar{N}_{ab} := \frac{\partial^2}{\partial\phi_a(0)\partial\phi_b(0)}\bar{N}, \ldots. \tag{3.85}$$

We obtain

$$f_{\rm NL} = \frac{5}{12}\frac{1}{\bar{N}} \sim \left(\frac{m}{H_0}\right)^2, \tag{3.86}$$

$$g_{\rm NL} = -\frac{25}{144}\frac{g\kappa^2}{m_1^2 + m_2^2}\phi_1^2(0)\phi_2^2(0)\frac{1}{\bar{N}^3} \sim \frac{U_{\rm int}}{U}\left(\frac{m}{H_0}\right)^4. \tag{3.87}$$

So we can conclude that the nonlinear parameters $f_{\rm NL}$ and $g_{\rm NL}$ are suppressed by the slow rolling parameter $(m/H_0)^2$. For an arbitrary potential model satisfying assumptions (i), (ii), and (iii) presented above Proposition 6 and for the nonresonant masses as explained in Proposition 7, the N potential can be written as

$$\frac{1}{\kappa^2}\bar{N} = \frac{1}{4}\sum_a \phi_a^2(0) + \tilde{g}(\phi(0)), \tag{3.88}$$

where $\tilde{g}$ is the sum of mth order monomials of $\phi_a(0)$ ($m \geq 3$) and satisfies

$$\tilde{g}(\phi(0)) \leq \mu_c\frac{1}{\phi_0}|\phi(0)|^3 \tag{3.89}$$

for $|\phi(0)| \leq \phi_0$. Then we obtain the nonlinear parameters as

$$f_{\rm NL} = \frac{5}{12}\frac{1}{\bar{N}} \sim \left(\frac{m}{H_0}\right)^2, \qquad g_{\rm NL} \sim \mu_c\left(\frac{m}{H_0}\right)^4; \tag{3.90}$$

that is, the nonlinear parameters $f_{\rm NL}$ and $g_{\rm NL}$ are suppressed by the slow rolling parameters $(m/H_0)^2$.

E. Effect of the resonant interactions

According to Proposition 8 in the previous subsection, all of the nonresonant interaction terms can be eliminated by the field transformations $\phi_a \to \varphi_a$. But in the resonant interaction case, such linearization cannot be applied. In this subsection, we calculate the effect of the resonant interaction by the iteration method. Perturbatively at least, the resonant interactions do not generate any special effects in the N potential.

As the concrete example, we consider the model defined by

$$U_{\rm int} = \lambda\phi_1\phi_2^2, \qquad m_1^2 = 2m_2^2. \tag{3.91}$$

It can be solved as

$$\phi_1 = \phi_1(0)\exp(-m_1^2\tau) - \lambda\phi_2^2(0)\tau\exp(-m_1^2\tau) + \cdots, \tag{3.92}$$

$$\phi_2 = \phi_2(0)\exp(-m_2^2\tau) - \frac{2\lambda}{m_1^2}\phi_1(0)\phi_2(0)[\exp(-m_2^2\tau) - \exp\{-(m_1^2 + m_2^2)\tau)\}] + \cdots, \tag{3.93}$$

where in the right-hand side of ϕ_1 the second term proportional to $\lambda\tau$ appears because of the resonant interaction. The N potential is given by

$$\frac{1}{\kappa^2}\bar{N} = \frac{1}{4}\phi_1^2(0) + \frac{1}{4}\phi_2^2(0) - \frac{5\lambda}{8m_2^2}\phi_1(0)\phi_2^2(0) + \cdots. \tag{3.94}$$

So the resonant interaction terms generate no singular terms in the N potential $\bar{N}$.

IV. DISCUSSION

We presented the nonlinear LWL formula, and by using it we investigated the evolutionary behaviors of the nonlinear cosmological perturbations on superhorizon scales in the universe dominated by the multiple slow rolling scalar fields.

