Feynman parametrization and Mellin summation at finite temperature

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We show that the Mellin summation technique (MST) is a well-defined and useful tool to compute loop integrals at finite temperature in the imaginary-time formulation of thermal field theory, especially when interested in the infrared limit of such integrals. The method makes use of the Feynman parametrization which has been claimed to have problems when the analytical continuation from discrete to arbitrary complex values of the Matsubara frequency is performed. We show that without the use of the MST, such problems are not intrinsic to the Feynman parametrization but instead, they arise as a result of (a) not implementing the periodicity brought about by the possible values taken by the discrete Matsubara frequencies before the analytical continuation is made and (b) to the changing of the original domain of the Feynman parameter integration, which seemingly simplifies the expression but in practice introduces a spurious end point singularity. Using the MST, there are no problems related to the implementation of the periodicity but instead, care has to be taken when the sum of denominators of the original amplitude vanishes. We apply the method to the computation of loop integrals appearing when the effects of external weak magnetic fields on the propagation of scalar particles is considered.

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I. INTRODUCTION

At finite temperature, unlike in vacuum, momentum dependent loop integrals in general depend separately on the time (p_0) and space ($\mathbf{p}$) components of the momentum $P^\mu = (p_0, \mathbf{p})$, since Lorentz invariance is lost due to the presence of the medium. As a consequence, the limiting behavior of these integrals, as the momentum components approach given values, may depend on the way the limit is taken. For instance, in one-loop self-energy calculations, the *infrared limit* ($p_0 = 0$, $p = |\mathbf{p}| \to 0$) that accounts for the plasma screening properties does not necessarily coincide with the *static limit* ($p_0 \to 0$, $p = |\mathbf{p}| = 0$) that accounts for the long wavelength plasma oscillations. In physical terms, this nonanalyticity as $P^\mu \to 0$ is due to the cut structure of the self-energy at finite temperature, where branch cuts appear representing scattering processes not allowed in vacuum.

The above behavior was originally not fully recognized since there were results indicating that the aforementioned two limits commuted when computing the real part of the self-energy in a ϕ^3 theory [1,2]. In the imaginary-time formalism (ITF), the problem with these calculations was traced back to an incorrect analytic continuation when the discrete frequency takes on arbitrary complex values [3]. Since the erroneous result was obtained from an analysis based on the use of the Feynman parametrization at finite temperature, it is often thought that such a parametrization is also endemic to the source of the error.

In the ITF, the introduction of the Feynman parametrization for the computation of loop integrals containing two propagators can be avoided since there are techniques that allow to perform the sum over Matsubara frequencies in a straightforward manner. However, it has been recently shown that when studying the influence of weak magnetic fields over physical processes at finite temperature, the loop integrals that appear involve products of powers of two or more propagatorlike denominators [4]. Although it is possible to generalize the standard techniques to carry out the sum over Matsubara frequencies from the product of two propagators to the case of a product of powers of these, the calculations become extremely cumbersome. It is therefore desirable to have a more direct method to perform these calculations. One of such is the Mellin summation technique (MST). The method [5] calls for the use of Feynman parametrization, which allows to condense products of powers of propagators into a single propagatorlike factor raised to some power. This is particularly useful when one seeks an answer in terms of a power series involving a small parameter, for instance, at a high temperature T, the ratio m/T, where m is the particle's mass.

In light of the well-known mishaps with the use of the Feynman parametrization [3] in finite temperature calculations, it is important to establish that the MST in the ITF has the correct analytical properties when the discrete Matsubara frequency is continued to arbitrary complex values. In this work we undertake such a study. We perform an explicit calculation of the one-loop self-energy in a ϕ^3 theory as a guiding tool to find out how the Feynman parametrization should be used in the ITF of thermal field theory. The work is organized as follows: In Sec. II we give a brief summary of the previous analyses that have dealt with this problem and compare their results. In Sec. III, we give a detailed derivation of the standard calculation using conventional techniques. We explore the case when the external momentum approaches zero and, in particular, give the explicit result in the infrared limit. As we want to compare with the MST, the answer is given in terms of an expansion in powers of m/T for the case when $T \gg m$. We also point out the importance of considering that the external frequency takes on discrete values and therefore implementing the periodicity of the resulting expressions before taking the analytical continuation to arbitrary complex values. In Sec. IV we perform the calculation using the MST which involves the use of the Feynman parametrization. We underline the importance of carrying out the integral over the Feynman parameter $x \in [0, 1]$ to avoid the appearance of spurious end point singularities. We carefully show how the simple Feynman parametrization has to be corrected to account for the case where the sum of denominators in the Feynman formula vanishes and emphasize that this correction term accounts for the whole dependence on the way the momentum approaches zero, in agreement with the analysis in Ref. [3]. In Sec. V we apply the results for the computation of integrals describing the self-energy of a neutral scalar interacting with charged ones in the presence of a magnetic field. We finally present our conclusions and give an outlook in Sec. VI. We leave for the appendices the demonstration of important intermediate results and alternative derivations of the calculations arising in the discussion of Secs. III and IV.

II. NONANALYTICITY AND MISHAPS WITH FEYNMAN PARAMETRIZATION

The problem of the nonanalyticity of thermal field theory calculations as the momentum components approach zero, has been analyzed by several authors. The landscape of findings is, at first sight, rather blurred since there are many details in the calculations that are sources of more extended discussions. Among these we can mention: the implementation of derivative expansion techniques, the validity of perturbative and derivative expansions exchange, and the implementation of the external bosonic field periodicity. Other studies are concerned with the correct analytic continuation to arbitrary complex values of the external frequency, the soundness of some redefinition of variables inside potentially divergent integrals, the physical interpretation of the imaginary part of the thermal bubble, and the use of the Feynman parametrization.

In this work, our main purpose is to show that the use of Feynman parametrization, within the MST, is a well-defined procedure. Nevertheless, it is worth pausing to summarize, from a wider perspective, what has been found in the context of the nonanalyticity of thermal self-energies at the origin in calculations that do not resort to the use of the MST.

In general, this problem has been dealt with in terms of perturbative and nonperturbative approaches and both in the ITF and the real time formulation (RTF) of thermal field theory. The coincidence between perturbative and nonperturbative calculations is sometimes taken as a guide to decide on the correctness of the approach.

The discrepancy between the results in the infrared and the static limits caused a great deal of confusion, prompting a number of possible explanations. These ranged from assigning validity only to the infrared limit as a genuine result in thermal equilibrium [6], passing by suggesting that it is not necessary to assume that the external field is in thermal equilibrium [1,7,8], to dismissing the nonanalyticity by claiming that this is not present in an exact solution to the slow motion approximation of the Green's function [1,6]. In Ref. [1], the calculation in the ITF is done by extending the external frequency to the whole imaginary axis and then analytically continuing it to the entire complex plane. With this procedure, the periodicity of the functions in the external frequency—that was present before analytical continuation—is lost. This is how the erroneous result, that the infrared and static limits coincide, is obtained. This result seemingly confirmed the one in Ref. [2] which was performed in the RTF. Reference [9] points out that truncating the derivative expansion at the beginning of the calculation, either by keeping only the constant term [10] or at higher order [1,6], gives misleading results, in the first case, because the operator nature of the background field is lost; in the latter, because the periodicity is not considered.

However, Weldon [3] showed that the results in Refs. [1,2] go wrong, performing the calculation both in the ITF and the RTF. In addition to providing physical arguments for the inequivalence of the infrared and static limits he demonstrated that in the RTF calculation, the use of the Feynman parametrization, as is commonly implemented in $T = 0$ calculations, needs to be corrected. The correction accounts for the fact that the real time Feynman amplitude is not the boundary value of a single analytic function and thus it is necessary to perform one calculation for the real and another one for the imaginary part. However, for the discussion concerning the ITF, Weldon argued that starting from an expression that uses the Feynman parametrization, the analytical continuation is not unique and leads to a function containing branch points and an end point singularity that need to be removed by the addition of an extra term. The argument is based on an expression where the integration interval for the Feynman parameter x has been changed from $x \in [0, 1]$ to $x \in [0, 1/2]$ which uses the symmetry of the integrand about $x = 1/2$, *before* the analytical continuation is implemented.

In what follows, we will show that the Feynman parametrization can be implemented in the ITF (within the MST) without introducing spurious branch points and end point singularities. The key ingredients are the implementation of the periodicity in the expressions *before* the analytical continuation, the use of the original integration interval for the Feynman parameter x, and the accounting of the extra term that corrects the original Feynman formula when the sum of denominators vanishes and which happens naturally in the MST. Before proceeding with this analysis, it is convenient to set the stage and perform an explicit calculation using the standard technique in the ITF to have a reference to compare with the result obtained after introducing the MST.

III. STANDARD CALCULATION

We start the discussion with the explicit expression for the one-loop self-energy of a scalar field ϕ with a self-interaction of the form $\lambda\phi^3$. This is given by

$$\Pi(p_{0l}, p) = \frac{\lambda^2}{2} T \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \Delta(k_{0n}, E_k) \times \Delta(k_{0n} - p_{0l}, E_{k-p}), \tag{1}$$

where $E_x^2 = \mathbf{x}^2 + m^2$, $p_{0l} = 2i\pi lT$, $k_{0n} = 2i\pi nT$, with l, n being integers and

$$\Delta(p_{0l}, E_p) = -\frac{1}{p_{0l}^2 - E_p^2} = -\sum_{s=\pm 1} \frac{s}{2E_p} \frac{1}{p_{0l} - sE_p}. \tag{2}$$

The standard calculation [3] is done by first summing over the Matsubara frequencies. This is most easily accomplished by using Eq. (2) to write Eq. (1) as

$$\Pi(p_{0l}, p) = -\frac{\lambda^2}{2} T \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \sum_{r,s=\pm 1} \frac{rs}{4E_k E_{k-p}} \times \frac{1}{[2\pi nT + isE_k][2\pi nT + ip_{0l} + irE_{k-p}]}. \tag{3}$$

