Correlation functions of the energy-momentum tensor in SU(2) gauge theory at finite temperature

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We calculate correlation functions of the energy-momentum tensor in the vicinity of the deconfinement phase transition of $(3 + 1)$-dimensional SU(2) gauge theory and discuss their critical behavior in the vicinity of the second order deconfinement transition. We show that correlation functions of the trace of the energy-momentum tensor diverge uniformly at the critical point in proportion to the specific heat singularity. Correlation functions of the pressure, on the other hand, stay finite at the critical point. We discuss the consequences of these findings for the analysis of transport coefficients, in particular, the bulk viscosity, in the vicinity of a second order phase transition point.

I. INTRODUCTION

Experimental evidence for rapid thermalization of the dense matter created in heavy ion collisions at the Relativistic Heavy Ion Collider (RHIC) [1] provided strong motivation for a new look at the nonperturbative structure of the quark gluon plasma (QGP). The rapid thermalization of the QGP has been related to the smallness of the shear viscosity in the plasma phase [2], which may come close to the theoretical lower bound established for the ratio of shear viscosity to entropy density [3]. This gave rise to the interpretation of the quark gluon plasma above but close to the transition temperature as a strongly interacting medium that has properties of an almost perfect liquid.

These experimental findings also renewed the interest in determining transport properties of gauge theories through the calculation of correlation functions of the energy-momentum tensor on the lattice. This is by far not a straightforward calculation and requires a careful analysis of the low frequency structure of the spectral representation of these correlation functions [4]. Nonetheless progress has been made through high statistics calculations in SU(3) gauge theories [5–7].

Recently it has been argued that close to the transition from low temperature hadronic matter to the plasma phase of QCD bulk viscosity might play a much more important role than shear viscosity [8]; bulk viscosity has been related to a temperature derivative of the trace of the energy-momentum tensor. The latter does diverge at a second order critical point [9] and thus would force bulk viscosity to diverge too. This argument is based on a sum rule derived for correlation functions of the energy-momentum tensor in Minkowski space [8]. However, as noted recently [10], in Euclidean space the correlation function of the trace of the energy-momentum tensor picks up an additional constant contribution corresponding to an exact δ function at zero frequency in the spectral representation of this correlator. This contribution, in fact, cancels the leading singular behavior of the correlation function and thus

makes the relation between transport coefficients and properties of bulk thermodynamic observables more subtle.

In the following we will provide evidence for the presence of a contribution to the correlation function of the trace of energy-momentum tensor that is constant in Euclidean time but temperature dependent. We show that the temperature dependence of this contribution scales like the specific heat and thus dominates the singular behavior of this correlation function in the vicinity of T_c. This implies the presence of a delta function singularity at zero frequency in this correlation function with amplitude proportional to the specific heat. This delta function contribution has been ignored in the sum rule analysis of Ref. [8], which explains the inconsistency observed in the parametric dependence of bulk viscosity derived in high temperature perturbation theory and through the sum rule analysis [11].

The singular behavior of bulk viscosity in the vicinity of a critical point has long been known in statistical physics [12,13]. In particular, the divergence of bulk viscosity at the critical point of the liquid-gas transition has been analyzed extensively [14]. In this case the divergence of bulk viscosity (ζ) has been related to the critical behavior of the 3-dimensional Ising model [13,15,16], which is in the universality class for the critical point of the liquid-gas phase transition. It has been argued that the divergence of ζ is strong and, in fact, almost quadratic in the inverse reduced temperature, $t = |T - T_c|/T_c$. The singular behavior, $\zeta \sim t^{-z\nu+\alpha}$, with α, ν denoting static critical exponents of the 3-d Ising model and z being a dynamical exponent characterizing the equilibration of density fluctuations, goes along with a strong divergence of the relaxation time for density fluctuations, τ_R. Their ratio, $\zeta/\tau_R \sim t^\alpha$, however, is proportional to the inverse of the specific heat and thus vanishes slowly at the critical point.

While numerous numerical studies have confirmed that the critical behavior of bulk thermodynamics in $(d + 1)$-dimensional gauge theories is indeed controlled by d-dimensional universality classes corresponding to the

global symmetry of the relevant order parameter, little is known about the relation between dynamic properties and the related critical exponents [17]. The SU(2) gauge theory with its second order deconfinement phase transition seems to be an ideal model to explore critical behavior of dynamical properties, e.g. transport coefficients. The SU(2) gauge theory at finite temperature incorporates all the basic features one expects to be relevant for deconfinement in QCD. Moreover, it has a second order phase transition which belongs to the universality class of the 3-dimensional Ising model. It thus will allow to analyze in how far dynamic universal properties known for the Ising universality class are relevant for transport properties in a quantum field theory. It also may give insight into transport properties in the vicinity of the chiral critical point in QCD that may exist at nonzero baryon number density and also would belong to the Ising universality class [18]. In the vicinity of the SU(2) deconfinement phase transition a lot of information exists about the critical behavior of bulk thermodynamics that can be related to the critical behavior of correlation functions of the energy-momentum tensor. This will help to gain experience with lattice calculations of the bulk viscosity in pure gauge theories.

In this paper we will establish relations between the critical behavior of bulk thermodynamic observables in the vicinity of the SU(2) deconfinement transition and properties of correlation functions of the energy-momentum tensor. In particular, we will analyze the Euclidean correlation functions of the trace of the energy-momentum tensor and compare it to critical behavior of the specific heat. We present evidence that the strength of the divergence of this correlation function at T_c is independent of Euclidean time and is controlled by the universal structure of the divergence of the specific heat at a critical point belonging to the 3-dimensional Ising universality class. In the next section we review the calculation of bulk thermodynamic observables in SU(2) gauge theory [19] and present some new results on the diagonal components of the energy-momentum tensor, i.e. energy density and pressure, in the vicinity of the deconfinement transition temperature. In Sec. III we introduce local operators for energy density and pressure and discuss basic properties of their finite temperature Euclidean-time correlation functions at vanishing momentum. Section IV is devoted to a discussion of critical behavior extracted from correlation functions of the trace anomaly and presents a calculation of the pressure-pressure correlator. We analyze consequences for the determination of bulk viscosity from these correlation functions in Sec. V. Finally, we conclude in Sec. VI.

II. THERMODYNAMICS OF SU(2) GAUGE THEORIES ON THE LATTICE

As preparation for the analysis of correlation functions of diagonal elements of the energy-momentum tensor we

want to discuss here the calculation of the relevant bulk thermodynamic observables, energy density (ϵ), and pressure (p), on the lattice. In order to analyze Euclidean space-time correlation functions of ϵ, p, or the trace of the energy-momentum tensor, $\Theta^{\mu\mu} = \epsilon - 3p$, one obviously needs to use an approach to bulk thermodynamics that allows to define local operators for energy density and pressure. We will call this approach, which in fact was the basis for the original formulation of bulk thermodynamics on the lattice [20], the differential formalism. It is in contrast to the so-called integral method which nowadays is most commonly used in lattice calculations to extract energy density and pressure from the trace anomaly [21]. The differential formalism has not been used recently in studies of bulk thermodynamic observables as it is more involved and requires additional nonperturbative calculations of derivatives of the bare gauge couplings with respect to the anisotropy parameter, $\xi = a_\sigma/a_\tau$, that controls the spatial ($a \equiv a_\sigma$) and temporal (a_τ) lattice spacings [22]. Having control over the anisotropy is needed to derive thermodynamic quantities as function of temperature, $T = 1/N_\tau a_\tau$, and volume, $V = (N_\sigma a_\sigma)^3$ through appropriate partial derivatives with respect to a_τ and a_σ [20], although numerical calculations will generally be performed on isotropic lattice ($\xi = 1$). In particular, for the study of the singular behavior of the energy density in the vicinity of the SU(2) deconfinement transition this approach has been used successfully [19]. In this context also the necessary derivatives of the gauge coupling with respect to ξ had been analyzed in detail. We will follow this analysis here closely.

