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Comparison of approximate gravitational lens equations and a proposal for an improved new one

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Keeping the exact general relativistic treatment of light bending as a reference, we compare the accuracy of commonly used approximate lens equations. We conclude that the best approximate lens equation is the Ohanian lens equation, for which we present a new expression in terms of distances between observer, lens, and source planes. We also examine a realistic gravitational lensing case, showing that the precision of the Ohanian lens equation might be required for a reliable treatment of gravitational lensing and a correct extraction of the full information about gravitational physics.

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I. INTRODUCTION

Gravitational lensing is a well-established research subject treating the bending of light trajectories by gravitational fields. Its methodology is traditionally developed within the weak field approximation of general relativity, which describes photon trajectories through the geodesics equation. In this context, the true position of a source in the sky and its apparent position after deflection of the light by a massive body are related by the so-called lens equation [1,2] (for textbook reviews, see [3–5]). This relation and its mathematical properties have been extensively studied for generic lens models and represents the basis of the whole gravitational lensing theory.

The lens equation is typically established in the small angle approximation, in conjunction with the weak deflection hypothesis. In more recent years, there has been a renewed interest in astrophysical situations in which the deflection angle is not small. For example, if the bending of light emitted by sources near a black hole is considered, the deflection angle may reach arbitrary large values [6]. For such gravitational lenses the old small angles lens equation must be obviously revised. Several proposals for generalized lens equations have then appeared in the literature and have been applied to specific cases [7–13]. Such lens equations principally differ among each other for what concerns the variables in which they are expressed. However they also lie at different approximation levels below the full general relativistic description of the photon motion.

In this work, we review all these lens equations for spherically symmetric bodies that have appeared in the literature and introduce a new lens equation that fills a gap in the present taxonomy. We also present a detailed discussion of the order of magnitude of the errors committed in the use of different lens equations, giving a complete interpretation for their origin. It is worth mentioning that a critical review of the approximations leading to the weak deflection lens equation has recently appeared [14]. The present work shares the same spirit by testing different approximations leading to large deflection lens equations, instead.

The plan of the paper is as follows. In Sec. II we establish the notations by describing the basic geometric configuration for gravitational lensing. In Sec. III, we review the exact lens equation by Frittelli, Kling, and Newman, derived in a fully general relativistic context. In Sec. IV, we discuss the asymptotic approximation and all lens equations making use of this approximation. We also introduce our new proposal for an improved lens equation. In Sec. V we present a numerical example about a realistic gravitational lensing situation, in order to compare the different approximations in the lens equations previously discussed. Section VI contains a discussion about the precision needed in gravitational lensing observations and the conclusions of the work.

II. BASIC LENSING GEOMETRY

Let us consider a spherically symmetric spacetime, whose metric is

$$ds^2 = A(r)dt^2 - B(r)dr^2 - C(r)^2(d\vartheta^2 + \sin\vartheta^2 d\phi^2). \tag{1}$$

Let us put a static source at radial coordinate d_{LS} and a static observer at radial coordinate d_{OL}. The orientation of the polar coordinates is chosen in such a way that both the source and the observer lie on the equatorial plane $\vartheta = \pi/2$. As a consequence of the spherical symmetry, the whole photon motion takes place on this plane. We also assume that the metric is asymptotically flat, so that the coordinates (t, r, ϑ, ϕ) become Minkowski polar coordinates very far from the center (we neglect any effects due to cosmological expansion in our discussion).

A very useful structure for understanding several lens equations proposed in the literature is the Minkowski space associated with (1). This structure can be introduced by assuming that the metric depends on one or more parameters continuously so that, once we tune these parameters to zero, the metric becomes Minkowski and the coordinates

(t, r, ϑ, ϕ) become Minkowski polar coordinates on the whole spacetime. The parameters of the metric may be the mass of the lens, the electric charge, the scalar charge or any other parameter dictated by any kind of gravitational theory. Many definitions need to be given by referring to this associated Minkowski space obtained by tuning all these parameters to zero. Of course, the original curved spacetime and the associated Minkowski space coincide in the asymptotically flat region.

A typical geometric configuration of gravitational lensing is shown in Fig. 1, with a photon emitted by a source S, curved by the gravitational field generated by the lens L and detected by the observer in O. The observer sees the photon at an angle θ with respect to the optical axis OLM, whereas, if there were no lens, he would directly observe the source at an angle β. β is thus the first quantity that is well-defined only in the associated Minkowski space, as we must tune the mass of the lens to zero to define it.

The emission direction of the photon is SC, whereas the detection direction is CO. In the associated Minkowski space, we can also measure the angle between these two directions and define it as the deflection angle α (we cannot compare directions from different points in a curved space).

It is common practice to define the lens plane and the source plane as those planes orthogonal to the optical axis passing through the lens and the source, respectively. Of course, also these definitions can be given in the associated Minkowski space and then extended to the original curved space. Then, one can define the distances from the observer to the lens plane as D_{OL}, the distance between the lens plane and the source plane as D_{LS} and the distance between the observer and the source plane as D_{OS}. The usual relation

$$D_{OS} = D_{OL} + D_{LS} \tag{2}$$

holds.

These distances between planes are obviously different from the distances between the points O, L, and S. We have already defined d_{OL} and d_{LS} as the radial coordinates of observer and source in the original curved metric. Once we

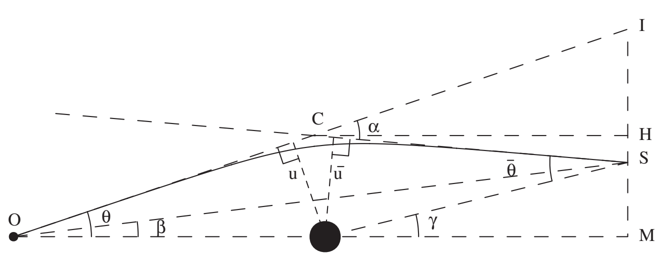

FIG. 1. Generic gravitational lensing configuration. Note that only the angles γ, θ, and $\bar{\theta}$ are well-defined in the curved spacetime, whereas all other geometrical quantities need to be defined referring to the Minkowski space associated with the original curved space.

report these coordinates in the associated Minkowski space, they coincide with the proper distances of the observer and the source from the lens. Then, in this space, we can establish simple geometrical relations with the distances between planes. In particular, we have

$$d_{OL} = D_{OL} \tag{3}$$

$$d_{LS} = D_{LS}/\cos\gamma \tag{4}$$

$$d_{OS} = D_{OS}/\cos\beta. \tag{5}$$

Of course, if the source is very close to the optical axis, β and γ are small and the differences between the uppercase distances and the lowercase distances is of second order in the angles. Therefore, in the classical weak deflection paradigm, the two notions are confused without consequences. Here, we must keep them distinct in order to avoid confusion.