The sum over the Matsubara frequencies can be computed by means of the identity

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n + ix)(n + iy)} = \left(\frac{\pi}{x - y}\right) \times [\coth(\pi x) - \coth(\pi y)], \tag{4}$$

which yields the expression for the self-energy

$$\Pi(p_{0l}, p) = -\frac{\lambda^2}{2} \int \frac{d^3k}{(2\pi)^3} \times \sum_{r,s=\pm 1} \left(\frac{1}{8E_k E_{k-p}}\right) \times \frac{1}{sE_k - (rE_{k-p} + p_{0l})} \times \left[r \coth\left(\frac{E_k}{2T}\right) - s \coth\left(\frac{E_{k-p}}{2T}\right)\right], \tag{5}$$

where we used that $\coth(x + il\pi) = \coth(x)$, in account of the fact that $p_{0l} = 2i\pi lT$, and furthermore that $s \coth(sx) = \coth(x)$. We emphasize that this is an important step necessary to implement the periodicity in the expression and that eventually allows the analytic continuation of the result to arbitrary complex values of p_{0l}. As discussed in Ref. [9] when this condition is not taken, the result cannot be interpreted on physical grounds. To stress the importance of this point, we show in Appendix A that, had this condition not been taken, the eventual analytical continuation would have led to an erroneous result that needs to be corrected precisely by the addition of the function Π_δ found in Ref. [3]. Simplifying Eq. (5) we get

$$\Pi(p_{0l}, p) = -\frac{\lambda^2}{2} \sum_{s=\pm 1} \int \frac{d^3k}{(2\pi)^3} \times \left\{\frac{\coth(\frac{E_k}{2T})}{4E_k[(E_k - sp_{0l})^2 - E_{k-p}^2]} + (E_k \leftrightarrow E_{k-p})\right\}. \tag{6}$$

Note that upon the change of variable $\mathbf{k} - \mathbf{p} \to \mathbf{k}$, the second term in Eq. (6) reduces to the first one and thus the complete expression for the self-energy is twice the first term in the above equation. Carrying out the angular integration we get

$$\Pi(p_{0l}, p) = -\frac{\lambda^2}{2(2\pi)^2} \sum_{s=\pm 1} \int_0^\infty kdk \frac{\coth(\frac{E_k}{2T})}{4pE_k} \times \ln\left(\frac{p_{0l}^2 - p^2 - 2sE_k p_{0l} + 2kp}{p_{0l}^2 - p^2 - 2sE_k p_{0l} - 2kp}\right). \tag{7}$$

At this point we take the analytical continuation in p_{0l} from discrete imaginary values to arbitrary complex ones, $p_{0l} \to p_0$ and explore the limiting behavior of Eq. (7) as the momentum components of the vector $P^\mu = (p_{0l}, \mathbf{p})$ approach zero. We specialize to the case where p_0 is real. Since the result depends on the way the limit is taken, we first set $p_0 = \alpha p$,

$$\Pi(\alpha p, p) = -\frac{\lambda^2}{2(2\pi)^2} \sum_{s=\pm 1} \int_0^\infty kdk \frac{\coth(\frac{E_k}{2T})}{4E_k p} \times \ln\left(\frac{\alpha^2 p^2 - p^2 - 2s\alpha E_k p + 2kp}{\alpha^2 p^2 - p^2 - 2s\alpha E_k p - 2kp}\right), \tag{8}$$

and take the limit $p \to 0$ by expanding the logarithm around $p = 0$,

$$\Pi(\alpha p, p) \stackrel{p\to 0}{=} -\frac{\lambda^2}{2(2\pi)^2}\sum_{s=\pm 1}\int_0^\infty kdk\frac{\coth(\frac{E_k}{2T})}{4E_k p} \times\left[\frac{kp(\alpha^2-1)}{k^2-\alpha^2E_k^2}+\log\left(\frac{s\alpha E_k+k}{s\alpha E_k-k}\right)\right]. \quad (9)$$

After carrying out the sum in Eq. (9) we get in the limit $p \to 0$,

$$\Pi(\alpha p, p) \stackrel{p\to 0}{=} -\frac{\lambda^2}{2(2\pi)^2}\int_0^\infty dk\frac{\coth(\frac{E_k}{2T})}{2E_k}\frac{k^2(\alpha^2-1)}{k^2-\alpha^2E_k^2}. \quad (10)$$

By using the identity

$$\coth\left(\frac{E_k}{2T}\right) = [1+2n(E_k)], \quad (11)$$

where $n(E_k)$ is the Bose-Einstein distribution, we can separate the vacuum and thermal contributions of the above equation. Keeping only the thermal part we obtain

$$\Pi^T(\alpha p, p\to 0) = -\frac{\lambda^2}{2(2\pi)^2}\int_0^\infty dk\frac{n(E_k)}{E_k}\frac{k^2(\alpha^2-1)}{k^2-\alpha^2E_k^2}. \quad (12)$$

We now follow Ref. [10] to find the explicit expression for Π in the infrared limit at high temperature. The result for arbitrary α is given in Appendix B. Setting $\alpha = 0$ in Eq. (12) we get

$$\Pi^T(0, p\to 0) = \frac{\lambda^2}{4(2\pi)^2}\mu^{1-d}\int d^dk\frac{n(E_k)}{E_k}, \quad (13)$$

where in order to write the integral in d dimensions, we have first extended the integration domain from $[0, \infty]$ to $[-\infty, \infty]$ and thus multiplied by $1/2$. The extension of the integral in Eq. (12) to d dimensions represents a way of handling the infinities involved in the explicit computation and we should keep in mind that in order to make contact with Eq. (12), the limit $d \to 1$ will be eventually taken. The extension to d dimensions also calls for the introduction of the mass scale μ. The angular integration in Eq. (13) can be done straightforward and the result is

$$\Pi^T(0, p\to 0) = \frac{\lambda^2}{4(2\pi)^2}\mu^{1-d}\frac{2\pi^{d/2}}{\Gamma(d/2)}\int_0^\infty k^{d-1}dk\frac{n(E_k)}{E_k}. \quad (14)$$

Using the identity

$$\frac{n(E_k)}{E_k} = -\frac{1}{2E_k}+\beta\sum_{n=-\infty}^{\infty}\frac{1}{(\beta E_k)^2+(2\pi n)^2}, \quad (15)$$

where $\beta = 1/T$, Eq. (14) can be written as

$$\Pi^T(0, p\to 0) = \frac{\lambda^2}{4(2\pi)^2}\mu^{1-d}\frac{\pi^{d/2}}{\Gamma(\frac{d}{2})}\int_0^\infty k^{d-2}dk^2 \times\left[-\frac{1}{2E_k}+T\sum_{n=-\infty}^{\infty}\frac{1}{E_k^2+(2\pi nT)^2}\right]. \quad (16)$$

Upon the change of variable

$$z = \frac{m^2}{k^2+m^2}, \quad (17)$$

we get

$$\Pi^T(0, p\to 0) = \frac{\lambda^2}{4(2\pi)^2}\mu^{1-d}\frac{\pi^{d/2}}{\Gamma(\frac{d}{2})}\left\{-\frac{1}{2}m^{d-1} \times\int_0^1 dz(1-z)^{(d/2)-1}z^{-(d+1)/2} + T\sum_{n=-\infty}^{\infty}(m^2+(2\pi nT)^2)^{d-2/2} \times\int_0^1 dz(1-z)^{(d/2)-1}z^{-(d/2)}\right\}. \quad (18)$$

The integrals in the last expression are well known and can be expressed in terms of ratios of gamma functions Γ, namely

$$\int_0^1 dz(1-z)^\rho z^\gamma = \frac{\Gamma(\rho+1)\Gamma(\gamma+1)}{\Gamma(\rho+\gamma+2)}. \quad (19)$$

Therefore, Eq. (18) becomes

$$\Pi^T(0, p\to 0) = \frac{\lambda^2}{4(2\pi)^2}\mu^{1-d}\frac{\pi^{d/2}}{\Gamma(\frac{d}{2})} \times\left\{-\frac{m^{d-1}}{2}\frac{\Gamma(\frac{d}{2})\Gamma(\frac{1-d}{2})}{\Gamma(\frac{1}{2})} + T\sum_{n=-\infty}^{\infty}(m^2+(2\pi nT)^2)^{d-2/2} \times\frac{\Gamma(\frac{d}{2})\Gamma(1-\frac{d}{2})}{\Gamma(1)}\right\}. \quad (20)$$

Separating the term with $n = 0$ in the sum and keeping in mind that the terms in the sum are even powers of n we get

$$\Pi^T(0, p\to 0) = \frac{\lambda^2}{4(2\pi)^2}2T\mu^{1-d}\pi^{d/2}\left\{-\frac{1}{4T}m^{d-1}\frac{\Gamma(\frac{1-d}{2})}{\Gamma(\frac{1}{2})} + \Gamma\left(\frac{2-d}{2}\right)\frac{m^{d-2}}{2}+\Gamma\left(\frac{2-d}{2}\right) \times\sum_{n=1}^{\infty}[m^2+(2\pi nT)^2]^{d-2/2}\right\}. \quad (21)$$

The first and third terms within the curly brackets in the right-hand side of the above equation have a singularity when $d = 1$ that should be isolated. The singularity in the first term arises as the argument of one of the gamma functions vanishes. The singularity in the third term is less obvious and we concentrate on it. Defining

$$
\begin{aligned}
S &\equiv \sum_{n=1}^{\infty}[(m^2+(2\pi nT)^2)^{(d-2)/2}-(2\pi nT)^{d-2}+(2\pi nT)^{d-2}] \\
&= (2\pi T)^{d-2}\zeta(2-d)+\sum_{n=1}^{\infty}[(m^2+(2\pi nT)^2)^{(d-2)/2} \\
&\quad -(2\pi nT)^{d-2}] \\
&= (2\pi T)^{d-2}\zeta(2-d)+\sum_{n=1}^{\infty}(2\pi nT)^{d-2} \\
&\quad \times\left[\left(\frac{m^2}{(2\pi nT)^2}+1\right)^{(d-2)/2}-1\right],
\end{aligned}
\tag{22}
$$

where ζ is the Riemann zeta function, which has a simple pole at $d = 1$. In the high temperature limit $T \gg m$, we can approximate the above expression as

$$
\begin{aligned}
S &\approx (2\pi T)^{d-2}\zeta(2-d)+\sum_{n=1}^{\infty}(2\pi nT)^{d-2}\left[\frac{d-2}{2}\frac{m^2}{(2\pi nT)^2}\right] \\
&= (2\pi T)^{d-2}\zeta(2-d)+\frac{d-2}{2}(2\pi T)^{d-4}m^2\zeta(4-d).
\end{aligned}
\tag{23}
$$

Substituting Eq. (23) into Eq. (21) we get

$$
\begin{aligned}
\Pi^T(0, p\to 0) &= \frac{\lambda^2}{4(2\pi)^2}\mu^{1-d}\pi^{d/2}\bigg\{-\frac{1}{2}m^{d-1}\frac{\Gamma(\frac{1-d}{2})}{\Gamma(\frac{1}{2})} \\
&\quad +\Gamma\left(1-\frac{d}{2}\right)Tm^{d-2}+2T\Gamma\left(1-\frac{d}{2}\right) \\
&\quad \times\bigg[(2\pi T)^{d-2}\zeta(2-d)+\frac{d-2}{2}(2\pi T)^{d-4} \\
&\quad \times m^2\zeta(4-d)\bigg]\bigg\}.
\end{aligned}
\tag{24}
$$

We now set $d = 1 - 2\epsilon$ and make a series for $\epsilon \to 0$. The ϵ-poles cancel and the expression for Π^T at high temperature and in the infrared limit is

$$
\Pi^T(0, p\to 0) = \frac{\lambda^2}{4(2\pi)^2}\left\{\frac{\pi T}{m}+\ln\left(\frac{m}{2T}\right)+\gamma_E-\frac{m^2\zeta(3)}{8\pi^2T^2}\right\},
\tag{25}
$$

where γ_E is Euler's gamma.

