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Mimicking the QCD equation of state with a dual black hole

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We present numerical and analytical studies of the equation of state of translationally invariant black hole solutions to five-dimensional gravity coupled to a single scalar. As an application, we construct a family of black holes that closely mimics the equation of state of quantum chromodynamics at zero chemical potential.

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I. INTRODUCTION

In the supergravity approximation, the near-extremal D3 brane has equation of state $s \propto T^3$, with a constant of proportionality that is $3/4$ of the free-field value for the dual $\mathcal{N} = 4$ super-Yang-Mills theory [1]. The speed of sound is $c_s = 1/\sqrt{3}$, as required by conformal invariance. On the other hand, the speed of sound of a thermal state in quantum chromodynamics (QCD) has an interesting and phenomenologically important dependence on temperature, with a minimum near the crossover temperature T_c. Lattice studies of the equation of state are too numerous to cite comprehensively, but they include [2] (for pure glue), [3] (a review article), and [4,5] (recent studies with $2 + 1$ flavors).

We would like to find a five-dimensional gravitational theory that has black hole solutions whose speed of sound as a function of temperature mimics that of QCD. We will not try to include chemical potentials or to account for chiral symmetry breaking. We will not try to include asymptotic freedom, but instead will limit our computation to $T \lesssim 4T_c$ and assume conformal behavior in the extreme UV. We will not even try to give an account of confinement, except insofar as the steep rise in the number of degrees of freedom near the crossover temperature T_c is recovered in our setup, corresponding to a minimum of c_s near T_c. We will not try to embed our construction in string theory, but instead adjust parameters in a five-dimensional gravitational action to recover approximately the dependence $c_s(T)$ found from the lattice. That action is

$$S = \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right], \quad (1)$$

up to total derivative terms which affect the evaluation of the free energy, but not the entropy or temperature. We will not include higher derivative corrections, which would arise from α' and loop corrections if the theory (1) were embedded explicitly in string theory.

The ansatz we will study is

$$ds^2 = e^{2A}(-h dt^2 + d\vec{x}^2) + e^{2B} \frac{dr^2}{h}, \quad (2)$$

where A, B, and h are functions of r, and ϕ is also some function of r. This ansatz is dictated by the symmetries: we want translation invariance in the $\mathbf{R}^{3,1}$ directions parametrized by $(t, \vec{x})$, and we want $SO(3)$ symmetry in the $\vec{x}$ directions but not $SO(3, 1)$ boost invariance—because boost invariance is broken by finite temperature. Assuming conformal behavior in the extreme UV means that we assume the geometry (2) is asymptotically anti-de Sitter. A regular horizon arises when h has a simple zero. Let us say the first such zero (that is, the one closest to the conformal boundary) is at $r = r_H$. It is assumed that A and B are finite and regular at $r = r_H$. Standard manipulations lead to the following formulas for entropy density and temperature:

$$s = \frac{2\pi}{\kappa_5^2} e^{3A(r_H)} \qquad T = \frac{e^{A(r_H) - B(r_H)} |h'(r_H)|}{4\pi}, \quad (3)$$

and once these quantities are known, the speed of sound can be read off from

$$c_s^2 = \frac{d \log T}{d \log s}. \quad (4)$$

The formula for the entropy density in (3) comes from the Bekenstein-Hawking result $S = A/4G_N$, where A is the area of the horizon (really a volume in our case) and $G_N = \kappa_5^2/8\pi$. The formula for the temperature comes from Hawking's result $T = \kappa/2\pi$, where κ is the surface gravity at the horizon.

By adjusting $V(\phi)$ one might expect to be able to recover any prespecified $c_s(T)$, at least within certain limits—perhaps including that $s(T)/T^3$ should be monotonic or some similar criterion (in this connection see [6]). The main aim of this paper is to characterize how $V(\phi)$ translates into $c_s(T)$ and vice versa. In Sec. II, we begin with the simplest possible case: $c_s(T)$ constant. It translates into $V(\phi) = V_0 e^{\gamma\phi}$ for some $V_0 < 0$ and γ related to c_s. In

Sec. III, we tackle the general case, exploiting a weak form of integrability of the equations resulting from plugging (2) into (1). In Sec. V, we exhibit several examples. These include a particular $V(\phi)$ whose corresponding $c_s(T)$ curve closely mimics that of QCD.[1] We close with a discussion in Sec. VI.

The results in this paper are based in large part on [9], and aspects of them will also be summarized in [10]. After this paper appeared on arXiv as a preprint, we received [11], which has some overlap with our results.

II. CHAMBLIN-REALL SOLUTIONS AND AN ADIABATIC GENERALIZATION OF THEM

In a D-dimensional conformal field theory (meaning a conformal field theory in $D-1$ spatial dimensions plus one time dimension), the entropy density must obey

$$s \propto T^{D-1}, \tag{5}$$

simply because this expression is dimensionally correct and there is no scale other than the temperature that would permit a more complicated dependence. So the speed of sound is $c_s = 1/\sqrt{D-1}$. If $D > 4$, then we could obtain a nonconformal theory in four dimensions by compactifying our CFT_D on a $D-4$-dimensional torus. (A similar idea has been considered in [12,13].) Doing so should not change the speed of sound: a planar sound wave in the resulting 4-dimensional theory would correspond to a planar sound wave in the original theory whose propagation is in the direction of the uncompactified directions.

The AdS_{D+1}-Schwarzschild solution is an extremum of the action

$$S = \frac{1}{2\kappa_{D+1}^2}\int d^{D+1}x\sqrt{-\hat{g}}\left[\hat{R} + \frac{D(D-1)}{L^2}\right], \tag{6}$$

and it takes the form

$$d\hat{s}^2 = \frac{L^2}{z^2}\left(-hdt^2 + d\hat{\vec{x}}^2 + \frac{dz^2}{h}\right), \tag{7}$$

where

$$h = 1 - \frac{z^D}{z_H^D}. \tag{8}$$

We use hats to distinguish $D+1$-dimensional quantities from 4-dimensional ones. It is easy to see that $T \propto 1/z_H$ and $s \propto 1/z_H^{D-1}$, so that $s \propto T^{D-1}$ as the conformal field theory requires. Suppose we now perform the dimensional reduction described in the previous paragraph on the solution (7). In slightly more generality than we need, the Kaluza-Klein ansatz is