III. EXACT LENS EQUATION

The lens equation is a relation among the source and observer coordinates and the angle θ at which the observer detects an image of the source S.

Remaining in a fully general relativistic context, it is possible to write down the exact equations governing the photon motion and consequently write an exact lens equation. This approach has been proposed by Frittelli and Newman in Ref. [7] and then applied to the Schwarzschild lens in Ref. [8]. Later on, it has been generalized to all spherically symmetric spacetimes in Ref. [15] (see also [16]).

A photon emitted from a source, deflected by the lens and detected by the observer experiences a change in the azimuthal coordinate given by

$$\Phi(J, d_{OL}, d_{LS}) = \left[\int_{r_0}^{d_{OL}} + \int_{r_0}^{d_{LS}}\right] \frac{\sqrt{B(r)}J}{C(r)\sqrt{\frac{1}{A(r)} - \frac{J^2}{C(r)}}} dr, \tag{6}$$

where J is the specific angular momentum of the photon, which is a constant of motion, and r_0 is the distance of closest approach to the lens. These two quantities are related by

$$J = \sqrt{\frac{C(r_0)}{A(r_0)}}. \tag{7}$$

The specific angular momentum is also related to the angle θ at which the observer detects the photon. Defining the angle by using the scalar product between the arrival direction and the direction specified by the optical axis OL (see, e.g., [17]), we find

$$\theta = \arcsin\left(J\sqrt{\frac{A(d_{OL})}{C(d_{OL})}}\right). \tag{8}$$

If the observer is very far from the lens, $A(d_{OL}) \to 1$ and $C(d_{OL}) \to d_{OL}^2$, so that one recovers the relation

$$\theta \simeq \arcsin J/d_{OL}, \tag{9}$$

which allows to identify J with the impact parameter u of the light ray trajectory (except for a speed of light factor). In general, however, this identification is only approximate (see next section).

The exact lens equation can finally be written by noting that the change in the azimuthal coordinate must be equal to the difference between the azimuthal coordinates of the observer and the source. Replacing all occurrences of J and r_0 in terms of θ by Eqs. (7) and (8), we get [7,8,15]

$$\Phi(\theta, d_{OL}, d_{LS}) = \pi - \gamma. \tag{10}$$

This equation is the exact general relativistic relation between the angle at which the image appears in the observer's sky and the relative positions of source, observer, and lens. All other lens equations represent approximate forms of this equation. Of course, most lens equations are derived under assumptions that are largely satisfied in realistic situations. Therefore, their simplifications are absolutely welcome, if the induced errors are below observational sensitivity or other sources of noise. In the following sections we will introduce some lens equations, describing their approximations.

IV. APPROXIMATE LENS EQUATIONS

The most popular approximation is what can be called as asymptotic approximation, which amounts to saying that the source and the observer are in the asymptotic flat region of the spacetime. Quantitatively, this is expressed by requiring $d_{OL}, d_{LS} \gg r_g$, where r_g is the gravitational radius of the lens, i.e., the typical scale in the curved spacetime metric controlling the range of the gravitational field (it is $2GM/c^2$ in the Schwarzschild metric). As a first consequence, Eq. (9) holds, so that the angular momentum of the photon J can be identified with the impact parameter $\bar{u}$ of the initial trajectory and the impact parameter u of the final trajectory, as depicted in Fig. 1. As a second consequence, the deflection angle can be calculated as

$$\alpha(u) \equiv \Phi(J = u, \infty, \infty) - \pi, \tag{11}$$

which is the azimuthal shift of a photon incoming from infinity with impact parameter J and escaping to infinity.

Once this approximation is accepted, it is possible to establish a relation among θ and the relative positions of source, lens, and observer using pure Euclidean geometry. In fact, all distances are defined in the asymptotic spacetime and the only input from general relativity is the precise expression of the deflection angle as a function of the impact parameter u.

Note that the asymptotic approximation does not necessary imply the small angle approximation. In fact, it just assumes that the source and the observer distances are much larger than the gravitational radius of the lens. This does not prevent the impact parameter to be very large.

In the following subsections, we will introduce several approximate lens equations that appeared in the literature, expressed in terms of different quantities.

A. Ohanian lens equation

The first lens equation we introduce is due to Ohanian [9], who proposed it in a study of gravitational lensing by a Schwarzschild black hole for arbitrary deflection angles.

Thanks to the asymptotic approximation, we can use Euclidean geometry to relate the various quantities and store all the relativistic input in the angle α. Let us define the angle $\zeta \equiv \widehat{OSL}$. Considering the triangles OLS and OSC, we can write down the relations

$$\beta + \zeta + \pi - \gamma = \pi \tag{12}$$

$$(\theta - \beta) + (\bar{\theta} - \zeta) + \pi - \alpha = \pi. \tag{13}$$

Summing up these two equalities, we obtain [18] (see also [13])

$$\theta + \bar{\theta} - \alpha = \gamma. \tag{14}$$

The angle $\bar{\theta}$ can be expressed in terms of θ recalling that the impact parameter of the incoming trajectory $\bar{u}$ is equal to the impact parameter of the outgoing trajectory u (in the asymptotic approximation). Therefore

$$\bar{\theta} = \arcsin\left(\frac{d_{OL}}{d_{LS}} \sin\theta\right). \tag{15}$$

Once Eq. (15) is used in Eq. (14), we obtain the lens equation as a relation involving the detection angle θ, the distance between the observer and the lens d_{OL}, the distance of the lens to the source d_{LS}, the source position angle γ and the deflection angle α, which is a function of θ through $u = d_{OL} \sin\theta$.