IV. MELLIN SUMMATION TECHNIQUE AND FEYNMAN PARAMETRIZATION

The MST is a useful tool to compute infinite sums [11] of the form encountered in finite temperature calculations in the ITF. The technique resorts to applying a Mellin transform over the discrete frequency to the expression involving the sum and afterwards applying the inverse transform to obtain an identity. In this fashion, the calculation of the sum becomes the easiest part and the problem reduces to computing the Mellin transform and its inverse of the remaining expression.

We now show how the combined use of the MST and the Feynman parametrization leads to the same result as the standard calculation obtained in Eq. (25). For this purpose, we start from the expression for the self-energy in Eq. (1) separating the sum over Matsubara frequencies as

$$
\sum_{n=-\infty}^{\infty} = \sum_{n=-\infty}^{-|l|-1}+\sum_{n=|l|+1}^{\infty}+\sum_{n=-|l|}^{+|l|}.
\tag{26}
$$

This expression has the advantage of separating the sum into pieces where the frequencies involved have a definite sign from the one where the frequencies have a mixed sign. The former is suited for the application of the MST since this is an integral transform over a continuous variable restricted to the positive real axis [see Eqs. (32)]. Since the combination of frequencies appears as a square, all that matters is that this has a definite sign, either positive or negative. For the last term in Eq. (26) this is not possible. Nevertheless the calculation can be performed making use of the Feynman parametrization. In addition, notice that by transforming the original discrete frequencies into a continuous variable, there are no problems associated to the implementation of periodicity conditions. However, care has to be taken when the sum of denominators of the original Feynman amplitude vanishes, leading to a correction term, as discussed by Weldon in Ref. [3]. In what follows we analyze these contributions separately.

A. Definite sign frequencies

To begin, let us concentrate on the first two terms arising from the separation of the sum over Matsubara frequencies in Eq. (26) and define

$$
\begin{aligned}
\Pi_1(p_{0l}, p) &= \frac{\lambda^2}{2}T\left(\sum_{n=-\infty}^{-|l|-1}+\sum_{n=|l|+1}^{\infty}\right) \\
&\quad \times\int\frac{d^3k}{(2\pi)^3}\Delta(k_{0n}, E_k)\Delta(k_{0n}-p_{0l}, E_{k-p}) \\
&= \frac{\lambda^2}{2}T\left(\sum_{n=-\infty}^{-|l|-1}+\sum_{n=|l|+1}^{\infty}\right)\int\frac{d^3k}{(2\pi)^3} \\
&\quad \times\int_0^1\frac{dx}{[(1-x)D_2+xD_1]^2},
\end{aligned}
\tag{27}
$$

where we have introduced the Feynman parametrization and thus the integral over the Feynman parameter x. In Eq. (27) $D_1 = \omega_n^2 + E_k^2$, $D_2 = (\omega_n - \omega_l)^2 + E_{k-p}^2$. $\omega_{n,l}$ are related to k_{0n} and p_{0l} by $k_{0n} = i\omega_n$, $p_{0l} = i\omega_l$ with $\omega_n = 2\pi nT$ and $\omega_l = 2\pi lT$. We shift the three momentum integration variable $\mathbf{k} \to \mathbf{k} - (1-x)\mathbf{p}$ and, after making the appropriate renaming of the summation index, the expression for Π_1 can be written as

$$\Pi_1(p_{0l}, p) = \frac{\lambda^2}{2} T \int_0^1 dx [S_+ + S_-], \tag{28}$$

where $S_\pm$ are defined as

$$S_\pm = \sum_{n=0}^{\infty} f(\omega_{n\pm}), \tag{29}$$

with

$$f(y) = \mu^{3-d} \times \int \frac{d^d k}{(2\pi)^d} \times \frac{1}{[y^2 + k^2 + x(1-x)(\omega_l^2 + p^2) + m^2]^2} \tag{30}$$

and

$$\omega_{n\pm}^2 = (2\pi T)^2 (n + |l| + 1 \pm (1-x)l)^2. \tag{31}$$

The quantities $\omega_\pm$ will become the variables over which the Mellin transform is computed, that is to say $\omega_\pm \to y$ in Eq. (32). Notice that the expression for f involves an integral that for later purposes has been extended to d dimensions, and thus the need to introduce the mass scale μ. In order to make contact with Eq. (27), this time, as opposed to the discussion after Eq. (12), we will be interested in taking the limit $d \to 3$.

We perform the sums $S_\pm$ by means of the Mellin summation technique. In general, the Mellin transform pair $f(y)$, $\mathcal{M}[f; s]$ is given by

$$\begin{aligned} \mathcal{M}[f; s] &= \int_0^\infty y^{s-1} f(y) dy, \qquad \alpha < \mathrm{Re}(s) < \beta, \\ f(y) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^{-s} \mathcal{M}[f; s] ds, \qquad \alpha < c < \beta, \end{aligned} \tag{32}$$

where α and β are determined by the condition that the first of the integrals in the above equations converges at $y = 0$ and $y = \infty$, respectively. The variable y contains all the dependence on the summation variable n and is treated as a continuous variable. By expressing $f(y)$ as an inverse Mellin transform, we can then perform the summation over n. The problem reduces then to finding the Mellin transform and its inverse of the remaining expression.

From Eq. (30), it is easy to see that $\alpha = 0$ and $\beta = 4 - d$. In terms of their Mellin transforms, $S_\pm$ can be expressed as

$$S_\pm = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \int_{c-i\infty}^{c+i\infty} \frac{1}{\omega_{n\pm}^s} \mathcal{M}[f; s] ds. \tag{33}$$

The sum over n can be explicitly evaluated, yielding

$$\sum_{n=0}^{\infty} \frac{1}{[n + |l| + 1 \pm (1-x)l]^s} = \zeta[s, |l| + 1 \pm (1-x)l], \tag{34}$$

where $\zeta(a, b)$ is the modified Riemann zeta function. To find out the Mellin transform of f, we observe that the integrand of the first of Eqs. (32), with f given by Eq. (30), can be thought of overall as an integral in $(s + d)$ dimensions of a function of the square of an $(s + d)$-dimensional vector

$$K^2 = \underbrace{\omega_{n\pm}^2}_{s-\mathrm{dim}} + \underbrace{k^2}_{d-\mathrm{dim}}. \tag{35}$$

Such integrals are well known [12] and the result, after compensating for the volume of the solid angle when extending the integral from d to $(s + d)$ dimensions, is

$$\mathcal{M}[f; s] = \frac{\mu^{3-d}}{(2\pi T)^s} \frac{\Gamma(s/2)}{2(4\pi)^{d/2}} \frac{\Gamma(2 - d/2 - s/2)}{\Gamma(2)} \times \frac{1}{[m^2 + x(1-x)(\omega_l^2 + p^2)]^{2-d/2-s/2}}, \tag{36}$$

where Γ is the gamma function. Combining the results in Eqs. (34) and (36) the explicit expression for $S_\pm$ is

$$\begin{aligned} S_\pm = {}& \mu^{3-d} \left(\frac{1}{2\pi i}\right) \left(\frac{1}{2(4\pi)^{d/2}}\right) \frac{1}{\Gamma(2)} \times [m^2 + x(1-x) \\ & \times (\omega_l^2 + p^2)]^{d/2-2} \times \int_{c-i\infty}^{c+i\infty} ds \zeta(s, |l| + 1 \\ & \pm (1-x)l) \Gamma(s/2) \times \Gamma\left(2 - \frac{d+s}{2}\right) \\ & \times \left[\frac{m^2 + x(1-x)(\omega_l^2 + p^2)}{(2\pi T)^2}\right]^{s/2}. \end{aligned} \tag{37}$$

In order to perform the integral over s in Eq. (37), we notice that it is necessary to know whether the term $[m^2 + x(1-x)(\omega_l^2 + p^2)]/(2\pi T)^2$ is larger or smaller than 1. For the present purposes where we work in the high temperature limit and want to explore the analytic properties of Π near the origin, we see that $[m^2 + x(1-x) \times (\omega_l^2 + p^2)]/(2\pi T)^2 < 1$. Notice that this assumption limits the range of values of the external index l to be $l = 0, \pm 1$. Taking $d = 3 - 2\epsilon$, $\epsilon \to 0^+$, we can choose c such that $1 < c < 1 + 2\epsilon$, in order to both comply with the upper bound requirement for the existence of the Mellin transform, Eq. (32), and to avoid the pole of ζ at $s = 1$. Therefore, the integration contour can be closed to the right by a half-circle at infinity. The only singularities within the integration contour are those of $\Gamma[2 - (3 - 2\epsilon + s)/2]$, namely, when $s = 1 + 2\epsilon + 2k$, $k = 0, 1, 2, \ldots$ and the integral over s in Eq. (37) can be computed by means of the Cauchy's residue theorem, yielding

$$\begin{aligned} S_\pm = {}& \lim_{\epsilon \to 0^+} \mu^{2\epsilon} \frac{(4\pi)^{-3/2+\epsilon}}{\Gamma(2)} (2\pi T)^{-(1+2\epsilon)} \times \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \\ & \times \zeta(1 + 2k + 2\epsilon, |l| + 1 \pm (1-x)l) \\ & \times \Gamma(1/2 + k + \epsilon) \left[\frac{m^2 + x(1-x)(\omega_l^2 + p^2)}{(2\pi T)^2}\right]^k. \end{aligned} \tag{38}$$

