Crossing the phantom divide with parametrized post-Friedmann dark energy

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Dark energy models with a single scalar field cannot cross the equation of state divide set by a cosmological constant. More general models that allow crossing require additional degrees of freedom to ensure gravitational stability. We show that a parameterized post-Friedmann description of cosmic accelerzation provides a simple but accurate description of multiple scalar field crossing models. Moreover the prescription provides a well-controlled approximation for a wide range of "smooth" dark energy models. It conserves energy and momentum and is exact in the metric evolution on scales well above and below the transition scale to relative smoothness. Standard linear perturbation tools have been altered to include this description and made publicly available for studies of the dark energy involving cosmological structure out to the horizon scale.

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I. INTRODUCTION

Observational constraints on the acceleration of the expansion have continued to close in on a dark energy equation of state of a cosmological constant $w_e = -1$ that delineates the phantom divide. Testing the small deviations from that value in the future requires a description of the dark energy that allows the equation of state to evolve across the phantom divide possibly multiple times.

It is well known that single scalar fields are gravitationally unstable to such a crossing of the phantom divide [1–3]. Dark energy that is minimally coupled to the matter requires additional degrees of freedom to cross the divide stably. While specific models with multiple fields can be constructed [2,4] they are cumbersome or impossible to implement in a general analysis of the dark energy.

The usual approach in the literature for finessing such cases is to artificially turn off the dark energy perturbations explicitly or implicitly by limiting the range of observables. Doing so violates energy-momentum conservation whenever $w_e \neq -1$ and leads to inconsistencies between the Einstein equations for the evolution of the metric due to the Bianchi identities which can persist even on small scales. Though the impact of perturbations tend to be small near $w_e = -1$, cosmological constraints often require the exploration of a large swath of parameter space around the maximum likelihood. Excising the instability around the transition provides another, albeit rather *ad hoc* approach [5].

In this paper, we show that the so-called parameterized post-Friedmann (PPF) approach to describing linear metric evolution in a Friedmann-Robertson-Walker (FRW) universe provides a simple solution to this dilemma. The PPF framework was ostensibly introduced for describing modified gravity theories under a metric framework with strict local conservation of energy and momentum [6]. As such it also applies to dark energy models [7] and, in particular, the class of models which have a well-defined Jeans scale under which the dark energy is smooth compared to the dark matter. This framework also has the benefit of being an exact description for the metric evolution well above and well below this scale and hence provides a very well-controlled approximation that is simple to implement in an Einstein-Boltzmann linear perturbation code.

II. PHANTOM DIVIDE AND SCALAR FIELDS

Minimally coupled scalar field dark energy models that evolve across the phantom divide require new internal degrees of freedom to maintain gravitational stability. To see this fact, consider the conservation equation for the momentum density $(\rho_e \mathbf{u}_e)_i \equiv T^0{}_i$ (see, e.g. [7] for an explicit derivation)

$$u_e' = (3w_e - 1)u_e + k_H \frac{\delta p_e}{\rho_e} + (1 + w_e)k_H A, \qquad (1)$$

where $' \equiv d/d\ln a$, $w_e = p_e/\rho_e$, $k_H = k/aH$, and A is the gravitational potential in an arbitrary gauge.

The relationship between the pressure and density fluctuation defines a sound speed. For a single scalar field with kinetic and potential degrees of freedom, this relationship is most simply described in a coordinate system that comoves with the dark energy such that the momentum density and transverse spatial metric fluctuations vanish [8,9]. From an arbitrary gauge, this quantity is obtained by a gauge transformation that changes the time slicing

$$\delta\rho^{(\mathrm{rest})} = \delta\rho_e + 3\rho_e \frac{u_e}{k_H},$$
$$\delta p^{(\mathrm{rest})} = \delta p_e + 3\frac{p_e'}{\rho_e'}\rho_e \frac{u_e}{k_H}, \qquad (2)$$

which defines a sound speed

$$c_s^2 \equiv \frac{\delta p_e^{(\mathrm{rest})}}{\delta \rho_e^{(\mathrm{rest})}}, \tag{3}$$

bringing the momentum conservation equation to

$$u_e' = 3\left(w_e + c_s^2 - \frac{p_e'}{\rho_e'} - \frac{1}{3}\right)u_e + k_H c_s^2 \delta_e + (1 + w_e)k_H A,$$

where $\delta_e = \delta\rho_e/\rho_e$.