A. The differential formalism

We start from the partition function for an SU(N_c) gauge theory at finite temperature, which we write in standard lattice notation [23],

$$Z(T, V) = \int \prod_{x,\mu} dU_{x,\mu} e^{-S(T,V)}, \qquad (1)$$

with

$$S(T, V) = \frac{2N_c}{g^2} \sum_{x,\mu,\nu} (1 - P_{\mu\nu}(x)). \qquad (2)$$

Here $x = (\vec{x}, x_4)$ labels the discrete set of space-time points on a four-dimensional hypercubic lattice of size $N_\sigma^3 N_\tau$.

In the standard Wilson discretization scheme the local gauge action in the $\mu\nu$ hyperplane is expressed in terms of plaquette variables $P_{\mu\nu}(x) \equiv \frac{1}{N_c} \text{Re Tr} \, U_{x,\mu} U_{x+\hat{e}_\mu,\nu} U^\dagger_{x+\hat{e}_\nu,\mu} U^\dagger_{x,\nu}$, with link variables $U_{x,\mu} \in \text{SU}(N_c)$. We write the action in terms of contributions arising only from entirely spacelike plaquettes (P_σ) and space-time like plaquettes (P_τ),

$$P_\sigma = \frac{1}{N_\sigma^3 N_\tau} \sum_x P_\sigma(x), \qquad P_\sigma(x) = \frac{1}{3} \sum_{i>j=1,2,3} P_{ij}(x),$$

$$P_\tau = \frac{1}{N_\sigma^3 N_\tau} \sum_x P_\tau(x), \qquad P_\tau(x) = \frac{1}{3} \sum_{j=1,2,3} P_{4,j}(x). \tag{3}$$

We also introduce $P_0 = (P_\sigma + P_\tau)/2$ as the average of spacelike and timelike plaquettes evaluated on a symmetric ($T \simeq 0$) lattice, i.e. for $N_\tau = N_\sigma$.

From the logarithm of the partition function one obtains pressure and energy density as derivatives with respect to a_σ and a_τ, respectively [20],

$$\frac{p}{T^4} = N_c N_\tau^4 ((2g^{-2} - (c_\sigma - c_\tau))(P_\sigma - P_\tau)$$

$$- 3(c_\sigma + c_\tau)(2P_0 - (P_\sigma + P_\tau))), \tag{4}$$

$$\frac{\epsilon}{T^4} = 3N_c N_\tau^4 ((2g^{-2} - (c_\sigma - c_\tau))(P_\sigma - P_\tau)$$

$$+ (c_\sigma + c_\tau)(2P_0 - (P_\sigma + P_\tau))). \tag{5}$$

Here c_σ and c_τ are given in terms of derivatives of the gauge coupling, $g^{-2}(a, \xi)$, with respect to the anisotropy at $\xi = 1$ [20,22]. Their sum is constraint by the β function,

$$B(g^{-2}) = a \left. \frac{dg^{-2}}{da} \right|_{\xi=1} = -2(c_\sigma + c_\tau). \tag{6}$$

With these relations for ϵ and p it is straightforward to obtain also the trace of the energy-momentum tensor and the entropy density,

$$\frac{\Theta^{\mu\mu}}{T^4} \equiv \frac{\epsilon - 3p}{T^4} = 6B(g^{-2})N_c N_\tau^4 [P_\sigma + P_\tau - 2P_0], \tag{7}$$

$$\frac{s}{T^3} \equiv \frac{\epsilon + p}{T^4} = 4N_c N_\tau^4 ([2g^{-2} - (c_\sigma - c_\tau)](P_\sigma - P_\tau)). \tag{8}$$

We note that these particular combinations of energy density and pressure are proportional to simple sums and differences of spacelike and timelike plaquette expectation values. They only require common multiplicative renormalizations for P_σ and P_τ. Extracting ϵ/T^4 or p/T^4 directly, however, involves different renormalization factors for spacelike and timelike plaquette expectation values. For this reason it is straightforward to analyze correlation functions of local densities of the entropy or trace anomaly. We note that in the vicinity of the SU(2) deconfinement transition the dominant singular behavior of $\Theta^{\mu\mu}/T^4$ as well as s/T^3 is controlled by the energy density. We will show later that in the vicinity of T_c the pressure is an order of magnitude smaller than the energy density. When one combines $\Theta^{\mu\mu}/T^4$ and s/T^3 to extract the pressure it thus requires good knowledge of the renormalization factors c_σ and c_τ in order to eliminate the dominant energy contri-

butions and arrive at a proper definition for a local operator for the pressure. This is a prerequisite for the analysis of pressure-pressure correlation functions and, as will become clear later, this becomes of importance for studies of transport coefficients, e.g. the bulk viscosity, in the vicinity of T_c.

B. Critical energy density and pressure

As an application of the differential formalism reviewed above we have performed a finite size scaling analysis of the critical energy density and the pressure at the deconfinement transition temperature of the SU(2) gauge theory. This extends earlier studies of the critical energy density performed in Ref. [19] to larger lattices and yields, for the first time, a determination of the pressure at the critical point.

The deconfinement transition of the SU(2) gauge theory in $(3 + 1)$ dimensions is in the same universality class as the 3-dimensional Ising model. Its critical behavior is well understood and has been confirmed in various finite size scaling studies. On a lattice of size $N_\sigma^3 N_\tau$ energy density and pressure at the critical point are expected to scale as

$$\left(\frac{p(T_c)}{T_c^4} \right)_{N_\tau, N_\sigma} = \left(\frac{p(T_c)}{T_c^4} \right)_{N_\tau, \infty} + a_p \left(\frac{N_\tau}{N_\sigma} \right)^3, \tag{9}$$

$$\left(\frac{\epsilon(T_c)}{T_c^4} \right)_{N_\tau, N_\sigma} = \left(\frac{\epsilon(T_c)}{T_c^4} \right)_{N_\tau, \infty} + a_\epsilon \left(\frac{N_\tau}{N_\sigma} \right)^{(1-\alpha)/\nu}. \tag{10}$$

Here $\alpha = 0.110(1)$ and $\nu = 0.6301(4)$ denote the critical exponents of the 3-dimensional Ising universality class, controlling the divergence of the specific heat and the correlation length, respectively.[1]

We have analyzed the scaling behavior of energy density and pressure on lattices with temporal extent $N_\tau = 4$, 6, and 8 for various spatial lattice sizes; covering the range $N_\sigma \in [16, 96]$ for $N_\tau = 4$, $N_\sigma \in [24, 96]$ for $N_\tau = 6$, and $N_\sigma \in [32, 96]$ for $N_\tau = 8$. For these values of N_τ the infinite volume critical couplings are known quite precisely [24–26]. We have listed them in Table I. At these values of the gauge coupling also the anisotropy coefficients c_σ and c_τ have been determined previously [19].