$$d\hat{s}^2 = \exp\left\{\sqrt{\frac{2}{3}\frac{D-4}{D-1}}\phi\right\}ds^2 + \exp\left\{-\sqrt{\frac{6}{(D-1)(D-4)}}\phi\right\}ds_{D-4}^2, \tag{9}$$

where ds^2 is a five-dimensional metric and ds_{D-4}^2 is the flat metric on a torus $\mathbf{T}^{D-4}$, whose shape we will assume to be square with side length ℓ, so that $\mathrm{Vol}\mathbf{T}^{D-4} = \ell^{D-4}$. All components of the metric, and also ϕ, are assumed to depend only on the five-dimensional coordinates. It is assumed that ℓ is a constant; variation of the size of the torus is taken care of by the exponential prefactor multiplying ds_{D-4}^2 in (9). The particular coefficients in the exponentials were chosen presciently to obtain a simple five-dimensional action. Comparing the general form (9) with the specific solution (7), one finds

$$\begin{aligned} ds^2 &= \left(\frac{L}{z}\right)^{(2/3)(D-1)}\left(-hdt^2 + d\vec{x}^2 + \frac{dz^2}{h}\right) \\ e^{\phi} &= \left(\frac{z}{L}\right)^{\sqrt{(2/3)(D-1)(D-4)}}, \end{aligned} \tag{10}$$

where $h = 1 - z^D/z_H^D$ as in (8). The line element (10) was obtained by the authors of [14], but not via Kaluza-Klein reduction; instead, they considered black hole solutions to the equations of motion from an action like (1) with potentials of the form

$$V(\phi) = V_0 e^{\gamma\phi}, \tag{11}$$

with $V_0 < 0$. To see that the solutions have to come out the same in either approach, let us carry through the Kaluza-Klein reduction at the level of the action by plugging (9) into (6). After performing the trivial integral over T^{D-4}, one obtains

$$S = \frac{\ell^{D-4}}{2\kappa_{D+1}^2}\int d^5x\sqrt{-g}\left[R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right], \tag{12}$$

where $V(\phi)$ has the form (11) with the identifications

$$V_0 = -\frac{D(D-1)}{L^2} \qquad \gamma = \sqrt{\frac{2}{3}\frac{D-4}{D-1}}. \tag{13}$$

Evidently, the length scale ℓ enters the action only as a prefactor, which can be absorbed into a definition of the five-dimensional gravitational constant: $\kappa_5^2 = \kappa_{D+1}^2/\ell^{D-4}$.

By comparing the expression for γ in (13) with the result $c_s = 1/\sqrt{D-1}$ for the speed of sound, we find

$$c_s^2 = \frac{1}{3} - \frac{\gamma^2}{2}. \tag{14}$$

This result can be derived more directly by showing that $s \propto T^{6/(2-3\gamma^2)}$ for Chamblin-Reall solutions: explicitly,

[1] Two earlier studies [7,8] of thermodynamic properties of putative holographic duals to QCD obtain $s \sim T^3 e^{-T_0^2/T^2}$ for some constant T_0 in a region above the deconfinement transition temperature. But the line elements considered in these studies are simply assumed, rather than derived starting from a fully specified classical action, as our solutions are.

$$s = \frac{1}{2\kappa_5^2}\left(\frac{L}{z_H}\right)^{D-1} = \frac{1}{2\kappa_5^2}\exp\left\{-\frac{\phi_H}{\gamma}\right\}$$
$$T = \frac{D}{4\pi z_H} = \frac{1}{4\pi L}\frac{8-3\gamma^2}{2-3\gamma^2}\exp\left\{\left(\frac{\gamma}{2}-\frac{1}{3\gamma}\right)\phi_H\right\}, \quad (15)$$

where ϕ_H is the value of ϕ at the horizon. The dimensional reduction we have described is well defined only for integer $D > 4$, but for the purposes of the computations presented here, it can be any real number greater than 4.

Suppose we rewrite the result (15) as

$$\log s = -\frac{\phi_H}{\gamma} + (\text{constant in } \phi_H)$$
$$\log T = \left(\frac{\gamma}{2} - \frac{1}{3\gamma}\right)\phi_H + (\text{constant in } \phi_H). \quad (16)$$

Given (16) and the formula $\gamma = V'(\phi)/V(\phi)$, a natural next step would be to guess the following dependence of s and T on ϕ_H when γ is a slowly varying function of ϕ rather than a constant:

$$\log s = -\int_{\phi_0}^{\phi_H} d\phi \frac{V(\phi)}{V'(\phi)} + (\text{slowly varying in } \phi_H)$$
$$\log T = \int_{\phi_0}^{\phi_H} d\phi \left(\frac{1}{2}\frac{V'(\phi)}{V(\phi)} - \frac{1}{3}\frac{V(\phi)}{V'(\phi)}\right) + (\text{slowly varying in } \phi_H). \quad (17)$$

The lower limit ϕ_0 in the integrals is an arbitrary cutoff. If we assume that $V(\phi)$ has a maximum at $\phi = 0$ and an expansion of the form (37), then $V(\phi)/V'(\phi) \approx -12/(m^2L^2\phi)$ near $\phi = 0$. So the integrals in (17) diverge if they are continued all the way to $\phi = 0$, and the cutoff ϕ_0 must be chosen to have the same sign as ϕ_H to avoid this divergence.

A consequence of the estimates (17) is a simple formula for the speed of sound:

$$c_s^2 = \frac{d\log T/d\phi_H}{d\log s/d\phi_H} \approx \frac{1}{3} - \frac{1}{2}\frac{V'(\phi_H)^2}{V(\phi_H)^2}. \quad (18)$$

Another consequence is

$$\log\frac{s}{T^3} = -\frac{3}{2}\int_{\phi_0}^{\phi_H} d\phi \frac{V'(\phi)}{V(\phi)} + (\text{slowly varying in } \phi_H)$$
$$= -\frac{3}{2}\log\frac{V(\phi_H)}{V(\phi_0)} + (\text{slowly varying in } \phi_H). \quad (19)$$

A simpler way of expressing (19) is

$$\frac{s}{T^3} \propto |V(\phi_H)|^{-3/2}, \quad (20)$$

up to corrections from slowly varying terms. This is interesting because s/T^3 is one way of defining the effective number of degrees of freedom available to a system, and we see from (20) that it is closely related to the potential evaluated at the horizon.