For a single scalar field, the rest or zero momentum gauge corresponds to time slicing where the field, and hence the potential energy, is constant leaving the energy density and pressure to be defined by fluctuations in the kinetic energy. For a canonical kinetic term $c_s^2 = 1$ representing the familiar kinetic energy dominated equation of state of such scalars.

The dark energy system is completed by the continuity equation

$$\delta_e' + 3(c_s^2 - w_e)\delta_e + 9\left(c_s^2 - \frac{p_e'}{\rho_e'}\right)\frac{u_e}{k_H} = -k_H u_e - (1 + w_e)(k_H B + 3H_L'), \tag{4}$$

where B is the space-time piece and H_L the space-space curvature piece of the metric fluctuations in an arbitrary gauge [10].

Taking $c_s^2 > 0$ makes dark energy perturbations Jeans stable in the regime $k_H c_s \gg 1$ so long as p_e'/ρ_e' remains finite. In the matter dominated epoch, matter density fluctuations continue to grow and so the Poisson equation for $\Phi \equiv H_L^{(\mathrm{newt})}$ in the Newtonian gauge

$$c_K k^2 \Phi = 4\pi G a^2 \sum_i \rho_i \delta_i^{(\mathrm{rest})} \tag{5}$$

becomes dominated by the matter component, i.e. the dark energy is relatively smooth compared with the matter

$$\rho_e \delta_e^{(\mathrm{rest})} \ll \rho_T \delta_T^{(\mathrm{rest})}, \tag{6}$$

where "T" denotes all other components excluding the dark energy. Here $c_K = 1 - 3K/k^2$ where K is the background curvature.

This condition for smoothness is not the same as setting all dark energy perturbations to zero which causes inconsistencies between the four scalar Einstein equations. In particular, in the synchronous gauge, where the dark matter momentum also vanishes, some care must be taken even at $k_H c_s \gg 1$ since the dark energy momentum is no longer negligible in comparison [7].

When $w_e = -1$, p_e'/ρ_e' will generally diverge leading to an instability in the evolution of perturbations if c_s^2 is held fixed [2]. The problem arises since the change in the time slicing required to reach the rest or constant field gauge becomes infinite when the field has no kinetic energy. Viewed as a fluid, the problem is that the relative fluid velocity $v_e = u_e/(1 + w_e)$ becomes undefined if the momentum remains finite.

If the dark energy is a composite of fields then c_s^2 need not itself be fixed by fundamental properties of the scalars at the crossing. For example if the dark energy were composed of the sum of minimally coupled fields each with sound speed c_e^2 then the pressure fluctuation is described by

$$\delta p_e = c_s^2 \rho_e \delta_e + 3\left(c_s^2 - \frac{p_e'}{\rho_e'}\right)\frac{\rho_e u_e}{k_H} = c_e^2 \rho_e \delta_e + 3\left(c_e^2 \frac{\rho_e u_e}{k_H} - \sum_\alpha \frac{p_{e\alpha}'}{\rho_{e\alpha}'}\frac{\rho_{e\alpha} u_{e\alpha}}{k_H}\right), \tag{7}$$

which implicitly defines c_s^2 as a function of the individual momenta. As long as no individual component crosses the phantom divide $w_{e\alpha} \neq -1$, the pressure fluctuations are no longer singular.

Simple two-field models which cross $w_e = -1$ were constructed in [2,4]. Unfortunately, this construction is cumbersome for obtaining a general function $w_e(\ln a)$ constrained to match cosmological distances.

The spirit of this construction is more broadly applicable. Models that cross the phantom divide must have internal degrees of freedom to ensure u_e remains finite through the crossing. Provided they do, energy-momentum conservation and the requirement that the dark energy is smooth compared with the matter for $c_e k_H \gg 1$ impose nearly unique constraints on their parameterization. We will use these requirements to construct a PPF description of dark energy crossing.