The original form proposed by Ohanian [9] was actually written by replacing θ and $\bar{\theta}$ by their small angle approximations u/d_{OL} and u/d_{LS}, respectively. This is an additional approximation, with respect to the asymptotic approximation, which we will not consider here.

B. Virbhadra, Narasimha, and Chitre lens equation

Another very simple expression for the lens equation was proposed by Virbhadra, Narasimha, and Chitre (VNC) in a paper studying the role of the scalar field in gravitational lensing [10]. It is just the result of the sine theorem applied to the triangle OCS

$$\sin(\theta - \beta) = \frac{d_{CS}}{d_{OS}} \sin\alpha, \tag{16}$$

where d_{CS} is the distance between the source S and the intersection point C between the incoming and outgoing trajectories. Equation (16) is valid for arbitrary deflections and impact parameters. As for the Ohanian lens equation, its only approximation with respect to the exact lens equation (10) is the asymptotic approximation, which is necessary to express the deflection angle α.

However, the lens equation (16) is not a closed relation among the detection angle θ and the relative positions of observer, lens, and source. In fact, it is expressed in terms of the position of point C, which is unknown *a priori* and needs to be estimated in some way. One possibility is to approximate d_{CS} by d_{LS}, which is reasonable if the source is at distances much larger than the impact parameter. However, this additional approximation would spoil the effectiveness of the lens equation in its original form. Therefore, the VNC lens equation is difficult to use in practical applications, since it demands an independent knowledge or at least an estimate of the position of the point C.

C. Virbhadra and Ellis lens equation

With their outbreaking paper about the possibility of observing higher order images around the black hole at the center of our Galaxy [11], Virbhadra and Ellis have attracted great attention on gravitational lensing beyond the weak deflection approximation, inspiring new vitality in black hole gravitational lensing. They have also proposed a new lens equation that has become very popular in the scientific literature. Their equation is written in terms of the distances between source, lens, and observer planes.

Starting from the relation among the segments

$$\overline{MS} = \overline{MI} - \overline{SI}, \tag{17}$$

we can write the relation

$$D_{OS} \tan\beta = D_{OS} \tan\theta - D_{CS}[\tan\theta + \tan(\alpha - \theta)], \tag{18}$$

where D_{CS} is the distance between point C and the source plane (namely the length of the segment $\overline{CH}$). Even in this case, the lens equation is expressed in terms of the position of point C, which should be estimated in some way. The proposal by Virbhadra and Ellis is to assume that C lies on the lens plane, so that $D_{CS} \simeq D_{LS}$. Then the final form of the lens equation is

$$D_{OS} \tan\beta = D_{OS} \tan\theta - D_{LS}[\tan\theta + \tan(\alpha - \theta)]. \tag{19}$$

At the cost of an additional approximation, the lens equation is put in a form very easy to use in many astrophysical applications, as it is finally expressed in terms of the positions of lens, source, and observer. We will discuss the error introduced by this approximation in Sec. V, while in Sec. IV F we will present an improvement of this equation which avoids this approximation and makes it equivalent to the Ohanian lens equation.

D. Dabrowski and Schunck lens equation

In a paper studying gravitational lensing by boson stars [12], Dabrowski and Schunck realized the difficulties of using the VNC lens equation (16) and derived the alternative lens equation

$$\sin(\theta - \beta) = \frac{d_{LS}}{d_{OS}} \cos\theta \cos\left\{\arcsin\left[\frac{d_{OS}}{d_{LS}} \sin\beta\right]\right\} \times [\tan\theta + \tan(\alpha - \theta)]. \tag{20}$$

This can be obtained from the Virbhadra and Ellis lens equation (19) replacing the distances between planes D_{OS} and D_{LS} by the distances between objects using Eqs. (4) and (5) and noting that

$$d_{LS} \sin\gamma = d_{OS} \sin\beta \tag{21}$$

in the associated Minkowski space.

In practice, the equation (20) by Dabrowski and Schunck is the same as Eq. (19) by Virbhadra and Ellis but expressed in terms of distances between objects instead of distances between planes. Of course, like the former equation, it contains the additional approximation that C lies on the lens plane.

E. Bozza and Sereno lens equation

The Ohanian lens Equation (14) has the advantage of being very simple and being the closest relative of the exact lens equation, since it only contains the asymptotic approximation and makes no additional assumptions. However, in several astrophysical applications one may prefer to have a lens equation directly written in terms of the angle β rather than γ. In fact, β is an angle with vertex in the observer and thus directly connected to coordinates in the observer's sky, whereas γ is an angle with vertex in the lens, thus being less close to observables. The relation between the two angles is given by Eq. (21). Therefore, if we take the sine of Eq. (14), using Eq. (15) and reordering terms we get

$$d_{OS} \sin\beta = d_{OL} \sin\theta \cos(\alpha - \theta) - \sqrt{d_{LS}^2 - d_{OL}^2 \sin^2\theta} \sin(\alpha - \theta), \tag{22}$$

which first appeared in Ref. [13]. Though being more complicated than Eq. (14), this lens equation has the advantage of being directly expressed in terms of the angle β and thus preferable for applications in which this angle is directly involved. Otherwise, it is completely equivalent to the Ohanian lens equation. Note that the distances involved in this equation are distances between the objects and not between their geometrical planes, as in the lens equations of Secs. IV C and IV F.

One may note that this lens equation is expressed in terms of the same variables appearing in the equation by Dabrowski and Schunck (20). However the two equations look different. Apart from the ordering of the terms, Eq.

(20) contains the additional approximation that C lies on the lens plane, whereas Eq. (22) does not. Therefore, Eq. (22) represents an improved version of Eq. (20). The same difference will arise in the new equation to be presented in the next section and the equation by Virbhadra and Ellis.

F. A new improved lens equations between planes

Equation (19) represents a very useful lens equation expressed in terms of distances between the observer, lens, and source planes. However, as we have explained in Sec. IV C, it is derived under the additional assumption that the intersection point C between the incoming and outgoing ray trajectories lies on the lens plane. It would be desirable to have a lens equation expressed in terms of the same quantities without this additional assumption.