The results (18) and (19) are a first attempt at solving the problem of translating an arbitrary $V(\phi)$ to an equation of state, or an arbitrary equation of state into $V(\phi)$. Here is how the latter process would work. Suppose one specifies the equation of state as $s = s(T)$. Ignoring corrections to (19), one has

$$f \equiv -\frac{2}{3}\log\frac{s}{T^3} = \log\frac{V}{V_0}, \quad (21)$$

where V_0 is some constant. Let us regard f as the independent variable. Because $V = V_0 e^f$, all we need is to find $\phi = \phi(f)$, and we will have a parametric representation of $V(\phi)$. One may rewrite (18) as

$$c_s^2 = \frac{1}{3} - \frac{1}{2(d\phi/df)^2}, \quad (22)$$

where corrections have again been ignored. Knowing $s(T)$ with good precision means one can express c_s^2 as a function of f. Then (22) can readily be integrated to give

$$\phi(f) = \int \frac{df}{\sqrt{2(\frac{1}{3} - c_s(f)^2)}}. \quad (23)$$

The integral is left in indefinite form because adding a constant to ϕ is obviously allowed.

We stress that the result of plugging (23) into the form $V = V_0 e^f$ will result in a $V(\phi)$ that only approximately reproduces the desired $s(T)$. If the speed of sound varies rapidly with T, the approximation may be poor. In Sec. IV we will show how to improve this approximation without resorting to differential equations that cannot be explicitly solved in terms of indefinite integrals.

III. A NONLINEAR MASTER EQUATION

There is a residual gauge freedom in the ansatz (2), namely, reparametrization of the radial direction. A convenient gauge choice, which should be at least piecewise valid in any geometry where the scalar is nonvanishing, is to set $r = \phi$. Then the line element becomes

$$ds^2 = e^{2A}(-hdt^2 + d\vec{x}^2) + e^{2B}\frac{d\phi^2}{h}, \quad (24)$$

and the equations of motion following from the action (1) take the form

$$A'' - A'B' + \tfrac{1}{6} = 0, \quad (25a)$$
$$h'' + (4A' - B')h' = 0, \quad (25b)$$
$$6A'h' + h(24A'^2 - 1) + 2e^{2B}V = 0, \quad (25c)$$
$$4A' - B' + \frac{h'}{h} - \frac{e^{2B}}{h}V' = 0, \quad (25d)$$

where primes denote $d/d\phi$. The first two of these equations come from the tt and x^1x^1 Einstein equations; the third comes from the $\phi\phi$ Einstein equation; and the last comes from the scalar equation of motion. There is typically some redundancy in equations obtained from classical gravity, with or without matter. In the case of (25), the redundancy is that the Φ derivative of the third equation follows algebraically from the four equations listed.

The ansatz (24) has one peculiar feature: e^{2B} must have dimensions of length squared. This is because ϕ is dimensionless.

The equations of motion (25) enjoy a weak form of integrability, in the following sense: If a smooth "generating function" $G(\phi)$ is specified, then it is possible to find a black hole solution where $A'(\phi) = G(\phi)$ in terms of indefinite integrals of simple functions of $G(\phi)$ and $G'(\phi)$. But $V(\phi)$ itself is expressed in terms of such integrals, and one cannot easily find all the possible $G(\phi)$ that lead to a specified $V(\phi)$. In other words, there can be simple analytic solutions to (25) for special $V(\phi)$ at a special value of the temperature, but as far as we know, there is no nontrivial $V(\phi)$ (i.e., none besides the exponential form) for which analytic solutions exist over a continuous range of temperatures.

To understand this claim of integrability, let us consider $A'(\phi) = G(\phi)$ to be fixed as a function of ϕ and work out $A(\phi)$, $B(\phi)$, $h(\phi)$, and $V(\phi)$. The first of these is trivial:

$$A(\phi) = A_0 + \int_{\phi_0}^{\phi} d\tilde{\phi} G(\tilde{\phi}). \tag{26}$$

Computing $B(\phi)$ is immediate once one solves (25a) for B':

$$B(\phi) = B_0 + \int_{\phi_0}^{\phi} d\tilde{\phi} \frac{G'(\tilde{\phi}) + 1/6}{G(\tilde{\phi})}. \tag{27}$$

Next, one observes that (25b) is straightforwardly solved once one knows $A(\phi)$ and $B(\phi)$:

$$h(\phi) = h_0 + h_1 \int_{\phi_0}^{\phi} d\tilde{\phi} e^{-4A(\tilde{\phi})+B(\tilde{\phi})}. \tag{28}$$

Now, (25c) can be solved for $V(\phi)$ in terms of known quantities:

$$V(\phi) = h(\phi) \frac{e^{-2B(\phi)}}{2} \left(1 - 24G(\phi)^2 - 6G(\phi)\frac{h'(\phi)}{h(\phi)}\right). \tag{29}$$

The constraint Eq. (25d) does not yield any new information.

If one chooses

$$G(\phi) = -\frac{1}{3\gamma}, \tag{30}$$

then by working through (26)–(29) one recovers the Chamblin-Reall solution in the form

$$\begin{aligned} A(\phi) &= A_0 - \frac{\phi - \phi_0}{3\gamma} \\ B(\phi) &= B_0 - \frac{\gamma}{2}(\phi - \phi_0) \\ h(\phi) &= h_0 + \tilde{h}_1 \exp\left\{\frac{8 - 3\gamma^2}{6\gamma}(\phi - \phi_0)\right\} \\ V(\phi) &= V_0 e^{\gamma\phi}, \end{aligned} \tag{31}$$

where

$$\tilde{h}_1 = \frac{6e^{-4A_0+B_0}\gamma}{8 - 3\gamma^2} h_1 \qquad V_0 = -\frac{8 - 3\gamma^2}{6\gamma} e^{-2B_0-\gamma\phi_0} h_0. \tag{32}$$

By choosing

$$\phi_0 = \frac{1}{\gamma}(\log h_0 - 2B_0), \tag{33}$$

one obtains $V(\phi)$ in a form that does not depend on any integration constants at all. This situation is very special and corresponds to the fact that for $V(\phi) \propto -e^{\gamma\phi}$ one can find a whole family of analytic solutions parametrized by ϕ_H.