III. PPF DESCRIPTION

The PPF description of dark energy replaces the density and momentum components with a single joint dynamical variable Γ but retains strict conservation of energy and momentum in its equation of motion.

Given the conservation laws, PPF and more generally any minimally coupled dark energy parameterization requires two closure conditions to complete the system [8]. The first can be taken as a condition on the anisotropic stress. For scalar fields this quantity vanishes for linear field perturbations.

In the discussion above, the second condition was taken to be the relationship between pressure and density fluctuations. We saw that this choice leads to difficulties in parameterizing models that cross the phantom divide due to the appearance of singularities in the equation of motion for the momentum density.

The PPF description replaces this condition on the pressure perturbations with a direct relationship between the momentum density of the dark energy and that of matter on large scales and a transition scale under which the dark energy explicitly becomes relatively smooth. The latter implicitly describes the momentum density on small scales.

The strategy for choosing these relationships is to match the evolution of the metric exactly for scales much larger and much smaller than the transition scale.

Let us start with the Γ variable. The conditions that the anisotropic stress of the dark energy vanishes and the Poisson equation is normal on small scales reduces the defining equation to (see [7] Eq. 30)

$$\Gamma \equiv \frac{4\pi G a^2}{c_K k^2}\rho_T\Delta_T - \Phi, \tag{8}$$

where

$$\Delta_T \equiv \delta_T^{(\mathrm{rest})} = \delta_T + 3u_T/k_H \tag{9}$$

is the density fluctuation in the zero momentum (total matter or comoving) gauge of the matter excluding the dark energy. Comparing this relationship with the Poisson equation (5) yields

$$\Gamma = -\frac{4\pi G a^2}{k^2 c_K}\rho_e\delta_e^{(\mathrm{rest})}. \tag{10}$$

The condition that the dark energy becomes smooth relative to the matter in their respective rest gauges then becomes a direct requirement on the evolution of Γ.

Now let us examine the second closure relation. On large scales, energy and momentum conservation determine that the curvature $\zeta \equiv H_L^{(T)}$ in the total matter gauge is conserved up to order k_H^2 in a flat universe with adiabatic fluctuations [10]. The corresponding evolution equation for the Newtonian potentials Φ and Ψ is closed by the anisotropic stress assumption [11,12].

The Einstein equation governing ζ reads

$$\zeta' = \xi - \frac{K}{(aH)^2}\frac{V_T}{k_H} - \frac{4\pi G}{H^2}\rho_e\frac{U_e}{k_H}, \tag{11}$$

where $V_T = B^{(T)}$, $\xi = A^{(T)}$, and $U_e = u_e^{(T)}$ in this gauge and

$$\xi = -\frac{\Delta p_T - \frac{2}{3}c_K p_T\pi_T}{\rho_T + p_T}, \tag{12}$$

with π_T as the anisotropic stress of the total matter and $\Delta p_T = \delta p_T^{(T)}$. Since $V_T = \mathcal{O}(k_H\zeta)$ we can enforce this condition on large scales by parameterizing a relationship between U_e and V_T at $k_H \ll 1$

$$\lim_{k_H\ll 1} U_e = -\frac{H^2}{12\pi G\rho_e}c_K k_H^2 V_T f_\zeta, \tag{13}$$

where $f_\zeta(\ln a)$ is a function of time only, i.e. $U_e = \mathcal{O}(k_H^3\zeta)$. Note that U_e is the dark energy momentum relative to the frame defined by zero matter momentum. The scaling requirement is that to first order in k_H, the dark energy and matter rest frames are the same at large scales. Both the single and multiple scalar field equations exhibit this property given that $\Delta p_e/\rho_e$ and ξ are $\mathcal{O}(k_H^2\zeta)$ (see [13] Eq. (115) for an explicit expression). Once U_e and its evolution are determined, δp_e follows by momentum conservation with no singularities encountered as w_e crosses the phantom divide.