We have calculated pressure, energy density, entropy density, and trace anomaly on lattices of temporal extent $N_\tau = 4$, 6, and 8 for various spatial lattice sizes at the critical couplings given in Table I. Data for the vacuum subtraction needed for all observables but the entropy density have been obtained through simulations on a 32^4 lattice. The number of configurations analyzed ranges from

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[1]Note that the leading finite volume correction that arises from the singular contribution to the pressure is inverse proportional to the volume as it is the case also for the leading corrections to regular terms. This is due to a hyperscaling relation, $2 - \alpha = d\nu$ with $d = 3$.

TABLE I.   Critical energy density and pressure calculated on lattices with temporal N_τ on lattices with aspect ratio $N_\sigma/N_\tau = 12$ and infinite volume extrapolated results. The second column gives the values of the gauge couplings at which these calculations have been performed. The error estimates are based on fits with the *Ansätze* given in Eqs. (9) and (10) and does not include errors that arise from uncertainties in the couplings c_σ, c_τ as well as the critical coupling β_c.

N_τ	β_c	$\epsilon(T_c)/T_c^4$	$p(T_c)/T_c^4$	$\epsilon(T_c)/T_c^4$	$p(T_c)/T_c^4$
		$N_\sigma = 12N_\tau$		$N_\sigma \to \infty$	
4	2.298 95(10)	0.3303(9)	0.0240(2)	0.2872(5)	0.0242(3)
6	2.4265(30)	0.3015(27)	0.0136(8)	0.2722(31)	0.0135(8)
8	2.5115(40)	0.3158(42)	0.0107(26)	$\cdots$	0.0107(15)

1×10^6 on our smaller lattices to 4×10^5 on the large lattices. For our calculations on $N_\tau = 4$ and 6 lattices we have used a standard heat-bath/overrelaxation algorithm, with a typical ratio of $1/4$ for all the points except the ones closest to the transition for which a ratio of $1/6$ was used. For calculations on the larger $N_\tau = 8$ lattices it became possible to use also a two-level algorithm [27] which turned out to be more efficient for the analysis of correlation functions. Autocorrelation times in the transition region range from $\mathcal{O}(1)$ on the small lattices to about 40 on the largest lattices.

In Fig. 1 we show results for the volume dependence of various bulk thermodynamic observables. It clearly can be seen that the scaling behavior of the pressure is different from that of the other three observables. Moreover, it is apparent that the pressure is more than an order of magnitude smaller than any of the observables that contain contributions of the energy density. This shows that the finite size scaling behavior of $(\epsilon + p)/T^4$ and $(\epsilon - 3p)/T^4$ is dominated by the nonanalytic behavior of the energy density and thus is expected to be controlled by the same ansatz as for the energy density given in Eq. (10). We also note that cutoff effects seem to be small already for lattices with temporal extent $N_\tau \gtrsim 6$ and that the volume

dependence starts being controlled by the scaling variable $V^{1/3}T = N_\sigma/N_\tau$. This may be expected as bulk thermodynamics in the vicinity of T_c is dominated by low frequency modes for which the underlying lattice cutoff becomes unimportant.

To check consistency with the scaling behavior of the Ising universality class we have fitted the data for the energy on the $N_\tau = 4$ lattice for all $N_\sigma/N_\tau > 4$ to the *Ansatz* given in Eq. (10) with a free scaling exponent $\Delta = (1 - \alpha)/\nu$. This fit yields $\Delta = 1.41(6)$ which is in good agreement with the value in the 3-dimensional Ising universality class, $\Delta = 1.412(1)$ [28]. Fits to the finite volume corrections for the pressure yield an exponent $d = 3.0 \pm 1.4$ in good agreement with the expected hyperscaling relation $d = 3$.

To extract the asymptotic, infinite volume values for the critical energy density and the pressure we then used the *Ansätze* given in Eqs. (9) and (10) with fixed scaling exponents 1.412 and 3, respectively. Results for $\epsilon(T_c)/T_c^4$ and $p(T_c)/T_c^4$ for $N_\sigma/N_\tau = 12$ as well as infinite volume extrapolated values are summarized in Table I; results for the critical energy density are consistent with those of Ref. [20] but have a statistical error that is an order of magnitude smaller. Comparison of the $N_\tau = 8$ results with
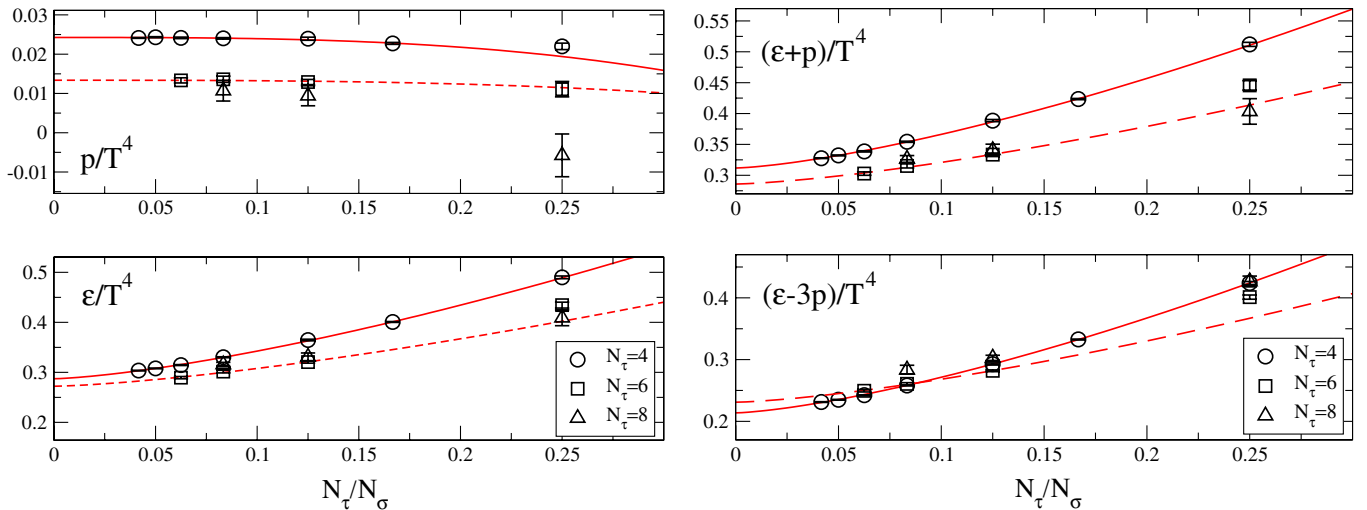

FIG. 1 (color online).   Finite size scaling analysis of bulk thermodynamic observables on lattices of size $N_\tau = 4$, 6, and 8. The curves show fits that have been performed for $N_\tau/N_\sigma \leq 1/6$.

those for $N_\tau = 6$ suggest that the remaining cutoff dependence is small.