In the excellent study as to the multiple slow rolling scalar fields [12,13], the multiple scalar fields were decomposed into the adiabatic field and the entropy fields instant by instant. By using this decomposition, it was pointed out that the growing of the Bardeen parameter corresponds to the curvature of the trajectory in the scalar field space. Further, in Ref. [12], the spectral index (the wave number dependences) of the Bardeen parameter was presented by calculating the differential coefficients at the instant of the first horizon crossing. But this study is a local investigation because the author did not calculate the S formulas connecting the final amplitudes of the adiabatic and the entropic field variables at the end of the slow rolling phase with the initial amplitudes of them at the first horizon crossing, and they calculated only the time derivative of the S formulas at the first horizon crossing by using the perturbation evolution equations. So the formulas of the spectral indices in Ref. [12] have black boxes (unknown factors) which come from the S formulas which could not be determined. Therefore, as the method for investigating the evolutions of the scalar field perturbations in the long time interval such as in the whole slow rolling phase analytically, we present the method using the τ function and the N potential. This method enables us to calculate the S formulas analytically. In fact, our expressions of spectral indices in Sec. III C can be expressed in terms of the initial field values and the masses of the scalar fields only, and they have no black boxes since we succeed in calculating the scalar field evolutions containing the S formulas completely. Our method will deepen the understanding of the dynamical evolutions of the multiple slow rolling scalar fields in the long time interval with high accuracy.

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APPENDIX A: DERIVATION OF (2.59)

In this appendix, we rewrite the momentum constraints (2.17) into the constraints which the solution constants of the solution of the locally homogeneous universe must satisfy.

By using (2.9) and (2.51), the left-hand side of the momentum constraint (2.17) is expressed as

$$D_i\tilde{A}^i_j - \frac{2}{3}D_jK = \frac{1}{\alpha}\bigg[\frac{1}{2}\frac{\partial_i\alpha}{\alpha}\tilde{\gamma}^{ik}\partial_t\tilde{\gamma}_{kj} + \frac{1}{2}\tilde{\gamma}^{il}\partial_i\tilde{\gamma}_{lm}\tilde{\gamma}^{mk}\partial_t\tilde{\gamma}_{kj} - \frac{1}{2}\tilde{\gamma}^{ik}\partial_i\partial_t\tilde{\gamma}_{kj} - \frac{3}{2}\frac{\partial_k a}{a}\tilde{\gamma}^{kl}\partial_t\tilde{\gamma}_{lj} + \frac{1}{2}{}^s\tilde{\Gamma}^k_{ij}\tilde{\gamma}^{il}\partial_t\tilde{\gamma}_{lk} + 2\alpha\partial_j\left(\frac{H}{\alpha}\right)\bigg], \tag{A1}$$

where ${}^s\tilde{\Gamma}^k_{ij}$ is the Christoffel symbol of $\tilde{\gamma}_{ij}$. As for the right-hand side of (A1),

$$\frac{1}{2}\frac{\partial_i\alpha}{\alpha}\tilde{\gamma}^{ik}\partial_t\tilde{\gamma}_{kj} - \frac{3}{2}\frac{\partial_k a}{a}\tilde{\gamma}^{kl}\partial_t\tilde{\gamma}_{lj} = -\frac{1}{2}\frac{\alpha}{a^3}\partial_i\left(\frac{a^3}{\alpha}\right)(M^{-1}\dot{M})^i_j, \tag{A2}$$

$${}^s\tilde{\Gamma}^k_{ij}\tilde{\gamma}^{il}\partial_t\tilde{\gamma}_{lk} = \tfrac{1}{2}\mathrm{tr}[M^{-1}\partial_jMM^{-1}\dot{M}], \tag{A3}$$

$$\tilde{\gamma}^{il}\partial_i\tilde{\gamma}_{lm}\tilde{\gamma}^{mk}\partial_t\tilde{\gamma}_{kj} = (M^{-1}\partial_iMM^{-1}\dot{M})^i_j, \tag{A4}$$

and

$$\tilde{\gamma}^{ik}\partial_i\partial_t\tilde{\gamma}_{kj} = (M^{-1}\partial_i\dot{M})^i_j. \tag{A5}$$