Notice that for $k > 0$, $\zeta(1 + 2k + 2\epsilon, |l| + 1 \pm (1 - x)l)$ has no singularities since $|l| + 1 \pm (1 - x)l \neq 0, -1, -2\ldots$ For $k = 0$, the singularity is regulated by ϵ. Also, the factor $[m^2 + x(1 - x)(\omega_l^2 + p^2)]^k$ is nonsingular. Therefore we can analytically continue the functions $S_\pm$ from discrete to arbitrary complex values $i\omega_l = 2\pi iTl \to p_0$. Upon this analytic continuation

$$(\omega_l^2 + p^2) \to -(p_0^2 - p^2), \qquad l \to -i\frac{p_0}{2\pi T}, \qquad |l| \to \frac{|p_0|}{2\pi T}. \tag{39}$$

We now explore the limiting behavior of Eq. (38) as the momentum components of the vector $P^\mu = (p_0, \mathbf{p})$ approach zero. Since the result depends on the way the limit is taken, again we restrict ourselves to real p_0 values, set $p_0 = \alpha p$, and take the limit $p \to 0$ for the argument of the ζ function,

$$\begin{aligned}\zeta[1 + 2(k + \epsilon), 1 + (|\alpha| \mp i(1 - x)\alpha)p/2\pi T] \\ \overset{p\to 0}{\longrightarrow} \zeta[1 + 2(k + \epsilon)] - (|\alpha| \mp i(1 - x)\alpha) \\ \times (1 + 2(k + \epsilon))\zeta[2 + 2(k + \epsilon)]\left(\frac{p}{2\pi T}\right).\end{aligned} \tag{40}$$

The integration over the Feynman parameter involves the computation of the integral

$$\begin{aligned}Z_\pm = \int_0^1 dx\bigg\{\zeta\left[1 + 2(k + \epsilon), 1 + (|\alpha| \mp i(1 - x)\alpha)\frac{p}{2\pi T}\right] \\ \times \left[\frac{m^2 - x(1 - x)(\alpha^2 - 1)p^2}{(2\pi T)^2}\right]^k\bigg\}.\end{aligned} \tag{41}$$

As we are interested in evaluating the result near the origin and in the high temperature limit, we make use of the expansion of ζ in Eq. (40) into Eq. (41) and evaluate for the first two terms in the series expansion of Eq. (38), namely $k = 0, 1$. Defining

$$Z_\pm \equiv \sum_{k=0}^{\infty} Z_\pm^k, \tag{42}$$

we get

$$\begin{aligned}Z_\pm^0 &= \zeta[1 + 2\epsilon] - (|\alpha| \mp i\alpha/2)(1 + 2\epsilon)\zeta[2 + 2\epsilon] \\ &\quad \times \left(\frac{p}{2\pi T}\right) + \mathcal{O}(p^2), \\ Z_\pm^1 &= \zeta[3 + 2\epsilon]\left(\frac{m}{2\pi T}\right)^2 - (|\alpha| \mp i\alpha/2)(3 + 2\epsilon) \\ &\quad \times \zeta[4 + 2\epsilon]\left(\frac{m}{2\pi T}\right)^2\left(\frac{p}{2\pi T}\right) + \mathcal{O}(p^2).\end{aligned} \tag{43}$$

Notice that in the limit $p \to 0$, the result coming from the sums $S_\pm$ is independent of α. Combining Eqs. (38), (42), and (43) into Eq. (28), we obtain

$$\begin{aligned}\Pi_1(0, p \to 0) = \frac{\lambda^2}{4(2\pi)^2} \times \bigg[\frac{1}{2\epsilon} + \ln\left(\frac{\mu}{2\sqrt{\pi}T}\right) \\ + \frac{\gamma_E}{2} - \frac{m^2\zeta(3)}{8\pi^2T^2}\bigg].\end{aligned} \tag{44}$$

B. Mixed sign frequencies

We now turn to the computation of the third term arising from the separation of the sum over Matsubara frequencies in Eq. (26). We define

$$\Pi_2(p_{0l}, p) = \frac{\lambda^2}{2}T\sum_{n=-|l|}^{+|l|}\int\frac{d^3k}{(2\pi)^3}\int_0^1\frac{dx}{[(\omega_n - x\omega_l)^2 + y]^2}, \tag{45}$$

where $y = (\mathbf{k} - x\mathbf{p})^2 - x(1 - x)(p_{0l}^2 - \mathbf{p}^2) + m^2$. In order to perform the sum over Matsubara frequencies, it is convenient to note that the degree in the denominator of Eq. (45) can be reduced since

$$\begin{aligned}\frac{1}{[(\omega_n - x\omega_l)^2 + y]^2} &= -\frac{\partial}{\partial m^2}\frac{1}{[(\omega_n - x\omega_l)^2 + y]} \\ &= -\frac{\partial}{\partial m^2}\sum_{s=\pm 1}\frac{is}{2y^{1/2}} \\ &\quad \times \frac{1}{[(\omega_n - x\omega_l) + isy^{1/2}]},\end{aligned} \tag{46}$$

where in the last line we have resorted to partial fractioning. With this reduction, the sum over Matsubara frequencies can be easily performed and we get

$$\begin{aligned}\Pi_2(p_{0l}, p) = -\frac{\lambda^2}{2}\int\frac{d^3k}{(2\pi)^3}\int_0^1 dx \times \frac{\partial}{\partial m^2}\sum_{s=\pm 1}\frac{1}{4y^{1/2}} \\ \times \bigg\{\coth\left(\frac{sxp_{0l}}{2T} + \frac{y^{1/2}}{2T}\right) \\ + \frac{is}{\pi}\psi\left(1 + |l| + \frac{is}{\pi}\left[\frac{sxp_{0l}}{2T} + \frac{y^{1/2}}{2T}\right]\right) \\ - \frac{is}{\pi}\psi\left(1 + |l| - \frac{is}{\pi}\left[\frac{sxp_{0l}}{2T} + \frac{y^{1/2}}{2T}\right]\right)\bigg\},\end{aligned} \tag{47}$$

where ψ is the digamma function. Before proceeding further, it is convenient to note that given a function F with argument $[sp_{0l}x + y(x, m)^{1/2}]/2T$, the identity

$$\begin{aligned}\frac{\partial}{\partial m^2}\left[\frac{F(\frac{sp_{0l}x}{2T} + \frac{y(x,m)^{1/2}}{2T})}{y(x, m)^{1/2}}\right] \\ = \frac{\partial}{\partial x}\left[\frac{F(\frac{sp_{0l}x}{2T} + \frac{y(x,m)^{1/2}}{2T})}{2y(x, m)(sp_{0l} + \frac{1}{2y(x,m)^{1/2}}\frac{\partial y(x,m)}{\partial x})}\right]\end{aligned} \tag{48}$$

is satisfied. Using Eq. (48) into Eq. (47), the integration over the Feynman parameter becomes trivial and we get, after performing the angular integration

$$\Pi_2(p_{0l}, p) = -\frac{\lambda^2}{2}\int_0^\infty \frac{k^2 dk}{(2\pi)^2}\sum_{s=-\pm 1} \times \frac{1}{8E_k pk}\ln\left(\frac{p_{0l}^2 - p^2 - 2sp_{0l}E_k + 2kp}{p_{0l}^2 - p^2 - 2sp_{0l}E_k - 2kp}\right)\times\Bigg[\coth\left(\frac{E_k}{2T}\right) + \coth\left(\frac{E_k}{2T} - \frac{sp_{0l}}{2T}\right) + \frac{is}{\pi}\psi\left(1 + |l| + \frac{is}{\pi}\left[\frac{E_k}{2T} - \frac{sp_{0l}}{2T}\right]\right) - \frac{is}{\pi}\psi\left(1 + |l| - \frac{is}{\pi}\left[\frac{E_k}{2T} - \frac{sp_{0l}}{2T}\right]\right) + \frac{is}{\pi}\psi\left(1 + |l| + \frac{is}{\pi}\frac{E_k}{2T}\right) - \frac{is}{\pi}\psi\left(1 + |l| - \frac{is}{\pi}\frac{E_k}{2T}\right)\Bigg]. \tag{49}$$

Notice that all along this part of the calculation, we have distinguished between l and p_{0l} which are in principle related through $p_{0l} = 2i\pi lT$. The reason is that the first term arises as a consequence of our treating this partial sum over Matsubara frequencies as having $|l|$ in the limits of the summation index, whereas the second is the discrete value taken by the external energy. If we now use that $p_{0l} = 2i\pi lT$ we notice, in particular, that the periodicity in Eq. (49), coming from the argument of the second coth, gets accounted for. However, there is an extra term that exhibits periodicity in p_{0l} and that is not that evident from the expression in Eq. (49). This comes from the difference of the functions ψ that have p_{0l} in their argument. This periodicity can be evidenced by resorting to the identity

$$\begin{aligned}\psi(x + iy) - \psi(x - iy) &= \sum_{k=0}^{\infty}\frac{2iy}{y^2 + (x + k)^2}\\ &= i\left(\pi\coth[\pi y] - \frac{1}{y}\right)\\ &\quad - \sum_{k=1}^{|l|}\frac{2iy}{y^2 + k^2}.\end{aligned} \tag{50}$$

After this simplification which allows to account for the periodicity in p_{0l}, the equation can be analytically continued from discrete to arbitrary complex values of $p_{0l} \to p_0$, since Eq. (49) is free from singularities. As before, we explore the limiting behavior of Eq. (49) as the momentum components of the vector $P^\mu = (p_0, \mathbf{p})$ approach zero. Again, we restrict ourselves to real values of p_0, set $p_0 = \alpha p$, and take the limit $p \to 0$ and we get

$$\Pi_2(\alpha p, p) \overset{p\to 0}{=} \frac{\lambda^2}{2(2\pi)^2}T\int_0^\infty dk\frac{1}{E_k^2} = \frac{\lambda^2}{2(2\pi)^2}T\left(\frac{\pi}{2m}\right), \tag{51}$$

which is independent of α. Upon combining the results of Eqs. (44) and (51), we obtain

$$\Pi(0, p\to 0) = \frac{\lambda^2}{4(2\pi)^2}\left[\frac{\pi T}{m} + \frac{1}{2\epsilon} + \ln\left(\frac{\mu}{2\sqrt{\pi}T}\right) + \frac{\gamma_E}{2} - \frac{m^2\zeta(3)}{8\pi^2T^2}\right]. \tag{52}$$

C. α dependence

We now proceed to discuss the α dependence of the result since, as we have seen in Secs. IV A and IV B, this dependence is absent in the terms calculated so far.