III. CORRELATION FUNCTIONS OF THE ENERGY-MOMENTUM TENSOR

Within the differential framework for bulk thermodynamics on the lattice, reviewed in the previous section, it is straightforward to define local operators for the energy density and pressure, as well as the entropy density and the trace of the energy-momentum tensor. Using $P_\tau(x)$ and $P_\sigma(x)$ as introduced in Eq. (3) one obtains from Eqs. (4) and (5) and Eqs. (7) and (8) local operators for pressure, energy, and entropy density as well as the trace anomaly, e.g.

$$\Theta^{\mu\mu}(\vec{x}, x_4) = 6N_cB(g^{-2})[P_\sigma(\vec{x}, x_4) + P_\tau(\vec{x}, x_4) - 2P_0] \tag{11}$$

for the trace anomaly and similar for other observables. The local plaquette variables, $P_\sigma(\vec{x}, x_4)$, $P_\tau(\vec{x}, x_4)$ have been defined in Eq. (3). We will denote this particular choice of local expressions for plaquette variables in the following as discretization scheme 1. However, when considering correlation functions in Euclidean time, one may want to account for the effect that timelike plaquettes connect two different time slices in a lattice and should be connected to a Euclidean time that is displaced from that of spacelike plaquettes. This ambiguity arises because gauge fields are introduced on links of the lattice and the field strength tensor thus is spread at least over a domain of the size of an elementary plaquette. These effects introduce additional $\mathcal{O}(a^2)$ discretization errors, i.e. errors that are of the same order as the discretization errors that exist anyhow in the definition of the Euclidean action or the energy-momentum tensor used in our calculations. They, of course, will disappear in the continuum limit. Nonetheless, comparing different discretization schemes will allow to estimate the magnitude of systematic $\mathcal{O}(a^2)$ effects. We therefore have introduced two other discretization schemes which symmetrize the contribution of spacelike plaquettes relative to a timelike plaquette or vice verse,

scheme 2: $P_{ij}(x) \to \frac{1}{2}(P_{ij}(x) + P_{ij}(x + \hat{e}_4))$,

scheme 3: $P_{4j}(x) \to \frac{1}{2}(P_{4j}(x) + P_{4j}(x - \hat{e}_4))$.

Note that there is no need for a local definition for P_0 which enters the definition of $\Theta^{\mu\mu}(x)$. P_0 is evaluated at zero temperature and is a constant, identical at all space-time points. We furthermore introduce zero-momentum projected operators and their fluctuations,

$$Y(\tau) = \frac{1}{N_\sigma^3}\sum_{\vec{x}} Y(\vec{x}, \tau), \qquad \bar{Y}(\tau) = Y(\tau) - \left\langle \frac{1}{N_\tau}\sum_{\tau=1}^{N_\tau} Y(\tau) \right\rangle. \tag{12}$$

Correlation functions are then obtained as thermal aver-

ages over products of fluctuation operators, i.e. we consider connected correlation functions,

$$\frac{G_{XY}(\tau, T)}{T^5} = N_\tau^5 \langle \bar{X}(\tau)\bar{Y}(0) \rangle. \tag{13}$$

Because of the periodic boundary conditions in Euclidean time points in the hyperplane at $\tau = 0$ are identical to those at $\tau = N_\tau$. In the following we also will discuss properties of the midpoint-subtracted correlation functions,

$$\frac{\Delta G_{XY}(\tau, T)}{T^5} = N_\tau^5 (\langle X(\tau)Y(0) \rangle - \langle X(1/2T)Y(0) \rangle). \tag{14}$$

We used the three discretization schemes introduced above for diagonal components of the energy-momentum tensor to analyze the cutoff dependence of correlation functions. In Fig. 2 (left) we show results for correlation functions of the trace of the energy-momentum tensor, $G_{\Theta\Theta}$ and the energy, $G_{\epsilon\epsilon}$. These correlators have been evaluated on lattices with the largest temporal extent, $N_\tau = 8$, used in this study, and at the largest Euclidean-time separation possible at finite temperature, i.e. the midpoint of the temporal direction $\tau = 1/2T$, at the critical temperature T_c. It is apparent from these figures that cutoff effects in $G_{\Theta\Theta}$ are only of the order of 10%; within the current errors different discretization schemes yield consistent results on the $N_\tau = 8$ lattices. For $G_{\epsilon\epsilon}$, however, discretization errors are large even on the largest lattices.

In the right-hand part of Fig. 2 we show the energy-energy and energy-pressure correlation functions as a function of Euclidean time, τT, evaluated at the critical temperature. As has been pointed out in Ref. [10] these correlation functions will, in the continuum limit, be independent of Euclidean time as the energy operator is time independent,

$$G_{\epsilon X}(\tau, T) = T^2 \frac{\partial}{\partial T}\langle X(0) \rangle. \tag{15}$$

At finite lattice spacing, however, this is not yet the case. It is apparent from Fig. 2 (right) that the influence of a finite lattice cutoff is more severe at shorter distances, and correlation functions $G_{\epsilon X}$ thus show a significant dependence on Euclidean time. As a consequence, the simple relation between midpoint-subtracted correlation functions of $\Theta^{\mu\mu}$ and pressure-pressure correlators,

$$\Delta G_{\Theta\Theta}(\tau, T) = \Delta G_{\epsilon\epsilon}(\tau, T) + 9\Delta G_{PP}(\tau, T) - 6\Delta G_{\epsilon P}(\tau, T) \tag{16}$$

$$\equiv 9\Delta G_{PP}(\tau, T) + \mathcal{O}(a^2), \tag{17}$$

which holds in the continuum limit, is violated by $\mathcal{O}(a^2)$ corrections at finite lattice spacing. The left-hand side of Eq. (16) as well as the various components of the right-
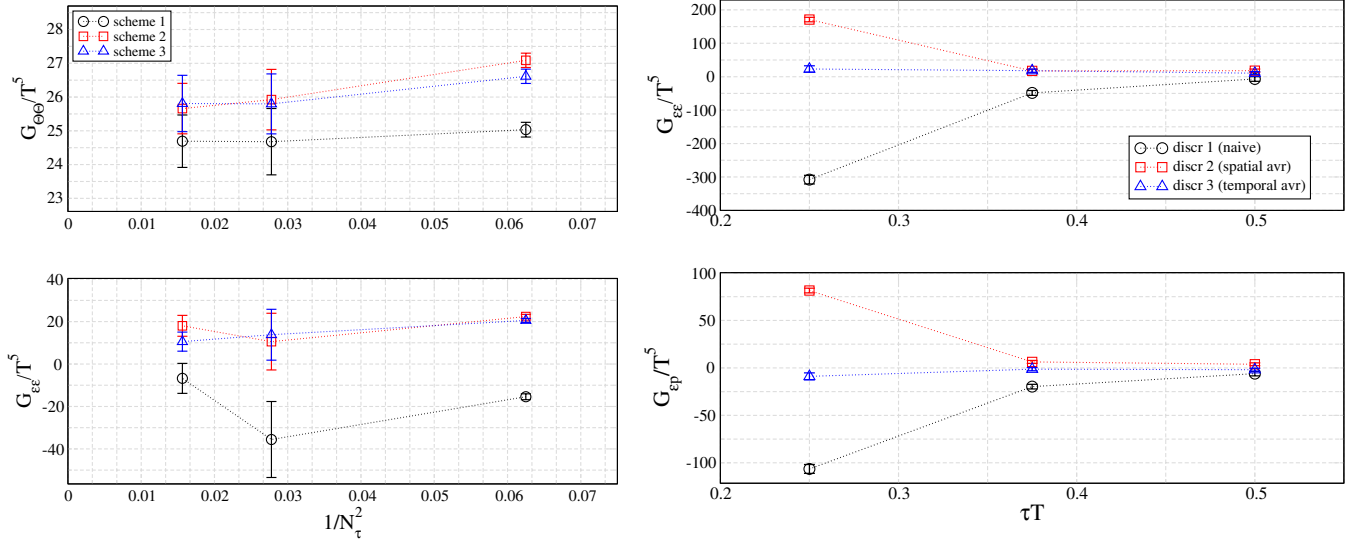
FIG. 2 (color online).   Cutoff dependence in different discretization schemes of the energy-momentum tensor. The left-hand part of the figure shows results for the trace-trace and energy-energy correlation functions for different values of the lattice cutoff $aT = 1/N_\tau$ evaluated on lattices with spatial extent $N_\sigma = 12N_\tau$. The right-hand part of the figure shows results for the energy-energy and energy-pressure correlation functions versus Euclidean time, τT.
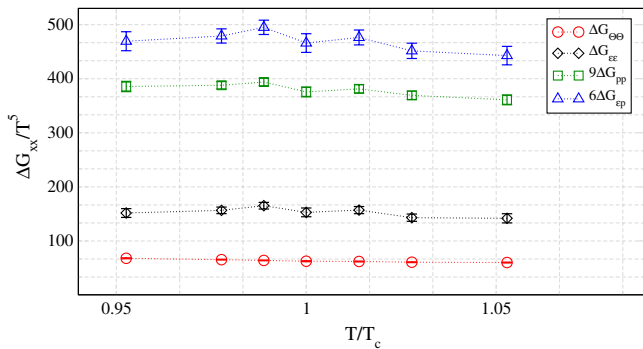