By differentiating combinations of (26)–(29), one can derive the following "nonlinear master equation:"

$$\frac{G'}{G + V/3V'} = \frac{d}{d\phi} \log\left(\frac{G'}{G} + \frac{1}{6G} - 4G - \frac{G'}{G + V/3V'}\right). \tag{34}$$

Describing (34) as the master equation is appropriate because if one starts knowing $V(\phi)$ and manages to solve (34), then to obtain a black hole solution one only needs to perform the "trivial" integrations (26)–(28). A numerically efficient strategy for obtaining an equation of state given $V(\phi)$ centers around solving (34) numerically. In more detail, the procedure is

(1) Choose the value ϕ_H of the scalar at the horizon.
(2) Find a series solution of the nonlinear master equation around $\phi = \phi_H$.
(3) Seed a numerical integrator like Mathematica's `NDSolve` close to $\phi = \phi_H$ using the series solution.
(4) Integrate the nonlinear master equation up to a value of ϕ close to a maximum, corresponding to the asymptotically anti-de Sitter part of the geometry.
(5) Extract s and T from integrals of simple functions of $G(\phi)$.

Most of these steps require further explanation, which will occupy the remainder of this section.

At the horizon, h has a simple zero, and the other quantities appearing in (25c) and (25d) are finite. Evaluating these two equations at the horizon gives

$$V(\phi_H) = -3e^{-2B(\phi_H)}G(\phi_H)h'(\phi_H)$$
$$V'(\phi_H) = e^{-2B(\phi_H)}h'(\phi_H), \tag{35}$$

which implies that $G + V/3V'$ vanishes at the horizon. Starting from this condition, one may develop a power series solution around the horizon

$$G(\phi) = -\frac{1}{3}\frac{V(\phi_H)}{V'(\phi_H)} + \frac{1}{6}\left(\frac{V(\phi_H)V''(\phi_H)}{V'(\phi_H)^2} - 1\right)(\phi - \phi_H) + O[(\phi - \phi_H)^2]. \tag{36}$$

This series solution can be developed to any desired order without encountering arbitrary integration constants.

To understand the asymptotic behavior far from the horizon, let us specialize to the case where $V(\phi)$ has a maximum at $\phi = 0$

$$V(\phi) = -\frac{12}{L^2} + \frac{1}{2}m^2\phi^2 + O(\phi^3), \tag{37}$$

where $m^2 < 0$ in order for $\phi = 0$ to be a maximum. The gauge theory operator $\mathcal{O}_\phi$ dual to ϕ has dimension Δ, where

$$\Delta(\Delta - 4) = m^2L^2. \tag{38}$$

We will be primarily interested in the case where Δ is close to 4. It helps our intuition at this point to pass to a more standard gauge: instead of setting $r = \phi$, we can set $B = 0$ to obtain

$$ds^2 = e^{2A}(-hdt^2 + d\vec{x}^2) + \frac{dr^2}{h}. \tag{39}$$

We note that A and h appearing in (39) are precisely the same as when we use the $r = \phi$ gauge, only expressed as functions of r rather than ϕ. Large r corresponds to the region far from the horizon, and the leading asymptotic behavior of solutions there is

$$A \approx \frac{r}{L} \qquad h \approx 1 \qquad \phi \approx (\Lambda L)^{4-\Delta}e^{(\Delta-4)A}. \tag{40}$$

Each approximate equality in (40) means that the ratio of the two expressions on each side approaches 1 as r becomes large. The behavior of ϕ indicates a relevant deformation of the conformal field theory

$$\mathcal{L} = \mathcal{L}_{\text{CFT}} + \Lambda^{4-\Delta}O_\phi. \tag{41}$$

As we vary temperature to compute the equation of state, we should of course keep Λ fixed. A simple way to do this is to set $\Lambda L = 1$. Then the last equation in (40) becomes

$$A(\phi) = \frac{\log\phi}{\Delta - 4} + o(1) \tag{42}$$

for small ϕ, where $o(1)$ means a quantity that is parametrically smaller than 1 in the limit under consideration—so in the limit $\phi \to 0$ in the case of (42).

In order to compute the entropy density using (3), we need to know $A(\phi_H)$. This can be extracted by comparing (42) to (26) with ϕ_0 set equal to ϕ_H and A_0 set equal to $A_H = A(\phi_H)$

$$A(\phi) = A_H + \int_{\phi_H}^{\phi} d\tilde{\phi}G(\tilde{\phi}) = \frac{\log\phi}{\Delta - 4} + o(1). \tag{43}$$

Solving for A_H and then taking $\phi \to 0$, one finds

$$A_H = \frac{\log\phi_H}{\Delta - 4} + \int_0^{\phi_H} d\phi\left[G(\phi) - \frac{1}{(\Delta - 4)\phi}\right]. \tag{44}$$

The integral converges because the explicit $1/\phi$ term cancels against the leading behavior of $G(\phi)$ for small ϕ. Plugging (44) into the expression for entropy density from (3), we find at last

$$s = \frac{2\pi}{\kappa_5^2}\phi_H^{3/(\Delta-4)}\exp\left\{3\int_H^{\phi_H} d\phi\left[G(\phi) - \frac{1}{(\Delta - 4)\phi}\right]\right\}. \tag{45}$$

A similar formula for the temperature may be derived starting with the observation that

$$\frac{dr}{d\phi} = -e^B, \tag{46}$$

where B is the function controlling the $g_{\phi\phi}$ metric component in $r = \phi$ gauge. One obtains (46) by comparing (24) with (39). The sign is based on assuming that ϕ increases from 0 to positive values as r decreases from $+\infty$ to finite values. The first equation in (40) implies $dA/dr \to 1/L$ as $r \to \infty$. Combining this with (46) gives

$$G = \frac{dA}{d\phi} = \frac{dr}{d\phi}\frac{dA}{dr} \approx -e^B\frac{1}{L}, \tag{47}$$

where the approximate equality means that the ratio of the last expression to the previous ones approaches 1 as ϕ goes to 0. In summary,

$$1 \approx -LG(\phi)e^{-B(\phi)}, \tag{48}$$

using the same sense of approximate equalities. (Recall that e^{-B} has dimensions of inverse length, while $G(\phi)$ is dimensionless, so (48) is dimensionally correct.) We may cast the expression for temperature from (3) in terms of $G(\phi)$ by multiplying by 1 in the form indicated in (48):

$$\begin{aligned}T &= \frac{e^{A_H - B(\phi_H)}|h'(\phi_H)|}{4\pi} \approx \frac{e^{A_H - B(\phi_H)}h'(\phi_H)}{4\pi}LG(\phi)e^{-B(\phi)} \\ &= \frac{Le^{-2B(\phi_H)}G(\phi_H)h'(\phi_H)}{4\pi} \\ &\quad \times \exp\left\{A_H + B(\phi_H) - B(\phi) - \log\frac{G(\phi_H)}{G(\phi)}\right\} \\ &= -\frac{LV(\phi_H)}{12\pi}\exp\left\{A_H + \int_\phi^{\phi_H}\frac{d\tilde{\phi}}{6G(\tilde{\phi})}\right\}.\end{aligned} \tag{49}$$