Therefore we obtain

$$D_i\tilde{A}^i_j - \frac{2}{3}D_jK = \frac{1}{\alpha}\bigg[-\frac{1}{2}\frac{\alpha}{a^3}\partial_iT^i_j + \frac{1}{4}\frac{\alpha}{a^3}\mathrm{tr}(R^{-1}\partial_jRT) + \frac{\alpha}{a^3}\int_{t_0}dt\partial_j\left(\frac{\alpha}{a^3}\right)c_T + \frac{1}{2}\frac{\alpha}{a^3}\int_{t_0}dt \times\frac{\alpha}{a^3}\partial_jc_T + 2\alpha\partial_j\left(\frac{H}{\alpha}\right)\bigg], \tag{A6}$$

where we used the solution of M (2.53).

The right-hand side of (2.17) is expressed as

$$\kappa^2J_j = \frac{1}{\alpha}[\kappa^2h(v_j + \beta_j)]. \tag{A7}$$

By summing (2.44) with fluid indices α, we obtain

$$[h(\beta_i + v_i)]_f = \frac{\alpha}{a^3}\sum_\alpha C_{\alpha i} + \frac{\alpha}{a^3}\int_{t_0}dt\alpha a^3\bigg[-\partial_iP_f - \frac{1}{\alpha}D_i\alpha h_f - \sum_a S_a\partial_i\phi_a\bigg] + O(\epsilon^3), \tag{A8}$$

where we used $\sum_\alpha Q_{\alpha i} = -(Q_i)_S = -\sum_a S_a\partial_i\phi_a$, and as for the scalar field components we obtain

$$[h_a(\beta_i + v_{ai})]_S = -\sum_a\dot{\phi}_a\partial_i\phi_a + O(\epsilon^3). \tag{A9}$$

We substitute the sum of (A8) and (A9) into $h(\beta_i + v_i)$ in (A7).

Then through simple calculations we obtain (2.59).

APPENDIX B: LIE DERIVATIVE IN THE LEADING ORDER OF THE GRADIENT EXPANSION

The Lie derivatives of the quantity of the upper index and the quantity with the lower index are expressed as

$$L(T)X^\mu = T^\alpha\partial_\alpha X^\mu - \partial_\alpha T^\mu X^\alpha, \tag{B1}$$

$$L(T)X_\mu = T^\alpha\partial_\alpha X_\mu + X_\alpha\partial_\mu T^\alpha, \tag{B2}$$

respectively. By the Leibniz rule, these definitions are expanded into the tensor of an arbitrary rank. For example, the Lie derivative of the metric is given by

$$L(T)g_{\mu\nu} = T^\alpha\partial_\alpha g_{\mu\nu} + g_{\alpha\nu}\partial_\mu T^\alpha + g_{\mu\alpha}\partial_\nu T^\alpha. \tag{B3}$$

From now on, we consider the gradient expansion scheme defined by

$$\partial_i = O(\epsilon), \qquad T^i = O(\epsilon), \qquad g_{i0} = O(\epsilon), \tag{B4}$$

and the $O(\epsilon^2)$ order corrections are dropped. For an arbitrary scalar quantity A, in our scheme (B4) the Lie derivative of A is written by

$$L(T)A = T^0 \frac{d}{dt} A + O(\epsilon^2). \tag{B5}$$

By using (B3) to $g_{00} = -\alpha^2 + \beta_k \beta^k$ and $g_{ij} = \gamma_{ij}$, we obtain

$$L(T)\alpha = T^0 \frac{d}{dt} \alpha + \alpha \frac{d}{dt} T^0 + O(\epsilon^2), \tag{B6}$$

$$L(T)\gamma_{ij} = T^0 \frac{d}{dt} \gamma_{ij} + O(\epsilon^2). \tag{B7}$$

From (B5) and (B6), we obtain

$$L(T)\left(\frac{\dot{A}}{\alpha}\right) = T^0 \frac{d}{dt} \left(\frac{\dot{A}}{\alpha}\right) + O(\epsilon^2). \tag{B8}$$