Starting again from the Ohanian lens equation (14), we can solve it in terms of $\bar{\theta}$ and plug it in Eq. (15). Recalling Eq. (4) and the relation between γ and β (21), we can put the lens equation in the form

$$D_{OS}\tan\beta = \frac{D_{OL}\sin\theta - D_{LS}\sin(\alpha-\theta)}{\cos(\alpha-\theta)}, \qquad (23)$$

which represents the improved version of Eq. (19) by Virbhadra and Ellis, in the same sense as Eq. (22) by Bozza and Sereno represents the improved version of Eq. (20) by Dabrowski and Schunck. In Sec. V we make a thorough discussion and comparison of the two equations.

G. Small angles lens equation

We finally recall the classical lens equation obtained in the hypothesis of small angles $\alpha, \theta, \beta \ll 1$. This hypothesis has nothing to do with the weak deflection approximation, since the deflection angle is typically expressed in powers of r_g/u, whereas the corrections to the small angles approximation are expressed in powers of θ, α, β. So it makes sense to consider an exact deflection angle while performing the small angles approximation in all trigonometric functions.

The small angles lens equation can be obtained from any of the Eqs. (19), (20), and (22) or (23). The well-known result [1–5] is

$$\beta = \theta - \frac{D_{LS}}{D_{OS}}\alpha. \qquad (24)$$

The small angles approximation is the most rude approximation that can be done on the lens equation. Nevertheless, it is a useful reference approximate equation lying on the other extremum of the approximation ladder with respect to the exact lens equation (10), with all other lens equations staying in the middle steps between the two equations.

It is interesting to note that a similar equation can be deduced when the deflection angle is very close to multiples of 2π. This case corresponds to higher order images generated by photons performing one or more loops around the lens before emerging. The only difference is that the deflection angle α in Eq. (24) must be replaced by $\alpha - 2n\pi$, where n is the number of loops.

H. Conclusions on approximate lens equations

To summarize this section, we can say that the asymptotic approximation allows the use of the relations of Euclidean geometry, confining general relativity to the derivation of the precise expression of the deflection angle in terms of the impact parameter.

In order to write the lens equation in terms of θ, α, and the relative positions of observer, source and lens, an additional relation is needed between the incoming and the outgoing branches of the photon trajectory. The lens equation by Ohanian (14), and the related Eqs. (22) by Bozza and Sereno and (23) presented in this paper, use the equality of the impact parameters of the incoming and outgoing trajectories, expressed by Eq. (15). This equality descends from the time-reversal symmetry of the photon geodesic in general relativity along with the asymptotic approximation.

On the other hand, the lens equations by Virbhadra and Ellis (19) and Dabrowski and Schunck (20) use an alternative relation between the incoming and outgoing trajectories by imposing that the intersection point C lies on the lens plane. In the next section we will quantify the accuracy of these lens equations in a realistic astrophysical situation.

V. LENS EQUATIONS AT WORK IN A REALISTIC EXAMPLE

As explained in the previous section, in the treatment of gravitational lensing by spherically symmetric lenses with arbitrary deflection angles, we can distinguish an exact lens equation (10), a family of lens equations with the asymptotic approximation only (14), (22), and (23), and a family of lens equations with the additional geometrical approximation that the intersection point C lies on the lens plane (19) and (20). For simplicity, we shall refer to these two families of approximate lens equations as the Ohanian family and the Virbhadra and Ellis family, respectively. Furthermore, we have the small angles lens equation, in which all trigonometric functions are approximated by their first order expansions. Finally, we can also consider the effect of the weak deflection approximation on the deflection angle, by retaining only the lowest order term in the weak deflection expansion. We will not consider here the VNC lens equation (16) since it is not expressed in terms of the relative configuration of source, lens, and observer but demands additional information on the position of point C.

As pointed out in Ref. [11], the best candidate for the observation of gravitational lensing in the regime of large deflections is the black hole at the center of our Galaxy, named Sgr A*. Its mass is estimated to $M = 3.6 \times 10^6 M_\odot$ and its distance is $D_{OL} = 8$ kpc [19]. The Schwarzschild radius is therefore

$$r_g = \frac{2GM}{c^2} = 10^{10}\ \mathrm{m}. \tag{25}$$

In order to test the accuracy of the different lens equations, we consider the source plane to be at distance $D_{LS} = 1000 r_g$ from the black hole. At this distance, the source is just at the margin of the accretion disk of the black hole, so it may be a star spiralling towards the black hole (the star S2 is known to reach a distance of $1700 r_g$ at its periapse [19,20]) or a bright spot of hot material rotating around the black hole. Apart from the physical meaning of a source at such distance, this choice has been made with the aim of putting in better numerical evidence the differences in the performances of the lens equations, since the expected order of magnitude of the errors scales with powers of r_g/D_{LS} and r_g/D_{OL}.

We are going to study the position of the images as a function of the position of the source, keeping the distance to the source plane D_{OS} fixed. Therefore, we will compare the results of the exact lens equation (10) with the results of the Virbhadra and Ellis (VE) lens equation (19), the improved lens equation (23), which is equivalent to Ohanian lens equation (we shall refer to it as the OB lens equation), and the small angles lens equation (24). The last three equations are already expressed in terms of β, D_{LS}, and D_{OS}, whereas the exact lens equation (10) is expressed in terms of γ and d_{LS}. We will use Eqs. (4) and (21) to obtain these quantities as functions of β and D_{LS}, referring to the associated Minkowski spacetime.

The azimuthal shift Φ is calculated assuming that the spacetime metric around the black hole is Schwarzschild. Therefore we have

$$A(r) = \left(1 - \frac{r_g}{r}\right); \qquad B(r) = \left(1 - \frac{r_g}{r}\right)^{-1}; \qquad C(r) = r^2. \tag{26}$$

We remind that the detection angle θ is related to the angular momentum of the photon J by Eq. (8). In the approximate lens equations (19), (23), and (24) the deflection angle is obtained by Eq. (11) as a function of θ through Eq. (8). For any given value of the source position β, we find the corresponding position of the images θ numerically in all lens equations (see Sec. V C for details).