The PPF description can be made an exact match at large scales to any given system of scalar fields with an arbitrary equation of state evolution $w_e(\ln a)$ by solving the full equations at $k_H \to 0$ and inferring f_ζ for the evolution of all other finite k modes. However, for the purpose of obtaining the correct evolution for the *metric* or gravitational potentials, even this is not necessary as long as $f_\zeta \lesssim \rho_e/(\rho_T + \rho_e)$. By construction, the metric condition $\zeta' = \mathcal{O}(k_H^2)$ is satisfied and the specific value chosen just determines the ratio of the dark energy to matter contributions to the metric fluctuations. Since ultra large scales where the dark energy is not smooth are generally probed gravitationally via gravitational redshifts and perhaps lensing in the future, it suffices for most purposes to simply take $f_\zeta = 0$.

The final piece in the construction is to assure that the dark energy becomes smooth relative to the matter inside a transition scale $c_e k_H = 1$ while exactly conserving energy and momentum locally by taking [6,7]

$$(1 + c_\Gamma^2 k_H^2)[\Gamma' + \Gamma + c_\Gamma^2 k_H^2\Gamma] = S, \tag{14}$$

where

$$S = -\frac{4\pi G}{H^2}[f_\zeta(\rho_T + p_T) - (\rho_e + p_e)]\frac{V_T}{k_H}. \tag{15}$$

This relation explicitly guarantees that $\Gamma \ll V_T/k_H = \mathcal{O}(\Phi)$ for $c_\Gamma k_H \gg 1$. Comparison with Eq. (8) shows that this condition requires the dark energy to be smooth relative to the matter [see Eq. (6)]. While the specifics of how rapidly the dark energy becomes negligible in contributing

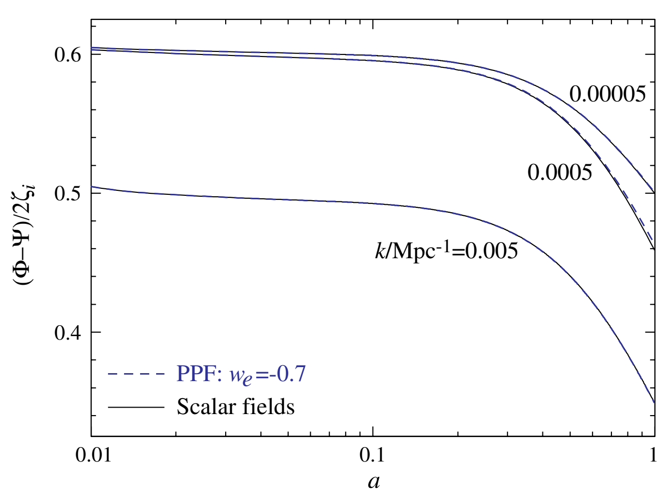

FIG. 1 (color online). PPF vs scalar field calculation of the evolution of the potential responsible for gravitational redshifts and lensing $(\Phi - \Psi)/2$ for a $w_e = -0.7$ model (flat, with $\Omega_m = 0.31$ and $h = 0.64$). Curves are normalized to the initial curvature ζ_i.

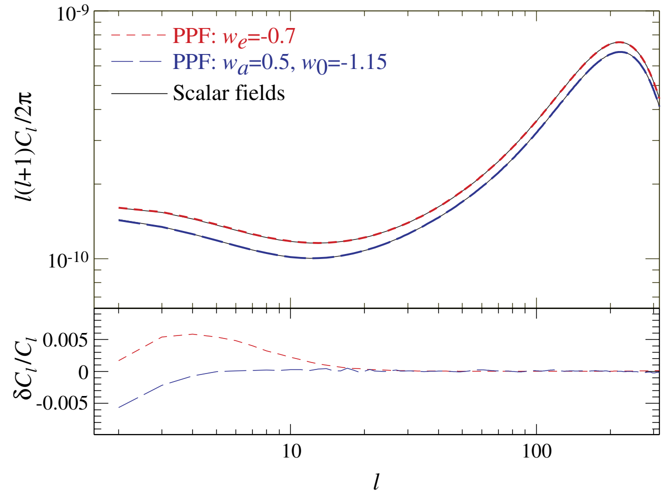

FIG. 2 (color online). PPF vs scalar field calculation of the CMB anisotropy power spectrum for the $w_e = -0.7$ model of Fig. 1 and a two-field crossing model that approximates $w_0 = -1.15$ and $w_a = 0.5$ (flat, with $\Omega_m = 0.26$ and $h = 0.74$).

to gravitational potentials below this scale depend on the specific form of Eq. (14), the net impact on observable quantities of this choice is small as we shall see below.