From this equation, we can see that the Lie derivative of $\dot{A}/\alpha$, where A is a scalar quantity, has the same form as the Lie derivative of the scalar quantity (B5). By using (B7) to $\det(\gamma_{ij}) = a^6$, we obtain

$$L(T)a = T^0 \frac{d}{dt} a + O(\epsilon^2), \tag{B9}$$

and therefore we obtain

$$L(T)\tilde{\gamma}_{ij} = T^0 \frac{d}{dt} \tilde{\gamma}_{ij} + O(\epsilon^2). \tag{B10}$$

In our gradient expansion scheme, the scale factor a and $\tilde{\gamma}_{ij}$ can be regarded as the scalar quantity.

APPENDIX C: PROOF OF PROPOSITION 1

The gauge transformation is described by the differential equation (2.70). So we can consider the μ derivative as the gauge transformation. By differentiating (2.70) with respect to λ, we obtain

$$\frac{d}{d\mu} \frac{dA}{d\lambda} = L\left(\frac{dT}{d\lambda}\right) A + L(T) \frac{dA}{d\lambda}, \tag{C1}$$

$$\frac{d}{d\mu} \frac{d^2A}{d\lambda^2} = L\left(\frac{d^2T}{d\lambda^2}\right) A + 2L\left(\frac{dT}{d\lambda}\right) \frac{dA}{d\lambda} + L(T) \frac{d^2A}{d\lambda^2}, \tag{C2}$$

and so on. As for D^nA, where D is defined by (2.72) and A is an arbitrary scalar quantity, we can prove

$$\frac{d}{d\mu} D^n A = L(T) D^n A \tag{C3}$$

for an arbitrary natural number n.

Proof.—If we interpret $D^0 A := A$, (C3) holds evidently for $n = 0$. We assume that (C3) holds for $n = k - 1$, where $k = 1, 2, \ldots$. Then for $n = k$

$$\begin{aligned} \frac{d}{d\mu} D^k A &= \frac{d}{d\mu} \left\{ \left(\frac{d}{d\lambda} - \frac{da}{d\lambda} \frac{1}{\dot{a}} \frac{d}{dt} \right) D^{k-1} A \right\} = \frac{d}{d\lambda} \frac{d}{d\mu} D^{k-1} A - \frac{d}{d\lambda} \left(\frac{da}{d\mu} \right) \frac{1}{\dot{a}} (D^{k-1} A)^{\cdot} - \frac{da}{d\lambda} \frac{d}{d\mu} \left\{ \frac{1}{\dot{a}} (D^{k-1} A)^{\cdot} \right\} \\ &= L\left(\frac{dT}{d\lambda}\right) D^{k-1} A + L(T) \frac{d}{d\lambda} D^{k-1} A - \left\{ L\left(\frac{dT}{d\lambda}\right) a + L(T) \frac{da}{d\lambda} \right\} \frac{1}{\dot{a}} (D^{k-1} A)^{\cdot} - \frac{da}{d\lambda} L(T) \left\{ \frac{1}{\dot{a}} (D^{k-1} A)^{\cdot} \right\} \\ &= L(T) \left\{ \left(\frac{d}{d\lambda} - \frac{da}{d\lambda} \frac{1}{\dot{a}} \frac{d}{dt} \right) D^{k-1} A \right\} = L(T) D^k A, \end{aligned} \tag{C4}$$

where we use the fact that both $D^{k-1}A$ and the scale factor a have the Lie derivatives of the scalar quantity type (B5). For $n = k$, (C3) holds. By induction, we complete the proof. ■