As is shown in Ref. [3], the usual Feynman parametrization formula at finite temperature has to be corrected when the sum of denominators can vanish. The correct expression is in this case

$$\frac{1}{D_1D_2} = \int_0^1\frac{dx}{[(1-x)D_2 + xD_1]^2} + 4\pi i\frac{\delta(D_1 + D_2)}{D_1 - D_2}, \tag{53}$$

where the first term is taken as the principal value. Since within the MST, the Matsubara frequencies in the sum are first treated as continuous upon the Mellin transform, and when an analytic continuation is required, there is the possibility that the second term contributes. We proceed to show that this is indeed the case and that this last term carries the full α dependence of the result. In Appendix B we show that the term here computed coincides with the one obtained by using the standard procedure.

We first look at the contribution from the positive sign frequencies which can be written as

$$\Pi_\alpha^{n>0} = \left(\frac{\lambda^2}{2}\right)4\pi iT\sum_{n=1}^{\infty}\int\frac{d^3k}{(2\pi)^3} \times\frac{\delta[(\omega_n - \omega_l)^2 + E_{k-p}^2 + \omega_n^2 + E_p^2]}{(\omega_n - \omega_l)^2 + E_{k-p}^2 - \omega_n^2 - E_p^2}. \tag{54}$$

Upon the change of variable

$$\mathbf{k} \to \mathbf{k} + \mathbf{p}/2, \tag{55}$$

and the introduction of the Mellin transform and its inverse, Eq. (54) can be written as

$$\Pi_\alpha^{n>0} = \left(\frac{\lambda^2}{2}\right)4\pi iT\sum_{n=1}^{\infty}\int_{c-i\infty}^{c+i\infty}\frac{ds}{\omega_n^s}(-i)^s\int_0^\infty du u^{s-1} \times\int\frac{d^3k}{(2\pi)^3} \times\frac{\delta[(-iu - \omega_l)^2 - u^2 + 2E_k^2 + p^2/2]}{(-iu - \omega_l)^2 + u^2 - 2\mathbf{p}\cdot\mathbf{k}}, \tag{56}$$

where the Mellin transform has been taken from the discrete Matsubara frequency $\omega_n \to -iu$, in anticipation for the taking of the analytical continuation $i\omega_l \to p_0$. Also, for convergence of the integral, $0 < c < 3$. Taking the analytic continuation $i\omega_l \to p_0$, setting $p_0 = \alpha p$, the integral over k can be performed straightforward. In the limit $p \to 0$ we get

$$\int \frac{d^3k}{(2\pi)^3} \frac{\delta[-(u-\alpha p)^2 - u^2 + 2E_k^2 + p^2/2]}{-(u-\alpha p)^2 + u^2 - 2\mathbf{p}\cdot\mathbf{k}} = \frac{1}{(2\pi)^2 8p}\left\{\ln\left[\frac{\sqrt{u^2-m^2}+\alpha u}{-\sqrt{u^2-m^2}+\alpha u}\right] - \frac{\alpha^2 m^2 p}{\sqrt{u^2-m^2}(m^2-u^2(1-\alpha^2))}\right\}. \quad (57)$$

It is easy to show that the potentially dangerous first term in the above equation for $p \to 0$ is canceled from a similar contribution arising from the sum over negative Matsubara frequencies. We thus just concentrate on the second term of Eq. (57). We give explicit results for the case $\alpha > 1$; the case $\alpha < 1$ can be worked out by resorting to the transformation formulas for the hypergeometric function. Here we just point out that when $0 < \alpha < 1$, the high temperature expansion contains an imaginary part. Integration over the variable u gives

$$-\int_0^\infty (-i)^s du u^{s-1} \frac{\alpha^2 m^2}{\sqrt{u^2-m^2}[m^2-u^2(1-\alpha^2)]} = (-i)^s \frac{m^{s-1}\sqrt{\pi}}{2}\frac{\alpha^2}{(1-\alpha^2)} \times \left\{\frac{\Gamma(\frac{3-s}{2})}{\Gamma(\frac{4-s}{2})}\, {}_2F_1\left(1,\frac{3-s}{2},\frac{4-s}{2},\frac{1}{1-\alpha^2}\right) - i\frac{\Gamma(\frac{s-2}{2})}{\Gamma(\frac{s-1}{2})}\, {}_2F_1\left(1,\frac{3-s}{2},\frac{4-s}{2},\frac{1}{1-\alpha^2}\right) + i\frac{\Gamma(\frac{s}{2})\Gamma(\frac{2-s}{2})}{\Gamma(\frac{1}{2})}\frac{(-1)^{s/2}}{(1-\alpha^2)^{s/2-1}} \times {}_2F_1\left(\frac{s}{2},\frac{1}{2},\frac{s}{2},\frac{1}{1-\alpha^2}\right)\right\}, \quad (58)$$

where ${}_2F_1$ is a hypergeometric function. The remaining s-dependent factor comes from the summation over the Matsubara frequencies in Eq. (56) which yields

$$\sum_{n=1}^{\infty} \frac{1}{\omega_n^s} = \frac{1}{(2\pi T)^s}\zeta(s). \quad (59)$$

Therefore, the integral over s involves the terms

$$L_1 = \frac{T}{m}\frac{\alpha^2\sqrt{\pi}}{2(2\pi i)}\int_{c-i\infty}^{c+i\infty} ds\left(\frac{-im}{2\pi T}\right)^s \frac{\zeta(s)}{(1-\alpha^2)}\frac{\Gamma(\frac{3-s}{2})}{\Gamma(\frac{4-s}{2})} \times {}_2F_1\left(1,\frac{3-s}{2},\frac{4-s}{2},\frac{1}{1-\alpha^2}\right),$$

$$L_2 = i\frac{T}{m}\frac{\alpha^2\sqrt{\pi}}{2(2\pi i)}\int_{c-i\infty}^{c+i\infty} ds\left(\frac{-im}{2\pi T}\right)^s \frac{(-1)^{s/2}\zeta(s)}{(1-\alpha^2)^{s/2}} \times \frac{\Gamma(\frac{s}{2})\Gamma(\frac{2-s}{2})}{\Gamma(\frac{1}{2})}\sqrt{\frac{\alpha^2-1}{\alpha^2}}, \quad (60)$$

$$L_3 = i\frac{T}{m}\frac{\alpha^2\sqrt{\pi}}{2(2\pi i)}\int_{c-i\infty}^{c+i\infty} ds\left(\frac{-im}{2\pi T}\right)^s \frac{\zeta(s)}{(1-\alpha^2)}\frac{\Gamma(\frac{s-2}{2})}{\Gamma(\frac{s-1}{2})} \times {}_2F_1\left(1,\frac{3-s}{2},\frac{4-s}{2},\frac{1}{1-\alpha^2}\right),$$

where we used that

$${}_2F_1\left(\frac{s}{2},\frac{1}{2},\frac{s}{2},\frac{1}{1-\alpha^2}\right) = \sqrt{\frac{\alpha^2-1}{\alpha^2}}. \quad (61)$$

In order to compute $L_{1,2}$ we can close the contour of integration by a half-circle at $\mathrm{Re}(s) \to \infty$ since the convergence of the integrals is controlled by the ratio m/T which is taken to be less than one. The integral over this half-circle vanishes. $L_{1,2}$ are given by the residue of the poles of $\Gamma[(3-s)/2]$ and $\Gamma[(2-s)/2]$, which are located at $s = 2k+3$, and at $s = 2(k+1)$, $k = 0, 1, 2\ldots$, respectively. Choosing $c > 1$ we can avoid the pole of $\zeta(s)$ at $s = 1$. Working with this choice we get

$$L_1 = -\frac{i}{2\sqrt{\pi}}\left(\frac{\alpha^2}{\alpha^2-1}\right)\sum_{k=0}^{\infty}\frac{1}{k!}\left(\frac{m}{2\pi T}\right)^{2k+2} \times \zeta(2k+3) \times \frac{{}_2F_1(1,-k,1/2-k,\frac{1}{1-\alpha^2})}{\Gamma(1/2-k)},$$

$$L_2 = \frac{i}{2\pi}\left(\frac{\alpha^2}{\alpha^2-1}\right)^{1/2}\sum_{k=0}^{\infty}\frac{1}{(\alpha^2-1)^k}\left(\frac{m}{2\pi T}\right)^{2k+1} \times \zeta[2(k+1)]. \quad (62)$$

For L_3 we notice that the contour of integration can be closed by a half-circle at $\mathrm{Re}(s) \to -\infty$ since, as we will show, the contribution will be proportional to T/m which we take to be larger than 1. The integral over this half-circle vanishes. The integral is given by the residue of the poles of $\Gamma[(s-2)/2]$ which are located at $s=-2k$, $k=0,1,2\ldots$. However, since $\zeta(-2k)$ vanishes for $k=1,2\ldots$, the only pole that contributes is the one at $s=0$. We thus get

$$L_3 = i\left(\frac{T}{m}\right)\left(\frac{\alpha}{2}\right)[\alpha - \sqrt{\alpha^2-1}]. \tag{63}$$

By changing $\omega_n \to -\omega_n$ in Eq. (54), or equivalently, $u \to -u$ in Eqs. (56) and (57), it is straightforward to show that for the modes with $n<0$ the contribution from the logarithmic term in Eq. (57) cancels. Therefore, the contribution from the modes with $n \neq 0$ is just twice the above discussed contribution from the modes with $n>0$. The remaining term to compute is the one coming from the mode with $n=0$. It is easy to show that the delta function in this case does not have support and thus this contribution vanishes.

Writing all together, the final result expressed as an explicit power series in the ratio $m/2\pi T$ can be written as

$$\begin{aligned}\Pi_\alpha = {}&\left(\frac{\lambda^2}{8\pi^2}\right)\Bigg\{\left(\frac{\pi T}{m}\right)\alpha(\sqrt{\alpha^2-1}-\alpha)\\ &+\sqrt{\frac{\alpha^2}{\alpha^2-1}}\Bigg[-\left(\frac{m}{2\pi T}\right)\zeta(2)-\frac{1}{(\alpha^2-1)}\left(\frac{m}{2\pi T}\right)^3\zeta(4)\\ &-\frac{1}{(\alpha^2-1)^2}\left(\frac{m}{2\pi T}\right)^5\zeta(6)-\ldots\Bigg]\\ &+\left(\frac{\alpha^2}{\alpha^2-1}\right)\Bigg[\frac{1}{2}\left(\frac{m}{2\pi T}\right)^2\zeta(3)+\frac{(3-\alpha^2)}{4(\alpha^2-1)}\left(\frac{m}{2\pi T}\right)^4\zeta(5)\\ &+\frac{(3\alpha^4-10\alpha^2+15)}{16(\alpha^2-1)^2}\left(\frac{m}{2\pi T}\right)^6\zeta(7)+\ldots\Bigg]\Bigg\}.\end{aligned} \tag{64}$$

As we show in Appendix B, this result coincides with the one computed with the standard method.