In the second step, we assumed that $h'(\phi_H) < 0$, which is the expected sign when ϕ vanishes on the boundary and is positive at the horizon. In the last step, we used the first equation from (35) to simplify the prefactor and also

$$B(\phi_H) - B(\phi) = \log\frac{G(\phi_H)}{G(\phi)} + \int_{\phi}^{\phi_H} \frac{d\tilde{\phi}}{6G(\tilde{\phi})}, \qquad (50)$$

which is a consequence of (27). The integral in the last expression of (51) converges even when $\phi \to 0$. We use (44) to eliminate A_H from (51) and obtain at last

$$T = \frac{\phi_H^{1/(\Delta-4)}}{\pi L}\frac{V(\phi_H)}{V(0)}\exp\left\{\int_0^{\phi_H} d\phi\left[G(\phi) - \frac{1}{(\Delta-4)\phi} + \frac{1}{6G(\phi)}\right]\right\}. \qquad (51)$$

The measure of the number of degrees of freedom that is easiest for us to access is

$$\frac{s}{T^3} = 2\pi^4\frac{L^3}{\kappa_5^2}\frac{V(0)^3}{V(\phi_H)^3}\exp\left\{-3\int_0^{\phi_H}\frac{d\phi}{6G(\phi)}\right\}, \qquad (52)$$

where to obtain the right-hand side we simply combined (45) and (51). When ϕ_H is small, entropy and temperature become large because of the factors $\phi_H^{3/(\Delta-4)}$ and $\phi_H^{1/(\Delta-4)}$ in (45) and (51). In this limit, the integrals in the exponent become negligible, and (52) becomes

$$\frac{s}{T^3} \approx 2\pi^4\frac{L^3}{\kappa_5^2}. \qquad (53)$$

So we recover conformal behavior in the ultraviolet, as expected.

IV. AN APPROXIMATE DETERMINATION OF THE EQUATION OF STATE

The adiabatic formulas (17) work well when ϕ_H is in a region where $V(\phi)$ is nearly exponential, but they do not work well for small ϕ_H, where $V(\phi)$ is close to attaining a maximum. This is shown in Fig. 1 for $V(\phi) = -(12/L^2)\cosh(\phi/2)$. On the other hand, it is easy to extract asymptotic formulas valid in the $\phi_H \to 0$ limit from (45) and (51): using the expansion (37), one finds

$$T = \frac{1}{\pi L}\phi_H^{1/(\Delta-4)} \qquad s = \frac{2\pi}{\kappa_5^2}\phi_H^{3/(\Delta-4)}. \qquad (54)$$

We wish to find formulas that interpolate smoothly between (17) and (54) and involve at most indefinite integrals, not solutions to a difficult, nonlinear, second-order differential equation such as (34). We start by noting that the formulas (54) together with (37) imply

$$\frac{d\log T}{d\phi_H} \approx \frac{\Delta}{4}\left(\frac{1}{2}\frac{V'(\phi_H)}{V(\phi_H)} - \frac{1}{3}\frac{V(\phi_H)}{V'(\phi_H)}\right)$$
$$\frac{d\log s}{d\phi_H} \approx \frac{\Delta}{4}\left(-\frac{V(\phi_H)}{V'(\phi_H)}\right) \quad \text{for small } \phi_H, \qquad (55)$$

where approximate equality means that the ratio of the two sides tends to 1 as $\phi_H \to 0$. On the other hand, provided $V(\phi)$ tends to an exponential form $V_0 e^{\gamma\phi}$ for large ϕ, the adiabatic approximation becomes good if ϕ_H is large. So for such potentials, (17) can be rephrased as

$$\frac{d\log T}{d\phi_H} \approx \frac{1}{2}\frac{V'(\phi_H)}{V(\phi_H)} - \frac{1}{3}\frac{V(\phi_H)}{V'(\phi_H)}$$
$$\frac{d\log s}{d\phi_H} \approx -\frac{V(\phi_H)}{V'(\phi_H)} \quad \text{for large } \phi_H. \qquad (56)$$

Comparing (55) and (56), we are led to introduce "fudge factors" $\rho_s(\phi_H)$ and $\rho_T(\phi_H)$ such that

$$\frac{d\log T}{d\phi_H} = \rho_T(\phi_H)\left(\frac{1}{2}\frac{V'(\phi_H)}{V(\phi_H)} - \frac{1}{3}\frac{V(\phi_H)}{V'(\phi_H)}\right)$$
$$\frac{d\log s}{d\phi_H} = \rho_s(\phi_H)\left(-\frac{V(\phi_H)}{V'(\phi_H)}\right). \qquad (57)$$

We can rephrase (55) and (56) as the statements that both $\rho_T(\phi_H)$ and $\rho_s(\phi_H)$ interpolate between $\Delta/4$ at small ϕ_H