By putting $\lambda = 0$ in (C3) and by noticing $T(\lambda = 0) = 0$,

$$\frac{d}{d\mu} D^n A|_{\lambda=0} = 0. \tag{C5}$$

Then we proved that $D^n A$, where A is an arbitrary scalar quantity, is gauge invariant. Since $\dot{A}/\alpha$, where A is an arbitrary scalar quantity, has the same Lie derivative as that of the scalar quantity A, $D^n(\dot{A}/\alpha)$ is also gauge invariant. In a way similar to the above calculation, as for $\mathcal{A}_n$ defined by (2.73) we can prove

$$\frac{d}{d\mu} \mathcal{A}_n = L(T) \mathcal{A}_n. \tag{C6}$$

Then $\mathcal{A}_n$ is gauge invariant.

APPENDIX D: PROOF OF PROPOSITION 2

Since ζ_1 is conserved for arbitrary values of the integration constants $C(\lambda = 0)$, ζ_1 with $C(\lambda = 0)$ replaced with $C(\lambda)$ is also conserved; that is, ζ_1 is conserved for arbitrary values of λ:

$$\left(\frac{\partial}{\partial a} \zeta_1\right)_a = 0. \tag{D1}$$

Then by the expression of $\bar{\delta}$ (2.80) we obtain

$$\zeta_2 = \left(\frac{\partial}{\partial \lambda} \zeta_1\right)_a. \tag{D2}$$

Since $(\partial/\partial a)_a$ and $(\partial/\partial\lambda)_a$ are commutative, we obtain

$$\left(\frac{\partial}{\partial a}\zeta_2\right)_a = \left(\frac{\partial}{\partial a}\frac{\partial}{\partial\lambda}\zeta_1\right)_a = \left(\frac{\partial}{\partial\lambda}\frac{\partial}{\partial a}\zeta_1\right)_a = 0. \quad \text{(D3)}$$

By iterating the same process, we obtain

$$\zeta_n = \left(\frac{\partial}{\partial\lambda}\right)_a^{n-1}\zeta_1, \qquad \left(\frac{\partial}{\partial a}\zeta_n\right)_a = 0 \quad \text{(D4)}$$

for $n \geq 2$.

APPENDIX E: PROOF OF PROPOSITION 3

When $P = P(\rho)$,

$$a\frac{\partial\rho}{\partial a} = -3(\rho + P). \quad \text{(E1)}$$

Then

$$\zeta_1 = -\left\{\left(\frac{\partial\rho}{\partial\lambda}\right)_a \Big/ \left(\frac{\partial\rho}{\partial a}\right)_a\right\}\frac{1}{a} = \frac{1}{3}\frac{1}{\rho + P}\left(\frac{\partial\rho}{\partial\lambda}\right)_a = \left(\frac{\partial}{\partial\lambda}\right)_a\left[\frac{1}{3}\int d\rho\frac{1}{\rho + P(\rho)}\right] \quad \text{(E2)}$$

and

$$\left(\frac{\partial}{\partial a}\zeta_1\right)_a = \left(\frac{\partial}{\partial\lambda}\frac{\partial}{\partial a}\right)_a\left[\frac{1}{3}\int d\rho\frac{1}{\rho + P(\rho)}\right] = \left(\frac{\partial}{\partial\lambda}\right)_a\left(-\frac{1}{a}\right) = 0 \quad \text{(E3)}$$

since $(\partial/\partial\lambda)_a$ and $(\partial/\partial a)_a$ are commutative. Then we can use Proposition 2, and we complete the proof.