D. Infrared limit

As can be seen from Eq. (57) [which is written before computing the integral over u for which we have assumed $\alpha>1$] in the infrared limit ($\alpha=0$), the self-energy does not depend on α. In order to compare the result in Eq. (52) with Eq. (25), which is computed in the infrared limit, we need to subtract from Eq. (52) the vacuum contribution, since the MST does not explicitly separate this from the thermal contribution. From Eqs. (10) and (11) the vacuum contribution to Π in the infrared limit is

$$\Pi^{\rm vac}(0, p\to 0) = \frac{\lambda^2}{8(2\pi)^2}\mu^{1-d}\int\frac{d^dk}{E_k}, \tag{65}$$

where we have extended the integral to d dimensions. Notice that in this case, in order to make contact with Eq. (10), the limit $d\to 1$ will be eventually taken. Explicit evaluation of Eq. (65) yields

$$\Pi^{\rm vac}(0, p\to 0) = \frac{\lambda^2}{8(2\pi)^2}\left\{\frac{1}{\epsilon}+\ln\left(\frac{\mu^2}{\pi m^2}\right)-\gamma_E\right\}. \tag{66}$$

Therefore, the thermal contribution is obtained by subtracting Eq. (66) from Eq. (52) and this is given by

$$\begin{aligned}\Pi^T(0,p\to 0) &= \Pi(0,p\to 0) - \Pi^{\rm vac}(0,p\to 0)\\ &= \frac{\lambda^2}{4(2\pi)^2}\left\{\frac{\pi T}{m}+\ln\left(\frac{m}{2T}\right)+\gamma_E-\frac{m^2\zeta(3)}{8\pi^2T^2}\right\},\end{aligned} \tag{67}$$

which coincides with Eq. (25).

At this point it is important to underline that the ingredient making possible that the standard procedure reproduced in Sec. III and the MST described in Sec. IV lead to the same result is the implementation of the periodicity of the expressions—by appealing to the fact that the external frequency is discrete—*before* the analytical continuation to arbitrary complex values of the external frequency is taken. Also, when the Feynman parametrization is used and afterwards the periodicity implemented, the procedure leads to the well-known result, *provided the integration domain for the Feynman parameter x is $x\in[0,1]$*. Nevertheless, as is discussed in Refs. [1,3], before the analytic continuation, the integrand is symmetric about $x=1/2$ and thus it is seemingly possible to get the Feynman integral as twice the result when $x\in[0,1/2]$. We show in Appendix C that this introduces the extra complication of a spurious end point singularity and thus leads to the well-known mishaps with the use of the Feynman parametrization in the ITF.

We also point out that, as mentioned in Ref. [1], for practical purposes, the result in Eq. (67), that is to say, in the infrared limit, can be directly obtained from Eq. (1) by setting $p_{0l}, p=0$ right from the start. In this case, in the context of the MST, the mixed frequency sum in Eq. (45) collapses to the computation of the contribution of the $n=0$ Matsubara frequency and the definite sign frequency sums in Eq. (27) can be condensed into a single sum over positive definite frequencies. For calculations involving propagators raised to higher powers, where one seeks an answer in the infrared limit, this simplification makes the MST a rather convenient technique, particularly in the high temperature limit $T\gg m$ since it gives the final answer in terms of a series in m/T. We proceed to show that this is the case when computing the self-energy of a scalar particle interacting with charged scalar particles in the presence of an external magnetic field.

V. APPLICATION: SCALAR SELF-ENERGY IN A MAGNETIC FIELD

In the standard model, after symmetry breaking, there is an interaction term of the physical Higgs ϕ with the charged ones $\varphi^\pm$ of the form $\lambda\varphi^\dagger\varphi\phi$. In the presence of an external magnetic field, the propagators for the charged modes are affected, becoming, in the weak field limit and at finite temperature [4]

$$D^B(\omega_n, k) = \frac{1}{(\omega_n^2 + E_k^2)} \times \left[1 - \frac{(eB)^2}{(\omega_n^2 + E_k^2)^2} + \frac{2(eB)^2 k_\perp^2}{(\omega_n^2 + E_k^2)^3} \right], \tag{68}$$

where eB is the coupling of the charged scalars to the external magnetic field. One of the diagrams contributing to the physical Higgs self-energy at one loop, depicted in Fig. 1, is given explicitly by

$$\Pi^B(\omega_l, p) = \lambda^2 T \sum_n \int \frac{d^3k}{(2\pi)^3} \times D^B(\omega_n, k) \times D^B(\omega_n - \omega_l, k - p). \tag{69}$$

To lowest order in eB, this self-energy becomes

$$\Pi^B(\omega_l, p) = \lambda^2 T \sum_n \int \frac{d^3k}{(2\pi)^3} \{ I_{11} - (eB)^2 [I_{31} + I_{13} - 2k_\perp^2 I_{41} - 2(k-p)_\perp^2 I_{14}] \}, \tag{70}$$

where we define

$$I_{nm} = \frac{1}{[\omega_n^2 + E_k^2]^n [(\omega_n - \omega_l)^2 + E_{k-p}^2]^m}. \tag{71}$$

When interested in describing the infrared properties of this self-energy, we look at the infrared limit which, as previously discussed can be obtained in a straightforward manner by setting $p_{0l}, p = 0$. In doing so, we get

$$\Pi^B(0, p \to 0) = \lambda^2 T \sum_n \int \frac{d^3k}{(2\pi)^3} \{ I_{20} - 2(eB)^2 \times [I_{40} - 2k_\perp^2 I_{50}] \}. \tag{72}$$

Notice that the functions I_{n0} are all related through

$$I_{n0} = \frac{(-1)^{n-1}}{(n-1)!} \frac{\partial^{n-1}}{\partial (m^2)^{n-1}} I_{10}. \tag{73}$$

The MST technique discussed in Sec. IV can be generalized to the computation of I_{10} and from Eq. (73) to all the expressions involved in Eq. (72). The interested reader is referred to Refs. [4] for details and the result at high temperature and to lowest order in the magnetic field strength is

$$\Pi^B(0, p \to 0) = \lambda^2 \left\{ \frac{2}{(4\pi)^2} \left[\frac{1}{2\epsilon} + \gamma_E + \ln\left(\frac{\mu}{4\pi T} \right) \right] + \frac{T}{8\pi m} - \frac{(eB)^2}{64} \left(\frac{T}{\pi m^5} + \frac{1}{T^4} \frac{\zeta(5)}{16\pi^6} \right) \right\}, \tag{74}$$

where we have not subtracted the vacuum contribution.

We emphasize that the MST is suited to obtain an expression such as Eq. (74), namely, an expansion at high temperature starting from the original expression for the self-energy in the ITF. This is so since the sum over Matsubara frequencies and the integration over the spatial components of the momentum can be carried out together in a single step, in a straightforward manner, right from the very beginning.

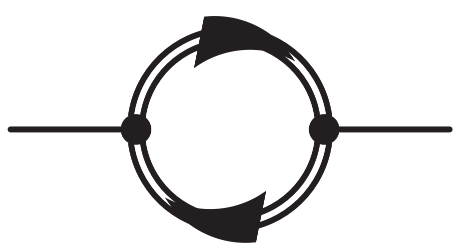

FIG. 1. One-loop Feynman diagram contributing to the self-energy of the physical Higgs, represented by the single line, interacting with the charged Higgs components, represented by the double lines, in the presence of an external weak magnetic field.

VI. SUMMARY AND CONCLUSIONS

In this work we have shown that the MST is a well-defined method to compute Feynman integrals in the ITF of finite temperature field theory. The MST is particularly useful to find the explicit result as a series in a small parameter, for instance, the ratio m/T at high temperature and in the infrared limit. The method calls for the use of the Feynman parametrization which in the past has been linked to problems in the ITF when the analytical continuation from discrete Matsubara frequencies to arbitrary complex values is performed. We have also shown that these problems are not endemic to the Feynman parametrization and have traced back their origin to (a) not implementing the periodicity of the expressions before analytical continuation and to (b) changing the domain of integration in the Feynman parameter from $x \in [0, 1]$ to $x \in [0, 1/2]$ which introduces a spurious end point singularity. We have explicitly shown that when using the MST and the calculation is properly carried out, it leads to the same result obtained by means of the standard technique in the ITF in the infrared limit. In particular, we have shown the need to take into account the correction term to the usual Feynman formula, in order to consider the case when the sum of denominators vanishes, and that this term is the source of the full α dependence of the result, in agreement with Ref. [3]. The usefulness of the method is illustrated by the computation of the one-loop self-energy in the standard model of the physical Higgs field interacting with the charged components in the presence of a weak external magnetic field, in the infrared limit.

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APPENDIX A: EVALUATION OF $\Pi(p_{0l}, p)$ WITHOUT IMPLEMENTING THE PERIODICITY

In this appendix, we aim at furthering the argument on the importance of having implemented the periodicity in the function coth in Eq. (5), to achieve the proper analytic continuation to arbitrary complex values of p_{0l} in the evaluation of $\Pi(p_{0l}, \mathbf{p})$. We show here that when p_{0l} *is not* taken initially as i times a discrete Matsubara frequency, then, when $p_{0l} \to p_0$, where p_0 is a continuous arbitrary complex number, one is bound to obtain a spurious term which needs to be canceled precisely by the addition of the quantity π_δ of Ref. [3].

Without using that p_{0l} takes on discrete integer values, instead of arriving at that equation we would have

$$\Pi(p_{0l}, p) = -\frac{\lambda^2}{2}\int \frac{d^3k}{(2\pi)^3} \times \sum_{r,s=\pm1}\left(\frac{rs}{8E_kE_{k-p}}\right) \times \frac{1}{sE_k - (rE_{k-p} + p_{0l})} \times \left[\coth\left(\frac{sE_k}{2T}\right) - \coth\left(\frac{rE_{k-p} + p_{0l}}{2T}\right)\right], \quad (A1)$$

which can be rewritten as

$$\Pi(p_{0l}, p) = -\frac{\lambda^2}{2}\int \frac{d^3k}{(2\pi)^3} \times \sum_{r,s=\pm1}\left(\frac{rs}{8E_kE_{k-p}}\right) \times \frac{1}{sE_k - (rE_{k-p} + p_{0l})} \times \left[\coth\left(\frac{sE_k}{2T}\right) - \coth\left(\frac{rE_{k-p}}{2T}\right) + \frac{\mathrm{csch}^2(\frac{rE_{k-p}}{2T})}{\coth(\frac{rE_{k-p}}{2T}) + \coth(\frac{p_{0l}}{2T})}\right], \quad (A2)$$

where we used $\coth(a+b) = \coth(a) - \mathrm{csch}^2(a) \times (\coth(a) + \coth(b))^{-1}$ to separate the dependence on E_{k-p} and p_{0l} in the second hyperbolic function. Note that, compared to what we had in Eq. (5), we now have a third term as a result of not fully exploiting the periodic properties of the functions involved.