FIG. 3 (color online).   Midpoint-subtracted correlation functions on lattices with temporal extent $N_\tau = 8$ versus temperature. Shown are various contributions to the correlation function of the trace anomaly [Eq. (16)].

IV. CORRELATION FUNCTIONS AND CRITICAL BEHAVIOR

Fortunately, the cutoff dependence discussed in the previous section is not crucial for the analysis of critical behavior as it arises from short-distance effects. Close to T_c, however, the correlation length is large and thermal effects are not very sensitive to the underlying cutoff. The cutoff dependence thus is part of the smooth regular back-

ground that contributes to $G_{XY}(\tau, T)$. This is apparent from the weak temperature dependence of the differences $\Delta G_{XY}(\tau, T)$ shown in Fig. 3. Moreover, we find that correlators calculated in the three discretization schemes show a similar temperature dependence and differ, at fixed τT only by an almost temperature independent shift.

A. Correlation function of $\Theta^{\mu\mu}$

We will discuss here the temperature dependence of correlation functions of $\Theta^{\mu\mu}(\vec{x}, \tau)$ at vanishing momentum, i.e. we calculate $G_{\Theta\Theta}(\tau, T)/T^5$ as introduced in Eq. (13). This correlation function is the sum of three contributions corresponding to energy-energy, pressure-pressure, and energy-pressure correlations, respectively,

$$G_{\Theta\Theta}(\tau, T) = G_{\epsilon\epsilon}(\tau, T) + 9G_{PP}(\tau, T) - 6G_{\epsilon P}(\tau, T).$$

(18)

As discussed above in the continuum limit the entire τ dependence of $G_{\Theta\Theta}$ is expected to arise from the pressure-pressure correlations, while close to T_c the dominant temperature dependence will arise from the energy-energy correlator, which through the fluctuation-dissipation theorem, is proportional to the specific heat,

$$\frac{G_{\epsilon\epsilon}(\tau, T)}{T^5} \sim \frac{d\epsilon}{dT} \sim \frac{c_V}{T^3}.$$

(19)

Close to the deconfinement transition we therefore expect that $G_{\Theta\Theta}(\tau, T)$ will show critical behavior that coincides with that of the specific heat in a 3-dimensional Ising model,

hand side are shown in Fig. 3 for $\tau T = 1/4$ and $N_\tau = 8$. In the continuum limit the differences $\Delta G_{\epsilon\epsilon}(\tau, T)$ and $\Delta G_{\epsilon P}(\tau, T)$ should vanish and $\Delta G_{\Theta\Theta}(\tau, T)$ should be equal to $9\Delta G_{PP}(\tau, T)$. This is clearly not the case for finite values of the lattice cutoff.

$$\frac{G_{\Theta\Theta}(\tau, T)}{T^5} \sim \frac{c_V}{T^3}$$

$$\sim A_\pm \left| \frac{T - T_c}{T_c} \right|^{-\alpha} \left(1 + B_\pm \left| \frac{T - T_c}{T_c} \right|^\omega + \ldots \right)$$

$$\text{for } T \to T_c^\pm. \qquad (20)$$

Here α is the specific heat critical exponent and ω is the correction-to-scaling exponent which characterizes the leading nonanalytic scaling correction. Like α also the exponent ω is universal. For the SU(2) gauge theory in $(3 + 1)$ dimensions, which we are analyzing here, the relevant universality class is that of the 3-dimensional Ising model. In this case $\alpha = 0.110(1)$ and $\omega = 0.53(3)$ [28] which also has been verified in lattice calculations [19].

At nonzero lattice spacing the direct relation between correlation functions involving the energy operator and temperature derivatives of any observable is violated by cutoff effects. Nonetheless we expect that these are small in the vicinity of a second order phase transition. One thus may expect that at least the singular behavior of correlation functions that involve correlations with the energy operator will be independent of Euclidean time. We will analyze the critical behavior of $G_{\Theta\Theta}(\tau, T)$ in the next two subsections and show explicitly that the leading singular behavior indeed is independent of Euclidean time.

B. Finite size scaling of $G_{\Theta\Theta}$ at the critical point

We have calculated the correlation function $G_{\Theta\Theta}(\tau, T)$ close to the deconfinement transition point of the SU(2) gauge theory. In most of our simulation we use lattices of size $N_\sigma^3 N_\tau$ with $N_\tau = 4$. This gives us information on the correlation function at two nonzero values of Euclidean time, i.e. at $\tau T = 1/4$ and at the midpoint $\tau T = 1/2$. Most of our simulations have been performed at temperatures close to the phase transition where the correlation length becomes large. This required calculations on large spatial lattices in order to eliminate finite volume effects. We used spatial lattice sizes with aspect ratios N_σ/N_τ varying from 8 ($32^3 \times 4$ lattices) up to values as large as 32 ($128^3 \times 4$ lattices). This large aspect ratio made it possible to perform calculations at temperatures in the proximity of T_c without being affected by large finite volume effects. We checked that for all temperature values, except for calculations performed directly at T_c, the thermodynamic limit has been reached within our numerical accuracy. To reach the statistical accuracy needed for our scaling tests close to the critical point, where fluctuations are large, we have generated a large number of gauge field configurations ranging from 1×10^6 on our smaller lattices to 4×10^5 on the large lattices.

As mentioned in Sec. II B the location of the critical point is quite well known for lattices with temporal extent $N_\tau = 4$ [24]. The relation between gauge coupling and temperature has been determined in the vicinity of β_c

nonperturbatively from calculations of the transition temperature on lattices of temporal size $4 \leq N_\tau \leq 16$ [19]. We use these results to determine the reduced temperature t.

In various studies of the critical behavior of bulk thermodynamics of the $(3 + 1)$-dimensional SU(2) gauge theory it has been established that this gauge theory belongs to the universality class of the 3-dimensional Ising model. In our analysis of the critical behavior of correlation functions of the energy-momentum tensor we use established results on the critical exponents α, ν, and ω as well as the universal ratio of the specific heat amplitudes, A_+/A_-. These parameters are summarized in Table II.