APPENDIX F: PROOF OF PROPOSITION 4

For $k = 0$, by solving (3.11) and (3.12) we obtain

$$|p - e^{-3N}p(0)| \leq \eta^2, \quad \text{(F1)}$$

$$|\phi - \phi(0)| \leq \eta^2 N. \quad \text{(F2)}$$

If $p(0)$, $\phi(0) \sim 1$, Proposition 4 is right for $k = 0$. We assume that Proposition 4 holds for $k = 0, 1, 2, \ldots, k-1$:

$$\left|\left(\frac{\partial^i}{\partial\lambda^i}\right)_a\phi\right|, \qquad \left|\left(\frac{\partial^i}{\partial\lambda^i}\right)_a p\right| \sim \delta_c^i \quad (i = 0, 1, 2, \ldots, k-1), \quad \text{(F3)}$$

and therefore

$$\left|\left(\frac{\partial^i}{\partial\lambda^i}\right)_a f(\phi, p, c_T, N)\right| \sim \delta_c^i \qquad (i = 0, 1, 2, \ldots, k-1), \quad \text{(F4)}$$

if

$$|f(\phi, p, c_T, N)| \leq 1, \quad \text{(F5)}$$

for a proper complex domain containing the real interval which we consider. For k, the evolution equations are

$$\frac{d}{dN}\left(\frac{\partial^k}{\partial\lambda^k}\right)_a\phi = \eta^2\left(\frac{\partial^k}{\partial\lambda^k}\right)_a\phi + \eta^2\left(\frac{\partial^k}{\partial\lambda^k}\right)_a p + \eta^2\delta_c^k, \quad \text{(F6)}$$

$$\frac{d}{dN}\left(\frac{\partial^k}{\partial\lambda^k}\right)_a p = -3\left(\frac{\partial^k}{\partial\lambda^k}\right)_a p + \eta^2\left(\frac{\partial^k}{\partial\lambda^k}\right)_a\phi + \eta^2\left(\frac{\partial^k}{\partial\lambda^k}\right)_a p + \eta^2\delta_c^k, \quad \text{(F7)}$$

where all of the coefficients bounded by positive constants are omitted except -3 in (F7). By solving (F6) and (F7), for $0 \leq N \leq 1/\eta^2$, we obtain

$$\left|\left(\frac{\partial^k}{\partial\lambda^k}\right)_a\phi\right| \leq \left|\frac{\partial^k}{\partial\lambda^k}\phi(0)\right| + \eta^2\left|\frac{\partial^k}{\partial\lambda^k}p(0)\right| + \delta_c^k, \quad \text{(F8)}$$

$$\left|\left(\frac{\partial^k}{\partial\lambda^k}\right)_a p\right| \leq e^{-3N}\left|\frac{\partial^k}{\partial\lambda^k}p(0)\right| + \eta^2\left[\left|\frac{\partial^k}{\partial\lambda^k}\phi(0)\right| + \eta^2\left|\frac{\partial^k}{\partial\lambda^k}p(0)\right| + \delta_c^k\right]. \quad \text{(F9)}$$

Therefore for the initial conditions (3.17), for k, (3.18) and (3.19) hold. By induction, for all non-negative integers k, Proposition 4 holds.

APPENDIX G: PROOF OF PROPOSITION 5

First, we consider $k = 0$ case. By the mean value theorem, we obtain

$$\frac{d}{dN}\Delta\phi = \eta^2\Delta\phi + \eta^2 f, \quad \text{(G1)}$$

where

$$|f| \leq |p| + \eta^2 \leq e^{-3N} + \eta^2, \quad \text{(G2)}$$

where we used Proposition 4 in the last inequality. By solving (G1), we obtain

$$|\Delta\phi| \leq \exp(\eta^2 N)(|\Delta\phi(0)| + \eta^2 + \eta^4 N). \quad \text{(G3)}$$

Therefore for $0 \leq N \leq 1/\eta^2$, under the initial condition (3.21) and (3.22) holds for $k = 0$. Next, we consider a positive integer k case. In this case, we notice the following fact. By Proposition 4, if an complex analytic function f satisfies

$$|f(\phi, p, c_T, N)| \leq 1, \quad \text{(G4)}$$

then

$$\left|\left(\frac{\partial^k}{\partial\lambda^k}\right)_a f(\phi, p, c_T, N)\right| \leq \delta_c^k. \quad \text{(G5)}$$