A Schwarzschild black hole generates two infinite sequences of images on each side of the source [6]. The outermost pair of images is made of the classical primary and secondary images, which can also be described in the classical weak deflection limit. Section V A is devoted to their study. The second pair of images is generated by photons performing one complete loop around the black hole before reaching the observer. These higher order images are typically very faint, but their detection could be very important for a confirmation of general relativity. We examine them in Sec. V B.

A. Primary and secondary images

For any values of β, we look for solutions of the lens equation such that $\pi/2 < \Phi(\theta, d_{OL}, d_{LS}) < 3\pi/2$. In this interval we always have one solution for any values of β.

In Fig. 2 we show the exact position of the image for β in the range [− 10 mas, 10 mas]. For positive β the image is on the same side of the source (primary image), whereas for negative β the image is on the opposite side (secondary image). We see that for large values of β the position of the primary image tends to coincide with the source position itself ($\theta \simeq \beta$). The secondary image, instead, becomes closer and closer to the black hole. The minimum angle $\theta_{\min}$ represents the border of the so-called shadow of the black hole. This angle is obtained by the minimum angular momentum through Eq. (8) with $J_{\min} = 3\sqrt{3} r_g/2$. For the black hole in Sgr A*, we have $\theta_{\min} = 23\ \mu$as. The value of θ for $\beta = 0$ represents the radius of the Einstein ring, which for our geometrical configuration is $\theta_E = 404\ \mu$as.

Figure 2 has been obtained with the exact lens equation. However, as anticipated before, we expect errors at most of the order r_g/D_{LS} using the approximate equations. So, the difference cannot be appreciated by superposing the solutions obtained with different lens equations. We choose, instead, to plot the relative error in the image positions with respect to the exact lens equation. These are defined as

$$\delta_{\mathrm{OB}} = \frac{\theta_{\mathrm{OB}}}{\theta_{\mathrm{ex}}} - 1 \tag{27}$$

$$\delta_{\mathrm{VE}} = \frac{\theta_{\mathrm{VE}}}{\theta_{\mathrm{ex}}} - 1 \tag{28}$$

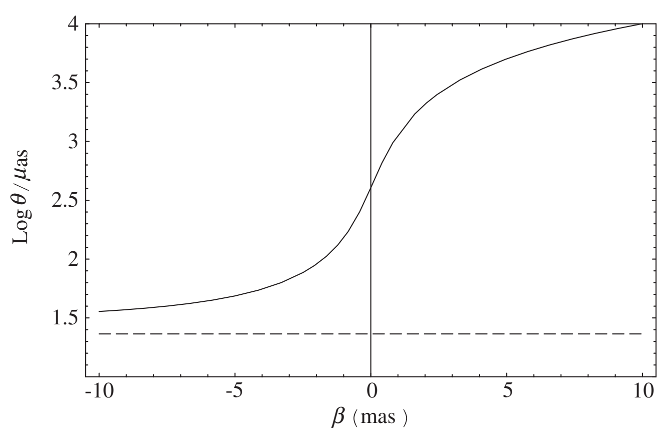

FIG. 2. The exact position of the image θ as a function of the source position β in a linear-log plot. The dashed line is the shadow of the black hole $\theta_{\min}$. For $\beta > 0$ the plot represents the primary image, for $\beta < 0$ it represents the secondary image.

$$\delta_{\rm SA} = \frac{\theta_{\rm SA}}{\theta_{\rm ex}} - 1 \tag{29}$$

$$\delta_{\rm WD} = \frac{\theta_{\rm WD}}{\theta_{\rm ex}} - 1 \tag{30}$$

where the subscript "ex" refers to the result of the exact lens equation (10), "OB" refers to the improved equation presented in this paper (23) derived from that by Ohanian, "VE" refers to the equation by Virbhadra and Ellis (19), "SA" refers to the small angles equation (24), and "WD" refers to the small angles lens equation with the Einstein approximation $\alpha \simeq 2r_g/D_{OL}\theta$ for weak deflection angles. Apart from $\theta_{\rm WD}$ and $\theta_{\rm ex}$, in all remaining cases the position of the images is computed picking α from Eq. (11) and relating u to θ by Eq. (8), as mentioned before.

Figure 3 shows a comparison between the accuracy of the four equations.

All lens equations have the same accuracy for very large β, when gravitational lensing becomes negligible. However, as β approaches θ_E the error of all equations increases, save for the OB lens equation. Noting that the plot is in a logarithmic scale, we see that $\delta_{\rm VE}$ tends to be of the order 10^{-3}, whereas $\delta_{\rm OB}$ drops to 10^{-7}, proving to be much more accurate. Going to $\beta < 0$, we see that the error stays nearly constant for the VE lens equation, whereas it continues to decrease for the OB lens equation to very tiny values. Note the change in sign in $\delta_{\rm VE}$ for positive β, which can be deduced from the negative spike in the logarithmic plot as $\delta_{\rm VE}$ crosses zero. This signals the fact that the sources of error dominating for large β and small β are different and have opposite signs. Surprisingly, the small angles lens equation SA has an error comparable to the VE lens equation in the most interesting lensing region ($|\beta| \lesssim \theta_E$), corresponding to a very good alignment of the source with the lens. The VE lens equation performs better than the SA equation at large negative β, where the small angles approximation is no longer tenable. On the other hand, the weak deflection approximation WD always stays at larger errors.

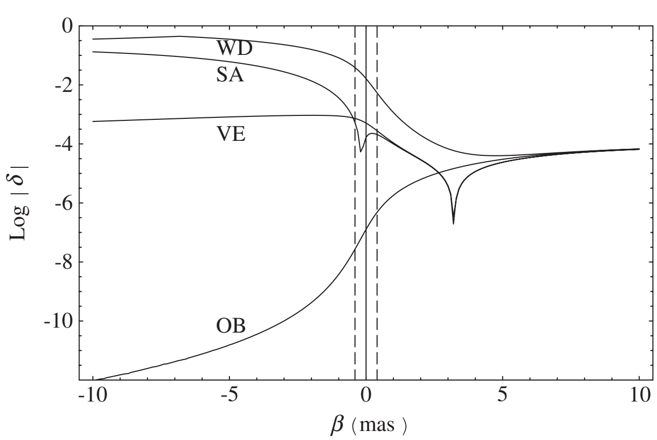

FIG. 3. Relative error in the position of the images for the weak deflection lens equation (WD), the small angles lens equation (SA), the Virbhadra and Ellis lens equation (VE), and the improved Ohanian lens equation (OB) in a log scale. The dashed vertical lines bound the region with $|\beta| < \theta_E$, in which gravitational lensing is mostly interesting.