In Fig. 4 (left) we show results for the correlation function $G_{\Theta\Theta}(\tau, T)/T^5$ calculated at the midpoint, $\tau T = 1/2$, on lattices with temporal extent $N_\tau = 4$ and for various spatial lattice sizes. As can be seen at all temperatures except, of course, at T_c the dependence on the spatial volume is well under control. The peak at the pseudocritical point close to T_c rises only slowly with increasing size of the volume as the specific heat exponent α is quite small. The right-hand part of Fig. 4 shows the volume dependence of the correlation function evaluated at T_c on lattices with temporal extent $N_\tau = 4$ at the two nonzero distances accessible in this case. At T_c we have fitted the volume dependence of $G_{\Theta\Theta}(\tau, T_c)/T_c^5$ to a finite size scaling *Ansatz*, which includes the leading singular behavior of the specific heat and a correction-to-scaling term,

$$G_{\Theta\Theta}(\tau, T_c)/T_c^5 = A_\sigma N_\sigma^{\alpha/\nu}(1 + B_\sigma N_\sigma^{-\omega/\nu}) + C_\sigma. \qquad (21)$$

Here A_σ, B_σ, C_σ are fit parameters, which all might depend on Euclidean time. We performed fits for the two nonzero Euclidean-time separations reachable on lattices with temporal extent $N_\tau = 4$, i.e. $\tau T = 1/4$ and $1/2$. In the fits we make use of the known Ising exponents α, ν, and ω which are given in Table II. This *Ansatz* gives excellent finite size scaling fits for both data sets shown in Fig. 4 (right). This confirms that the critical behavior is controlled by the exponents of the 3-dimensional Ising model. When performing fits of the data sets at distance $\tau T = 1/4$ and $1/2$ separately, we observe that the amplitudes A_σ and B_σ agree within errors. This also is apparent from the fact that the data at $\tau T = 1/4$ can be shifted on top of the data at $\tau T = 1/2$ by a N_σ independent constant [open squares in Fig. 4 (right)]. We then performed simultaneous fits of both data sets, still allowing for different amplitudes at both dis-

TABLE II. Specific heat exponent (α), correlation length exponent (ν), correction-to-scaling exponent ω, and the universal ratio of amplitudes $g_A = A_+/A_-$ of the 3-dimensional Ising model (from [28]).

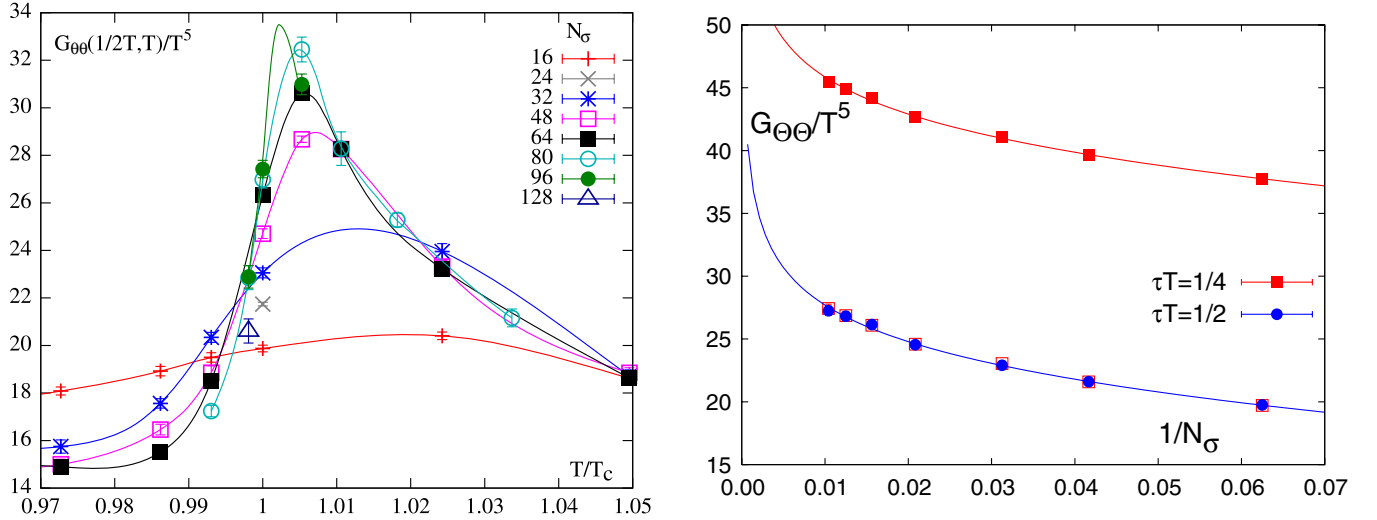| α	ν	ω/ν	g_A
0.110(1)	0.6301(4)	0.84(4)	0.54(1)

FIG. 4 (color online). The correlation function of the trace of the energy-momentum tensor, $G_{\Theta\Theta}(\tau, T)/T^5$ at distance $\tau T = 1/2$ calculated on lattices with temporal extent $N_\tau = 4$ and several spatial lattice sizes N_σ (left) as a function of the temperature in units of the infinite volume critical temperature T_c. The curves show spline interpolations. The right-hand part of the figure shows the volume dependence of $G_{\Theta\Theta}(\tau, T)/T^5$ at the critical point $T = T_c$. The curves show finite size scaling fits as explained in the text. The hardly visible open squares on top of the $\tau T = 1/2$ data are the data obtained for $\tau T = 1/4$ shifted by $\Delta C_\sigma = 18.0675$.

TABLE III. Fit parameters for fits to $G_{\Theta\Theta}(\tau, T)/T^5$ using Eq. (22) and for the volume dependence of $G_{\Theta\Theta}(\tau, T_c)/T_c^5$. The last row gives parameters of combined fits to the data for $\tau T = 1/4$ and $1/2$.

	τT	A_+	B_+	B_-	C	D	A_σ	B_σ	C_σ
free	1/4	16.3(2.1)	−0.59(19)	−1.9(1.4)	21.5(5.7)	−150(53)	9.2(1.2)	−2.4(1.0)	26.3(2.7)
fit	1/2	16.4(2.2)	−0.76(18)	−2.2(1.5)	3.7(3.4)	−95(32)	9.1(1.2)	−2.4(0.9)	8.4(2.9)
combined	1/4	16.5(1.3)	−0.74(10)	−2.12(86)	21.8(2.1)	−143(20)	9.15(73)	−2.39(59)	26.4(1.9)
fit	1/2				3.8(1.9)	−87(18)			8.3(1.7)

tances. From this combined fit we find that $A_\sigma(\tau T = 1/2)/A_\sigma(\tau T = 1/4) = 1.01 \pm 0.19$ and $B_\sigma(\tau T = 1/2)/B_\sigma(\tau T = 1/4) = 1.01 \pm 0.58$. The fit results[2] are summarized in Table III, which also includes the coefficients extracted from an equally good combined fit of both data sets obtained requiring the same coefficients A_+, B_+, and B_- for the two Euclidean distances.

The analysis presented above establishes that the correlation function of the trace of the energy-momentum tensor shows the expected universal singular structure of the specific heat in a 3-dimensional Ising model. In the vicinity of T_c the singular contributions to $G_{\Theta\Theta}(\tau, T)$ are found to be independent of Euclidean time. We have performed corresponding analyses for $N_\tau = 6$ and 8 which are con-

sistent with these findings, however statistical errors are larger in these cases.