By differentiating (G1) with respect to λ and by using the above fact and Proposition 4, we obtain

$$\frac{d}{dN}\left(\frac{\partial^k}{\partial\lambda^k}\right)_a \Delta\phi = \eta^2\left(\frac{\partial^k}{\partial\lambda^k}\right)_a \Delta\phi + \eta^2\delta_c^k(e^{-3N} + \eta^2), \tag{G6}$$

which is solved as

$$\left|\left(\frac{\partial^k}{\partial\lambda^k}\right)_a \Delta\phi\right| \leq \exp(\eta^2 N) \times \left(\left|\frac{\partial^k}{\partial\lambda^k}\Delta\phi(0)\right| + \eta^2\delta_c^k + \eta^4 N\delta_c^k\right). \tag{G7}$$

Therefore for $0 \leq N \leq 1/\eta^2$, under the initial condition (3.21) and (3.22) holds for a positive integer k.

APPENDIX H: PROOF OF PROPOSITION 6

Lemma 1.—

$$\phi_a = \phi_a(0)\exp[-m_a^2\tau] + \langle\mu_c\phi_0\exp[-m^2\tau]\rangle. \tag{H1}$$

*Proof.—*We consider the evolution equation

$$\frac{\partial}{\partial\tau}\phi_a = -m_a^2\phi_a + \tilde{f}_a(\phi), \tag{H2}$$

where $\tilde{f}_a(\phi)$ is the sum of mth order monomials ($m \geq 2$) satisfying

$$|\tilde{f}_a(\phi)| \leq m^2\mu_c\frac{1}{\phi_0}|\phi|^2 \tag{H3}$$

for $|\phi| \leq \phi_0$. From (H2), we obtain

$$\frac{\partial}{\partial\tau}|\phi| \leq -m^2|\phi| + \frac{m^2\mu_c}{\phi_0}|\phi|^2, \tag{H4}$$

which is solved as

$$|\phi| \leq \phi_0\exp[-m^2\tau](1 + O(\mu_c)). \tag{H5}$$

We consider $\bar{\phi}_a$ satisfying

$$\frac{\partial}{\partial\tau}\bar{\phi}_a = -m_a^2\bar{\phi}_a, \tag{H6}$$

$$\bar{\phi}_a(0) = \phi_a(0). \tag{H7}$$

The difference $\Delta\phi_a := \phi_a - \bar{\phi}_a$ satisfies

$$\frac{\partial}{\partial\tau}\Delta\phi_a = -m_a^2\Delta\phi_a + \tilde{f}_a(\phi), \tag{H8}$$

$$\Delta\phi_a(0) = 0, \tag{H9}$$

where

$$|\tilde{f}_a(\phi)| \leq m^2\mu_c\phi_0\exp[-m^2\tau], \tag{H10}$$

where we used (H5). By solving the above evolution equation under the above initial condition, we obtain

$$|\Delta\phi| \leq O(\mu_c)\phi_0\exp[-m^2\tau]. \tag{H11}$$

We complete the proof. ■

Lemma 2.—

$$\left(\frac{\partial^k}{\partial\lambda^k}\right)_\tau \phi_a = \frac{\partial^k}{\partial\lambda^k}\phi_a(0)\exp[-m_a^2\tau] + \langle\delta_c^k\mu_c\phi_0\exp[-m^2\tau]\rangle. \tag{H12}$$