We now concentrate in the last term in Eq. (A2) and show that, according to Ref. [3] and in the limit when $p \to 0$, this corresponds to minus the function one needs to add to correct the result. Let us then call $\Pi_X(p_{0l}, p)$ the contribution from the aforementioned term

$$\Pi_X(p_{0l}, p) = -\frac{\lambda^2}{2}\int \frac{d^3k}{(2\pi)^3} \sum_{r,s=\pm1}\left(\frac{rs}{8E_kE_{k-p}}\right) \times \frac{1}{sE_k - (rE_{k-p} + p_{0l})} \times \left[\frac{\mathrm{csch}^2(\frac{rE_{k-p}}{2T})}{\coth(\frac{rE_{k-p}}{2T}) + \coth(\frac{p_{0l}}{2T})}\right], \quad (A3)$$

where, upon summing over s we have

$$\Pi_X(p_{0l}, p) = -\frac{\lambda^2}{2}\int \frac{d^3k}{(2\pi)^3} \sum_{r=\pm1}\left(\frac{-r}{4E_{k-p}}\right) \times \frac{1}{(rE_{k-p} + p_{0l})^2 - E_k^2} \times \left[\frac{\mathrm{csch}^2(\frac{rE_{k-p}}{2T})}{\coth(\frac{rE_{k-p}}{2T}) + \coth(\frac{p_{0l}}{2T})}\right]. \quad (A4)$$

In order to integrate out the angular contribution, we can perform the momentum shift $\mathbf{k} - \mathbf{p} \to \mathbf{k}$ so that all the angular dependence will be in the coefficient rather than in the hyperbolic functions. This allows for a straightforward integration and we arrive at

$$\Pi_X(p_{0l}, p) = -\frac{\lambda^2}{2}\int_0^\infty \frac{dk}{(2\pi)^2} \sum_{r=\pm1}\left(\frac{-rk^2}{4E_k}\right) \times \left[\frac{\mathrm{csch}^2(\frac{rE_k}{2T})}{\coth(\frac{rE_k}{2T}) + \coth(\frac{p_{0l}}{2T})}\right] \times \frac{1}{2kp} \times \ln\left(\frac{p_{0l}^2 - p^2 + 2rE_kp_{0l} - 2kp}{p_{0l}^2 - p^2 + 2rE_kp_{0l} + 2kp}\right). \quad (A5)$$

We now proceed as in the main body of the paper after Eq. (7). We take the analytical continuation in p_{0l} from discrete imaginary values to arbitrary complex ones $p_{0l} \to p_0$. Since the result depends on how the limit is explored, we first set $p_0 = \alpha p$. To analyze the behavior near the origin, we expand the function coth and the logarithm around $p = 0$ and, up to linear terms, we obtain

$$\Pi_X(\alpha p, p) \overset{p\to0}{=} -\frac{\lambda^2}{2}\int_0^\infty \frac{dk}{(2\pi)^2} \sum_{r=\pm1}\left(\frac{-rk}{8E_kp}\right) \times \left[\frac{\alpha p}{2T}\mathrm{csch}^2\left(\frac{rE_k}{2T}\right)\right] \times \ln\left(1 - \frac{2k}{k + r\alpha E_k}\right), \quad (A6)$$

which, after summing over r gives

$$\Pi_X(\alpha p, p) \overset{p\to0}{=} -\frac{\lambda^2}{2}\int_0^\infty \frac{dk}{(2\pi)^2}\left[\frac{\alpha k}{16E_kT}\mathrm{csch}^2\left(\frac{E_k}{2T}\right)\right] \times \ln\left(\frac{k - \alpha E_k}{k + \alpha E_k}\right)^2. \quad (A7)$$

Now, just as we did in Sec. III, we are interested in having an explicit functional dependence on α of Π_X. We can then easily extract the thermal contributions thereby knowing how Π_X modifies Π, as was discussed in Eq. (12). For this purpose, it is convenient to note that the term in the square brackets of Eq. (A7) can be written in terms of a partial derivative ($\partial_k \coth E_k = kE_k^{-1}\mathrm{csch}^2 E_k$), so that we can complete a total derivative through integration by parts, to have

$$\Pi_X(\alpha p, p) \overset{p\to 0}{=} -\frac{\lambda^2}{2}\int_0^\infty \frac{dk}{(2\pi)^2}\left\{-\frac{\alpha}{8}\times\frac{\partial}{\partial k}\left[\coth\left(\frac{E_k}{2T}\right)\times\ln\left(\frac{k-\alpha E_k}{k+\alpha E_k}\right)^2\right]+\frac{\alpha^2(E_k^2-k^2)}{2E_k(k^2-\alpha^2E_k^2)}\times\coth\left(\frac{E_k}{2T}\right)\right\}. \quad \text{(A8)}$$

Finally we can separate the vacuum and the thermal contributions using the identity in Eq. (11), so that the thermal part is

$$\Pi_X^T(\alpha p, p) \overset{p\to 0}{=} -\frac{\lambda^2}{2(2\pi)^2}\int_0^\infty dk\frac{n(E_k)}{E_k}\frac{\alpha^2 m^2}{(k^2-\alpha^2E_k^2)}. \quad \text{(A9)}$$

The function Π_X^T in Eq. (A9) is precisely $\lim_{p\to 0}\pi_\delta(\alpha p, p)$ found in Eq. (30) of Ref. [3], *but with the opposite sign*. We can see that in the event of not implementing the periodicity, as we have analyzed in this appendix, inevitably we will end up with a contribution stemming from the extra term Π_X. The situation is corrected, as noted in Ref. [3], if one adds a function that behaves just as π_δ in the limit considered. This turns out to be an important observation, since we are presenting evidence that neglecting the implementation of the periodicity in the external frequency is linked to the need of such correcting function. Further developments on this argument are presented in the rest of this work.

APPENDIX B: EVALUATION OF $\Pi^T(\alpha p, p)$ FOR $p \to 0$ AND ARBITRARY α

We start from Eq. (12) rewriting it as

$$\Pi^T(\alpha p, p) = \frac{\lambda^2}{2(2\pi)^2}\int_0^\infty k^2 dk\frac{n(E_k)}{E_k}\frac{1}{E_k^2-\alpha'^2m^2}, \quad \text{(B1)}$$

where $\alpha'^2 = \frac{1}{1-\alpha^2}$. We follow again Ref. [10] and use the identity in Eq. (15) into Eq. (B1), that is

$$\Pi^T(\alpha p, p) = \frac{\lambda^2}{2(2\pi)^2}\frac{\mu^{3-d}}{4\pi}\int d^dk\frac{1}{E_k^2-\alpha'^2m^2}\times\left\{-\frac{1}{2E_k}+T\sum_{n=-\infty}^{\infty}\frac{1}{(E_k)^2+(2\pi nT)^2}\right\}, \quad \text{(B2)}$$

where we have written the integral in d dimensions. The first structure in Eq. (B2) is

$$J_1 = -\mu^{3-d}\frac{\lambda^2}{2(2\pi)^2}\frac{1}{4\pi}\int d^dk\frac{1}{2E_k}\frac{1}{E_k^2-\alpha'^2m^2}. \quad \text{(B3)}$$

Carrying out the angular integration and upon the change of variable $z = \frac{m^2}{k^2+m^2}$, we get

$$J_1 = -\mu^{3-d}m^{d-3}\frac{\lambda^2}{2(2\pi)^2}\frac{1}{8\pi}\frac{\pi^{d/2}}{\Gamma(\frac{d}{2})}\int_0^1 dz(1-z)^{(d/2)-1}\times z^{(1-d)/2}(1-\alpha'^2z)^{-1}. \quad \text{(B4)}$$

Using the identity

$$_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 dt(1-t)^{c-b-1}\times t^{b-1}(1-zt)^{-a}, \quad \text{(B5)}$$

where $_2F_1$ is the hypergeometric function, we get

$$J_1 = -\mu^{3-d}m^{d-3}\frac{\lambda^2}{2(2\pi)^2}\frac{\pi^{d/2}}{8\pi}\frac{\Gamma(\frac{3-d}{2})}{\Gamma(\frac{3}{2})}\times{}_2F_1\left(1,\frac{3-d}{2},\frac{3}{2};\alpha'^2\right). \quad \text{(B6)}$$

For the second structure in Eq. (B2), a similar procedure leads to

$$J_2 = \mu^{3-d}\frac{\lambda^2}{2(2\pi)^2}\frac{\pi^{d/2}}{4\pi}\Gamma\left(2-\frac{d}{2}\right)\times T\sum_{n=-\infty}^{\infty}(m^2+\omega_n^2)^{(d/2)-2}\times{}_2F_1\left(1,2-\frac{d}{2},2;\frac{\omega_n^2+\alpha'^2m^2}{\omega_n^2+m^2}\right). \quad \text{(B7)}$$

Note that for the term $n = 0$ in Eq. (B7), the argument of the hypergeometric function becomes independent of m and T. The result for $\Pi(\alpha p, p)$ is thus

$$\Pi^T(\alpha p, p) = J_1 + J_2. \quad \text{(B8)}$$

In order to verify this result in the limit $\alpha = 0$ ($\alpha' = 1$) we recall the identity

$$_2F_1(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)}, \quad \text{(B9)}$$

that can be used to write

$$J_1 \overset{\alpha=0}{\longrightarrow} -\frac{\lambda^2}{2(2\pi)^2}\frac{1}{8\pi}\left(\frac{\mu}{m}\right)^{3-d}\frac{\pi^{(d-1)/2}\Gamma(\frac{3-d}{2})\Gamma(\frac{d}{2}-1)}{\Gamma(\frac{d}{2})},$$

$$J_2 \overset{\alpha=0}{\longrightarrow} \mu^{3-d}\frac{\lambda^2}{2(2\pi)^2}\frac{1}{4\pi}\frac{\pi^{d/2}\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2}-1)}{\Gamma(\frac{d}{2})}\times T\sum_{n=-\infty}^{\infty}(m^2+\omega_n^2)^{(d/2)-2}. \quad \text{(B10)}$$

Using the procedure as in Eqs. (20)–(23) to obtain the high temperature limit, we get for J_2

$$J_2 = \frac{\lambda^2}{2(2\pi)^2}\frac{1}{4\pi}\mu^{3-d}\frac{\pi^{d/2}\Gamma(2-\frac{d}{2})\Gamma(\frac{d}{2}-1)}{\Gamma(\frac{d}{2})} \times \bigg(Tm^{d-4} + 2T(2\pi T)^{d-4}\zeta(4-d) + 2T\Big(\frac{d}{2}-2\Big)(2\pi T)^{d-6}m^2\zeta(6-d).\bigg) \tag{B11}$$

Taking $d \to 3 - 2\epsilon$ and $\alpha = 0$, the result in the infrared limit is

$$\Pi^T(0, p\to 0) = \frac{\lambda^2}{4(2\pi)^2}\bigg\{\frac{\pi T}{m} + \ln\Big(\frac{m}{2T}\Big) + \gamma_E - \frac{m^2\zeta(3)}{8\pi^2T^2}\bigg\}, \tag{B12}$$

which coincides with Eq. (25).