Now let us find the interpretation of the errors of the considered lens equations. We start from the OB lens equation, which only contains the asymptotic approximation. This lens equation is completely equivalent to the Ohanian lens equation, so we can use this equation for the evaluation of the error and translate the results in terms of the OB equation.

If we expand the azimuthal shift Φ (6) in inverse powers of D_{OL} and D_{LS}, at zero order we obtain the deflection angle $\alpha + \pi$. After this, we obtain some terms that reproduce the power expansions of the geometrical terms $-\theta$ and $-\bar{\theta}$, which are explicitly present in the Ohanian lens equation (14) and finally, the first term that is not present in the Ohanian lens equation is

$$\Delta\gamma_{\rm asym} = \frac{1}{8}\left[\frac{r_g}{d_{LS}}\bar{\theta}^3 + \frac{r_g}{d_{OL}}\theta^3\right]. \tag{31}$$

Taking this term to the right-hand side of the Ohanian lens equation, we can interpret it as an effective change in γ. Then we can estimate the corresponding change in θ simply by

$$\Delta\theta_{\rm asym} = \frac{\partial\theta}{\partial\gamma}\Delta\gamma_{\rm asym}. \tag{32}$$

Evaluating the derivative of θ numerically, and taking for $\Delta\gamma_{\rm asym}$ the expression in Eq. (31), we exactly reproduce the error $\epsilon_{\rm OB}$ in the OB lens equation, as can be appreciated from Fig. 4. In this figure we have also shown the VE error, already presented in Fig. 3, together with the error deriving from the approximation that C lies on the lens plane. This error can be simply calculated by taking

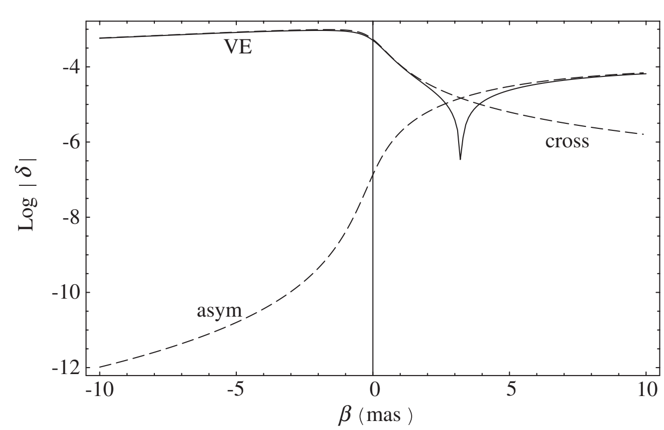

FIG. 4. Comparison among the relative error in the position of the images for the Virbhadra and Ellis lens equation (VE) and what we expect from the asymptotic approximation (asym) and the approximation that the intersection point C lies on the lens plane (cross).

the differences of the right-hand sides of Eqs. (19) and (23)

$$\Delta\theta_{\rm cross} = \frac{\partial\theta}{\partial \tan\beta}\Delta(\tan\beta)_{\rm cross} \qquad (33)$$

$$\Delta(\tan\beta)_{\rm cross} = \frac{D_{OL}}{D_{OS}}\left[\tan\theta - \frac{\sin\theta}{\cos(\alpha-\theta)}\right]. \qquad (34)$$

We can explicitly see that the error due to the approximation that C lies on the lens plane dominates the error due to the asymptotic approximation in the calculation of the secondary image ($\beta < 0$) and for the primary image up to moderately large β. For $\beta \gg \theta_E$, the error due to the asymptotic approximation becomes more relevant than the error due to the position of the intersection point. However, as already mentioned before, the last regime is the least important for gravitational lensing.

The error induced by the small angles approximation can be obtained by considering the next-to-leading order in the power expansions of the trigonometric functions present in Eq. (23). It amounts to

$$\Delta\beta_{\rm small} = -\frac{(D_{OS}\theta - D_{LS}\alpha)^3}{3D_{OS}^3} + \frac{D_{OL}\theta(3\alpha^2 - 6\alpha\theta + 2\theta^2) - 2D_{LS}(\alpha-\theta)^3}{6D_{OS}}. \qquad (35)$$

To conclude this subsection, it is interesting to give approximate estimates of the order of magnitude of the errors in terms of the geometric distances involved in the problem in different limits. For example, we can consider the two limits $\beta \gg \theta_E$ (primary image) and $-\beta \gg \theta_E$ (secondary image). In the first case we have $\theta \simeq \beta$, in the second case $\theta \simeq -2r_g D_{LS}/(D_{OL}D_{OS}\beta)$ from the weak deflection approximation. Taking also $D_{OS} \simeq D_{OL}$, we have the simple estimates

$$\delta_{\rm asym+} \simeq -\frac{r_g}{8D_{LS}}\frac{D_{OL}^2\beta^2}{d_{LS}^2} \qquad (36)$$

$$\delta_{\rm asym-} \simeq -\left(\frac{r_g}{D_{OL}\beta}\right)^4\frac{D_{LS}^2}{d_{LS}^2} \qquad (37)$$

$$\delta_{\rm cross+} \simeq \frac{2r_g^2}{D_{OL}^2\beta^2} \qquad (38)$$

$$\delta_{\rm cross-} \simeq \frac{r_g}{D_{LS}} \qquad (39)$$

$$\delta_{\rm small+} \simeq -\frac{2r_g^2}{D_{OL}} \qquad (40)$$

$$\delta_{\rm small-} \simeq -\frac{D_{OL}^2\beta^2}{3D_{LS}^2}, \qquad (41)$$

which reproduce the curves in Figs. 3 and 4 fairly well outside the lensing zone and are useful to realize the order of magnitude of the errors. In particular, we can appreciate that the error due to the position of the intersection point has opposite sign with respect to the others.