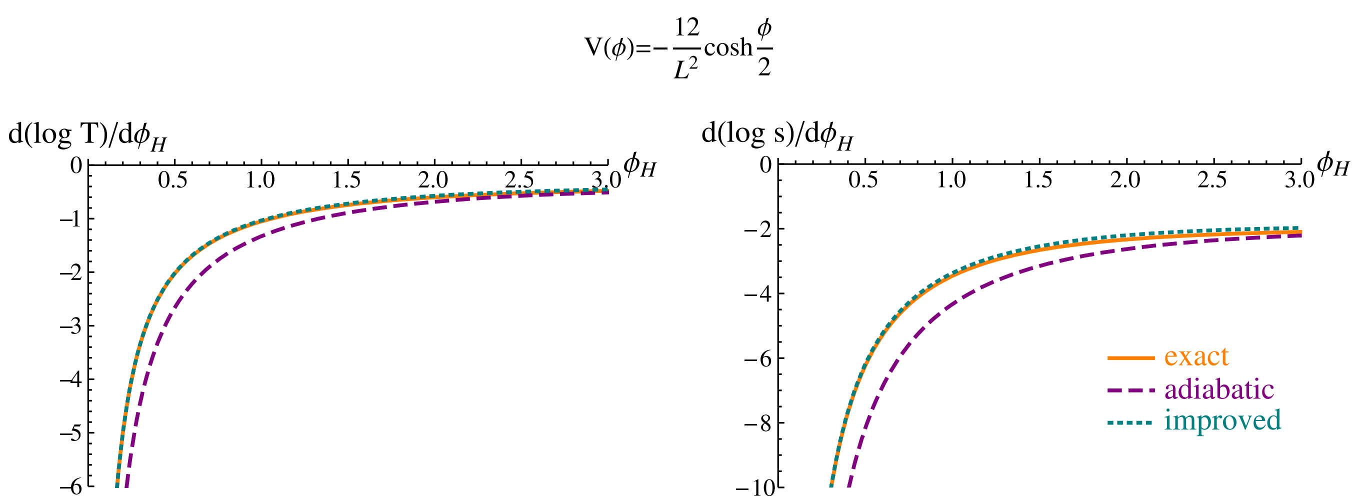

FIG. 1 (color online). A comparison of the exact $\frac{d(\log s)}{d\phi_H}$ and $\frac{d(\log T)}{d\phi_H}$ for $V(\phi) = -\frac{12}{L^2}\cosh\frac{\phi}{2}$ with the adiabatic approximation (17), and the improved approximation scheme, (57) with the choice (58).

and 1 at large ϕ_H. Our improved estimate of the equation of state consists simply of guessing an interpolating function with these properties. The guess is

$$\rho_T(\phi_H) \approx \rho_s(\phi_H) \approx \rho(\phi_H) \equiv 1 + \frac{V(0)}{V(\phi_H)}\left(\frac{\Delta}{4} - 1\right). \tag{58}$$

In (58) approximate equality means that ρ_T, ρ_s, and ρ are supposed to be nearly equal for all ϕ_H. But away from the small ϕ_H and large ϕ_H limits, (58) is not a controlled approximation, in the sense that there is not a parameter that we can tune to make it better. It is nevertheless useful for quickly determining the qualitative features of an equation of state given $V(\phi)$, as illustrated in Fig. 1. There might be a better choice of $\rho_T(\phi_H)$ and $\rho_s(\phi_H)$, or perhaps even a systematic expansion for them in terms of powers of the potential and its derivatives.

Starting from (57), we have immediately

$$c_s^2 = \frac{d\log T/d\phi_H}{d\log s/d\phi_H} = \frac{\rho_T(\phi_H)}{\rho_s(\phi_H)}\left(\frac{1}{3} - \frac{1}{2}\frac{V'(\phi_H)^2}{V(\phi_H)^2}\right). \tag{59}$$

Thus, assuming $\rho_T(\phi_H) \approx \rho_s(\phi_H)$ is the same as assuming that the speed of sound, as a function of ϕ_H, is well approximated by the adiabatic formula (18).

V. EXAMPLES

In this section, we will present results for c_s^2 or s/T^3 versus T based on numerical integration of the nonlinear master Eq. (34), for several different choices of the scalar potential $V(\phi)$.

The simplest analytical form that interpolates between a maximum at $\phi = 0$ and exponential behavior for large ϕ is

$$V(\phi) = V_{\cosh}(\phi) \equiv -\frac{12}{L^2}\cosh\gamma\phi. \tag{60}$$

The adiabatic treatment discussed in Sec. II leads us to expect that the speed of sound will be $c_s = \sqrt{\frac{1}{3} - \frac{\gamma^2}{2}}$ for large ϕ_H. So in order to have stable black holes in this regime, we must have $\gamma \leq \sqrt{2/3}$. This bound can be regarded as an application of the correlated stability conjecture (CSC) [15,16]. But there is a tighter bound on γ coming from the behavior near $\phi = 0$

$$V_{\cosh}(\phi) = -\frac{12}{L^2} - \frac{6\gamma^2}{L^2}\phi^2 + O(\phi^4), \tag{61}$$

so $m^2 = -12\gamma^2/L^2$. In order to satisfy the Breitenlohner-Freedman bound $m^2L^2 \geq -4$ [17–19], we must have $\gamma \leq 1/\sqrt{3}$. This means that the minimum speed of sound we can arrange at large ϕ_H using the pure cosh potential (60) is $c_s = 1/\sqrt{6} \approx 0.41$. The behavior of c_s^2 as a function of T is shown in Fig. 2 for $\gamma = 1/\sqrt{6}$.

If one uses the potential (60), then c_s in the infrared is tied to the dimension Δ of the operator that breaks conformal invariance in the ultraviolet. Let us consider a minimal generalization that loosens this artificial constraint:

$$V(\phi) = V(\gamma, b; \phi) \equiv -\frac{12}{L^2}\cosh\gamma\phi + b\phi^2. \tag{62}$$

A parameter equivalent to γ, as before, is the speed of sound in the infrared $c_s^2 = \frac{1}{3} - \frac{\gamma^2}{2}$. With γ fixed, a parameter equivalent to b is the dimension Δ of the operator dual to ϕ near the UV fixed point

$$b = \frac{6\gamma^2}{L^2} + \frac{\Delta(\Delta - 4)}{2L^2}. \tag{63}$$

Note that taking Δ close to 4 amounts to making $V(\phi)$ nearly quartic near its maximum. As we will report in more

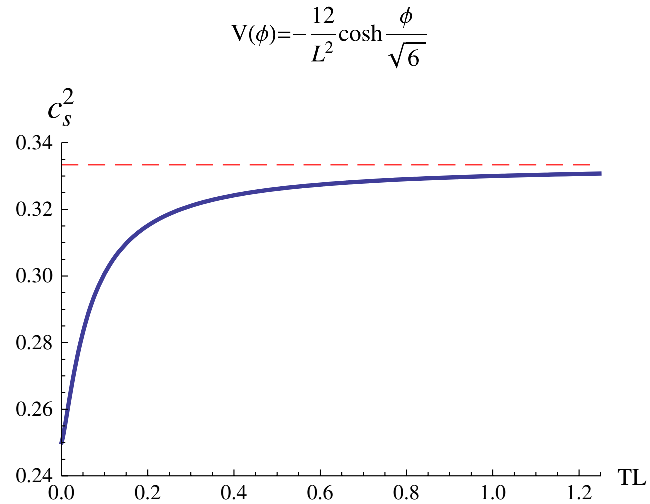

FIG. 2 (color online). The speed of sound for $V(\phi) = -\frac{12}{L^2} \times \cosh\frac{\phi}{\sqrt{6}}$.