*Proof.—*As for the evolution equation as

$$\frac{\partial}{\partial\tau}\left(\frac{\partial^k}{\partial\lambda^k}\right)_\tau \phi_a = -m_a^2\left(\frac{\partial^k}{\partial\lambda^k}\right)_\tau \phi_a + \left(\frac{\partial^k}{\partial\lambda^k}\right)_\tau \tilde{f}_a(\phi), \tag{H13}$$

where

$$\left|\left(\frac{\partial^k}{\partial\lambda^k}\right)_\tau \tilde{f}_a(\phi)\right| \leq m^2\frac{\mu_c}{\phi_0}|\phi|\left|\left(\frac{\partial^k}{\partial\lambda^k}\right)_\tau \phi_a\right| + m^2\frac{\mu_c}{\phi_0}\delta_c^k|\phi|^2 \leq m^2\mu_c\exp[-m^2\tau]\left|\left(\frac{\partial^k}{\partial\lambda^k}\right)_\tau \phi_a\right| + \delta_c^k\mu_c m^2\phi_0\exp[-2m^2\tau]. \tag{H14}$$

We perform the calculations similar to the proof of Lemma 1. We complete the proof. ■

Lemma 3.—

$$\left(\frac{\partial^k}{\partial\lambda^k}\right)_\tau\left(\prod_{l=1}^N \phi_{a(l)}\right) = \frac{\partial^k}{\partial\lambda^k}\left(\prod_{l=1}^N \phi_{a(l)}(0)\right)\exp\left[-\sum_{l=1}^N m_{a(l)}^2\tau\right] + \langle\delta_c^k\mu_c\phi_0^N\exp[-Nm^2\tau]\rangle. \tag{H15}$$

*Proof.—*We use the Leibniz rule for the λ derivative and use Lemmas 1 and 2. We complete the proof. ■

*Lemma 4.—*For $n \geq 1$

$$A(2n, 0) = \sum_a (m_a^2)^n\phi_a^2 + F_{2n}(\phi), \tag{H16}$$

where $F_{2n}(\phi)$ is the sum of mth order monomials of ϕ ($m \geq 3$) and satisfies

$$|F_{2n}(\phi)| \leq \mu_c m^{2n}\frac{1}{\phi_0}|\phi|^3 \tag{H17}$$

for $|\phi| \leq \phi_0$.

*Proof.—*We use (H2). We complete the proof. ■

As for

$$\left(\frac{\partial^k}{\partial\lambda^k}\right)_\tau \bar{N} = \kappa^2\int_0^\infty d\tau\left(\frac{\partial^k}{\partial\lambda^k}\right)_\tau U, \tag{H18}$$

$$A(0, k) = 4\int_\tau^\infty d\tau\left(\frac{\partial^k}{\partial\lambda^k}\right)_\tau U, \tag{H19}$$

and

$$A(2n, k) = \left(\frac{\partial^k}{\partial\lambda^k}\right)_\tau A(2n, 0), \tag{H20}$$

for $n \geq 1$, where $A(2n, 0)$ is expressed as a polynomial of

$\phi(\tau)$ (H16), we use Lemma 3. Then we complete the proof of Proposition 6.

APPENDIX I: PROOF OF PROPOSITION 7

By α_M and α_m, we mean

$$\alpha_M := \max_a \{m_a^2\}, \tag{I1}$$

$$\alpha_m := \min_a \{m_a^2\}. \tag{I2}$$

Then we obtain

$$|(k \cdot \alpha) - \alpha_a| \geq |k|\alpha_m - \alpha_M. \tag{I3}$$

Therefore for k satisfying

$$|k| \geq \frac{\delta_1 + \alpha_M}{\alpha_m} =: L, \tag{I4}$$

where δ_1 is a positive constant, we obtain

$$|(k \cdot \alpha) - \alpha_a| \geq \delta_1. \tag{I5}$$

Since the number of k satisfying $|k| < L$ is finite, there exists a positive constant δ_2 satisfying

$$|(k \cdot \alpha) - \alpha_a| \geq \delta_2 \tag{I6}$$

for $|k| < L$. Therefore for δ_m defined by

$$\delta_m := \min\{\delta_1, \delta_2\}, \tag{I7}$$

the inequality (3.71) holds. We complete the proof.

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