Finally, a very interesting limit is the case $\beta = 0$, corresponding to a source perfectly aligned with the black hole and the two images merged into an Einstein ring of radius θ. The errors in the estimate of the radius of the Einstein ring are

$$\delta_{\rm asym_0} \simeq -\frac{r_g^2}{8D_{LS}^2} \qquad (42)$$

$$\delta_{\rm cross_0} \simeq \frac{r_g}{2D_{LS}} \qquad (43)$$

$$\delta_{\rm small_0} \simeq \frac{r_g}{6D_{LS}} \qquad (44)$$

$$\delta_{\rm weak_0} \simeq -\frac{15\pi}{64}\sqrt{\frac{r_g}{2D_{LS}}}, \qquad (45)$$

where the error due to the weak deflection approximation comes from the first term neglected in the weak deflection expansion (see, for example, Refs. [21–23]).

These relations are very useful to understand the accuracy of the various lens equations in the most interesting regime, that is when the gravitational lensing images are most prominent and eventually form an Einstein ring. The weak deflection lens equation has an error of the order $(r_g/D_{LS})^{1/2}$. In the case under examination, this would translate into an error of 6.6 μas in the radius of the Einstein ring, which is $\theta_E = 404$ μas in the situation imagined in this calculation. The small angles lens equation employed with the exact deflection angle has an error of order (r_g/D_{LS}), which would amount to 0.07 μas. The VE lens equation has an error of the same order of magnitude, but with a slightly larger numerical coefficient, thus leading to an error of 0.2 μas. Finally, the OB lens equation has an error of order $(r_g/D_{LS})^2$, which amounts to 5×10^{-5} μas.

B. Higher order images

An analysis similar to Sec. VA can be repeated for the higher order images, whose position and magnification has been analyzed in Ref. [11] for the case of the black hole in the center of our Galaxy, and then revisited in many papers with different methods. The first pair of higher order images can be found in the interval $5\pi/2 < \Phi(\theta, d_{OL}, d_{LS}) < 7\pi/2$, corresponding to photons performing one loop before reaching an observer on the other side of the lens with respect to the source. Since these images appear very close to the shadow border $\theta_{\rm min}$, we think it is more instructive to discuss the fractional displacement of the image from the shadow border, defined as

$$\epsilon = \frac{\theta}{\theta_{\min}} - 1. \tag{46}$$

This quantity is shown in Fig. 5 for the geometric configuration examined in this paper. The plot has been calculated using the exact lens equation (10). For $\beta > 0$ the curve represents the fractional displacement of the positive parity image on the same side of the source and for $\beta < 0$ the curve represents the fractional displacement of the negative parity image on the opposite side. For $\beta = 0$, we obtain the displacement of the first order Einstein ring $\theta_{E,1}$ from the shadow border.

In order to compare the performances of the different lens equations, in Fig. 6 we plot the relative error in this fractional displacement with respect to the exact lens equation

$$\delta_{\epsilon,\mathrm{OB}} = \frac{\epsilon_{\mathrm{OB}}}{\epsilon_{\mathrm{ex}}} - 1 \tag{47}$$

$$\delta_{\epsilon,\mathrm{VE}} = \frac{\epsilon_{\mathrm{VE}}}{\epsilon_{\mathrm{ex}}} - 1 \tag{48}$$

$$\delta_{\epsilon,\mathrm{SA}} = \frac{\epsilon_{\mathrm{SA}}}{\epsilon_{\mathrm{ex}}} - 1. \tag{49}$$

The SA lens equation, in this case, is Eq. (24) with α replaced by $\alpha - 2\pi$, as explained in Sec. IV G.

Again we see that the OB lens equation has much lower errors in the estimate of the position of the images than any other lens equations. The VE lens equation performs slightly better than the small angles equation for $|\beta| \gtrsim \theta_{E,1}$, whereas the opposite occurs in the good alignment regime $|\beta| \lesssim \theta_{E,1}$. It is interesting to note that there is a value for β such that $\alpha = 2\pi$ and $\beta = \theta$. In this particular point, all lens equations are equivalent and are affected by the same small error, determined by the asymptotic approximation.

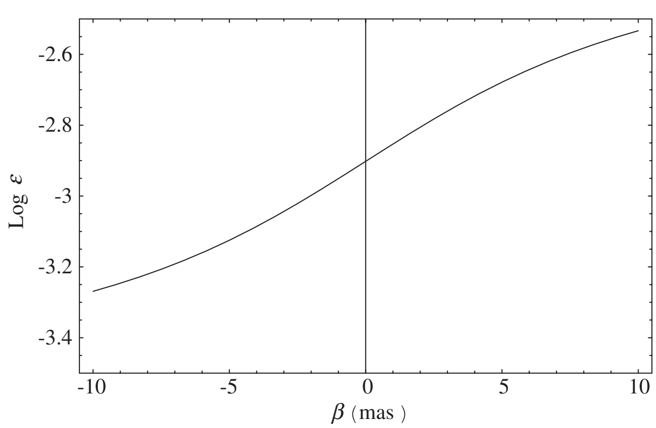

FIG. 5. Fractional displacement of the first pair of higher order images from the shadow border as a function of β as defined in Eq. (46).

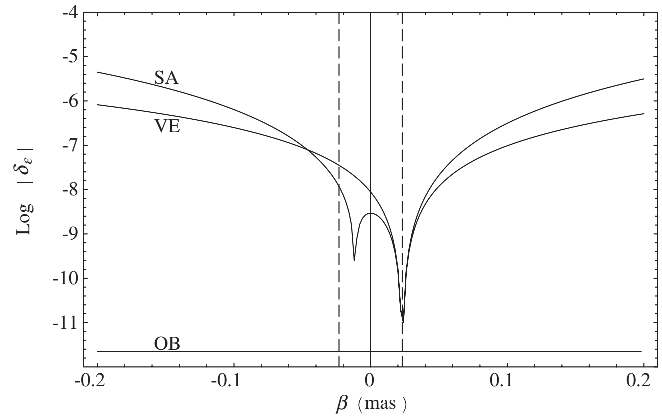

FIG. 6. Errors in the estimate of the fractional displacement of the first pair of higher order images from the shadow borders. SA refers to the small angles lens equation, VE to the Virbhadra and Ellis lens equation, OB to the improved Ohanian lens equation. The dashed lines bound the range in which $|\beta| < \theta_{E,1}$, where $\theta_{E,1}$ is the angular radius of the first order Einstein ring.