The analysis presented above confirms that the correlation function $G_{\Theta\Theta}$ contains a constant contribution that arises from low frequency modes that lead to the singular behavior of the energy-energy correlation function at T_c. In the continuum limit this constant will reflect the τ independence of the correlators $G_{\epsilon\epsilon}$ and $G_{\epsilon P}$.

C. Critical behavior of $G_{\Theta\Theta}(\tau, T)$ in the vicinity of T_c

In the previous subsection we established through a finite size scaling analysis at T_c that the singular terms in $G_{\Theta\Theta}(\tau, T)$ are independent of Euclidean time. We will show here that this is also the case at temperatures in the vicinity of T_c at which the universal scaling behavior allows to separate the long-distance singular behavior of $G_{\Theta\Theta}(\tau, T)$ from regular contributions.

In Fig. 5 (left) we show infinite volume extrapolated results for $G_{\Theta\Theta}(\tau, T)/T^5$ at two distances, $\tau T = 1/4$ and $1/2$. Here we left out the data at T_c. At both distances

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[2]Note that the coefficients of the correction-to-scaling term, $B_\pm$, are negative. Although these amplitudes are nonuniversal it has been noted previously that for the 3-dimensional Ising model on various lattice geometries this correction term comes out to be negative [29].
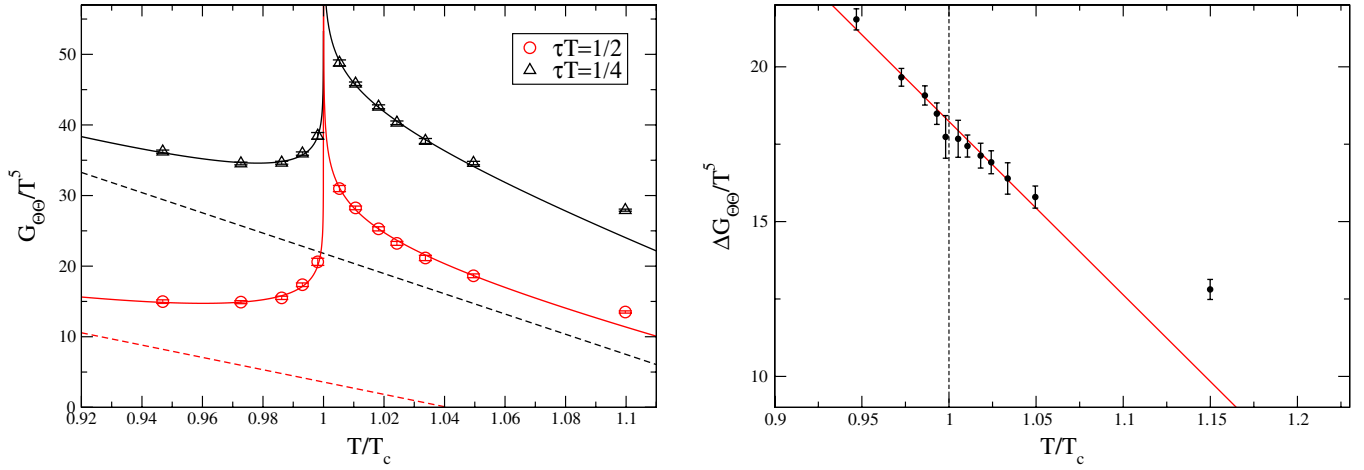
FIG. 5 (color online).   The correlation function of the trace of the energy-momentum tensor, $G_{\Theta\Theta}(\tau, T)/T^5$ at distances $\tau T = 1/4$ and $1/2$ (left). The curve shows the combined fit of both data sets performed with the *Ansatz* given in Eq. (22). The dashed straight lines give the regular part of the fit. The right-hand part of this figure shows the difference $\Delta G_{\Theta\Theta} = G_{\Theta\Theta}(1/4T, T) - G_{\Theta\Theta}(1/2T, T)$.

$G_{\Theta\Theta}(\tau, T)/T^5$ increases rapidly when T approaches T_c; above T_c the correlation function changes by a factor 2 in the temperature interval $[1.01T_c, 1.1T_c]$.

In the vicinity of T_c we expect that the data are well described by the scaling ansatz,

$$G_{\Theta\Theta}(\tau, T)/T^5 = A_{\pm} t^{-\alpha}(1 + B_{\pm} t^{\omega}) + C + Dt, \quad (22)$$

where A_+, $B_{\pm}$, C, D are free parameters, which again all may depend on Euclidean time; $A_+ = g_A A_-$ and α, ω, and g_A are the known Ising exponents given in Table II. We again have performed fits for the two data sets at $\tau T = 1/2$ and $1/4$ separately as well as simultaneously. The ansatz given in Eq. (22) gives good fits in the interval $T/T_c \in [0.94, 1.05]$. As expected the parameters C extracted for $\tau T = 1/2$ and $1/4$, are consistent with the constant terms obtained from the finite size scaling analysis at T_c.

Again we find that within errors the fit parameters A_+ and $B_{\pm}$ are independent of τ; a combined fit of $G_{\Theta\Theta}(1/4T, T)/T^5$ and $G_{\Theta\Theta}(1/2T, T)/T^5$ with common amplitudes A_+ and $B_{\pm}$ also gives a $\chi^2/\text{dof} = 1.1$. The resulting fit parameters are also given in Table III.

We thus conclude that in the vicinity of the critical point of the SU(2) gauge theory the temperature dependence of the correlation function of the energy-momentum tensor reflects the singular behavior of the specific heat of the 3-dimensional Ising model. More striking may be the fact that the amplitudes for the singular terms are to a good approximation independent of τ. However, as discussed above this is consistent with the expected τ independence of the energy-energy correlation function [10].

In order to eliminate the leading singular behavior from $G_{\Theta\Theta}(\tau, T)$ it thus suffices to consider differences of the correlation function evaluated for different Euclidean-time separations, e.g.

$$\frac{\Delta G_{\Theta\Theta}}{T^5} = \frac{G_{\Theta\Theta}(1/4T, T)}{T^5} - \frac{G_{\Theta\Theta}(1/2T, T)}{T^5}. \quad (23)$$

Figure 5 (right) shows that this difference indeed stays finite at T_c. However, as shown in Fig. 2, at nonzero value of the lattice spacing $\Delta G_{\Theta\Theta}$ does not yet coincide with $9\Delta G_{PP}$ as it will in the continuum limit.

We finally show in Fig. 6 the temperature dependence of the different correlation functions, $G_{XX}(\tau, T)$, contributing to $G_{\Theta\Theta}$. As can be seen the entire strong temperature dependence of $G_{\Theta\Theta}$ in the vicinity of T_c indeed arises from $G_{\epsilon\epsilon}$. All other correlators show only a weak temperature dependence in the vicinity of T_c. In particular, we find that the pressure-pressure correlation function stays finite at T_c and varies little in its vicinity. As will be discussed in the next section this has immediate consequences for the determination of the bulk viscosity from correlation functions of the energy-momentum tensor.
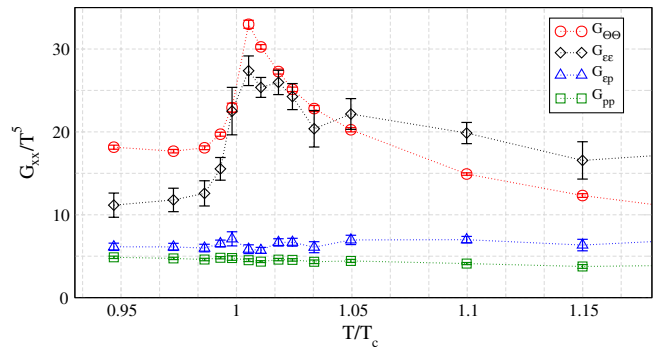

FIG. 6 (color online).   Correlation functions $G_{XX}(1/2T, T)$ contributing to the correlator of the trace anomaly. Shown are results obtained from calculations on lattices with $N_\tau = 4$ in the infinite volume limit.