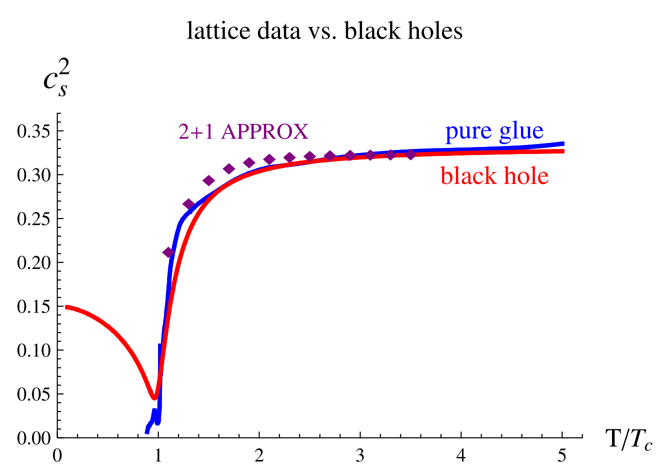

FIG. 3 (color online). The equation of state of a black hole (red) compared to the lattice equation of state for pure glue (blue) and 2 + 1-flavor QCD. The pure glue curve is based on [2] and private communications from F. Karsch. The 2 + 1-flavor QCD points are based on [5].

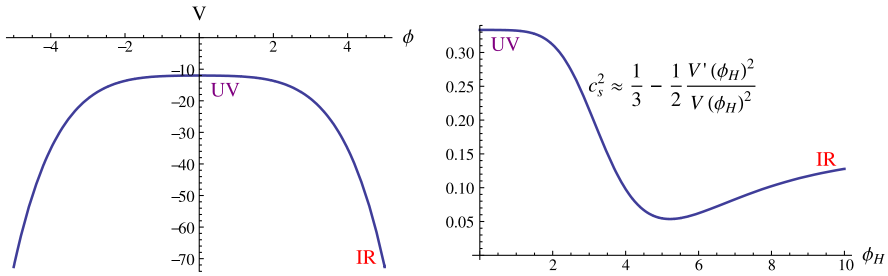

FIG. 4 (color online). Left: The potential (62) with the parameter choices (63) that give an equation of state resembling QCD's. Right: Although $V(\phi)$ is relatively featureless, the adiabatic formula $c_s^2 \approx \frac{1}{3} - \frac{1}{2}\frac{V'(\phi_H)^2}{V(\phi_H)^2}$ suggests that the equation of state resulting from it will indeed exhibit a low minimum for the speed of sound.

detail in [10], the choice

$$\gamma = 0.606 \qquad b = \frac{2.06}{L^2}, \tag{64}$$

corresponding to $c_s^2 = 0.15$ in the infrared and $\Delta = 3.93$, leads to an equation of state that bears a striking resemblance to the one expected for QCD: see Fig. 3. It may seem surprising that the most distinctive feature of the equation of state of QCD, namely, a smooth but rapid crossover, emerges from a potential that is nearly featureless. To gain some intuition about why this happened, consider again the adiabatic approximation (18) to the speed of sound. When ϕ_H is close to where the nearly quartic behavior of $V(\phi)$ rolls over into nearly exponential behavior, this approximate formula predicts that c_s^2 dips to a fairly low value, only to rise back up again for larger ϕ_H toward its infrared limit, 0.15. See Fig. 4.

Other behaviors emerge from the potential (62) for other choices of b and γ. For example, if $\gamma > \sqrt{2/3}$, the adiabatic approximation suggests that there is a minimum temperature $T_{\rm min}$ for black hole solutions. A particular case is illustrated in Fig. 5. Solutions with very low entropy have high temperature and negative specific heat, and they are always thermodynamically disfavored compared with a branch of high-entropy solutions. Presumably there is a first-order transition to geometries with no horizon, similar to the Hawking-Page transition [20]. This transition probably happens at a temperature above $T_{\rm min}$. It is worth noting that the specific heat diverges at $T_{\rm min}$, because T reaches a minimum as a function of ϕ_H, while S is varying smoothly.

It is also possible to have a first-order transition between high entropy and low entropy black holes. An example where this happens is illustrated in Fig. 6. For a finite range of ϕ_H, the speed of sound is imaginary, indicating a Gregory-Laflamme instability. This touches once again on the CSC, so let us pause to review it. It was proposed in [15,16] and further argued in [21] that, in the absence of conserved charges related to gauge symmetries, existence of a Gregory-Laflamme instability [22,23] is equivalent to positivity of the specific heat $C = T\partial S/\partial T$. According to a more general version of the CSC, dynamical stability of a horizon is equivalent to positivity of an appropriate

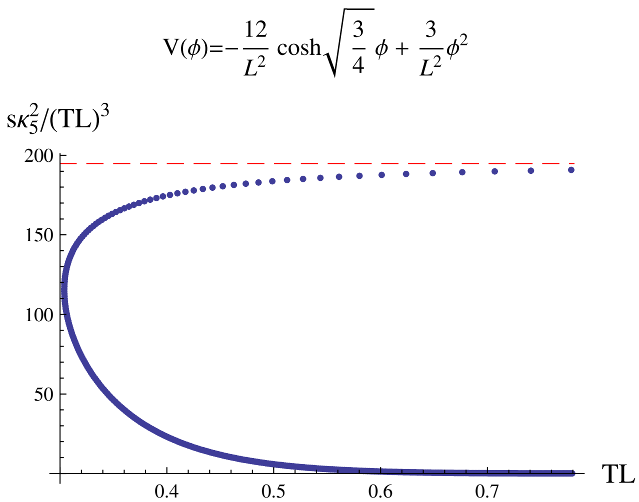

FIG. 5 (color online). The equation of state for $V(\phi) = -\frac{12}{L^2}\cosh\sqrt{\frac{3}{4}}\phi + \frac{3}{L^2}\phi^2$.