We can also use the former analysis to give an explicit expression for the α dependence of the self-energy in the high temperature limit. We first separate from Eq. (B1) all α dependence. In terms of the parameter α', we get

$$\Pi(\alpha p, p) = \frac{\lambda^2}{2(2\pi)^2}\int_0^\infty dk\frac{n(\omega_k)}{\omega_k}\bigg[1 + \frac{(\alpha'^2-1)m^2}{\omega_k^2 - \alpha'^2m^2}\bigg] \equiv \Pi_0 + \Pi_\alpha, \tag{B13}$$

where

$$\Pi_\alpha \equiv \frac{\lambda^2(\alpha'^2-1)m^2}{2(2\pi)^2}\int_0^\infty dk\frac{n(\omega_k)}{\omega_k}\frac{1}{\omega_k^2-\alpha'^2m^2}. \tag{B14}$$

Notice that the above integral can be obtained from Eq. (B1) taking $d = 1 - 2\epsilon$ and with the changes

$$\mu^{3-d} \to \mu^{1-d}, \qquad \frac{1}{4\pi} \to \frac{1}{2} \tag{B15}$$

from where we get

$$\begin{aligned} J_1 &\to -\frac{\lambda^2(\alpha'^2-1)}{2(2\pi)^2}\frac{1}{4}\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{3}{2})}{}_2F_1\Big(1,1,\frac{3}{2};\alpha'^2\Big) \\ &= -\frac{\lambda^2(\alpha'^2-1)}{2(2\pi)^2}\frac{1}{2}\frac{\sin^{-1}(\alpha')}{\alpha'\sqrt{1-\alpha'^2}}, \\ J_2 &\to +\frac{\lambda^2(\alpha'^2-1)}{2(2\pi)^2}\frac{\pi T m^{-1}}{2}\bigg[{}_2F_1\Big(1,\frac{3}{2},2;\alpha'^2\Big) + 2\sum_{n=1}^\infty x_n^3(x_n^2+1)^{-(3/2)}{}_2F_1\Big(1,\frac{3}{2},2;\frac{1+\alpha'^2x_n^2}{1+x_n^2}\Big)\bigg], \\ \Pi_\alpha &\equiv J_1 + J_2, \end{aligned} \tag{B16}$$

where $x_n = m/2\pi nT$. Notice that J_1 in Eq. (B16) yields a T-independent term and therefore contributes only to the vacuum part. This can be shown to correspond to considering the pole of $\zeta(s)$ at $s = 1$ in L_2 given in Eq. (60). We can thus ignore this term.

In the high temperature limit the parameter $x_n \ll 1$, thus we can perform a series expansion in J_2, yielding

$$\begin{aligned} J_2 = \Big(\frac{\lambda^2}{8\pi^2}\Big)\bigg\{&\Big(\frac{\pi T}{m}\Big)\alpha(\sqrt{\alpha^2-1}-\alpha) \\ &+ \sqrt{\frac{\alpha^2}{\alpha^2-1}}\bigg[-\Big(\frac{m}{2\pi T}\Big)\zeta(2) - \frac{1}{(\alpha^2-1)}\Big(\frac{m}{2\pi T}\Big)^3\zeta(4) - \frac{1}{(\alpha^2-1)^2}\Big(\frac{m}{2\pi T}\Big)^5\zeta(6) - \ldots\bigg] \\ &+ \Big(\frac{\alpha^2}{\alpha^2-1}\Big)\bigg[\frac{1}{2}\Big(\frac{m}{2\pi T}\Big)^2\zeta(3) + \frac{(3-\alpha^2)}{4(\alpha^2-1)}\Big(\frac{m}{2\pi T}\Big)^4\zeta(5) + \frac{(3\alpha^4-10\alpha^2+15)}{16(\alpha^2-1)^2}\Big(\frac{m}{2\pi T}\Big)^6\zeta(7) + \ldots\bigg]\bigg\}, \end{aligned} \tag{B17}$$

which coincides with Eq. (64).

APPENDIX C: π_δ IN THE ITF

In this appendix we show that the function π_δ found in Ref. [3] emerges in the ITF making use of the Feynman parametrization only when the limits of integration are replaced from $x \in [0, 1]$ to $x \in [0, 1/2]$.

We start from Eq. (3.33) in Ref. [1]

$$\Pi_x(p_{0l}, p) = -\frac{\lambda^2}{8}\int\frac{d^3k}{(2\pi)^3}\int_0^1 dx \times \frac{\partial}{\partial m^2}\sum_{r=\pm1}\frac{\coth\frac{\beta}{2}(rxp_{0l}+y^{1/2})}{y^{1/2}}, \tag{C1}$$

where y is defined as

$$y = E_k^2 + x(E_{k-p}^2 - E_k^2) - x(1-x)p_{0l}^2. \tag{C2}$$

It is worth noticing that in Ref. [3] the change of variable $\mathbf{k} - x\mathbf{p} \to \mathbf{k}$ is performed in Eq. (C1), but this change is not allowed in this case since the integral is divergent, unless the divergence is regulated by using for instance, dimensional regularization.

Using the identity in Eq. (48) into Eq. (C1), we get

$$\Pi_x(p_{0l}, p) = -\frac{\lambda^2}{8}\sum_{r=\pm1}\int\frac{d^3k}{(2\pi)^3}\int_0^1 dx \times \frac{\partial}{\partial x} \times \bigg[\frac{\coth\frac{\beta}{2}(rxp_{0l}+y^{1/2})}{2y(rp_{0l}+\frac{\partial y^{1/2}}{\partial x})}\bigg]. \tag{C3}$$

The integral over x becomes trivial and when evaluating in the integration limits $x = 1$, $x = 0$ we obtain Eq. (5). This is what is done in Ref. [1] which leads to the correct result, provided the periodicity in coth is imposed, as discussed in Sec. III. However, if we instead follow Ref. [3] and use that the integrand is symmetric about $x = 1/2$ and thus that the integral over x in the interval $x \in [0, 1]$ is twice the integral in the interval $x \in [0, 1/2]$, we get

$$\Pi_x(p_{0l}, p) = -\frac{\lambda^2}{4}\sum_{r=\pm1}\int\frac{d^3k}{(2\pi)^3}\int_0^{1/2} dx \times \frac{\partial}{\partial x} \times \bigg[\frac{\coth\frac{\beta}{2}(rxp_{0l}+y^{1/2})}{2y(rp_{0l}+\frac{\partial y^{1/2}}{\partial x})}\bigg]. \tag{C4}$$

Notice that Eq. (C4) is valid when p_{0l} is imaginary and discrete, since only in this case, coth is periodic. Evaluating the integral over x in Eq. (C4) we get

$$\Pi_x(p_{0l}, p) = -\frac{\lambda^2}{4}\sum_{r=\pm1}\int\frac{d^3k}{(2\pi)^3}\times\bigg[-\frac{\coth\frac{\beta}{2}E_k}{E_k(2rp_{0l}E_k + E_{k-p}^2 - E_k^2 - p_{0l}^2)} + \frac{\coth\frac{\beta}{2}(rxp_{0l} + y^{1/2})}{2y(rp_{0l} + \frac{\partial y^{1/2}}{\partial x})}\bigg|_{x=1/2}\bigg], \tag{C5}$$

where the first term results from evaluating in the lower limit of the x integral and in the second one we have left indicated that x is evaluated in $1/2$. Notice that when completing the square in the denominator of the first term in Eq. (C5), this becomes identical to the result in Eq. (6), which is the correct result, thus leaving Eq. (C5) with an extra term, which in fact, as we proceed to show, corresponds to the function $-\pi_\delta$ in Ref. [3]. To show this we must carry out the angular integration in Eq. (C5). Defining

$$\Pi_\Delta(p_{0l}, p) \equiv -\frac{\lambda^2}{4}\sum_{r=\pm1}\int\frac{d^3k}{(2\pi)^3}\times\frac{\coth\frac{\beta}{2}(rxp_{0l} + y^{1/2})}{y^{1/2}[2ry^{1/2}p_{0l} + E_{k-p}^2 - E_k^2 - (1-2x)p_{0l}^2]}, \tag{C6}$$

where x should be evaluated in $1/2$. Upon the change of variable $\mathbf{k} - x\mathbf{p} \rightarrow \mathbf{k}$, the angular dependence inside the function coth is removed and we get

$$\Pi_\Delta(p_{0l}, p) = -\frac{\lambda^2}{4}\sum_{r=\pm1}\int\frac{d^3k}{(2\pi)^3}\times\bigg[\frac{\coth\frac{\beta}{2}(rxp_{0l} + \phi^{1/2})}{\phi^{1/2}[2r\phi^{1/2}p_{0l} - 2\mathbf{k}\cdot\mathbf{p}]}\bigg], \tag{C7}$$

where $\phi = k^2 + m^2 - x(1-x)(p_{0l}^2 - p^2)$. The remaining angular integration is readily performed and the result is

$$\Pi_\Delta(p_{0l}, p) = -\frac{\lambda^2}{4(2\pi)^2}\int_0^\infty\frac{kdk}{\phi^{1/2}p}\ln\bigg(\frac{p_{0l}\phi^{1/2} + kp}{p_{0l}\phi^{1/2} - kp}\bigg)\times[n(xp_{0l} + \phi^{1/2}) - n(-xp_{0l} + \phi^{1/2})]|_{x=1/2}, \tag{C8}$$

where n is the Bose-Einstein distribution. Notice that if in Eq. (C8) we use that p_{0l} is purely imaginary and discrete, the function Π_Δ vanishes. However, if p_{0l} is analytically continued to arbitrary complex values, the correct result is obtained only by the addition of the function π_δ found in Ref. [3], which exactly cancels Π_Δ.

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