The main task is to calibrate the scale of the transition, i.e. a relationship between c_Γ and c_e. We find that $c_\Gamma = 0.4c_e$ matches the evolution of scalar field models. We show an example with $w_e = -0.7$ of the evolution of the quantity $(\Phi - \Psi)/2$ that is responsible for gravitational redshifts and lensing in Fig. 1. Metric evolution for scales $c_e k_H \ll 1$ and $c_e k_H \gg 1$ show exact agreement between the PPF prescription and the direct scalar field calculation by construction. In this model the two limits differ by 44% in the fractional change in the gravitational potential during the acceleration epoch.

In Fig. 2, we compare the cosmic microwave background (CMB) temperature power spectrum in the PPF approximation to the direct scalar field calculation for the $w_e = -0.7$ model and a two-field model that approximates $w_e(\ln a) = w_0 + (1 - a)w_a$ with $w_0 = -1.15$ and $w_a = 0.5$ [2]. The latter model has w_e evolving across -1.

In summary, we have shown that the PPF prescription for describing the evolution of metric perturbations in a FRW universe is sufficiently general to encompass multiple scalar field models whose joint equation of state evolves across the phantom divide at $w_e = -1$. This description is accurate to well below the cosmic variance limit as long as the transition scale to relative smoothness is comparable to the horizon. Moreover it is in fact exact for the metric evolution well above and well below the transition scale. As such it provides a well-controlled approximation for any model where the energy and momentum of the dark energy is separately conserved and features a transition of this type.

This prescription is useful for the joint analysis of growth and distance measures of the dark energy, especially those involving horizon scale perturbations like the integrated Sachs-Wolfe effect in the CMB. The CAMB Einstein-Boltzmann package has been altered to include PPF [14] and a version for the dark energy has been made publicly available [15]. Potential future uses include principal component approaches to dark energy constraints where w_e is allowed to cross the phantom divide multiple times (see e.g. [16]). Here explicit matching to multiple scalar fields is cumbersome if not impossible. The PPF prescription provides a simple but general approach that explicitly enforces conservation of energy and momentum and all of the Einstein equations removing potential ambiguities to the meaning of a "smooth" dark energy component.

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[1] A. Vikman, Phys. Rev. D **71**, 023515 (2005).
[2] W. Hu, Phys. Rev. D **71**, 047301 (2005).
[3] R. R. Caldwell and M. Doran, Phys. Rev. D **72**, 043527 (2005).
[4] B. Feng, X.-L. Wang, and X.-M. Zhang, Phys. Lett. B **607**, 35 (2005).
[5] G. Huey, arXiv:astro-ph/0411102.
[6] W. Hu and I. Sawicki, Phys. Rev. D **76**, 104043 (2007).
[7] W. Hu, Phys. Rev. D **77**, 103524 (2008).
[8] W. Hu, Astrophys. J. **506**, 485 (1998).
[9] C. Armendariz-Picon, V. Mukhanov, and P. Steinhardt, Phys. Rev. Lett. **85**, 4438 (2000).
[10] J. M. Bardeen, Phys. Rev. D **22**, 1882 (1980).
[11] W. Hu and D. J. Eisenstein, Phys. Rev. D **59**, 083509 (1999).
[12] E. Bertschinger, Astrophys. J. **648**, 797 (2006).
[13] W. Hu, astro-ph/0402060.
[14] W. Fang *et al.*, arXiv:0808.2208 [Phys. Rev. D (to be published)].
[15] http://camb.info/ppf/ (PPF module for CAMB).
[16] D. Huterer and G. Starkman, Phys. Rev. Lett. **90**, 031301 (2003).