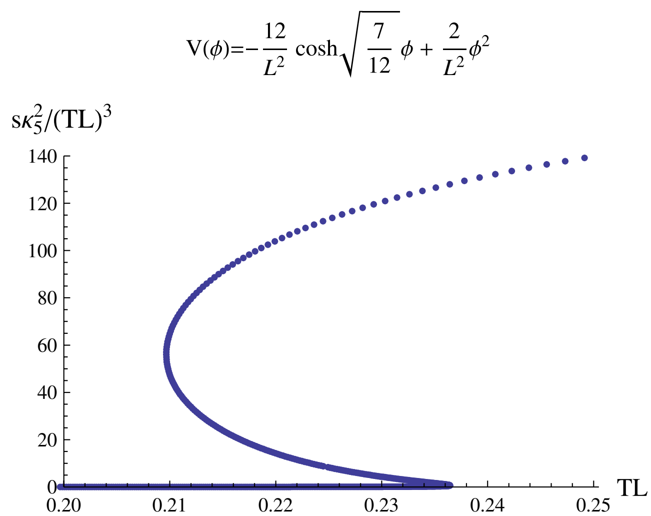

FIG. 6 (color online). The equation of state for $V(\phi) = -\frac{12}{L^2}\cosh\sqrt{\frac{7}{12}}\phi + \frac{2}{L^2}\phi^2$.

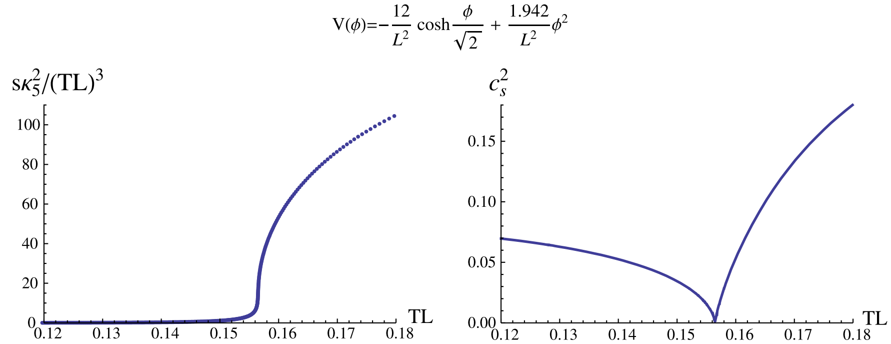

FIG. 7 (color online). The equation of state and the speed of sound for $V(\phi) = -\frac{12}{L^2}\cosh\frac{\phi}{\sqrt{2}} + \frac{1.942}{L^2}\phi^2$. The point where $c_s^2 = 0$ is a second-order phase transition. If one considers instead $V(\phi) = -\frac{12}{L^2}\cosh\frac{\phi}{\sqrt{2}} + b\phi^2$ for $b > 1.942$, the transition becomes first order, while if $b < 1.942$, it is a crossover.

Hessian matrix of susceptibilities, one of which is the specific heat [15,16]. As pointed out in [24], $C > 0$ is equivalent to $c_s^2 > 0$, which makes the CSC seem inevitable, at least in the absence of conserved charges and in the presence of some kind of holographic dual. The argument of [24] can probably be extended to cover the general case by considering the dispersion relations for all the hydrodynamical modes, including those arising from the dual conserved currents. However, the CSC remains a conjecture, and there appears to be room for violations: see for example [25–27].

The CSC relates only to the existence of a linearized instability around a static or stationary horizon. Considerable work has been devoted to the question of what the endpoint of the evolution of the Gregory-Laflamme instability might be: see for example [28–33]. When there are thermodynamically stable horizons both with larger and smaller entropy, it seems to us likely that the endpoint of the evolution is a mixed phase with uniformly small curvatures outside the horizon, which remains unbroken. A mixed phase is a configuration where high entropy and low entropy regions with the same temperature are separated by domain walls. Typical solutions may not be static, but may instead evolve slowly according toward larger domains according to an effective theory with domain walls whose width is eventually negligible compared with the size of the domains. Mixed phases were previously suggested in connection with the Gregory-Laflamme instability in [34].

Finally, it is possible to arrange second-order behavior by tuning the potential $V(\phi)$ so that c_s^2 goes to 0 at some value of ϕ_H but never becomes negative: see Fig. 7. There is a corresponding critical temperature, and near it the equation of state typically takes the form

$$s \approx s_0 + s_{1/3} t^{1/3} \quad \text{where } t = \frac{T - T_c}{T_c}. \qquad (65)$$

The specific heat diverges as $C \sim t^{-2/3}$, and consequently the speed of sound behaves as $|t|^{-1/3}$.

VI. DISCUSSION

Since the inception of the anti-de Sitter/conformal field theory correspondence [35–37], it has been hoped that it would help solve QCD. This hope was articulated most clearly in the early literature in [38]. Subsequently, a large and somewhat heterogeneous literature has grown up around the idea of "AdS/QCD." Points of entry into this literature include [39–42].

The first thermodynamic question one might ask of a putative dual to QCD is whether the equation of state is right. We have shown that the equation of state can be built into the construction by choosing an appropriate potential $V(\phi)$ for a scalar field that describes the breaking of conformal invariance. Indeed, within certain limitations, any equation of state $s = s(T)$ can be translated into a choice of $V(\phi)$, and vice versa. The limitations include that we use the supergravity approximation. This immediately points to a weakness of our approach: the shear viscosity will always satisfy $\eta/s = 1/4\pi$, regardless of temperature [43–46]. Low shear viscosity is in conflict with expectations for the low-temperature phase of QCD, where the mean free path becomes large. Another reason to be suspicious of any attempt to describe the low-temperature phase using a black hole horizon is that at large N, entropy of a horizon scales as N^2, whereas the number of degrees of freedom in the confined phase of an $SU(N)$ gauge theory scales as N^0. A black hole description may be approximately valid above T_c, and its validity may fail only gradually as one passes through the crossover. But sufficiently far below the transition, the paradigm of weakly interacting hadrons should take over, and that is not part of our construction. One might imagine improvements on our construction, where, for example, higher curvature corrections significantly increase η/s, especially around or below the transition temperature. Eventually—perhaps when curvatures near the horizon become sufficiently large compared with the string scale—there could be a crossover to a gas of strings in a curved spacetime.

Our methods for constructing black holes are more general than the particular problem of mimicking the equation of state of QCD. Smooth crossovers, second-order transitions, first-order transitions, and perhaps even mixed phases may all be accommodated within the framework we have proposed. Our nonlinear master equation approach is special to the case of a single scalar, and it takes advantage of a weak form of integrability of the underlying equations. However, it is straightforward in principle to work with multiple scalars as well as with gauge fields: in this connection see for example [47,48]. It seems likely that black holes in suitably designed theories exhibit a remarkable diversity of phase transitions.

ACKNOWLEDGMENTS

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