V. BULK VISCOSITY AND RELAXATION TIME SCALE

Correlation functions of the energy-momentum tensor are widely used to gain information on transport coefficients. The spectral representation of these Euclidean-time correlation functions and their relation to real-time Kubo formulas allow, in principle, to extract transport coefficients from the zero frequency limit of spectral functions [4]. We want to discuss here some consequences for the calculation of transport coefficients, in particular, the bulk viscosity, that can be drawn from the analysis of correlation functions presented in the previous sections.

The bulk viscosity can be extracted from the spectral representation of pressure-pressure correlation functions [4],

$$G_{PP}(\tau, T) = \int_0^\infty d\omega \rho_{PP}(\omega, T) K(\tau, \omega, T), \qquad (24)$$

where

$$K(\tau, \omega, T) = \frac{\cosh(\omega(\tau - \frac{1}{2T}))}{\sinh(\frac{\omega}{2T})} \qquad (25)$$

and $\rho_{PP}(\omega, T)$ is the spectral density at vanishing momentum. Making use of the definition of transport coefficients via the Kubo formulas it is readily seen that the bulk viscosity ζ is related to the low frequency limit of this spectral function [30],

$$\zeta(T) = \pi \lim_{\omega \to 0} \frac{\rho_{PP}(\omega, T)}{\omega}. \qquad (26)$$

From Eq. (18) it is obvious that this can also be obtained from a spectral analysis of the correlator of the trace of the energy-momentum tensor. However, as we have verified explicitly in the previous sections this correlation function receives a constant contribution from correlation functions containing the energy operator. This constant term is reflected in zero frequency contribution to $\rho_{\Theta\Theta}$, i.e. a δ function at vanishing frequencies,

$$G_{\Theta\Theta}(\tau, T) = \int_0^\infty d\omega \rho_{\Theta\Theta}(\omega, T) K(\tau, \omega, T), \qquad (27)$$

with

$$\rho_{\Theta\Theta}(\omega, T) = 9\rho_{PP}(\omega, T) + T \partial_T(\epsilon - 6P)\omega\delta(\omega). \qquad (28)$$

As the bulk viscosity is obtained from the spectral function in the limit of vanishing frequency one should eliminate the contribution arising from $\rho_{\Theta\Theta}(0)$. This can, of course, easily be done by analyzing the subtracted correlation function $\Delta G_{\Theta\Theta}(\tau, T)$ introduced in Eq. (23). From Fig. 5 (right) we know that this correlation function will stay finite at T_c. Its spectral representation is given by

$$\Delta G_{\Theta\Theta}(\tau, T) = 9 \int_0^\infty d\omega \rho_{PP}(\omega, T) \frac{\cosh(\omega(\tau - \frac{1}{2T})) - 1}{\sinh(\frac{\omega}{2T})}. \qquad (29)$$

The low frequency part of the spectral function, $\rho_{PP}(\omega, T)$, is expected to be linear, the slope being proportional to the bulk viscosity. The rise of the spectral functions ends at a characteristic frequency ω_0, which is identified as a characteristic relaxation time scale, $\tau_R \equiv 1/\omega_0$, for density fluctuations. Of course, ω_0 will be temperature dependent and, in fact, is expected to vanish at a second order phase transition point where fluctuations grow and relaxation times will diverge.

We may split the spectral function ρ_{PP} into a low frequency part and a large frequency part, $\rho_{PP}(\omega, T) \equiv f(\omega, \omega_0) + \rho_>(\omega, T)$. The low frequency behavior of the spectral function may be modeled by a Breit-Wigner ansatz,

$$f(\omega, \omega_0) = \frac{9}{\pi} \zeta(T)\omega \frac{\omega_0^2}{\omega^2 + \omega_0^2} \qquad . \qquad (30)$$

With this we can write the low frequency contribution to the pressure correlator, $G_{PP}(\tau T, T)$, as

$$\begin{aligned} G_{PP}^{\text{low}}(\tau, T) &= \frac{9}{\pi} \zeta \omega_0 \int_0^\infty dx \frac{\omega_0 x}{1 + x^2} \\ &\quad \times \frac{\cosh(\omega_0 x(\tau - 1/2T)) - 1}{\sinh(\omega_0 x/2T)} \\ &= 9T\zeta(T)\omega_0(T)\Big(1 - \frac{\omega_0(T)}{2T\pi} \\ &\quad \times \ln[2 - 2\cos(2\pi T\tau)] + \mathcal{O}(\omega_0^2)\Big). \quad (31) \end{aligned}$$

Here we have assumed in the second equality that ω_0 becomes small in the vicinity of T_c and an expansion in ω_0 thus is appropriate. Similarly we can obtain the low frequency contribution to $\Delta G_{\Theta\Theta}(\tau, T)$. Both correlation functions have the same functional dependence on Euclidean time for $\tau T \simeq 1/2$. It is this region, where one can hope that the low frequency part of the spectral function dominates the τ dependence of the correlation functions and thus will allow a determination of transport properties.

From Eq. (31) we find that the low frequency part of the spectral function, which is proportional to bulk viscosity, contributes a term proportional to $\zeta\omega_0$ to the value of the correlation function at the midpoint, $\tau T = 1/2$. As discussed in the introduction, universality arguments given for the scaling of the bulk viscosity in the vicinity of a critical point in the Ising universality class suggest, however, that the ratio of bulk viscosity and relaxation time, $\zeta/\tau_R \equiv \zeta\omega_0$, vanishes as the inverse of the specific heat, i.e. it is expected to be proportional to the velocity of sound. In the vicinity of T_c the correlation function $G_{\Theta\Theta}(1/2T, T)$ thus is dominated by contributions from the large frequency,

regular part of the spectral function. The sensitivity to bulk viscosity also is small away from the midpoint where contributions from the low frequency part of the spectral function are proportional to $\zeta \omega_0^2$. To separate this contribution from the large frequency contributions it would be necessary to become sensitive to the characteristic logarithmic dependence on Euclidean time that arises from the low frequency part of the spectral function.

VI. CONCLUSIONS

We have shown that the correlation function of the trace of the energy-momentum tensor, $G_{\Theta\Theta}$, diverges at the critical point of the finite temperature SU(2) gauge theory in $(3 + 1)$ dimensions. The divergence is controlled by the critical exponent α of the 3-dimensional Ising model that also controls the divergence of the specific heat. Furthermore, we have shown that $G_{\Theta\Theta}$ becomes independent of Euclidean time at the critical point, which indicates that its spectral representation receives a contribution from

a δ function at zero frequency. This confirms the observations made in Ref. [10].

To get direct access to the bulk viscosity ζ and the characteristic frequency ω_0 that controls the relaxation time for density fluctuations will require much more detailed studies of the dependence of correlation functions on Euclidean time. In fact, we conclude that a determination of bulk viscosity from correlation functions of the energy-momentum tensor is not possible without a simultaneous determination of the relevant relaxation time scale.

ACKNOWLEDGMENTS

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