Similarly to the case of the primary and secondary images, we can make analytical estimates of the errors in the positions of the higher order images. We follow the same procedure described in the previous subsection and use the analytical approximation

$$\epsilon \simeq \epsilon_{SDL} = 216(7 - 4\sqrt{3})e^{-3\pi+\gamma}, \tag{50}$$

derived in the strong deflection limit [6,9,24], in order to have fully analytical results. Then, the errors in the displacement of the first order Einstein ring are

$$\delta_{\epsilon,\mathrm{OB},0} = -\frac{81\sqrt{3}}{64}\left(\frac{r_g}{D_{LS}}\right)^4 \tag{51}$$

$$\delta_{\epsilon,\mathrm{VE},0} = \frac{81\sqrt{3}}{16}\left(\frac{r_g}{D_{LS}}\right)^3 \tag{52}$$

$$\delta_{\epsilon,\mathrm{SA},0} = \frac{27\sqrt{3}}{16}\left(\frac{r_g}{D_{LS}}\right)^3. \tag{53}$$

Again the error in the VE lens equation is of the same order as the error in the small angles lens equation with a slightly larger numerical coefficient.

C. Notes on numerical integration

In the previous subsections we have compared the images obtained by solving different lens equations. The relative error has proved to be quite small, particularly for the OB lens equation, whose error goes down to 10^{-12} in some plots. In order to investigate such tiny differences, we must push the numerical precision of our calculations sufficiently far.

The crucial step in the computations is represented by the evaluation of the integral (6), which contains an integrable singularity at r_0. We have used the NINTEGRATE routine by MATHEMATICA with the default method (adaptive Gaussian quadratures with error estimation based on Kronrod points [25]). In order to reduce the errors, we have worked with the variable z, defined by

$$r = \frac{r_0}{1-z}, \tag{54}$$

z ranges from 0 to 1 as r ranges from r_0 to $+\infty$. In the Schwarzschild metric, the integrand in Eq. (6) expressed in terms of z becomes

$$\frac{\sqrt{r_0}}{\sqrt{z}\sqrt{2r_0 - 3 + (3 - r_0)z - z^2}}. \tag{55}$$

We have checked that increasing the precision goal and the working precision makes the numerical results converge to an asymptotic limit, with a controllable precision (we have reached a 10^{-25} precision). Furthermore, we have also rewritten the integral as an ordinary differential equation to be solved by the NDSOLVE routine, which switches between a nonstiff Adams method and a stiff Gear backward differentiation formula method. Solving the differential equation, we get the primitive function, which can be evaluated at both ends of the domain. In this way, we have double-checked our results obtained by NINTEGRATE finding that the two methods converge to the same result. In particular, we find that the differential equation method converges more slowly than the numerical integration with the Gaussian quadratures, as the precision goal is increased. For this reason we tend to give a preference to the NINTEGRATE routine. The results presented in the plots are calculated with a 10^{-18} precision.

The accuracy in the numerical calculation is finally confirmed by the very good agreement with the analytical estimates for the errors at leading order.

VI. DISCUSSION AND CONCLUSIONS

Of course, an approximation can be valid or not depending on the precision of the observational instrumentation, the level of environmental noise and the accuracy needed for the extraction of the interesting information.

The best resolution we can expect to reach in a reasonably near future is that of the MAXIM project [26], which amounts to a fraction of μas. Then, in the example considered in this section, the weak deflection lens equation would not be able to ensure such a precision. The VE and the small angles lens equations would be marginally adequate, whereas the Ohanian lens equation would be largely sufficient, without need to resort to the exact lens equation. So, in order to exploit the resolution of future instruments, it might be necessary to use the Ohanian lens equation or one of its variants.

Environmental noise can be provided by scattering and absorption processes affecting the photons involved in gravitational lensing. Such processes should be very effective in the environment of the supermassive black hole at the Galactic center. However, they act in a statistical way and a good average process should be able to restore the original information regarding gravitational lensing. More subtle are the effects that introduce unwanted systematics, such as gravitational lensing by secondary objects. However, the gravitational lensing optical depth towards the center of the bulge is of the order 10^{-6} [5], which indicates that lensing or microlensing effects by stars or other compact objects on the line of sight is absolutely negligible (see also [27]). In the absence of relevant systematics, the position of the centroids of the images can be used for a clean reconstruction of the gravitational lensing event at the angular resolution level reached by the instruments.

Finally, we note that there are many theoretical reasons to require a reconstruction of a gravitational lensing event as precise as possible. As anticipated, the study of higher order terms in the deflection angle requires a precision better than $(r_g/D_{LS})^{1/2}$ at second PPN order and (r_g/D_{LS}) at third PPN order. Therefore, if one wants to compare the results of PPN formalism with different black hole metrics, it is mandatory to use a lens equation that guarantees the necessary accuracy. In this respect the VE lens equation does not represent an improvement with respect to the small angles approximation, as it is affected by an error of the same order of magnitude.

In this work we have compared the level of accuracy of several lens equations, keeping the exact general relativistic lens equation introduced by Frittelli, Kling, and Newman [7,8] as the reference equation. We have shown that the Ohanian lens equation [9] and its close relatives [13,18] are the best approximations of the exact lens equation, in that they adopt the asymptotic approximation only. Other lens equations are not expressed in terms of relative positions of source, lens and, observer [10] or introduce additional approximations [11,12].

Furthermore, we have presented a new formulation of the Ohanian lens equation in terms of the distances between the observer, lens, and source planes, which fills a gap in the lens equation zoo. As shown by a numerical example describing a realistic gravitational lensing event by the black hole at the center of our Galaxy, such an equation represents a noteworthy improvement with respect to previous commonly used lens equations expressed in terms of the same quantities.

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